Introducing Pseudo Primes For The Analysis In Prime Number Theory

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Abstract

This article introduces the concept of pseudo primes and shows how it could be used as an useful tool to examine the prime number properties such as distribution of prime numbers and twin primes.

1 Introduction

Prime numbers demonstrates what essentially looks like random patterns when one attempts to examine their various properties. In this paper the concept of Pseudo Primes will be introduced which could be an extremely helpful tool in the analysis of prime numbers for visualizing and formulate their properties.

We'll denote pure prime number with the symbol p and pseudo prime number with the symbol ρ.

Let's define pseudo prime series of k^{th} degree, ρ_i^k, as the i^{th} number which is not divisible any of the primes in the $\{p_0, p_1, \ldots, p_k\}$.

Thus pseudo prime of degree 0 is simply a series of numbers which are not divisible by 2.

Pseudo prime of degree 1 is simply a series of numbers which are not divisible by either 2 or 3, i.e.,

$$\rho_i^1 = \{5, 7, 11, 13, 17, 19, \ldots\}$$

Similarly pseudo prime series of degree 2 would be series of all numbers not divisible by either 2,3 or 5, i.e.,

$$\rho_i^2 = \{7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59, 61, 67, 71, 73, \ldots\}$$

It could be seen that the series ρ_i^k demonstrates very regular patterns in properties such as pseudo prime density and twin primes. Also this series approches the series of prime numbers as $k \to \infty$.

2 Series Equations For Pseudo Primes

Notice that ρ_i^1 in (1.1) can be represented using two simple linear equations:

$$\rho_i^1 = \begin{cases} 5 + 6n \\ 7 + 6n \end{cases}$$

Similarly equations for the pseudo primes of degree 2 are given by:

$$\rho_i^2 = \begin{cases} 7 + 30n \\ 11 + 30n \\ 13 + 30n \\ \vdots \\ 31 + 30n \end{cases}$$

Generalizing,

$$\rho_i^k = \begin{cases} \rho_0^k + \Phi_k n \\ \rho_1^k + \Phi_k n \\ \vdots \\ \rho_{\Theta_k}^k + \Phi_k n \end{cases} \tag{2.1}$$

Where,

$$\Theta_k = \prod_{j=0}^{k} p_j - 1 \tag{2.2}$$

And,

$$\Phi_k = \prod_{j=0}^{k} p_j \tag{2.3}$$

As the degree $k \to \infty$, above set of equations represent series of pure prime numbers.

3 Exact Expression For Pseudo Prime Density

Unlike pure primes, the density of the pseudo primes can be expressed using exact algebric formula.

Let's denote density of pseudo primes of degree k by Υ_k.

Of all integers, $\frac{1}{2}$ of them are divisible by 2. Hence, Υ_0 is simply $\frac{1}{2}$.

Similarly we can find the value of Υ_1 by calculating how many numbers are divisible 2 and 3 in any given set consequitive integers. This can be simply calculated as follows,

$$\Upsilon_1 = 1 - (\frac{1}{2} + \frac{1}{3} - \frac{1}{6}) = \frac{2}{6}$$

Above equation tells us that in any given set of N consequtive integers we'll always find $\frac{2}{6}^{\text{th}}$ of them that can neither be divisible by 2 or 3. Infect, this statement is true not only for set of N consequitive integers but *also* for any given set of N pure random numbers too. If N is not multiple of 6 then $Upsilon_1 N$ would be a real number. In this case, the integer part of it gives the *minimum* number of members in that set that aren't divisible by 2 or 3 and the decimal part gives the *probability* of an additional member that is not divisible by 2 or 3 in that set compared to all of such set of size N that might exist. For example, consider the randomly choosen set $\{40, 41, 42, 43, 44, 45, 46, 47, 48\}$ which is of size $N = 4$. The number of members that aren't divisible by 2 or 3 is given by,

$$\Upsilon_1 . N = \frac{2}{6} . 4 \approx 1.3333$$

which means that in this set we are gurenteed have atleast 1 number that isn't divisible by 2 or 3 *and* that there is about 33.33% chance that this set contains one more number not divisible by 2 and 3.

Similarly,

$$\Upsilon_2 = 1 - (\frac{1}{2} + \frac{1}{3} - \frac{1}{6} + \frac{1}{5} - \frac{1}{10} - \frac{1}{15} + \frac{1}{30}) = \frac{8}{30}$$

The generalized equation for Υ_k is given by,

$$\begin{aligned}\Upsilon_k &= \frac{(p_0-1)\,.\,(p_1-1)\ldots(p_k-1)}{p_0.p_1\ldots p_k} \\ &= \frac{\prod\limits_{j=0}^{k} p_j - 1}{\prod\limits_{j=0}^{k} p_j} \\ &= \prod_{j=0}^{k} 1 - \frac{1}{p_j} \qquad (3.1)\end{aligned}$$

The equation (3.1) is the generalized equation of pseudo prime density.

4 Twin Pseudo Primes

Consider the pseudo prime series of degree 0,

$$\rho_i^0 = \{3, 5, 7, 9, 11, \ldots\} \qquad (4.1)$$

We notice that we have abundunt number of twin pseudo primes (TPP) in this series; all adjucent pairs in ρ_i^0 are infect twin pseudo primes. As we go higher up in the degree by eliminating more divisible numbers from this series, we'll end with lesser and lesser TPPs.

Let's number each of these pairs as follows:

$$\begin{aligned} 3:&\quad 3,5 \\ 4:&\quad 5,7 \\ 5:&\quad 7,9 \\ &\vdots \end{aligned} \qquad (4.2)$$

Now we have each of the TPP pairs of degree 0 in (4.1) numbered with an unique integer: $3, 4, 5, \ldots$. To generate the series of degree 1, we simply eliminate all the numbers divisible by 3 in (4.1). In doing so, we will be eliminating pairs in (4.2) numbered $3 + 3n$ and $5 + 3n$. Hence we'll be eliminating $2.\frac{1}{3}$ of all pairs in (4.2). Thus for the degree 1, density of TPP is $1 - 2.\frac{1}{3} = \frac{1}{3}$ which simply means that if we take N consecutive odd number pairs, we'll find that $\frac{1}{3}^{\text{rd}}$ of these pairs have neither of the members divisible by 3. Let's denote the TPP density for degree k by, τ_k. Hence,

$$\tau_1 = 1 - 2.\frac{1}{3}$$

Following the same procedure for the series of degree 2, we eliminate all pairs in (4.2) which has a member divisible by 5. The TPP density τ_2 could be obtained as follows,

$$\begin{aligned}
\tau_2 &= 1 - 2.\frac{1}{3} - 2.\frac{1}{5} + \frac{2.2}{15} \\
\text{or,} & \\
&= 1 - \frac{2^1}{3} - \frac{2^1}{5} + \frac{2^2}{15}
\end{aligned}$$

Extending the pair elimination process for the degree k, we obtain the generalized expression,

$$\begin{aligned}
\tau_k &= 1 - \frac{2^1}{3} - \frac{2^1}{5} + \frac{2^2}{3.5} - \frac{2^1}{7} + \frac{2^2}{3.7} + \frac{2^2}{5.7} - \frac{2^3}{3.5.7} - \ldots \\
\text{or,} & \\
&= \frac{(p_1 - 2)\,.\,(p_2 - 2)\ldots(p_k - 2)}{p_1 \ldots p_k} \\
&= \frac{\prod_{j=1}^{k} p_j - 2}{\prod_{j=1}^{k} p_j} \\
&= \prod_{j=1}^{k} 1 - \frac{2}{p_j} \qquad (4.3)
\end{aligned}$$

This is the general expression for the density of twin pseudo primes. It tells us how many twin pairs of pseudo primes we may find in given set of N consecutive numbers which are not divisible by any of the primes $p_0, p_1, \ldots, p_k$.