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Optimal error correction of the absolute value equation using a genetic algorithm

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ABSTRACT

In this paper we have studied the optimum correction of the absolute value equation through making minimal changes in the coefficient matrix and the right-hand side using the l_2 norm. Solving this problem is equal to solving a nonconvex and fractional quadratic problem. To solve this problem, we use a genetic algorithm. Our computational results show that this method is efficient, with high accuracy.

1. Introduction

Many mathematical programming problems can be reduced to the NP-hard linear complementarity problem [1–7]. In [8], Mangasarian showed that this problem is equivalent to an absolute value equation (AVE) as follows:

$$Ax - |x| = b, \tag{1}$$

where $A \in R^{n\times n}$ and $b \in R^n$ are given, and $|\cdot|$ denotes absolute value.

However, in many models obtained, we often encounter problems which present themselves as systems of infeasible AVEs. We could argue numerous reasons for the infeasibility of an AVE, including errors in the data, the complexity of the model, optimistic objectives, and lack of communication between different decision makers. As remodeling of a problem and finding the errors of a system might take a remarkable amount of time and expense, and also we might eventually get an infeasible system again, we do not do so.

We therefore focus on optimal correction of the given system. In this work, to correct an infeasible AVE, we will make changes in both the coefficient matrix and the right-hand side vector and obtain a fractional objective function of two quadratic functions, but is not necessarily convex itself, as follows.

$$\min_{x\in R^n} \frac{\|Ax - |x| - b\|^2}{1 + \|x\|^2}. \tag{2}$$

A genetic algorithm applied to solve (2), and results show that this method is efficient, with high accuracy.

This paper is organized as follows. Correction of an infeasible AVE is reviewed in Section 2. In Section 3, we describe the genetic algorithm. In Section 4, some numerical examples are tested. Section 5 concludes the paper.

In this paper, all vectors will be column vectors, and we denote the n-dimensional real space by R^n. The notation $A \in R^{m\times n}$ will signify a real $m \times n$ matrix. By $A^\top$, $\|\cdot\|$, and $\|\cdot\|_\infty$ we mean the transpose of matrix A, the Euclidean norm, and the infinity norm, respectively.

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2. Correction of the absolute value equation

First, we consider the following propositions. (Nonexistence of solution.)

Proposition 2.1. *Let $0 \neq b \geq 0$ and $\|A\| < 1$. Then the AVE (1) has no solution.*

Proof. See [9]. $\square$

Proposition 2.2. *The AVE (1) has no solution for any A, b if $r \geq A^T r \geq -r,\ b^T r > 0$ has solution $r \in R^n$.*

Proof. See [9]. $\square$

In these cases, to correct the AVE, we can make the changes in both the coefficient matrix and the right-hand side vector. Therefore, we must consider the following minimization problem:

$$\begin{aligned} &\min_x \min_{E,r} (\|E\|^2 + \|r\|^2) \\ &\text{s.t. } (A+E)x - |x| = b + r, \end{aligned} \tag{3}$$

where $E \in R^{n\times n}$ is a perturbation matrix and $r \in R^n$ is a perturbation vector.

In order to simplify problem (3), we consider the following inner minimization problem:

$$\begin{aligned} &\min_{E,r} (\|E\|^2 + \|r\|^2) \\ &\text{s.t. } (A+E)x - |x| = b + r, \end{aligned} \tag{4}$$

which is a constrained convex problem.

We know that the function $g(x) = (A+E)x - |x|$ is a piecewise linear vector function and that a generalized Jacobian $\partial g(x)$ of $g(x)$ is given by $\partial g(x) = (A+E) - D(x)$, where $D(x) = \partial|x| = \mathrm{diag}(\mathrm{sign}(x))$ [10].

Theorem 2.3. *Suppose that (E^*, r^*) denotes the optimal pair to problem (3). Then*

$$r^* = \frac{Ax^* - |x^*| - b}{1 + \|x^*\|^2}, \qquad E^* = -\frac{Ax^* - |x^*| - b}{1 + \|x^*\|^2} x^{*T},$$

where x^* is an optimal solution of $\min_x \frac{\|Ax-|x|-b\|}{1+\|x\|^2}^2$.

Proof. The Lagrangian of problem (4) is given by

$$L(E, r, \lambda) = \|E\|^2 + \|r\|^2 - \lambda^T((A+E)x - |x| - (b+r)).$$

Since problem (4) is convex, then the Karush–Kuhn–Tucker (KKT) necessary conditions are also sufficient, and any (E, r) satisfying the KKT conditions is a global minimum. The KKT conditions of (4) give (see [4])

$$\frac{\partial L}{\partial E} = 2E - \lambda x^T = 0, \tag{5}$$

$$\frac{\partial L}{\partial r} = 2r + \lambda = 0, \tag{6}$$

$$\frac{\partial L}{\partial \lambda} = (A+E)x - |x| - (b+r) = 0. \tag{7}$$

From Eqs. (5)–(7), we have that $2E = \lambda x^T$ and $\lambda = -2r$. Therefore, we obtain $E = -rx^T$ and $r = Ax + Ex - |x| - b$. By combining these expressions, we find that $r^* + r^*\|x\|^2 = Ax - |x| - b$, and we conclude that

$$r = \frac{Ax - |x| - b}{1 + \|x\|^2}, \qquad E = -\frac{Ax - |x| - b}{1 + \|x\|^2} x^T. \tag{8}$$

From (8), we obtain $\|E\|^2 + \|r\|^2 = \frac{\|Ax-|x|-b\|}{1+\|x\|^2}$, and the objective function value in (4) at optimal solution is equal to

$$\|E^*\|^2 + \|r^*\|^2 = \min_{x\in R^n} \frac{\|Ax - |x| - b\|^2}{1 + \|x\|^2}. \tag{9}$$

This completes the proof. $\square$

3. Genetic algorithm

Genetic algorithms (GAs) are adaptive heuristic search algorithms premised on the evolutionary ideas of natural selection and genetics. In comparison to the conventional searching algorithms, GAs have the following characteristics: (a) GAs work directly with the discrete points coded by finite length strings (chromosomes), not the real parameters themselves; (b) GAs consider a group of points (called a population size) in the search space in every iteration, not a single point; (c) GAs use fitness function information instead of derivatives or other auxiliary knowledge; and (d) GAs use probabilistic transition rules instead of deterministic rules. Generally, a simple GA consists of the three basic genetic operators: (a) reproduction, (b) crossover, and (c) mutation. They are described as follows.

3.1. Reproduction

Reproduction is a process to decide how many copies of individual strings should be produced in the mating pool according to their fitness value. The reproduction operation allows strings with higher fitness value to have larger number of copies, and strings with lower fitness values have a relatively smaller number of copies or even none at all. This is an artificial version of natural selection (strings with higher fitness values will have more chances to survive).

3.2. Crossover

Crossover is a recombined operator for two high-fitness strings (parents) to produce two offspring by matching their desirable qualities through a random process. In this paper, the uniform crossover method is adopted. The procedure is to select a pair of strings from the mating pool at random; then, a mark is selected at random. Finally, two new strings are generated by swapping all characters corresponding to the position of the mark where the bit is 1. Although the crossover is done by random selection, this is not the same as a random search through the search space. Since it is based on the reproduction process, it is an effective means of exchanging information and combining portions of high-fitness solutions.

3.3. Mutation

Mutation is a process to provide an occasional random alteration of the value at a particular string position. In the case of a binary string, this simply means changing the state of a bit from 1 to 0 and vice versa. In this paper, we provide a uniform mutation method. This method first produces a mask and selects a string randomly, and then it complements the selected string value corresponding to the position of the mask where the bit value is 1. Mutation is needed because some digits at particular positions in all strings may be eliminated during the reproduction and crossover operations. So mutation plays the role of a safeguard in GAs. It can help GAs to avoid the possibility of mistaking a local optimum for a global optimum. For more details, see [11,12].

4. Numerical testing

In this section, we present numerical results for correction of AVE on various randomly generated problems using a GA. The algorithm has been tested using MATLAB 7.9.0 on a Core 2 Duo 2.53 GHz processor with main memory 4 GB. Test problems are generated using the following MATLAB code:

```
%Sgen: Generate random infeasible system A*x-|x|=b;
n=input('Enter n: ')
A=rand(n,n);
A=100*(A-(1/2)*A);
A=A/(norm(A)+10);
b=10*rand(n,1);
```

The numerical experiments are shown in Table 1. In this table, the first column indicates the size of matrix A, $\|x^*\|_\infty$ is the infinity norm of the optimal solution of (9), *fval* is the optimal objective value of this problem, R indicates $\|(A+E^*)x^* - |x^*| - b - r^*\|_\infty$, and the final column indicates the time taken.

This results show that the GA for correcting the AVE is efficient, with high accuracy.

Table 1 illustrates the effectiveness and performance behavior of our proposed method.

5. Conclusion

In this paper, we have studied the optimal correction of an inconsistent absolute value equation, making minimal changes in both the coefficient matrix and the right-hand side vector using the Euclidean norm. We obtain a nonconvex fractional minimization problem. To solve this problem, we used a genetic algorithm. We have presented some numerical results using this process on various randomly generated systems (see Table 1) to illustrate the performance of the proposed algorithm.

Table 1
GA for optimal error correction of the AVE.

n	$\|x^*\|_\infty$	*fval*	R	Time (s)
10	45.1275	0.3181	3.6603e−15	3
50	67.2755	0.4702	1.0257e−14	7
100	85.5191	0.7304	1.2417e−14	10
150	101.1172	0.8357	1.5010e−14	16
200	118.9800	0.8594	2.2144e−14	33
250	120.5404	1.0817	2.0558e−14	45
300	120.8621	1.3481	2.1905e−14	50
350	125.6230	1.5874	3.6842e−14	59
400	132.3176	1.6778	2.8536e−14	66
450	137.1306	1.9786	2.6362e−14	75
500	145.7350	2.0819	4.7960e−14	84
550	133.7429	2.3735	3.9602e−14	95
600	128.7068	2.6048	2.7640e−14	108
650	142.6812	2.6683	4.0123e−14	115
700	145.2608	2.7396	3.7328e−14	124
750	140.7905	3.0398	3.6896e−14	135
800	151.2449	3.1857	3.9702e−14	140
850	145.6701	3.2358	3.6814e−14	153
900	145.6056	3.3976	4.5390e−14	162
950	139.7615	3.4513	3.4835e−14	178
1000	146.3419	3.6243	4.4540e−14	202

References

[1] P.M. Pardalos, S. Ketabchi, H. Moosaei, Minimum norm solution to the positive semidefinite linear complementarity problem, Optimization (2011) (in press).
[2] P.M. Pardalos, J.B. Rosen, Global optimization approach to the linear complementarity problem, SIAM Journal on Scientific and Statistical Computing 9 (1988) 341–353.
[3] P.M. Pardalos, Y. Ye, Class of linear complementarity problems solvable in polynomial time, Linear Algebra and its Applications 152 (1991) 3–17.
[4] M.S. Bazaraa, H.D. Sherali, C.M. Shetty, Nonlinear Programming Theory and Algorithms, John Wiely & Sons, 1999.
[5] R.W. Cottle, G. Dantzig, Complementary pivot theory of mathematical programming, Linear Algebra and its Applications 1 (1968) 103–125.
[6] R.W. Cottle, J.S. Pang, R.E. Stone, The Linear Complementarity Problem, Academic Press, New York, 1992.
[7] L. Vandenberghe, B. Demoor, J. Vandewalle, The generalized linear complementarity problem applied to the complete analysis of resistive piecewise-linear circuits, IEEE Transactions on Circuits and Systems 36 (1989) 1382–1391.
[8] O.L. Mangasarian, Absolute value programming, Computational Optimization and Applications 36 (2007) 43–53.
[9] O.L. Mangasarian, R.R. Meyer, Absolute value equations, Linear Algebra and its Applications 419 (2006) 359–367.
[10] O.L. Mangasarian, A generalized Newton method for absolute value equations, Optimization Letters 3 (2009) 101–108.
[11] M. Mitchell, An Introduction to Genetic Algorithms, MIT Press, Cambridge, 1997.
[12] Y. Rahmat-Samii, E. Michielssen, Electromagnetic Optimisation by Genetic Algorithms, John Wiely & Sons, 1999.