Computers and Mathematics with Applications 64 (2012) 1882–1885

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

An efficient method for optimal correcting of absolute value equations by minimal changes in the right hand side[☆]

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ARTICLE INFO

Article history:
Received 11 November 2011
Received in revised form 9 March 2012
Accepted 10 March 2012

Keywords:
Absolute value equations
Generalized Hessian matrix
Generalized Newton method
Infeasible systems

ABSTRACT

Our goal in this work is to give an optimum correction of the infeasible absolute value equations (AVE). In order to make the mentioned system feasible, we apply the minimal correction using the l_2 norm by changing just the right hand vector. We will show that this problem can be formulated as an unconstrained optimization problem with a quadratic objective function. We propose an extension of Newton's method for solving unconstrained objective optimization. Some examples are provided to illustrate the efficiency and validity of our proposed method.

1. Introduction

One of the frequently encountered issues in applied science is how to deal with infeasible systems [1,2]. We could argue numerous reasons for the infeasibility of a system, including errors in data, errors in modeling, and many other reasons. Because the remodeling of a problem, finding its errors, and generally removing its obstacles to feasibility might require a considerable amount of time and expense, and might result in yet another infeasible system, we are reluctant to do so. We therefore focus on optimal correction of the given system. In fact, we would like to reach the feasible systems with the least changes in data.

In this work, we study non-necessary feasible absolute value equations of the type [3–5]:

$$Ax - |x| = b, \tag{1}$$

where $A \in R^{n\times n}$, $b \in R^n$ and $|\cdot|$ denotes absolute value. In order to make the above-mentioned system feasible, we apply the change in the right-hand side vector b. In fact, we would like to study the flow problem under certain special assumptions on A (Section 2):

$$\min_{x\in R^n} \|Ax - |x| - b\|^2. \tag{2}$$

In Section 3, we present a generalized Newton method. Numerical results are presented in Section 4.

We now describe our notation. By A^T we mean the transpose of the matrix A, and $\partial g(x_0)$ denotes the Jacobian of g at x_0. For two vectors x and y in the n-dimensional real space, $x^T y$ will denote the scalar product. For $x \in R^n$, $\|x\|$ denotes 2-norm and $|x|$ will denote the vector in R^n of absolute values of components of x.

☆ The paper has been evaluated according to old Aims and Scope of the journal.

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doi:10.1016/j.camwa.2012.03.015

2. Correction of absolute value equations

In this section, we discuss the inconsistent absolute value equations $Ax - |x| = b$, where $A \in R^{n\times n}$ and $b \in R^n$.
First, we consider the following minimization problem:

$$\min_x \min_r \|r\|^2$$
$$\text{s.t. } Ax - |x| = b + r, \tag{3}$$

where $r \in R^m$ is a perturbation vector. The problem (3) is a nonconvex problem and to solve it, we consider the following inner minimization problem

$$\min_r \|r\|^2$$
$$\text{s.t. } Ax - |x| = b + r, \tag{4}$$

which is a constrained convex problem.

We know that the function $g(x) = Ax - |x|$ is piece-wise linear vector function and a generalized Jacobian $\partial g(x)$ of $g(x)$ is given by $\partial g(x) = A - D(x)$, where $D(x) = \partial|x| = \text{diag}(\text{sign}(x))$ [6].

Theorem 2.1. *Suppose that r^* denotes the optimal solution of problem (3). Then*

$$r^* = Ax^* - |x^*| - b,$$

where x^* is an optimal solution of $\min_x \|Ax - |x| - b\|^2$.

Proof. The Lagrangian of the problem (4) is given by

$$L(r, \lambda) = \|r\|^2 - \lambda^T(Ax - |x| - (b + r)).$$

Since the problem (4) is convex, then the KKT necessary conditions are also sufficient and any r satisfying the KKT conditions is a global minimum. The KKT conditions of (4) give (see [7,8])

$$\frac{\partial L}{\partial r} = 2r + \lambda = 0, \tag{5}$$
$$\frac{\partial L}{\partial \lambda} = Ax - |x| - (b + r) = 0. \tag{6}$$

From the Eqs. (5) and (6), we obtain $r = Ax - |x| - b$. Then the value of problem at optimal solution is equal to

$$\|r^*\|^2 = \|Ax^* - |x^*| - b\|^2, \tag{7}$$

where x^* is obtained by solving the following problem:

$$\min_{x\in R^n} \|Ax - |x| - b\|^2.$$

This completes our proof. □

We note that under certain assumptions on A, the quadratic program (2) is a convex problem (see [4]).

Theorem 2.2. *Suppose that the singular values of A are greater than or equal to* 1*, then the quadratic program* (2) *is a convex problem.*

Proof. Since the singular values of A are greater than or equal to 1, we have $\min\ \text{eig}(A^TA) \geq 1$ which is equivalent to

$$\min\ \text{eig}(A^TA - (\text{diag}(\text{sign}(x)))^2) \geq 0,$$

where min eig denotes the least eigenvalue. Therefore, the generalized Hessian of the objective function of (2) is positive semidefinite. □

Sometimes solving the problem (2) leads to solutions with very large norms which are practically impossible to use. In such cases, we usually use Tikhonov regularizing of problems and instead of solving problem (2), we consider the following problem:

$$\min_{x\in R^n} \|Ax - |x| - b\|^2 + \rho\|x\|^2,$$

where ρ is a positive constant value; it is called the regularizing parameter [9].

3. Generalized Newton method

In this section for solving unconstrained optimization problems (2), we used generalized Newton method. Let us suppose that

$$Q = A - \operatorname{diag}(\operatorname{sign}(x)),$$

Then the problem of solving (2) is equivalent to the problem of minimizing the quadratic function defined by

$$u(x) = \min_{x \in R^n} \|Qx - b\|^2. \tag{8}$$

The generalized gradient of $u(x)$ is $\partial u(x) = 2Q^T(Qx - b)$ and the generalized Hessian matrix can be defined as follows [6]:

$$\partial^2 u(x) = 2Q^T Q.$$

Since the generalized Hessian matrix can be singular, the following modified Newton direction is used [7].

$$-(\partial^2 u(x) + \delta I_n)^{-1} \partial u(x),$$

where δ is the small positive number (in our calculations, $\delta = 10^{-4}$) and I_n is the identity matrix of order n. In this case, the modified Newton method has the form

$$t_{n+1} = t_n - (\partial^2 u(t_n) + \delta I_n)^{-1} \partial u(t_n).$$

The starting vector is $t_0 = 0$. Also, our stopping criterion for this method is as follows: (in our calculations, $tol = 10^{-12}$) $\|t_{n+1} - t_n\| \le tol$.

In the following algorithm we apply the generalized Newton method with a line-search based on the Armijo rule [10].

Algorithm : Generalized Newton Method with the Armijo Rule

Choose any $p_0 \in R^n$ and $\epsilon > 0$, $i = 0$;
while $\|\partial u(p_i)\|_\infty \ge \epsilon$
Choose $\alpha_i = max\{s, s\delta, s\delta^2, \ldots\}$ *such that*
$u(p_i) - u(p_i + \alpha_i d_i) \ge -u_i \mu \partial u(p_i) d_i$,
where $d_i = -\partial^2 u(p_i)^{-1} \partial u(p_i)$, $s > 0$ is a constant, $\delta \in (0, 1)$ and $\mu \in (0, 1)$.
$p_{i+1} = p_i + \alpha_i d_i$
$i = i + 1$;
end

The proof of the finite-step global convergence of this method can be found in [11].

4. Numerical testing

In this section, we present numerical results for the correction of absolute value equations on various randomly generated problems. For solving unconstrained optimization problems (2), we used a generalized Newton method. The algorithm has been tested using MATLAB 7.11.0 on a Core 2 Duo 2.53 GHz with a main memory of 4 GB. Test random infeasible absolute value equations are generated using the result from [4], where it was shown that

Proposition 4.1 (*Nonexistence of Solution for AVE*). *If* $\{x | (A + I)x - b \ge 0\} = \emptyset$ *or* $\{x | (A - I)x - b \ge 0\} = \emptyset$, *then AVE has no solution.*

```
%Sgen:Generate random infeasible system (AVE) Ax − |x| = b
based on Proposition 4.1;
n = input('Enter n :')
pl = inline('(abs(x) + x)/2');
x = 10 * (rand(n, 1) − rand(n, 1));
u = 1 * pl(x); A = null(u');
A = A * A'; A = A + eye(n);
b = 5 * rand(n, 1);
```

Computational results for the test problems are given in the following table.

In this table the first column indicates the size of matrix A, $\|x^*\|_\infty$ is the norm infinity of optimal point and $\|r^*\|_\infty = \|Ax^* - |x^*| - b\|_\infty$ is the norm infinity of minimal changes on the right-hand side. $\|\partial u(x^*)\|_\infty$ indicates the norm infinity of generalized gradient of u at the point x^* and the final column indicates time.

The numerical results reported in Table 1 show good results for this approach on small, medium and large problems.

5. Conclusion

In this article, the corrections of infeasible absolute value equations were studied by applying minimal changes to the right-hand vector, using 2-norm. Numerical results coming from the examples show that the suggested algorithm is correct and efficient on small, medium and large problems.

Table 1
Numerical results on randomly generated problems.

n	$\|x^*\|_\infty$	$\|r^*\|_\infty$	$\|\partial u(x^*)\|_\infty$	Time (s)
100	4.9064	3.3630	6.0041e−013	0.4542
200	4.9887	4.2557	5.6413e−015	0.1011
300	4.9703	4.4866	6.2172e−013	0.0839
400	4.9738	4.4214	6.2172e−013	0.1592
500	4.9936	4.3660	1.8801e−014	0.2886
600	4.9978	4.9712	1.7500e−014	1.3263
700	4.9885	4.8058	1.9027e−014	0.7263
800	4.9970	4.2975	6.2528e−013	0.7339
900	4.9966	4.7412	6.2528e−013	0.9544
1000	4.9998	4.7894	2.2574e−014	2.3388
1500	4.9998	4.5048	3.7985e−014	4.1054
2000	4.9950	4.6921	6.3392e−013	7.3602
2500	4.9956	4.7665	4.2397e−014	18.4650
3000	4.9999	4.6780	4.3271e−014	32.0610
3500	4.9997	4.9890	3.5145e−014	47.8707
4000	4.9987	4.9245	6.2616e−013	49.9481
4500	4.9989	4.8448	5.7195e−014	85.6039
5000	4.9990	4.9148	6.4442e−013	95.7521

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