Optimization Letters (2007) 1:3–8
DOI 10.1007/s11590-006-0005-6

ORIGINAL PAPER

Absolute value equation solution via concave minimization

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Accepted: 12 April 2006 / Published online: 12 August 2006

Abstract The NP-hard absolute value equation (AVE) $Ax - |x| = b$ where $A \in R^{n\times n}$ and $b \in R^n$ is solved by a succession of linear programs. The linear programs arise from a reformulation of the AVE as the minimization of a piecewise-linear concave function on a polyhedral set and solving the latter by successive linearization. A simple MATLAB implementation of the successive linearization algorithm solved 100 consecutively generated $1,000$-dimensional random instances of the AVE with only five violated equations out of a total of $100,000$ equations.

Keywords Absolute value equation · Concave minimization · Successive linear programming

1 Introduction

We consider the absolute value equation (AVE):

$$Ax - |x| = b, \tag{1}$$

where $A \in R^{n\times n}, b \in R^n$ and $|\cdot|$ denotes absolute value. A slightly more general form of the AVE, $Ax + B|x| = b$ was introduced in [11] and investigated in a more general context in [6]. The AVE (1) was investigated in detail theoretically

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in [7] and a bilinear program was prescribed there for the special case when the singular values of A are not less than one. No computational results were given in either [7] or [6] where a parametric successive linearization algorithm (SLA) was proposed in the latter. As was shown in [7], the general NP-hard linear complementarity problem (LCP) [1–3] which subsumes many mathematical programming problems can be formulated as an AVE (1). This implies that (1) is NP-hard in its general form.

The basic contribution of the present work is a finite computational algorithm based on a new reformulation of the AVE (1) as the minimization of a parameter-free piecewise linear concave minimization problem on a polyhedral that is solved by a finite succession of linear programs. We prove that the algorithm terminates at a point satisfying a necessary optimality condition. This turns out to be an effective way for solving the AVE as indicated by computational results on one hundred consecutively generated random 1,000-dimensional AVEs, 95 of which were solved exactly to an accuracy of 10^{-6}, while each of the remaining five instances had only one equation out of 1,000 violated.

In Sect. 2 of the paper we formulate the AVE as the minimization of a piecewise-linear concave function on a polyhedral set and establish the existence of a vertex solution. In Sect. 3 we give a SLA for the solution of the AVE (1) and establish its finite termination at a vertex satisfying a necessary optimality condition. Section 4 gives our computational results. Section 5 concludes the paper.

A word about our notation and background material. The scalar product of two vectors x and y in the n-dimensional real space will be denoted by $x'y$ in conformity with MATLAB [8] notation. For $x \in R^n$, the norm $\|x\|$ will denote the two-norm $(x'x)^{1/2}$, while $|x|$ will denote the vector in R^n of absolute values of components of x and $\text{sign}(x)$ will denote a vector with components equal to $1, 0$ or -1 depending on whether the corresponding component of x is positive, zero or negative. For an $m \times n$ matrix A, A_i will denote the ith row of A. The identity matrix in a real space of arbitrary dimension will be denoted by I, while a column vector of ones of arbitrary dimension will be denoted by e. For a concave function $f : R^n \to R$ the supergradient $\partial f(x)$ of f at x is a vector in R^n satisfying

$$f(y) - f(x) \leq \partial f(x)(y - x) \tag{2}$$

for any $y \in R^n$. The set $D(f(x))$ of supergradients of f at the point x is nonempty, convex, compact and reduces to the ordinary gradient $\nabla f(x)$, when f is differentiable at x [9,10].

2 The AVE as a piecewise-linear concave minimization problem

We show now the equivalence of the AVE to the following concave minimization problem.

Proposition 1 (AVE as concave minimization) *The AVE (1) is equivalent to the following minimization problem:*

$$\begin{aligned} \min_{x\in R^n}\ & e'(Ax-b)-e'|x| \\ \text{s.t.}\ & -(A+I)x \le -b, \\ & (-A+I)x \le -b, \end{aligned} \tag{3}$$

having a zero minimum.

Proof The linear constraints of the concave minimization problem (3) are equivalent to:

$$-Ax+b \ \le x \ \le Ax-b, \tag{4}$$

which in turn are equivalent to:

$$|x| \ \le Ax-b. \tag{5}$$

Hence (3) is feasible if the AVE (1) is solvable and its concave objective is bounded below by zero. Thus, by [10, Corollary 32.3.3] the minimization problem (3) has a solution. This minimum is zero if and only if the AVE (1) is solvable. □

We now establish that the minimization problem (3) has a vertex solution when it is solvable.

Proposition 2 *The minimization problem* (3) *has a vertex solution if its feasible region is nonempty, which is the case if the AVE* (1) *is solvable.*

Proof By [10, Corollary 32.3.4] the minimization problem (3) has a vertex solution if its nonempty feasible region has no straight lines going to infinity in both directions. We will show that this is indeed the case for the feasible region of (3) in its equivalent form (4). If such a line exists then for some fixed points x and $y \neq 0$ in R^n and for $\lambda \uparrow \infty$ the constraints (4) must be satisfied as:

$$-A(x\pm\lambda y)+b \le x\pm\lambda y \le A(x\pm\lambda y)-b. \tag{6}$$

Dividing by λ and letting $\lambda \uparrow \infty$ we have that:

$$\mp Ay \le \pm y \le \pm Ay, \tag{7}$$

which is equivalent to:

$$\begin{aligned} (A+I)y &= 0, \\ (A-I)y &= 0. \end{aligned} \tag{8}$$

Subtracting the last equation from the previous one gives that $y = 0$, which contradicts $y \neq 0$. Hence there are no lines going to $\pm\infty$ in the feasible region of (3) and the minimization problem (3) must have a vertex solution. □

We describe now our SLA for solving the concave minimization problem (3).

3 The successive linearization algorithm

Alogorithm 1 (SLA Algorithm) *Start with a random $x^0 \in R^n$. Having x^i determine x^{i+1} as a vertex solution of the following linear program:*

$$\begin{aligned} \min_{x \in R^n} \; & e'(Ax-b) - \operatorname{sign}(x^i)'x \\ \text{s.t.} \; & -(A+I)x \leq -b, \\ & (-A+I)x \leq -b. \end{aligned} \tag{9}$$

Stop when $(e'A - sign(x^i)')(x^{i+1} - x^i) = 0$.

Remark 1 Note that minimizing the objective function of (9) is equivalent to minimizing a supporting plane of the convex hypograph of the concave objective function of (3). This is given by the scalar product of a supergradient of the concave objective function of (3) at the point x^i times $(x - x^i)$ added to the function value at x^i as follows:

$$e'(Ax^i - b) - e'|x^i| + (e'A - \operatorname{sign}(x^i)')(x - x^i). \tag{10}$$

The finite termination of Algorithm 1 follows from the following theorem.

Theorem 1 (SLA finite termination theorem) *Let f denote the concave function of the minimization problem* (3) *which is bounded below by zero on the feasible region of* (3) *when the AVE (1) is solvable. The SLA 1 generates a finite sequence of feasible vertices $\{x^1, x^2, \ldots, x^{\bar{i}}\}$ with strictly decreasing objective function values: $f(x^1) > f(x^2) > \cdots > f(x^{\bar{i}})$, such that $x^{\bar{i}}$ satisfies the minimum principle necessary optimality condition:*

$$(e'A - \operatorname{sign}(x^{\bar{i}}))(x - x^{\bar{i}}) \geq 0, \quad \forall x \in \{x \mid -Ax + b \leq x \leq Ax - b\}. \tag{11}$$

Proof Follows from Theorem 3 of [5]. □

We turn now to our computational testing.

4 Computational results

We used MATLAB 7.1 to test Algorithm 1 on 100 consecutively generated solvable random problems with fully dense matrices $A \in R^{1{,}000 \times 1{,}000}$ as follows. We first chose a random A from a uniform distribution on [−10,10], then we chose a random x from a uniform distribution on $[-1,1]$. Finally we computed $b = Ax - |x|$. We broke the 100 problems into ten groups each containing ten problems (instances). Code 1 gives the MATLAB code for generating "imax" problems and solving at most "itmax" (typically 40) iterations of of Algorithm 1 by using the CPLEX 9.0 [4] linear programming solver. The maximum number of iterations used per instance was itmax=40. An AVE component i is considered satisfied provided $|A_i x - |x_i| - b_i| \leq 10^{-6}$. This was used as a alternate stopping criterion when satisfied for $i = 1, \ldots, n$ instead of that of the SLA Algorithm 1 of $|(e'A - \mathrm{sign}(x^i)')(x^{i+1} - x^i)| = 0$ in order to ensure a 10^{-6} accuracy in satisfying the AVE (1).

Code 1 SLA: Successive linearization algorithm code

```
%generate imax instances of solvable AVE Ax-|x|=b & solve by SLA:
%min  e'(Ax-b)-e'|x| s.t. -Ax+b =< x =< Ax-b
%input: n,imax,itmax (itmax=maximum no. of iterations typically 40)
%output: total number of violated equations for all instnances (nnztot),
%max no. of violated equations per instance (nnzx), total LPs & time (toc)
%lpxt(c,B,d) is CPLEX call to solve  min c'x s.t. Bx =< d
I=eye(n);e=ones(n,1);
nnztot=0;nnzx=0;k=0;tic; %k is total number of LPs for all imax instances
for i=1:imax
A=10*(rand(n,n)-rand(n,n));x=rand(n,1)-rand(n,1);b=A*x-abs(x);%generate AVE
  j=1;y=rand(n,1)-rand(n,1);%generate initial point y
  while(j<itmax & norm(A*y-abs(y)-b)>1e-6)%{\ul itmax} or AVE error stop
  c=[A'*e-sign(y)];B=[-(A+I) ;(-A+I) ];d=[-b;-b];[t,z,u]=lpxt(c,B,d);
  y=z; j=j+1;
  end
err=(A*y-abs(y)-b);nnz1=nnz(find(abs(err)>1e-6));
nnztot=nnztot+nnz1;nnzx=max([nnz1 nnzx]);k=k+j-1;
end
[nnztot nnzx k toc]
```

Computational results are summarized in Table 1 and were obtained using a 3.00 GHz Pentium 4 processor running i386_tao10 Linux. We note the following:

1. Out of 100 instances, 95 instances were solved to an accuracy of 10^{-6}.
2. For each of the five unsolved instances, only one equation out of a thousand equations was violated per instance.
3. The average number of LPs per instance varied from 2.6 to 9.1 with the overall average for the 100 problems being 4.81.
4. The overall average time for solving each instance of 1,000 fully dense equations was 248 s.
5. The average time to solve each LP was 52 s.

5 Conclusion

We have proposed a finite successive linear programming algorithm for solving the the NP-hard AVE $Ax - |x| = b$. The effectiveness of the algorithm is demonstrated by its ability to solve 95 out of 100 consecutively generated random problems each consisting of 1,000 equations in 1,000 unknowns. One

Table 1 Results from Algorithm 1 on 100 consecutive random AVEs with n=1,000, shown in groups of 10

Instances	nztot	nnzx	k	toc(seconds)	seconds per LP
1–10	1	1	80	4,236	53
11–20	0	0	26	1,361	52
21–30	1	1	74	3,804	51
31–40	0	0	26	1,351	52
41–50	0	0	26	1,354	52
51–60	0	0	26	1,364	52
61–70	1	1	80	4,207	53
71–80	2	1	91	4,379	48
81–90	0	0	26	1,358	52
91–100	0	0	26	1,359	52

nnztot denotes the total number of violated equations in each group of ten problems. nnzx denotes the maximum number of violated equations per individual problem. k denotes the total number of iterations (LPs) utilized for each group of ten problems. toc denotes the total time for solving each group of ten problems. seconds per LP denotes the time for each LP obtained by dividing the previous column by the column preceding it

fascinating feature of the AVE is its simplicity despite its NP-hardness. Possible future work may consist of investigating other algorithms and improvement of the proposed algorithm here and those in [6,7].

Acknowledgments The research described in this Data Mining Institute Report 06-02, March 2006, was supported by National Science Foundation Grants CCR-0138308 and IIS-0511905. Thanks to Michael Thompson for helpful comments on this paper.

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