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ORIGINAL PAPER

A new concave minimization algorithm for the absolute value equation solution

Moslem Zamani[1,2] · Milan Hladík[3]

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Abstract
In this paper, we study the absolute value equation (AVE) $Ax - b = |x|$. One effective approach to handle AVE is by using concave minimization methods. We propose a new method based on concave minimization methods. We establish its finite convergence under mild conditions. We also study some classes of AVEs which are polynomial time solvable.

Keywords Absolute value equation · Concave minimization algorithms · Linear complementarity problem

1 Introduction

We consider the absolute value equation problem of finding an $x \in \mathbb{R}^n$ such that

$$Ax - b = |x|, \tag{AVE}$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and $|\cdot|$ denotes absolute value. In general, (AVE) is an NP-hard problem [16].

Since a general linear complementarity problem can be formulated as an absolute value equation, several methods, such as Newton-like methods [3,15,31] or concave optimization methods [20,21], have been proposed for solving (AVE).

✉ Moslem Zamani
zamani.moslem@tdt.edu.vn

Milan Hladík
hladik@kam.mff.cuni.cz

1 Parametric MultiObjective Optimization Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam

2 Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam

3 Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Malostranské nám. 25, 11800 Prague, Czech Republic

Concave optimization methods for solving (AVE) are based on the fact that (AVE) is equivalent to the following minimization problem,

$$\min\ e^T(Ax - b - |x|) \text{ s.t. } |x| - Ax \leq -b, \tag{1}$$

or equivalently,

$$\begin{aligned} &\min\ e^T(Ax - b - |x|) \\ &\text{s.t.}(I - A)x \leq -b, \\ &\quad - (A + I)x \leq -b. \end{aligned} \tag{2}$$

Indeed, (AVE) has a solution if and only if the optimal value of problem (2) is zero; see Proposition 1 in [20]. We denote the objective function of (1) by $f(x) = e^T(Ax - b - |x|)$ and the feasible set by $S = \{x : |x| - Ax \leq b\}$. As the objective function of (2) is concave, solving this problem is not easy [28]. In fact, some sufficient optimality conditions have been proposed for concave programs [4,9], but they are not verifiable in polynomial time.

The paper is structured as follows. After reviewing notations, in Sect. 2, we give some sufficient conditions under which some concave methods for solving AVEs may be convergent. We propose a new algorithm for solving AVEs in Sect. 3. Thanks to the results in Sect. 2, we prove its finite convergence under some mild conditions. We report numerical experiments in Sect. 4. Section 5 is devoted to some classes AVEs which are polynomial time solvable.

Notation Vectors are considered to be column vectors and the superscript T represents the transposition. We denote the vector of ones and identity matrix by e and I, respectively. For $d \in \mathbb{R}^n$, diag(d) denotes the diagonal matrix whose diagonal is d. The sign function is defined as follows: $\text{sign}(x) = 1$ if $x > 0$, $\text{sign}(x) = 0$ if $x = 0$, and $\text{sign}(x) = -1$ otherwise; for vectors it is applied entrywise. In addition, d_i denotes i-th component of d. For $A \in \mathbb{R}^{n\times n}$, the interval matrix $[A - I, A + I]$ is defined as $[A - I, A + I] = \{B : A - I \leq B \leq A + I\}$. An interval matrix $[A - I, A + I]$ is called regular if each $B \in [A - I, A + I]$ is nonsingular. Eventually, A_i denotes the ith row of A.

AVE contains the absolute value, which is not necessarily differentiable on $\mathbb{R}^n$. Hence we employ the generalized Jacobian matrices [1] for our analysis. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz function. The generalized gradient of f at $\hat{x}$ is defined as

$$\partial f(\hat{x}) := \text{co}\left\{\lim_{n\to\infty} \nabla f(x_n) : x_n \to \hat{x}, x_n \notin X_f\right\},$$

where X_f is the set of points at which f is not differentiable and co(X) denotes the convex hull of a set X.

2 Concave minimization

It is well-known that (AVE) has a unique solution for each b if and only if the interval matrix $[A-I, A+I]$ is regular; see Theorem 3.3 in [29]. When $[A-I, A+I]$ is regular we denote the global solution of (1) by $x^\star$. In this section, we show that, under this assumption, if $\bar{x}$ is a local optimal solution of problem (1), then it is global optimal. Before we get to the proof, we need to present an auxiliary lemma first.

Lemma 1 *Let $[A-I, A+I]$ be regular. The function $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ defined as $\phi(x) = Ax - |x|$ is a Lipschitz homeomorphism.*

Proof First, we show that ϕ is injective. Suppose to the contrary that there exist two different points $\hat{x}, \bar{x}$ with $\phi(\hat{x}) = \phi(\bar{x})$. By Theorem 8 in [8],

$$\phi(\hat{x}) - \phi(\bar{x}) = \left(\sum_{i=1}^{n} \lambda_i A_i\right)(\hat{x} - \bar{x}),$$

where $A_i \in \partial\phi(x_i)$, $x_i \in co(\{\hat{x}, \bar{x}\})$, $\lambda_i \geq 0$, $i = 1, \ldots, n$ and $\sum_{i=1}^{n} \lambda_i = 1$. Hence, we obtain $\bar{A}(\hat{x} - \bar{x}) = 0$ for some $\bar{A} \in [A-I, A+I]$, which contradicts regularity of $[A-I, A+I]$. Hence, ϕ is an injection.

Suppose that $d \neq 0$. As ϕ is piece-wise linear, there exists t_0 such that for $t \geq t_0$, $\phi(td) = td_1 + d_0$ for some $d_0, d_1 \in \mathbb{R}^n$. Due to the injectivity of ϕ, d_1 cannot be zero. Hence, $\lim_{\|x\|\to\infty} \|\phi(x)\| = \infty$. Since for each x, $\partial\phi(x)$ contains no singular matrix, we can infer from Theorem 7.1.1 in [1] that ϕ is a local homeomorphism. As ϕ is a local homeomorphism with $\lim_{\|x\|\to\infty} \|\phi(x)\| = \infty$, Palais' Theorem [24] implies that ϕ is a homeomorphism. Lipschitzian property of ϕ is straightforward. Proposition 2.8 in [30] implies that ϕ^{-1} is Lipschitz, which completes the proof. □

Note that Lemma 1 can be inferred from some results in [5]; see Chapter 4. For completeness, we proved it here with a different method. In the next theorem, we show that each local optimal solution of problem (1) is global optimal.

Theorem 1 *Let $[A-I, A+I]$ be regular. If $\bar{x}$ is a local optimal solution of problem (1), then it is global optimal.*

Proof As $[A-I, A+I]$ is regular, problem (1) has a unique global solution; recall we denote it by $x^\star$. Suppose to the contrary that $\bar{x} \neq x^\star$, that is $|\bar{x}| \neq A\bar{x} - b$. Since $\bar{x}$ is feasible for (1), there exists j such that $|\bar{x}_j| < A_j\bar{x} - b_j$. Let $\bar{b} := A\bar{x} - |\bar{x}|$. By Lemma 1, there exists $\epsilon > 0$ such that

$$\phi^{-1}(\bar{b} + te_j) \in S, \quad \forall |t| \leq \epsilon.$$

Hence,

$$f(\phi^{-1}(\bar{b} + te_j)) = e^T(\bar{b} + te_j - b) = f(\bar{x}) + t,$$

which contradicts the local optimality of $\bar{x}$. Therefore, $\bar{x} = x^\star$. □

One interesting consequence of Theorem 1, under its assumption, is that each method (including extreme point ranking methods [6,28]) that is able to obtain a local minima for concave optimization problems can solve (AVE).

We say that the generalized Jacobian of ϕ is non-singular at x if $\partial\phi(x)$ does not contain a singular matrix. It is not difficult to see if we have the condition that $\partial\phi(x)$ is non-singular at $x \in S$, then ϕ will be a locally Lipschitz homeomorphism around x [1], and the argument of Theorem 1 is valid. In the next corollary we state this point.

Corollary 1 *Let $\partial\phi(\cdot)$ be non-singular at $\bar{x}$. If $\bar{x}$ is a local optimal solution of problem* (1)*, then it is global optimal.*

Let $E \subseteq S$ denote the vertices of S and $\mathrm{Adj}_S(x) \subseteq E$ denotes the adjacent vertices to the vertex $x \in E$.

Proposition 1 *Let $\bar{x} \in E$ and $\partial\phi(\bar{x})$ be non-singular. If*

$$f(\bar{x}) \leq f(x), \quad \forall x \in \mathrm{Adj}_S(\bar{x})$$

then $\bar{x}$ is a solution of (AVE).

Proof As $\bar{x} \in E$, there exists $r > 0$ such that

$$S \cap B(\bar{x}, r) \subseteq \mathrm{co}(\mathrm{Adj}_S(\bar{x}) \cup \{\bar{x}\}) + D,$$

where $D = \{d : Ad - |d| \geq 0\}$ denotes the set of directions of S. Since $\phi(x + d) = A(x + d) - |x + d| \geq Ax + Ad - |x| - |d|$,

$$f(x + d) \geq f(x), \quad \forall x \in S, \ \forall d \in D. \tag{3}$$

Suppose that $x \in S \cap B(\bar{x}, r)$. The vector x may be written as $x = \lambda_0\bar{x} + \sum_{x_i \in \mathrm{Adj}_S(\bar{x})} \lambda_i x_i + d$, where $d \in D$, $\lambda_i \geq 0$ and $\lambda_0 + \sum_{x_i \in \mathrm{Adj}_S(\bar{x})} \lambda_i = 1$. As f is a concave function, together with inequality (3), we obtain

$$f(x) = f\left(\lambda_0\bar{x} + \sum_{x_i \in \mathrm{Adj}_S(\bar{x})} \lambda_i x_i\right) \geq \lambda_0 f(\bar{x}) + \sum_{x_i \in \mathrm{Adj}_S(\bar{x})} \lambda_i f(x_i) \geq f(\bar{x}).$$

The last inequality follows from the assumptions. Hence, $\bar{x}$ is a local minimum of (2). Corollary 1 implies that $\bar{x}$ is a solution of (AVE). □

In the following proposition we give another condition under which Proposition 1 holds. Let $T_S(\bar{x})$ denote the tangent cone of S at $\bar{x}$ and let $D_{\bar{x}}$ denote the extreme directions of $T_S(\bar{x})$. We use $f^{'}(\bar{x}; d)$ to denote the directional derivative of f at $\bar{x}$ along d, that is,

$$f^{'}(\bar{x}; d) := \lim_{t \downarrow 0} \frac{f(\bar{x} + td) - f(\bar{x})}{t} < 0.$$

As f and ϕ are concave, both of them are directionally differentiable [25].

Algorithm 1

Pick $x^0 \in \mathbb{R}^n$ arbitrarily.
for $i = 0, 1, \ldots$ **do**
  Set x^{i+1} a vertex solution of the linear program

$$\min \; e^T(A - \operatorname{sign}(x_i))x \;\; \text{s.t.} \; -(I + A)x \le -b, \;\; (I - A)x \le -b.$$

  if $e^T(A - \operatorname{sign}(x_i))(x_{i+1} - x_i) = 0$ **then**
    Stop.
  end if
end for

Proposition 2 *Let $\bar{x} \in E$ and let $\partial\phi(\bar{x})$ be non-singular. If*

$$f'(\bar{x}; d) \ge 0, \quad \forall d \in D_{\bar{x}},$$

then $\bar{x}$ is a solution of (AVE).

Proof Suppose to the contrary that $\bar{x}$ is not a solution of (AVE). Similarly to the proof of Theorem 1, one can show that there exists $\bar{d} \in T_S(\bar{x})$ with $f'(\bar{x}; \bar{d}) < 0$. Vector $\bar{d}$ may be written as $\bar{d} = \sum_{d_i \in D_{\bar{x}}} d_i$. By super-additive property directional derivative of concave functions [25], we have

$$0 > f'(\bar{x}; \bar{d}) \ge \sum_{d_i \in D_{\bar{x}}} f'(\bar{x}; d_i).$$

Hence, $f'(\bar{x}; d_j) < 0$ for some $d_j \in D_{\bar{x}}$, which contradicts the assumptions. □

Proposition 2 can be exploited for proposing a pivoting method for (AVE).

3 A new algorithm

Mangasarian [14,20] proposed Algorithm 1 for (AVE). Mangasarian showed that Algorithm 1 stops in a finite number of steps at a point satisfying the necessary optimality condition for problem (2); see [14,20,21]. Note that this point is not necessarily a solution of (AVE). In the following example, we show that Algorithm 1 is not necessarily convergent to a solution of (AVE), even when $[A - I, A + I]$ is regular.

Example 1 Let

$$A = \begin{pmatrix} 3 & 1 \\ 6 & 5 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 10 \end{pmatrix}, \quad \bar{x} = \begin{pmatrix} \frac{5}{3} \\ 0 \end{pmatrix}.$$

Algorithm 2

1: Pick $x^0 \in \mathbb{R}^n$ arbitrarily.
2: **for** $i = 0, 1, \ldots$ **do**
3: Set z^i the solution of $(A - \mathrm{diag}(\mathrm{sign}(x_i)))x = b$.
4: Set (x^{i+1}, y^{i+1}) a solution of the linear program

$$\min \ -e^T A - \mathrm{sign}(z_i)x + 2e^T y \ \text{ s.t. } x - y \leq 0, \ \ -x - y \leq 0.$$

5: **if** $f(x^{i+1}) \leq \epsilon$ **or** $i \geq itmax$ **then**
6: Stop.
7: **end if**
8: **end for**

It is seen that $\bar{x}$ is a vertex of S. Moreover, with a little algebra, it is seen that $\bar{x}$ is the unique optimal solution of the following linear program,

$$\begin{aligned} &\min \ e^T(A - \mathrm{diag}(\mathrm{sign}(\bar{x})))x \\ &\text{ s.t. } -(I + A)x \leq -b, \ \ (I - A)x \leq -b. \end{aligned}$$

Note that, in constrast, $x^\star = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the unique solution of AVE.

In Example 1 we have $\sigma_{\min}(A) > 1$, which implies that $[A - I, A + I]$ is regular. In addition, under this condition (AVE) is polynomially time solvable because it is equivalent to the following convex quadratic program

$$\begin{aligned} &\min \ ((A - I)x - b)^T((A + I)x - b) \\ &\ \text{s.t.} (I - A)x \leq -b, \\ &\quad\ -(A + I)x \leq -b; \end{aligned} \tag{4}$$

see [16] for more details. To solve this deficiency of Algorithm 1, Mangasarian [21] proposed Algorithm 2, which is in a sense a compromise between Algorithm 1 and a Newton method.

In Algorithm 2, ϵ and $itmax$ denote the accuracy tolerance and the maximum number of iterations, respectively. As seen, Algorithm 2 involves solving one linear system (Newton step) and one linear program. This method does not necessarily generate a decreasing sequence, that is, $f(x^{i+1}) < f(x^i)$. Hence, the algorithm may be divergent. Indeed, it is probable to fall in a loop. The following example shows this behavior.

Example 2 Let

$$A = \begin{pmatrix} -5 & -7 & 10 \\ -7 & 8 & 4 \\ 1 & -2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix}.$$

Algorithm 3

Pick $x^0 \in \mathbb{R}^n$ arbitrarily.
for $i = 0, 1, \ldots$ **do**
  Set x^{i+1} a vertex solution of the linear program,

$$\begin{aligned} \min\ & e^T(A - \mathrm{diag}(\mathrm{sign}(x^i)))x \\ \text{s.t.}\ & -(I + A)x \leq -b, \quad (I - A)x \leq -b. \end{aligned} \tag{5}$$

  if $e^T(A - \mathrm{diag}(\mathrm{sign}(x^i)))(x^{i+1} - x^i) = 0$ **then**
    if $f(x^i) = 0$ **then**
      Stop.
    else
      Compute $\mathrm{Adj}_S(\bar{x})$ and set $x^{i+1} \in \mathrm{argmin}_{x \in \mathrm{Adj}_S(\bar{x})} f(x)$.
      if $f(x^{i+1}) \geq f(x^i)$ **then**
        Stop.
      end if
    end if
  end if
end for

One solution of the problem is the vector of ones. Suppose that $x^0 = 0$. One can verify that Algorithm 2 generates alternatively $\bar{x} = (-0.0149, 0.522, 0.0597)^T$ and $\hat{x} = (-0.0909, 0.4805, 0)^T$, while $\partial\phi(x)$ is non-singular for each $x \in E$.

To the best knowledge of the authors there is no sufficient condition in the literature for the convergence of Algorithm 2. In the sequel, we propose Algorithm 3 for (AVE) based on concave optimization methods. Moreover, we prove its finite convergence under mild conditions.

We have the following result about the convergence of Algorithm 3.

Theorem 2 *If $\partial\phi(x)$ is non-singular for each $x \in E$, then Algorithm* 3 *terminates in a finite number of steps at a solution of* (AVE).

Proof It is seen that Algorithm generates a decreasing sequence of vertices of S and terminates at a vertex whose objective value is equal or less than that of its adjacent vertices. By Proposition 1, under the assumption, Algorithm 3 terminates at a solution of (AVE). □

The number of adjacent vertices can be high. The following observation gives a simple sufficient condition under which a vertex is nondegenerate and so the number of adjacent vertices is n.

Proposition 3 *Let $\bar{x}$ be a vertex such that $\bar{x}_i \neq 0$ for each i. If $\bar{x}$ is not a solution of* (AVE)*, then x^* is nondegenerate.*

Proof Suppose to the contrary that $\bar{x}$ is degenerate. Then there is i such that both inequalities $(-\bar{x}-A\bar{x})_i \leq -b_i$ and $(\bar{x}-A\bar{x})_i \leq -b_i$ in (5) are active. This gives $(-\bar{x}-A\bar{x})_i = -b_i$ and $(\bar{x} - A\bar{x})_i = -b_i$, from which we derive $\bar{x}_i = 0$; a contradiction. □

We conclude the section by a short discussion on concave minimization methods for linear complementarity problems. Concave minimization methods have also proposed for linear complementarity problems [13,19,22,23]. Consider a general linear complementarity problem

$$Mz + q \geq 0, \quad z \geq 0, \quad z^T(Mz + q) = 0, \tag{LCP}$$

where $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$. Concave optimization methods for (LCP) are formulated as follows

$$\begin{aligned} \min \ & e^T(\min(Mz + q, z)) \\ \text{s.t.} & Mz + q \geq 0 \\ & z \geq 0. \end{aligned} \tag{6}$$

In fact, (LCP) has a solution if and only if the optimal value of problem (6) is zero. Let $\psi(z) = \min(Mz + q, z)$ and $T = \{z : Mz + q \geq 0, z \geq 0\}$. Similar to the arguments in previous section, one can infer the following corollary.

Corollary 2 *Let $\partial\psi(\cdot)$ be non-singular at $\bar{z}$. If $\bar{z}$ is a local optimal solution of problem* (6)*, then it is global optimal.*

If M is a P-matrix, then $\partial\psi(z)$ is non-singular for each $z \in T$. Hence, problem (6) has only one local optimal solution, which is also global optimal. This follows from the facts that $\cup_{z\in\mathbb{R}^n}\partial\psi(z) \subseteq \{DM + I - D : D \in [0, I]\}$ and $\{DM + I - D : D \in [0, I]\}$ contains no singular matrix when M is a P-matrix; see Theorem 4.2 in [7].

4 Computational results

In this section, we report numerical performance of Algorithm 2 and Algorithm 3 on some solvable AVEs. We considered six groups of test problems with 50, 100, 200, 500, 750 and 1000 variables. Both algorithms were implemented in MATLAB 2019a. To generate matrix A, we used MATLAB's function randn. We set $b = Ax_s - |x_s|$, where the random vector x_s was also obtained by randn function.

We generated six groups of test problems with 50, 100, 200, 500, 750 and 1000 variables. Moreover, each group included two hundreds problems. We utilized cplexlp in CPLEX package [11] to solve linear programs. The computations were run on a Windows PC with Intel Core i7 CPU, 3.4 GHz, and 24GB of RAM. The performance of both algorithms are summarized in Table 1. Therein, n shows the dimension of test problems, LP and std denote the average and the standard deviation of the number of solved linear programs for successful instances, respectively, F denotes the number of instances out of 200 the algorithm was unsuccessful, and eventually $succ$ stands for the success rate (which is the complement of F displayed in percentages).

As Table 1 shows, the number instances for which both algorithms failed are somehow the same. In fact, the success rate for both methods is more than 90%. However, as the difference between std columns is not considerable for most groups of instances,

Table 1 Computational results

Instance	n	Algorithm 2				Algorithm 3			
		LP	std	F	$succ$ (in %)	LP	std	F	$succ$ (in %)
Group 1	50	3.9	1.9	10	95.0	3.8	1.8	10	95.0
Group 2	100	4.7	2.4	14	93.0	4.4	2.2	13	93.5
Group 3	200	5.5	2.5	16	92.0	4.9	2.6	17	91.5
Group 4	500	7.0	4.3	17	91.5	6.1	3.4	15	92.5
Group 5	750	7.1	5.6	17	91.5	6.3	4.1	18	91.0
Group 6	1000	8.1	4.7	19	90.5	7.4	5.4	18	91.0

one can infer that Algorithm 3 outperforms Algorithm 2 in terms of the average number of solved linear programs. Moreover, unlike Algorithm 2, we provide some mild conditions under which Algorithm 3 is convergent. This property makes this method suitable for some classes of AVEs.

5 Some polynomial time solvable classes of AVEs

In this concluding section, we study some classes of AVEs which are polynomial time solvable. Indeed, we investigate some classes of AVEs whose solution can be obtained by solving one linear program. Moreover, we introduce an iterative method for solving AVEs with $\rho(|A^{-1}|) < 1$.

By our discussion, if (AVE) is solvable, then one vertex of S is a solution. Therefore, a vertex optimal solution of the following linear program for some $c \in \mathbb{R}^n$ can give a solution of (AVE)

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & -(I + A)x \le -b, \quad (I - A)x \le -b. \end{aligned} \tag{7}$$

However, the characterization of such c is not straightforward. This question in the context of LCP has been discussed extensively in the literature [2,17,18]. In the next lemma we provide a sufficient condition under which the linear program (7) provides a solution of (AVE). Note that the dual of linear program (7) may be written as

$$\begin{aligned} \max\ & b^T(v + w) \\ \text{s.t.}\ & (A - I)^T v + (A + I)^T w = c, \\ & v, w \ge 0. \end{aligned} \tag{8}$$

Lemma 2 *Let $\bar{x}$ and $(\bar{v}, \bar{w})$ be respectively a solution of (7) and (8). If $\bar{v} + \bar{w} > 0$, then $\bar{x}$ is a solution of (AVE).*

Proof By complementary slackness theorem, for $i = 1, \ldots, n$, $\bar{v}_i + \bar{w}_i > 0$ implies that $((A + I)\bar{x})_i = b_i$ or $((A - I)\bar{x})_i = b_i$. Hence, $A\bar{x} - |\bar{x}| = b$, which shows that $\bar{x}$ is a solution of (AVE). $\square$

In the rest of the section, thanks to Lemma 2, we characterize some classes of matrices for which vector c can be determined easily.

Proposition 4 *Let* $(A-I)^{-1} \geq 0$ *and* $(A+I)^{-1} \geq 0$*. If* $c > 0$*, then each solution of* (7) *is a solution of* (AVE).

Proof Kuttler's theorem [12] implies that interval matrix $[A-I, A+I]$ is regular and inverse nonnegative. Hence, (AVE) has a unique solution and consequently (7) is feasible. We show that the linear program (7) has a solution. To this end, it suffices to establish that if d is an unbounded direction of polyhedral set S (i.e., $(A-I)d \geq 0$ and $(A+I)d \geq 0$), then $d \geq 0$. Since $(A-I)^{-1} \geq 0$ and $(A-I)d \geq 0$, we have $d \geq 0$.

Now, let $(\bar{v}, \bar{w})$ be a solution of (8). As $(A-I)^T\bar{v} + (A+I)^T\bar{w} = c$, there exists $\bar{A} \in [A-I, A+I]$ with $\bar{A}^T(\bar{v}+\bar{w}) = c$. Because $c > 0$ and $\bar{A}^{-1} \geq 0$, we have $\bar{v}+\bar{w} > 0$. Therefore, the proof follows from Lemma 2. □

Proposition 5 *Let* A *be an M-matrix with* $\rho(A^{-1}) < 1$*. If* $c > 0$*, then each solution of* (7) *is a solution of* (AVE).

Proof By Proposition 2.19 in [30], the interval matrix $[A-I, A+I]$ is regular and inverse nonnegative. Hence, the proof is similar to the proof of Proposition 4. □

Proposition 6 *If* $\rho(|A^{-1}|) < \frac{1}{2}$*, then the linear program*

$$\min \quad c^T Ax \ \text{ s.t. } \ (I-A)x \leq -b, \ -(I+A)x \leq -b$$

yields the solution of (AVE)*, where* c *is the Perron-Frobenius eigenvector of any positive matrix* B *such that* $|A^{-T}| \leq B$ *and* $\rho(B) < \frac{1}{2}$.

Proof Let $x^\star$ be the unique solution of (AVE). Denote $s := \operatorname{sign}(x^\star)$. Then $(A - \operatorname{diag}(s))x^\star = b$, so $x^\star$ is a vertex of (9) and the optimal solution of the linear program

$$\min \ c^T(A - \operatorname{diag}(s))x \ \text{ s.t. } \ (I-A)x \leq -b, \ -(A+I)x \leq -b$$

since its objective vector is the sum of negative normal vectors of active constraints. Thus it is sufficient to show that $A^Tc = (A^T - \operatorname{diag}(s))v$ for some $v > 0$. In other words, $c = \big(I - A^{-T}\operatorname{diag}(s)\big)v$.

By the assumption, $\rho(A^{-T}\operatorname{diag}(s)) \leq \rho(|A^{-T}|) < 1$, so by the Neumann Series Theorem the matrix $(I - A^{-T}\operatorname{diag}(s))$ is nonsingular and from the equation we estimate v from below

$$\begin{aligned} v &= \big(I - A^{-T}\operatorname{diag}(s)\big)^{-1}c = \sum_{k=0}^{\infty}\big(A^{-T}\operatorname{diag}(s)\big)^k c \geq c - \sum_{k=1}^{\infty} B^k c \\ &= \left(1 - \frac{\rho(B)}{1-\rho(B)}\right)c > 0, \end{aligned}$$

where the last inequality follows from the facts that $\frac{\rho(B)}{1-\rho(B)} < 1$ and $c > 0$. Hence, the proposition follows from Lemma 2.

□

Note that in Proposition 6, due to the continuity of the eigenvalues with respect to the elements, such a matrix B exists. We denote by $w \gneqq 0$ the condition $w \geq 0$, $w \neq 0$. The following result shows that the objective of the linear program can be taken as a nonnegative combination of rows of A.

Theorem 3 *Let $[A - I, A + I]$ be regular. Then there exists $w \gneqq 0$ such that $x^\star$ is the optimal solution of the linear program*

$$\min\ w^T Ax \ \text{ s.t. }\ (I - A)x \leq -b,\ -(A + I)x \leq -b. \tag{9}$$

Proof Denote $s := \operatorname{sign}(x^\star)$. Then $(A - \operatorname{diag}(s))x^\star = b$, so $x^\star$ is a vertex of (9) and the optimal solution of the linear program

$$\min\ e^T(A - \operatorname{diag}(s))x \ \text{ s.t. }\ (I - A)x \leq -b,\ -(A + I)x \leq -b$$

since its objective vector is the sum of negative normal vectors of active constraints. Therefore, it is sufficient to find $w \gneqq 0$ such that $A^T w = (A^T - \operatorname{diag}(s))v$ for some $v \gneqq 0$. Suppose to the contrary that the system

$$A^T w = (A^T - \operatorname{diag}(s))v,\ v, w \gneqq 0$$

is infeasible. By the Farkas lemma, the dual system

$$Ax \gneqq 0,\ (A - \operatorname{diag}(s))x \lneqq 0$$

is feasible. Thus there is x such that $0 \lneqq Ax \lneqq \operatorname{diag}(s)x$, from which $|x| = \operatorname{diag}(s)x$, $x \neq 0$. Therefore x fulfills $|Ax| \leq |x|$, which contradicts regularity of $[A - I, A + I]$; see Rohn [26]. □

Recall that $[A - I, A + I]$ is strongly regular if $\rho(|A^{-1}|) < 1$. Denote by $x^\star$ the solution of (AVE).

Lemma 3 *If $[A - I, A + I]$ is strongly regular, then the sequence $x_{i+1} = b + A^{-1}|x_i|$, $i = 1, \ldots,$ converges to $x^\star$ for any $x_0 \in \mathbb{R}^n$. Moreover, for every $i = 1, \ldots,$ we have*

$$\|x^\star - x_i\| \leq \rho^i \frac{1}{1-\rho} \|x_1 - x_0\|$$

for some $0 < \rho < 1$ and a certain norm.

Proof We have

$$x_{i+1} - x_i = b + A^{-1}|x_i| - b - A^{-1}|x_{i-1}| = A^{-1}(|x_i| - |x_{i-1}|),$$

from which

$$|x_{i+1} - x_i| \le |A^{-1}| \cdot ||x_i| - |x_{i-1}|| \le |A^{-1}| \cdot |x_i - x_{i-1}|.$$

Define $B := |A^{-1}| + \varepsilon ee^T$, where $\varepsilon > 0$ is sufficiently small. Then $B > 0$ and $\rho := \rho(B) < 1$. Let $v > 0$ be the corresponding Perron-Frobenius eigenvector, that is, $Bv = \rho v$. We define a vector norm $\|x\| := v^T|x|$ and for matrices we use the induced matrix norm. Then $\||A^{-1}|\| < \|B\| = \rho$ and we can write

$$\|x_{i+1} - x_i\| < \rho\|x_i - x_{i-1}\| < \cdots < \rho^i\|x_1 - x_0\|,$$

from which the statement follows. □

Besides (4), we can formulate another polynomially solvable subclass of (AVE).

Theorem 4 *Let $[A - I, A + I]$ be strongly regular. Then the solution $x^\star$ of (AVE) can be found in polynomial time.*

Proof The outer bounds from [10] provide an initial bound on the elements of $x^\star$, and the bound has polynomial size with respect to the input size. The solution $x^\star$ of (AVE) satisfies $(A - \operatorname{diag}(s))x = b$, where $s = \operatorname{sign}(x^\star)$. Thus the elements of $x^\star$ have polynomially bounded size, which can be bounded a priori; cf. Schrijver [27]. Therefore we have polynomial lower and upper bounds for $x^\star$ and the number of iterations of the process from Lemma 3 can be polynomially bounded, too. After this number of iterations the signs s of $x^\star$ are uniquely determined and we compute $x^\star$ from the system $(A - \operatorname{diag}(s))x = b$.

It remains to show that we can find a value of ρ from Lemma 3 having a polynomial size (notice that $\rho(|A^{-1}|)$ need not be rational and thus not be of polynomial size). To this end, we set up a linear program

$$\max \ r \ \text{ s.t. } \ |A^{-1}|x \le re + x, \quad x \ge 0, \ e^T x = 1.$$

It has always an optimal solution of polynomial size, denote it by r°, x°. Since $\rho(|A^{-1}|) < 1$, we have $r^\circ > 0$ and so $r^\circ e + x^\circ > 0$. By the Collatz formula,

$$\rho(|A^{-1}|) \le \rho := \max_{i=1,\dots,n} \frac{\big(|A^{-1}|(r^\circ e + x^\circ)\big)_i}{(r^\circ e + x^\circ)_i} \le \max_{i=1,\dots,n} \frac{r^\circ + x_i^\circ + (|A^{-1}|r^\circ e)_i}{r^\circ + x_i^\circ} < 1.$$

□

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