J Optim Theory Appl (2012) 154:209–216
DOI 10.1007/s10957-012-0009-6

Optimal Error Correction and Methods of Feasible Directions

Saeed Ketabchi · Hossein Moosaei

Received: 10 November 2010 / Accepted: 5 February 2012 / Published online: 18 February 2012
© Springer Science+Business Media, LLC 2012

Abstract The main objective of this study is to discuss the optimum correction of linear inequality systems and absolute value equations (AVE). In this work, a simple and efficient feasible direction method will be provided for solving two fractional nonconvex minimization problems that result from the optimal correction of a linear system. We will show that, in some special-but frequently encountered-cases, we can solve convex optimization problems instead of not-necessarily-convex fractional problems. And, by using the method of feasible directions, we solve the optimal correction problem. Some examples are provided to illustrate the efficiency and validity of the proposed method.

Keywords Absolute value equations · Convex optimization problem · Feasible directions · Fractional quadratic problem

1 Introduction

One of the frequently encountered issues in applied science is how to deal with infeasible systems [1–3]. We could argue numerous reasons for the infeasibility of a system, including errors in data, errors in modeling, and many other situations. As remodeling of a problem and finding the errors of a system might take a remarkable amount of time and expenses, and also we might eventually get an infeasible system again, we do not do so. We therefore focus on an optimal correction of the given

Communicated by Gianni Di Pillo.

S. Ketabchi · H. Moosaei (✉)
Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, P.O. Box 1914, Rasht, Iran
e-mail: hmoosaei@gmail.com

S. Ketabchi
e-mail: sketabchi@guilan.ac.ir

system. In fact, we would like to reach the feasible systems with the least changes in data.

In this work, we study two problems. The first is the infeasible linear systems of inequalities [4, 5]. The second is the infeasible absolute value equations (AVE) [6, 7].

In order to make the above-mentioned systems feasible, we apply the changes simultaneously in the entries of the matrix A and the right-hand side vector b, and in order to correct the linear systems of inequalities and AVE, we need to solve two unconstrained quadratic fractional problems, not necessarily convex [5, 8].

Sometimes, solving the mentioned problems leads to solutions with very large norms, which are practically impossible to use. Two optimization methods can be used to control the norm of the solutions: (i) Tikhonov regularizing of problems, where a quadratic penalty is appended to the unconstrained quadratic fractional problem [9–11], and (ii) regularized least squares method—which is a well-studied approach for the unconstrained quadratic fractional problem—where a quadratic constraint bounding the size of the solution is added [9, 12].

This article focuses on studying the two unconstrained quadratic fractional problems, in the case where the coefficient matrices are rank deficient, by using the regularized least squares method. And, according to their possible directions, instead of having to solve the fractional and not-necessarily-convex problem, we could solve a convex optimization problem. Some examples are provided to illustrate the efficiency and validity of the proposed method.

This paper is organized as follows. In Sect. 2, we describe correction of the non-necessarily feasible linear inequality systems. Correction of the absolute value equations is reviewed in Sect. 3. In Sect. 4, some numerical examples are tested. Finally, some conclusions are drawn.

2 Correction of Linear Inequality Systems

Let $a=[a_i]$ be a vector in $\mathbb{R}^n$. By a_+ we mean the vector in $\mathbb{R}^n$, whose ith entry is 0 if $a_i<0$ and equals a_i if $a_i \geq 0$. By A^T we mean the transpose of the matrix A, and $\nabla f(x_0)^T d$ is called the directional derivative of f at x_0 in the direction d where $\nabla f(x_0)$ is the gradient of f at x_0. For two vectors x and y in the n-dimensional real space, $x^T y$ will denote the scalar product. For $x \in \mathbb{R}^n$, $\|x\|$ denotes the 2-norm, and $|x|$ will denote the vector in $\mathbb{R}^n$ of absolute values of components of x.

We now consider the non-necessarily feasible linear inequality systems $Ax \leq b$ where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$. Our aim, in this section, is to discuss the following minimization problem:

$$\min_{x,E,r} \left(\|E\|^2+\|r\|^2\right) \quad \text{s.t.} \quad (A+E)x \leq b+r, \tag{1}$$

where $E \in \mathbb{R}^{m\times n}$ is a perturbation matrix and $r \in \mathbb{R}^m$ is a perturbation vector.

Note that the difficulty of this problem is its nonconvexity. Fix x and consider the inner minimization problem with respect to the variables E and r,

$$\min_{E,r}\left(\|E\|^2+\|r\|^2\right) \quad \text{s.t.} \quad (A+E)x \leq b+r. \tag{2}$$

It is obvious that the above problem is linearly constrained and convex. By using the Karush–Kuhn–Tucker (KKT) conditions for the problem (2), we have:

Theorem 2.1 *Suppose that* (E^*, r^*) *denotes the optimal pair for the problem* (1). *Then*

$$r^* = \frac{(Ax^* - b)_+}{1 + \|x^*\|^2}, \qquad E^* = -\frac{(Ax^* - b)_+}{1 + \|x^*\|^2} x^{*T},$$

where x^* *is an optimal solution of*

$$\min_{x \in \mathbb{R}^n} \frac{\|(Ax - b)_+\|^2}{1 + \|x\|^2}. \tag{3}$$

Proof See [13]. □

Now, instead of problem (3), we could deal with the following problem:

$$\min_{x \in \mathbb{R}^n} \frac{\|(Ax - b)_+\|^2}{1 + \|x\|^2} \quad \text{s.t.} \quad \|x\|^2 \le \beta. \tag{4}$$

To solve problem (4), we first focus on the following problem:

$$\min_{x \in \mathbb{R}^n} \left\|(Ax - b)_+\right\|^2. \tag{5}$$

Obviously, this is a convex but not everywhere differentiable optimization problem. Let us denote its solution set by X^*.

Lemma 2.1 *Let* x^* *be an optimal solution of* (5). *If there is a nonzero vector* $d \in \mathbb{R}^n$ *for which*

$$A_1 d \le 0, \qquad A_2 d = 0, \tag{6}$$

where A_1 *and* A_2 *are submatrices of* A *with*

$$\left(A_1 x^* - b_1\right)_+ = 0, \qquad \left(A_2 x^* - b_2\right)_+ = A_2 x^* - b_2,$$

then X^* *is unbounded.*

Proof Since x^* is an optimal solution of (5), then for any nonzero vector d that satisfies (6) and any $\alpha \ge 0$ we have

$$\begin{aligned} \left(A_1 x^* - b_1\right)_+ &= \left(A_1\left(x^* + \alpha d\right) - b_1\right)_+ = 0, \\ \left(A_2 x^* - b_2\right)_+ &= \left(A_2\left(x^* + \alpha d\right) - b_2\right)_+ = A_2 x^* - b_2. \end{aligned}$$

□

Corollary 2.1 *For rank deficient matrices* A, X^* *is unbounded.*

Corollary 2.2 *Let* $f(x)$ *denote the objective function of* (3), *and let* x^* *be a nonzero optimal solution of* (5). *If there exists a nonzero vector* $d \in \mathbb{R}^n$ *satisfying* (6), *we may find it by solving the following linear programming problem*:

$$\min_{d \in \mathbb{R}^n} \left(\nabla f\left(x^*\right)^T d\right) \quad \text{s.t.} \quad A_1 d \le 0, \quad A_2 d = 0, \quad \nabla f\left(x^*\right)^T d \ge -t, \tag{7}$$

where t *is a positive constant.*

Proposition 2.1 *If matrix A is rank deficient, then there exists a nonzero vector d that satisfies system* (6).

Proposition 2.2 *Let x^* be a solution of problem* (5) *and vector d be a solution of* (7). *Then, for any $\lambda \geq 0$, $x^* + \lambda d$ is also a solution of problem* (5).

Now, considering the fact that the numerator of the fraction is constant in problem (3), the denominator will tend towards infinity, so the objective function value of problem (3) will tend towards zero. Considering the fact that the norm of the solution is too large, this answer is not usable in general. Now in order to limit the solution, and considering the given β, we focus on studying problem (4).

Now, let x^* be a solution of (5) such that $\|x^*\|^2 \leq \beta$, and d be a solution of (7); then there are two choices of λ, which satisfy $\|x^* + \lambda d\|^2 = \beta$ (see [14]). One of these choices is

$$\lambda = \frac{\beta - \|x^*\|^2}{x^{*\top} z + \operatorname{sgn}(x^{*\top} z)[(x^{*\top} z)^2 + \beta - \|x^*\|^2]^{\frac{1}{2}}}, \tag{8}$$

where $z = d/\|d\|$ and sgn denotes the sign function. Also, for any λ satisfying (8), we have

$$\left\| \left(A\left(x^* + \lambda d\right) - b\right)_+ \right\|^2 = \left\| \left(Ax^* - b\right)_+ \right\|^2,$$

which means that $x^* + \lambda d$ is an optimal solution of (4). Therefore, we have proved the following proposition.

Proposition 2.3 *Let x^* be an optimal solution of problem* (5) *such that $\|x^*\|^2 \leq \beta$, and vector d be a solution of* (7). *Then there exists a $\lambda \in \mathbb{R}$ for which $\|x^* + \lambda d\|^2 = \beta$, and $x^* + \lambda d$ is an optimal solution of problem* (4).

3 Correction of Absolute Value Equations

In this section, we discuss the not-necessarily feasible absolute value equations $Ax - |x| = b$, where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. Our discussion in this section is similar to the one described in Sect. 2.

First, we consider the following minimization problem:

$$\min_x \min_{E,r} \left(\|E\|^2 + \|r\|^2\right) \quad \text{s.t.} \quad (A+E)x - |x| = b + r, \tag{9}$$

where $E \in \mathbb{R}^{n\times n}$ is a perturbation matrix and $r \in \mathbb{R}^n$ is a perturbation vector.

The problem (9) is nonconvex, and to solve it we consider the following inner minimization problem

$$\min_{E,r} \left(\|E\|^2 + \|r\|^2\right) \quad \text{s.t.} \quad (A+E)x - |x| = b + r, \tag{10}$$

which is a constrained convex problem.

We know that the function $g(x) = (A+E)x - |x|$ is piecewise linear vector function and a generalized Jacobian $\partial g(x)$ of $g(x)$ is given by $\partial g(x) = (A+E) - D(x)$, where $D(x) = \partial|x| = \operatorname{diag}(\operatorname{sgn}(x))$ [15].

Theorem 3.1 *Suppose that* (E^*, r^*) *denotes the optimal pair to the problem* (9). *Then*

$$r^* = \frac{Ax^* - |x^*| - b}{1 + \|x^*\|^2}, \qquad E^* = -\frac{Ax^* - |x^*| - b}{1 + \|x^*\|^2} x^{*T},$$

where x^* *is an optimal solution of* $\min_x \frac{\|Ax - |x| - b\|^2}{1+\|x\|^2}$.

Proof The Lagrangian of the problem (10) is given by

$$L(E, r, \lambda) := \|E\|^2 + \|r\|^2 - \lambda^T \big((A + E)x - |x| - (b + r)\big).$$

Since the problem (10) is convex, then the KKT necessary conditions are also sufficient and any (E, r) satisfying the KKT conditions is a global minimum. The KKT conditions of (10) give (see [16, 17])

$$\frac{\partial L}{\partial E} = 2E - \lambda x^T = 0, \tag{11}$$

$$\frac{\partial L}{\partial r} = 2r + \lambda = 0, \tag{12}$$

$$\frac{\partial L}{\partial \lambda} = (A + E)x - |x| - (b + r) = 0. \tag{13}$$

From (11)–(13), we have that $2E = \lambda x^T$, $\lambda = -2r$. Therefore, we obtain $E = -rx^T$ and $r = Ax + Ex - |x| - b$. By combining these expressions, we find that $r + r\|x\|^2 = Ax - |x| - b$, and we conclude that

$$r = \frac{Ax - |x| - b}{1 + \|x\|^2}, \qquad E = -\frac{Ax - |x| - b}{1 + \|x\|^2} x^T. \tag{14}$$

From (14) we obtain $\|E\|^2 + \|r\|^2 = \frac{\|Ax-|x|-b\|}{1+\|x\|^2}$, and the value of the objective function in (10) at the optimal solution is equal to $\|E\|^2 + \|r\|^2 = \frac{\|Ax-|x|-b\|}{1+\|x\|^2}$. Then, the value of problem at optimal solution is equal to

$$\|E^*\|^2 + \|r^*\|^2 = \min_{x\in\mathbb{R}^n} \frac{\|Ax - |x| - b\|^2}{1 + \|x\|^2}. \tag{15}$$

This completes the proof. □

Similar to Sect. 2, instead of problem (15), we could deal with the following problem:

$$\min_{x\in\mathbb{R}^n} \frac{\|Ax - |x| - b\|^2}{1 + \|x\|^2} \quad \text{s.t.} \quad \|x\|^2 \le \beta. \tag{16}$$

Now to solve problem (16), we consider the following problem:

$$\min_{x\in\mathbb{R}^n} \|Ax - |x| - b\|^2. \tag{17}$$

Let X^* be the solution set of (17) and $x^* \in X^*$. Consider the following system

$$Ad - |x^* + d| + |x^*| = 0. \tag{18}$$

Now, we do a simple rearrangement, and divide the vector x^* in to subvectors ${x_1}^*$ and ${x_2}^*$ as follows:

$$x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}, \tag{19}$$

where x_1^* and x_2^* are vectors whose components are all nonnegative and nonpositive, respectively. Also, d can be written in terms of subvectors as above. Then, $d = \left[\begin{smallmatrix} d_1 \\ d_2 \end{smallmatrix}\right]$, and by substituting it and x^* from (19) into (18) we obtain

$$A_1 d_1 + A_2 d_2 - \begin{bmatrix} x_1^* + d_1 \\ -x_2^* - d_2 \end{bmatrix} + \begin{bmatrix} x_1^* \\ -x_2^* \end{bmatrix} = 0, \quad d_1 \geq 0,\ d_2 \leq 0,$$

or

$$A_1 d_1 + A_2 d_2 + \begin{bmatrix} -d_1 \\ d_2 \end{bmatrix} = 0, \quad d_1 \geq 0,\ d_2 \leq 0.$$

If we replace d_2 by $-d_2$, then we have

$$[A_1, -A_2] \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} - I_n \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 0, \quad d_1 \geq 0,\ d_2 \geq 0.$$

The above system can be rewritten in the following form

$$Hd = 0, \quad d \geq 0, \tag{20}$$

where

$$H = \begin{bmatrix} A_1, -A_2 \end{bmatrix} - I_n, \qquad d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}.$$

For finding a nonzero solution of (20), we introduce the following linear program

$$\min_{d \in \mathbb{R}^n} \left(-e^T d\right) \quad \text{s.t.} \quad Hd = 0, \quad e^T d \leq 1,\ d \geq 0, \tag{21}$$

where e is an n-vector whose components are all ones.

Proposition 3.1 *If matrix H is rank deficient, then there exists a nonzero vector d that satisfies system* (20).

Proposition 3.2 *If x^* is a solution of problem* (17) *and vector d is a solution of* (21) *then, for any $\lambda \geq 0$, $x^* + \lambda d$ is also a solution of problem* (17).

4 Numerical Testing

In this section, we present two examples to illustrate our method. In the first example, we consider problem (4), and in the second example, we consider problem (16).

Example 4.1 Suppose that

$$A=\begin{bmatrix} 4 & 2 & 0 & 1 & -4 \\ -3 & -5 & 7 & -4 & 0 \\ 3 & -6 & -2 & 4 & 3 \\ -4 & -2 & 0 & -1 & 4 \\ 3 & 5 & -7 & 4 & 0 \\ -3 & 6 & 2 & -4 & -3 \end{bmatrix}, \qquad b=\begin{bmatrix} -9 \\ 6 \\ -3 \\ 9 \\ -7 \\ 2 \end{bmatrix}.$$

First, we solve the problem (5) and obtain

$$x^* = [-0.66426 \quad 0.79883 \quad 1.3348 \quad 0.21048 \quad 2.0378]^T.$$

The value of objective function will be 1. Then, by solving problem (7) for $t = 10$, we have the nonzero direction $d = [-1.5096 \ -1.9689 \ -2.6727 \ -1.0838 \ -2.7650]^T$. By using (8), we have $\|x^* + \lambda d\|^2 = \beta$. Therefore, $x^* + \lambda d$ is a solution of (4).

Example 4.2 Suppose that

$$A=\begin{bmatrix} 1 & -4 & 0 & 0 & 3 \\ -1 & 2 & 1 & 0 & -3 \\ 3 & -2 & 1 & -3 & 5 \\ 1 & 0 & -1 & 1 & -4 \\ 0 & 0 & -6 & 6 & 8 \end{bmatrix}, \qquad b=\begin{bmatrix} 1 \\ -1 \\ 0 \\ 5 \\ -3 \end{bmatrix}.$$

First, we solve the problem (17) and obtain

$$x^* = [1.6424 \quad -0.7999 \quad 0.0935 \quad 0.8291 \quad -0.8298]^T.$$

The value of objective function will be 0.3272. Then as can be seen, by solving problem (21), we have the nonzero direction $d = [1 \ \ 0 \ \ 1 \ \ 1 \ \ 0]^T$. By using (8), we have $\|x^* + \lambda d\|^2 = \beta$. Therefore, $x^* + \lambda d$ is a solution of (16).

5 Conclusion

In this article, correction of infeasible systems, linear inequality and AVE systems were studied by applying minimal changes to the coefficient matrix and the right-hand side vector, using the 2-norm. In both situations, we obtain fractional quadratic problems, but not necessarily convex. Sometimes, solving the above problems will lead to solutions with large norms. Therefore, in order to control the solution's norm, we employed the idea of feasible directions, and we obtained a solution for the system correction, according to a known solution from a convex quadratic problem, in such a way that the solution had an acceptable norm.

Numerical results in the two examples, provide an indication on the correctness of the algorithm.

Acknowledgements The authors are extremely thankful to Professor Richard W. Cottle for his careful reading of the manuscript. The paper has benefited from his suggestions and detailed comments.

The authors are also grateful to Gianni Di Pillo and two anonymous referees for their helpful comments and suggestions which led to the improvement of the originally submitted version of this work.

References

1. Censor, Y., Altschuler, M.D., Powlis, W.D.: A computational solution of the inverse problem in radiation therapy treatment planning. Appl. Math. Comput. **25**, 57–87 (1988)
2. Censor, Y., Altschuler, M.D., Powlis, W.D.: On the use of Cimmino's simultaneous projections method for computing a solution of the inverse problem in radiation therapy treatment planning. Inverse Probl. **4**, 607–623 (1988)
3. Censor, Y., Ben-Israel, A., Xiao, Y., Galvin, J.M.: On linear infeasibility arising in intensity-modulated radiation therapy inverse planning. Linear Algebra Appl. **428**, 1406–1420 (2008)
4. Salahi, M., Ketabchi, S.: Correcting an inconsistent set of linear inequalities by the generalized Newton method. Optim. Methods Softw. **25**, 457–465 (2010)
5. Amaral, P., Júdice, J., Sherali, H.D.: A reformulation–linearization–convexification algorithm for optimal correction of an inconsistent system of linear constraints. Comput. Oper. Res. **35**, 1494–1509 (2008)
6. Mangasarian, O.L.: Absolute value programming. Comput. Optim. Appl. **36**, 43–53 (2007)
7. Mangasarian, O.L., Meyer, R.R.: Absolute value equations. Linear Algebra Appl. **419**, 359–367 (2007)
8. Beck, A., Teboulle, M.A.: Convex optimization approach for minimizing the ratio of indefinite quadratic functions over an ellipsoid. Math. Program. **118**, 13–35 (2009)
9. Tikhonov, A.N., Arsenin, V.Y.: Solutions of Ill-Posed Problems. Halsted, New York (1977)
10. Golub, G.H., Hansen, P.C., O'Leary, D.P.: Tikhonov regularization and total least squares. SIAM J. Matrix Anal. Appl. **21**, 185–194 (1999)
11. Beck, A., Ben-Tal, A.: On the solution of the Tikhonov regularization of the total least squares problem. SIAM J. Optim. **17**, 98–118 (2006)
12. Renaut, R., Guo, A.H.: Efficient algorithms for solution of regularized total least squares. SIAM J. Matrix Anal. Appl. **26**, 457–476 (2005)
13. Amaral, P., Barahona, P.: A framework for optimal correction of inconsistent linear constraints. Constraints **10**, 67–86 (2005)
14. Moré, J., Sorensen, D.C.: Computing a trust region step. SIAM J. Sci. Stat. Comput. **4**, 553–572 (1983)
15. Mangasarian, O.L.: A generalized Newton method for absolute value equations. Optim. Lett. **3**, 101–108 (2009)
16. Bazaraa, M.S., Sherali, H.D., Shetty, C.M.: Nonlinear Programming Theory and Algorithms. Wiley, New York (1993)
17. Nocedal, J., Wright, S.J.: Numerical Optimization. Springer, Berlin (1999)