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A modified fixed point iteration method for solving the system of absolute value equations

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ABSTRACT

The fixed point iteration (FPI) method proposed by Ke [Appl Math Lett. 2020;99:105990] for solving the absolute value equations (AVE) with the form $Ax - |x| = b$ is interesting for its simplicity and efficiency. However, its convergence is only guaranteed for the case that $0 < \|A^{-1}\| < \frac{\sqrt{2}}{2}$, excluding the possible case that $\frac{\sqrt{2}}{2} \leq \|A^{-1}\| < 1$. To complete the gap, we develop a modified FPI (MFPI) method for solving the AVE with $0 < \|A^{-1}\| < 1$, which, besides keeping the simplicity of FPI, improves its efficiency by judiciously choosing the involving parameter. Under mild conditions, we prove its linear convergence. We present some preliminary numerical results for $0 < \|A^{-1}\| < 1$, demonstrating its convergence; and compare it with FPI when $0 < \|A^{-1}\| < \frac{\sqrt{2}}{2}$, illustrating its superiority.

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1. Introduction

Consider the solution of the system of absolute value equations (AVE)

$$Ax - |x| - b = 0, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, and $|x|$ denotes the vector with absolute values of each component of x. A more general form of the system of absolute value equations is $Ax + B|x| - b = 0$, which was introduced in [1] and studied in a more general background in [2–7]. Such type of absolute value equations are often encountered in the analysis of linear programming problem, bimatrix games, linear complementarity problem and many other applications, see [8–14] and the references therein. It was known that determining the existence of a solution to the AVE is NP-hard [15] and if solvable, the problem of checking whether the AVE has

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unique or multiple solutions is NP-complete [16]. Moreover, one sufficient condition for the AVE (1) being uniquely solvable for any b is described in the following proposition.

Proposition 1.1 ([17]): *Assume that $A \in \mathbb{R}^{n\times n}$ is invertible. If $\|A^{-1}\| < 1$, then the AVE (1) has a unique solution x^* for any $b \in \mathbb{R}^n$.*

The problem of finding the unique solution of the AVE (1) with $\|A^{-1}\| < 1$ has received much attention; see [17–20] and the references therein. A large number of iterative methods have been proposed, including the generalized Newton method [15], the SOR-like iteration method [8] and so forth, see [21–27] and the references therein.

Recently, by reformulating the AVE (1) as a two-by-two block nonlinear equation [8]

$$\begin{aligned} Ax - y &= b,\\ y - |x| &= 0, \end{aligned} \tag{2}$$

Ke in [24] suggested the following fixed point iteration (FPI) method for solving the AVE (1).

Algorithm 1.1 ([24], FPI method for the AVE (1)): Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix and $b \in \mathbb{R}^n$. Given initial vectors $x^{(0)} \in \mathbb{R}^n$ and $y^{(0)} \in \mathbb{R}^n$, for $k = 0, 1, 2, \ldots$ until the iteration sequence $\{(x^{(k)}, y^{(k)})\}_{k=0}^{\infty}$ is convergent, compute

$$\begin{aligned} x^{(k+1)} &= A^{-1}(y^{(k)} + b),\\ y^{(k+1)} &= (1-\tau)y^{(k)} + \tau|x^{(k+1)}|, \end{aligned}$$

where τ is a positive constant.

Let

$$e_k^x := x^{(k)} - x^* \quad \text{and} \quad e_k^y = y^{(k)} - y^*$$

denote the iteration error and denote

$$E^{(k+1)} := \begin{pmatrix} e_{k+1}^x \\ e_{k+1}^y \end{pmatrix}.$$

Ke [24] proved the following convergence result for the sequences generated by Algorithm 1.1.

Theorem 1.1 ([24]): *Assume that $A \in \mathbb{R}^{n\times n}$ is a nonsingular matrix and $b \in \mathbb{R}^n$. Denote $\nu = \|A^{-1}\|$. If*

$$0 < \nu < \frac{\sqrt{2}}{2} \quad \text{and} \quad \frac{1-\sqrt{1-\nu^2}}{1-\nu} < \tau < \frac{1+\sqrt{1-\nu^2}}{1+\nu}, \tag{3}$$

then $\|E^{(k+1)}\| < \|E^{(k)}\|$ for all $k = 0, 1, 2, \ldots$.

Comparing the results in Proposition 1.1 and Theorem 1.1, we can see that the convergence of Algorithm 1.1 is only guaranteed for the case that $0 < \nu = \|A^{-1}\| < \frac{\sqrt{2}}{2}$, while for the case that $\frac{\sqrt{2}}{2} \le \nu < 1$ the convergence result is absence. This greatly limits the application range of the interesting algorithm Algorithm 1.1. Hence, a natural question is arising, i.e. is there a fixed point iteration method that is suitable for solving AVE for any $0 < \nu < 1$? If so, can we further improve its efficiency?

This paper is devoting to the affirmative answers of these questions. To this end, we first introduce a nonsingular matrix $Q \in \mathbb{R}^{n\times n}$ and reformulate the AVE (1) as a new equivalent two-by-two nonlinear equations, which is slightly different from (2). Based on the new nonlinear form, we develop a modified fixed point iteration (MFPI) method. Under mild conditions, the MFPI method will linearly converge to the unique solution of the AVE (1) with $0 < \nu < 1$. The MFPI method can be seen as a generalization of the FPI method in some sense. Nevertheless, there are three differences between the MFPI method and the FPI method. Firstly, there is theoretical guarantee of the convergence for the MFPI method to solve the AVE (1) with $\frac{\sqrt{2}}{2} \le \nu < 1$ but no for the FPI method. Secondly, a nonsingular matrix $Q \in \mathbb{R}^{n\times n}$ is involved in the MFPI method which makes the enhanced convergent results possible. Thirdly, the range of the iteration parameter τ for the MFPI method is larger than that of the FPI method and, even when $0 < \|A^{-1}\| < \frac{\sqrt{2}}{2}$, the MFPI method can be superior to the FPI method in some situations. Our numerical results will confirm our claims.

The remainder of this paper is organized as follows: In Section 2, we present the MFPI method for solving the AVE (1), and explore the convergent conditions for the new method. In Section 3, some numerical experiments are provided to show the feasibility and effectiveness of our method. Finally, some concluding remarks are given in Section 4.

Notation: The set of all $n \times n$ real matrices is denoted by $\mathbb{R}^{n\times n}$ and $\mathbb{R}^n = \mathbb{R}^{n\times 1}$. The symbol I_m stands for an identity matrix of order m and I indicates an identity matrix with proper order. For $x \in \mathbb{R}^n$, x_i represents the ith entry of vector x for all $i = 1, 2, \ldots n$. $\mathrm{sgn}(x)$ denotes a vector with components equal to $-1, 0$ or 1 depending on whether the corresponding component of the vector x is negative, zero or positive, respectively. Denote $|x|$ the vector with ith component equal to $|x_i|$. For $x \in \mathbb{R}^n$, $\mathrm{diag}(x)$ represents a diagonal matrix with x_i as its diagonal entries for every $i = 1, 2, \ldots, n$. A matrix $A \in \mathbb{R}^{m\times n}$ is said to be nonnegative if its entries satisfy $a_{ij} \ge 0$ for all $1 \le i \le m$ and $1 \le j \le n$. $\|A\|$ denotes the spectral norm of A and is defined by the formula $\|A\| := \max\{\|Ax\| : x \in \mathbb{R}^n, \|x\| = 1\}$, where $\|x\|$ is the 2-norm of vector x. $\varnothing$ denotes the empty set.

2. The MFPI method

In this section, we develop the MPFI method for solving the AVE (1) and discuss its convergence.

Firstly, we present a new equivalent two-by-two block nonlinear form of the AVE (1). Let $Qy = |x|$. Then the AVE (1) is equivalent to

$$\begin{aligned} Ax - Qy &= b, \\ -|x| + Qy &= 0, \end{aligned}$$

that is

$$\begin{pmatrix} A & -Q \\ -D(x) & Q \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}. \tag{4}$$

Here, $Q \in \mathbb{R}^{n\times n}$ is invertible and $D(x) \doteq \mathrm{diag}(\mathrm{sgn}(x))$. Obviously, if $Q = I$, then (4) reduces to (2). When A is nonsingular, from (4), we obtain the following fixed point equation

$$\begin{aligned} x^* &= A^{-1}(Qy^* + b), \\ y^* &= (1-\tau)y^* + \tau Q^{-1}|x^*|, \end{aligned} \tag{5}$$

where $\tau > 0$ is a parameter and $y^* = Q^{-1}|x^*|$.

According to the fixed point Equation (5), we can develop the following MFPI method for solving the AVE (1).

Algorithm 2.1 (MFPI method for the AVE (1)): Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix and $b \in \mathbb{R}^n$. Given initial vectors $x^{(0)} \in \mathbb{R}^n$ and $y^{(0)} \in \mathbb{R}^n$ and a nonsingular matrix $Q \in \mathbb{R}^{n\times n}$, for $k = 0, 1, 2, \ldots$ until the iteration sequence $\{(x^{(k)}, y^{(k)})\}_{k=0}^{\infty}$ is convergent, compute

$$\begin{aligned} x^{(k+1)} &= A^{-1}(Qy^{(k)} + b), \\ y^{(k+1)} &= (1-\tau)y^{(k)} + \tau Q^{-1}|x^{(k+1)}|, \end{aligned} \tag{6}$$

where the iteration parameter $\tau > 0$.

It is obvious that, if $Q = I$, the MFPI method reduces to the FPI method. In this sense, the MFPI method is a generalization of the FPI method. Now we are in position to discuss the convergence of the MFPI method. The following lemma plays an important role in our covergence analysis.

Lemma 2.1 ([28]): *For any vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, the following results hold:*

(1) $\||x| - |y|\| \le \|x - y\|$;
(2) *if* $0 \le x \le y$, *then* $\|x\| \le \|y\|$;

(3) *if $x \le y$ and P is a nonnegative matrix, then $Px \le Py$.*

In the following, we are ready to prove the main result of this paper. The following proof is inspired by that of Theorem 2.1 in [24]. However, the existence of Q will make some difference.

Theorem 2.1: *Let $A, Q \in \mathbb{R}^{n\times n}$ be nonsingular matrices and $b \in \mathbb{R}^n$. Denote*

$$\nu = \|A^{-1}\|, \quad \alpha = \|Q\|, \quad \beta = \|Q^{-1}\|, \quad E^{(k+1)} = \begin{pmatrix} e^x_{k+1} \\ e^y_{k+1} \end{pmatrix}.$$

If

$$0 < \nu < \frac{1}{\alpha\sqrt{1+\beta^2}} \quad \text{and} \quad \frac{1-\sqrt{1-\alpha^2\nu^2}}{1-\alpha\beta\nu} < \tau < \frac{1+\sqrt{1-\alpha^2\nu^2}}{1+\alpha\beta\nu}, \tag{7}$$

then we have

$$\|E^{(k+1)}\| < \|E^{(k)}\| \tag{8}$$

for $k = 0, 1, 2, \ldots$.

Proof: It follows from (5) and (6) that

$$e^x_{k+1} = A^{-1}Qe^y_k, \tag{9}$$
$$e^y_{k+1} = (1-\tau)e^y_k + \tau Q^{-1}(|x^{(k+1)}| - |x^*|). \tag{10}$$

From (9), we get

$$\|e^x_{k+1}\| \le \|Q\|\|A^{-1}\|\|e^y_k\| = \alpha\nu\|e^y_k\|. \tag{11}$$

In light of (10), triangle inequality and the first result in Lemma 2.1, we obtain

$$\begin{aligned} \|e^y_{k+1}\| &\le |1-\tau|\|e^y_k\| + \tau\|Q^{-1}\|\||x^*| - |x^{(k+1)}|\| \\ &\le |1-\tau|\|e^y_k\| + \tau\|Q^{-1}\|\|x^* - x^{(k+1)}\| \\ &= |1-\tau|\|e^y_k\| + \beta\tau\|e^x_{k+1}\|. \end{aligned} \tag{12}$$

Consequently, from (11) and (12), we obtain

$$\begin{pmatrix} 1 & 0 \\ -\beta\tau & 1 \end{pmatrix} \begin{pmatrix} \|e^x_{k+1}\| \\ \|e^y_{k+1}\| \end{pmatrix} \le \begin{pmatrix} 0 & \alpha\nu \\ 0 & |1-\tau| \end{pmatrix} \begin{pmatrix} \|e^x_k\| \\ \|e^y_k\| \end{pmatrix}. \tag{13}$$

Since

$$\begin{pmatrix} 1 & 0 \\ -\beta\tau & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ \beta\tau & 1 \end{pmatrix} \doteq P \ge 0,$$

multiplying (13) from left by the nonnegative matrix P and according to the third rusult of Lemma 2.1, we get

$$\begin{pmatrix} \|e^x_{k+1}\| \\ \|e^y_{k+1}\| \end{pmatrix} \le \begin{pmatrix} 0 & \alpha\nu \\ 0 & \alpha\beta\tau\nu + |1-\tau| \end{pmatrix} \begin{pmatrix} \|e^x_k\| \\ \|e^y_k\| \end{pmatrix}. \tag{14}$$

Hence, it follows from (14) that

$$\|E^{(k+1)}\| \le \|W\| \cdot \|E^{(k)}\|$$

with

$$W = \begin{pmatrix} 0 & \alpha\nu \\ 0 & \alpha\beta\tau\nu + |1-\tau| \end{pmatrix}.$$

Next, we turn to consider the conditions of the parameters α, β and τ such that $\|W\| < 1$. Then the inequality (8) holds.

Since

$$W^T W = \begin{pmatrix} 0 & 0 \\ 0 & \alpha^2\nu^2 + (\alpha\beta\tau\nu + |1-\tau|)^2 \end{pmatrix},$$

then

$$\|W\| < 1 \Leftrightarrow \alpha^2\nu^2 + (\alpha\beta\tau\nu + |1-\tau|)^2 < 1. \tag{15}$$

We will continue our discussion as the following two cases.

- *Case 1:* when $0 < \tau \le 1$ and $0 < \alpha\nu < 1$. Then

$$\begin{aligned} &\alpha^2\nu^2 + (\alpha\beta\tau\nu + |1-\tau|)^2 < 1 \Leftrightarrow f(\tau) \\ &\quad \doteq (\alpha\beta\nu - 1)^2\tau^2 + 2(\alpha\beta\nu - 1)\tau + \alpha^2\nu^2 < 0. \end{aligned}$$

(1) If $\alpha\beta\nu = 1$, $f(\tau) < 0$ becomes $\alpha^2\nu^2 < 0$, a contradictory.

(2) If $\alpha\beta\nu > 1$, from $f(\tau) < 0$ we have

$$\frac{-1-\sqrt{1-\alpha^2\nu^2}}{\alpha\beta\nu - 1} < \tau < \frac{-1+\sqrt{1-\alpha^2\nu^2}}{\alpha\beta\nu - 1} < 0.$$

Obviously, it is against with $\tau > 0$.

(3) If $0 < \alpha\beta\nu < 1$, the two roots of $f(\tau)$ are

$$\tau_1 = \frac{1-\sqrt{1-\alpha^2\nu^2}}{1-\alpha\beta\nu}(>0), \quad \tau_2 = \frac{1+\sqrt{1-\alpha^2\nu^2}}{1-\alpha\beta\nu}.$$

(a) If $0 < \tau_1 < \tau_2 \le 1$, that is $1 + \sqrt{1-\alpha^2\nu^2} < 1 - \alpha\beta\nu$. This is a contradictory inequality.

(b) If $0 < \tau_1 < 1 < \tau_2$, that is

$$\frac{1-\sqrt{1-\alpha^2\nu^2}}{1-\alpha\beta\nu} < 1 < \frac{1+\sqrt{1-\alpha^2\nu^2}}{1-\alpha\beta\nu}.$$

It is equivalent to $0 < \nu < 1/\alpha\sqrt{1+\beta^2}$.

In conclusion, in Case 1, if $0 < \nu < 1/\alpha\sqrt{1+\beta^2}$ and $(1-\sqrt{1-\alpha^2\nu^2})/(1-\alpha\beta\nu) < \tau \leq 1$, it holds that $\|W\| < 1$.

- *Case 2:* when $\tau > 1$ and $0 < \alpha\nu < 1$. Then

$$\alpha^2\nu^2 + (\alpha\beta\tau\nu + |1-\tau|)^2 < 1 \Leftrightarrow \tilde{f}(\tau)$$
$$\doteq (\alpha\beta\nu+1)^2\tau^2 - 2(\alpha\beta\nu+1)\tau + \alpha^2\nu^2 < 0.$$

The two roots of $\tilde{f}(\tau)$ are

$$\tilde{\tau}_1 = \frac{1-\sqrt{1-\alpha^2\nu^2}}{1+\alpha\beta\nu}(>0), \quad \tilde{\tau}_2 = \frac{1+\sqrt{1-\alpha^2\nu^2}}{1+\alpha\beta\nu}.$$

(1) If $1 < \tilde{\tau}_1 < \tilde{\tau}_2$, that is $1-\sqrt{1-\alpha^2\nu^2} > 1+\alpha\beta\nu$. This is a contradictory inequality.

(2) If $0 < \tilde{\tau}_1 < 1 < \tilde{\tau}_2$, that is

$$\frac{1-\sqrt{1-\alpha^2\nu^2}}{1+\alpha\beta\nu} < 1 < \frac{1+\sqrt{1-\alpha^2\nu^2}}{1+\alpha\beta\nu}.$$

It is equivalent to $0 < \nu < 1/\alpha\sqrt{1+\beta^2}$.

Consequently, in Case 2, if $0 < \nu < 1/\alpha\sqrt{1+\beta^2}$ and $1 < \tau < (1+\sqrt{1-\alpha^2\nu^2})/(1+\alpha\beta\nu)$, it holds that $\|W\| < 1$.

According to Cases 1 and 2, from (15), we can conclude that if the conditions in (7) are satisfied, then $\|E^{(k+1)}\| < \|E^{(k)}\|$ holds for $k = 0, 1, 2, \ldots$. ■

It is easy to see that if the conditions of Theorem 2.1 are satisfied, then from (8) we have

$$0 \leq \|E^{(k)}\| \leq \|W\| \cdot \|E^{(k-1)}\| \leq \cdots \leq \|W\|^k \cdot \|E^{(0)}\|.$$

Since $\|W\| < 1$, we have $\lim_{k\to\infty} \|E^{(k)}\| = 0$. Using the definition of the 2-norm gives $\lim_{k\to\infty} \|e_k^x\| = 0$ and $\lim_{k\to\infty} \|e_k^y\| = 0$. According to Proposition 1.1, the sequence $\{x^{(k)}\}$ generated by (6) will linearly converge to the unique solution x^* of the AVE (1). In addition, if $Q = I$, the conditions in (7) reduce to those in (3).

Finally, we will go deeper in the choice of the nonsingular matrix Q in Algorithm 2.1. Since the inverse of Q is involved at each step of the MFPI method, it will increase the cost of each iteration step if Q is too complicated (for instance, if Q is dense or ill-conditioned). In order to preserve the cost of the MFPI method as the same level as that of the FPI method, one way to choose Q is let it be a diagonal matrix, especially be a scalar matrix, that is, let $Q = (1/q)I (q > 0)$. In this case, we specially have $\alpha = 1/\beta = 1/q$. Then conditions in (7) reduce to

$$0 < \nu < \frac{q}{\sqrt{1+q^2}} \quad \text{and} \quad \frac{1-\sqrt{1-\frac{\nu^2}{q^2}}}{1-\nu} < \tau < \frac{1+\sqrt{1-\frac{\nu^2}{q^2}}}{1+\nu}. \tag{16}$$

Since $h(q) = q/\sqrt{1+q^2} < 1$ is increasing with respect to $q > 0$, we can always find some $q > 0$ such that $h(q) \geq \nu$ with $0 < \nu < 1$. In particular, $\frac{\sqrt{2}}{2} \leq h(q) < 1$ whenever $q \geq 1$. Hence, we can take $Q = (1/q)I$ with suitable $q > 0$ such that the MFPI method is applicable for solving the AVE (1) with $0 < \|A^{-1}\| < 1$.

Remark 2.1: From the above discussion, one can find that the MFPI method is derived by subtlely modifying the equivalent nonlinear form of the AVE (1). However, the new equivalent nonlinear form is important since the involved nonsingular matrix Q enhances the convergent results, which makes the MFPI method converge when $\frac{\sqrt{2}}{2} \leq \nu < 1$ while the FPI does not (in the sense that the range of the iteration parameter τ is empty) in theory. Furthermore, when $0 < \nu < \frac{\sqrt{2}}{2}$, the MFPI method with suitable Q can behave better than the FPI method. Our numerical results in the next section will demonstrate our arguments.

Remark 2.2: We should note that (16) is a sufficient condition for the MFPI method converging to the unique solution of the AVE (1) with $0 < \nu < 1$ (a sufficient condition for the AVE (1) being uniquely solvable), but, theoretically, it is not applicable to the case $\nu \geq 1$. However, a weaker sufficient condition for the AVE (1) being uniquely solvable is proposed in [29, Proposition 2.1], in which $\nu \geq 1$ may arise. For instance, let

$$A = \begin{pmatrix} 1 & 3 \\ -0.01 & 1 \end{pmatrix},$$

then the AVE (1) is uniquely solvable for any $b \in \mathbb{R}^n$ [29, Example 2.3] and $\nu = 3.2058 > 1$. Numerically, we find that the MFPI (or FPI) method is still convergent for this AVE. For example, if we let $Q = I$, $b = Ax^* - |x^*|$ with $x^* = (1, -1)^T$, $\tau = 0.9$ and $x^{(0)} = y^{(0)} = (0, 0)^T$, then the MFPI (or FPI) method is convergent at iteration step 1015 under the stopping criterion $\|Ax^{(k)} - |x^{(k)}| - b\| \leq 10^{-8}$. The sufficient conditions which guarantee the convergence of the MFPI method for solving the AVE (1) with the interval matrix $[A - I, A + I]$ being regular and $\nu \geq 1$ in theory need further study.

3. Numerical experiments

In this section, we present some numerical results to demonstrate our theoretical analysis in the previous section. In our computations, all runs are implemented

Table 1. Parameters for Example 3.1.

μ	Parameters	m			
		40	60	80	100
1.2	ν	0.8253	0.8297	0.8312	0.8320
	q_l	1.4613	1.4862	1.4953	1.4997
	τ in (3)	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
	τ in (16)	$(1.89 \cdot 10^{-4}, 1.0957)$	$(1.96 \cdot 10^{-4}, 1.0931)$	$(1.99 \cdot 10^{-4}, 1.0921)$	$(2.00 \cdot 10^{-4}, 1.0917)$
	τ_{opt} in (16)	1.0957	1.0931	1.0921	1.0917
1.42	ν	0.6985	0.7016	0.7027	0.7033
	q_l	0.9760	0.9846	0.9878	0.9892
	τ in (3)	$(0.9429, 1.0101)$	$(0.9633, 1.0064)$	$(0.9707, 1.0051)$	$(0.9742, 1.0045)$
	τ in (16)	$(7.93 \cdot 10^{-5}, 1.1775)$	$(8.09 \cdot 10^{-5}, 1.1753)$	$(8.14 \cdot 10^{-5}, 1.1746)$	$(8.17 \cdot 10^{-5}, 1.1742)$
	τ_{opt} in (3)	1.01	0.99	0.99	0.99
	τ_{opt} in (16)	1.17	1.16	1.16	1.17
2.4	ν	0.4146	0.4157	0.4161	0.4163
	q_l	0.4557	0.4571	0.4577	0.4579
	τ in (3)	$(0.1538, 1.3502)$	$(0.1549, 1.3487)$	$(0.1553, 1.3482)$	$(0.1555, 1.3480)$
	τ in (16)	$(1.46 \cdot 10^{-5}, 1.4138)$	$(1.47 \cdot 10^{-5}, 1.4127)$	$(1.47 \times 10^{-5}, 1.4123)$	$(1.47 \cdot 10^{-5}, 1.4121)$
	τ_{opt} in (3)	1.33	1.31	1.32	1.32
	τ_{opt} in (16)	1.38	1.39	1.40	1.40
4	ν	0.2493	0.2497	0.2498	0.2499
	q_l	0.2574	0.2578	0.2580	0.2581
	τ in (3)	$(0.0420, 1.5757)$	$(0.0422, 1.5751)$	$(0.0423, 1.5749)$	$(0.0423, 1.5748)$
	τ in (16)	$(4.12 \cdot 10^{-6}, 1.6009)$	$(4.13 \cdot 10^{-6}, 1.6004)$	$(4.14 \cdot 10^{-6}, 1.6002)$	$(4.14 \cdot 10^{-6}, 1.6002)$
	τ_{opt} in (3) and (16)	1.24	1.24	1.25	1.25
8	ν	0.1248	0.1249	0.1250	0.1250
	q_l	0.1258	0.1259	0.1259	0.1260
	τ in (3)	$(0.0089, 1.7711)$	$(0.0090, 1.7709)$	$(0.0090, 1.7709)$	$(0.0090, 1.7709)$
	τ in (16)	$(8.88 \cdot 10^{-7}, 1.7781)$	$(8.89 \cdot 10^{-7}, 1.7779)$	$(8.90 \cdot 10^{-7}, 1.7779)$	$(8.90 \cdot 10^{-7}, 1.7778)$
	τ_{opt} in (3) and (16)	1.13	1.13	1.13	1.13

in MATLAB R2014b with a machine precision 2.22×10^{-16} on a personal computer with 2.50 GHz central processing unit (Intel(R) Core(TM) i7-6500U), 16 GB memory and Windows 10 operating system.

We compare the performances of the MFPI method (Algorithm 2.1), the FPI method [24] (Algorithm 1.1) and the generalized Newton method [15] (denoted by 'GN') for solving the AVE (1) in the sense of iteration step (denoted as 'IT'), elapsed CPU time in seconds (denoted as 'CPU') and residual error (denoted as 'RES') defined by

$$\text{RES} = \|Ax^{(k)} - |x^{(k)}| - b\|.$$

All tests are started from the initial zero vector and terminated if the current iteration satisfies $\text{RES} \le 10^{-8}$ or the number of prescribed maximal iteration steps $k_{\max} = 2000$ is exceeded.

Example 3.1 ([24]): Consider the AVE (1), where the matrix $A = \hat{A} + \mu I_{m^2} (\mu \ge 0)$ with

$$\hat{A} = \text{Tridiag}(-I_m, S_m, -I_m) \in \mathbb{R}^{m^2 \times m^2}, \quad S_m = \text{tridiag}(-1, 4, -1) \in \mathbb{R}^{m \times m},$$

and $b = \text{ones}(m^2, 1) \in \mathbb{R}^{m^2}$.

Table 2. Numerical results for Example 3.1.

μ	Method		m = 40	60	80	100
1.2	GN	IT	2	2	2	2
		CPU	**0.0369**	**0.1906**	**0.5134**	**1.2776**
		RES	1.5771×10^{-13}	2.6707×10^{-13}	2.7610×10^{-13}	3.4521×10^{-13}
	FPI	IT	104	109	112	113
		CPU	0.1559	0.4617	1.6307	2.8060
		RES	8.5302×10^{-9}	8.9109×10^{-9}	8.3419×10^{-9}	9.7051×10^{-9}
	MFPI	IT	104	109	112	113
		CPU	0.1531	0.4639	1.6906	2.9314
		RES	8.5302×10^{-9}	8.9109×10^{-9}	8.3419×10^{-9}	9.7051×10^{-9}
1.42	GN	IT	2	2	2	2
		CPU	**0.0336**	**0.1690**	**0.5274**	**1.2279**
		RES	8.3834×10^{-14}	1.4353×10^{-13}	1.4347×10^{-13}	1.8340×10^{-13}
	FPI	IT	61	64	65	66
		CPU	0.0903	0.2654	0.9869	1.6598
		RES	7.9735×10^{-9}	9.2782×10^{-9}	9.8354×10^{-9}	9.2712×10^{-9}
	MFPI	IT	51	53	54	54
		CPU	0.0779	0.2166	0.8357	1.2911
		RES	8.2721×10^{-9}	8.9575×10^{-9}	8.8472×10^{-9}	9.3054×10^{-9}
2.4	GN	IT	2	2	2	2
		CPU	0.0368	0.1569	0.5476	1.3430
		RES	3.6684×10^{-14}	5.8045×10^{-14}	6.6883×10^{-14}	8.3136×10^{-14}
	FPI	IT	15	16	16	16
		CPU	0.0265	0.0707	0.2428	0.4743
		RES	9.6641×10^{-9}	7.8702×10^{-9}	7.6054×10^{-9}	9.8274×10^{-9}
	MFPI	IT	14	14	14	14
		CPU	**0.0238**	**0.0577**	**0.2142**	**0.3932**
		RES	6.8836×10^{-9}	7.9089×10^{-9}	7.4391×10^{-9}	9.6418×10^{-9}
4	GN	IT	2	2	2	2
		CPU	0.0357	0.1584	0.5396	1.2266
		RES	2.2991×10^{-14}	3.7255×10^{-14}	4.1963×10^{-14}	5.3741×10^{-14}
	FPI	IT	9	9	9	9
		CPU	**0.0159**	0.0412	0.1484	0.2620
		RES	4.5325×10^{-9}	7.4079×10^{-9}	4.2489×10^{-9}	5.3965×10^{-9}
	MFPI	IT	9	9	9	9
		CPU	0.0163	**0.0398**	**0.1372**	**0.2214**
		RES	4.5325×10^{-9}	7.4079×10^{-9}	4.2489×10^{-9}	5.3965×10^{-9}
8	GN	IT	2	2	2	2
		CPU	0.0393	0.1713	0.5137	1.2542
		RES	1.6534×10^{-14}	2.7356×10^{-14}	2.9544×10^{-14}	3.7049×10^{-14}
	FPI	IT	6	6	6	6
		CPU	0.0124	0.0303	0.1080	0.1715
		RES	5.7729×10^{-9}	6.7102×10^{-9}	7.5585×10^{-9}	8.3452×10^{-9}
	MFPI	IT	6	6	6	6
		CPU	**0.0113**	**0.0289**	**0.1039**	**0.1455**
		RES	5.7729×10^{-9}	6.7102×10^{-9}	7.5585×10^{-9}	8.3452×10^{-9}

In Table 1, we list the value of $\nu = \|A^{-1}\|$, the lower bound q_l of q satisfies $\nu \leq q/\sqrt{1+q^2}$, the range of the iteration parameter τ according to (3) in Theorem 1.1 and (7) in Theorem 2.1 with $Q = (1/q)I$ and $q = q_l + 100$, and the experimentally optimal parameter τ_{opt} for Example 3.1 with different values of μ. We can see from Table 1 that as ν increasing, the range of τ in (3) becomes smaller and smaller and it will be empty when $\nu \geq \frac{\sqrt{2}}{2}$. In addition, the range of τ in (16) is larger than that in (3). However, as noted in [24], numerical tests indicate that conditions (3) can be relaxed. Thus, for test purpose, when the range of the parameter τ according to (3) is empty, the FPI method will use τ_{opt} of the MFPI method as the iteration parameter.

Numerical results are reported in Table 2. From Table 2, we find that the GN method is the best one when $\mu = 1.2$ and $\mu = 1.42$ in terms of IT and CPU time. However, for $\mu = 2.4, 4$ and 8, the GN method takes the most CPU time though it needs the least number of IT. When $\mu = 1.42$ and 2.4, the MFPI method is better than the FPI method for this example. In addition, when $\mu = 1.2$ ($\nu = 0.8253 > \frac{\sqrt{2}}{2}$), the FPI method with the τ_{opt} in (16) still converges (this is the case that there is still no theoretical guarantee; see for example [24]). In conclusion, the MFPI method with suitable parameters is linearly convergent for solving the AVE (1) with $0 < \nu = \|A^{-1}\| < 1$. In addition, when $0 < \nu = \|A^{-1}\| < \frac{\sqrt{2}}{2}$, the MFPI method with suitable Q can be superior to the FPI method.

4. Conclusions

In this paper, by reformulating the AVE (1) as a new two-by-two block non-linear equation, we present the MFPI method for solving the AVE (1) with $0 < \|A^{-1}\| < 1$. The linear convergence of the proposed method is studied. In addition, the selection strategy of the nonsingular matrix Q (which is involved in the MFPI method) is discussed. Numerical results are provided to demonstrate our claims and our method is better than the FPI method [24] in some case.

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Disclosure statement

No potential conflict of interest was reported by the authors.

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