NP-Completeness of the Linear Complementarity Problem[1]

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Abstract. We consider the linear complementarity problem (q, M) for which the data are the integer column vector $q \in R^n$ and the integer square matrix M of order n. GLCP is the decision problem: Does (q, M) have a solution? We show that GLCP is NP-complete in the strong sense.

Key Words. Linear complementarity problems, NP-completeness, NP-completeness in the strong sense, 0-1 integer programming problems, 0-1 knapsack problems.

1. Introduction

We use the following terms and abbreviations in this paper:

Decision Problem: a problem with some specified input data, for which the answer is either yes or no.

NP: abbreviation for nondeterministic polynomial; NP refers to the class of decision problems solvable in polynomial time by a nondeterministic algorithm.

NP-complete class: this is the subclass of problems in NP to which every problem in NP can be reduced by a polynomial time reduction.

LCP: linear complementarity problem.

GLCP: this is an abbreviation for the general linear complementarity problem.

FKP: the decision problem of checking the feasibility of a 0-1 equality constrained knapsack problem.

KLCP: an LCP equivalent to the FKP.

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FIP: the decision problem of checking the feasibility of a system of equality constraints in 0-1 variables.

IPLCP: an LCP equivalent to the FIP.

The LCP is an important problem in mathematical programming. The practical importance of the LCP is evidenced by the fact that linear programming problems, convex quadratic programming problems, and problems of computing equilibrium points of bimatrix games can be transformed into LCPs (see Chap. 16 in Ref. 1). Recent papers (Refs. 2 and 3) on the computational complexity of complementary pivot methods indicate that these methods take $O(2^n)$ pivot steps in the worst case, even for solving an LCP of order n with data possessing special nice properties.

In this paper, we employ complexity theory to show that the decision problem of determining whether the LCP with integer data has a solution is NP-complete in the strong sense (Ref. 4).

2. NP-Completeness of GLCP

Let I be the identity matrix of order n. In general, I_r denotes the unit matrix of order r.

We define the decision problem GLCP as follows.

Problem GLCP

Input: An arbitrary integer square matrix M of order n and an integer column vector $q=(q_1, q_2, \dots q_n)^T$.

Question: Whether there exists a rational solution $w=(w_i)\in R^n$ and $z=(z_i)\in R^n$ satisfying

$$Iw - Mz = q, \tag{1a}$$

$$w, z \geq 0, \tag{1b}$$

$$w_i z_i = 0, \qquad i=1 \text{ to } n. \tag{1c}$$

The problem of finding vectors w and z satisfying (1) is an LCP, denoted by (q, M). In this LCP, the pair of variables (w_i, z_i) is known as the ith complementary pair of variables. A vector $y=(y_1, y_2, \dots, y_n)$, with $y_j \in \{w_j, z_j\}$, $j=1$ to n, is called a complementarity vector of variables in this LCP.

Theorem 2.1. Problem GLCP is NP-complete.

Proof. A nondeterministic algorithm can guess a complementarity basic vector and then check its feasibility in polynomial time, since the feasibility check amounts to solving a system of linear equations in nonnegative variables. Hence, GLCP $\in$ NP.

We now want to show that a known NP-complete problem reduces to GLCP in polynomial time. Consider FKP, a decision problem of checking the feasibility of a 0-1 equality constrained knapsack problem, which we describe below.

Problem FKP

Input: $n+1$ positive integer values $a_1, a_2, \ldots, a_n$ and b.
Problem: Does $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ have a 0-1 solution?

Problem FKP is known to be NP-complete (Ref. 5).

We construct an LCP $(\tilde{q}, \tilde{M})$ of order $n+2$ corresponding to this FKP, where $\tilde{M}$ is an $(n+2)\times(n+2)$ matrix and $\tilde{q}$ is an $(n+2)$-dimensional column vector defined as follows:

$$\tilde{q}_i = a_i, \qquad i = 1 \text{ to } n,$$

$$\tilde{q}_{n+1} = -b, \qquad \tilde{q}_{n+2} = b;$$

$$\tilde{M} = (\tilde{m}_{ij}),$$

$$\tilde{m}_{ij} = \begin{cases} -1, & i = j = 1 \text{ to } n+2, \\ 1, & j = 1 \text{ to } n \text{ with } i = n+1, \\ -1, & j = 1 \text{ to } n \text{ with } i = n+2, \\ 0, & \text{otherwise.} \end{cases}$$

We then get the following problem, which we call KLCP.

Problem KLCP

$$\begin{bmatrix} \\ I_{n+2} \\ \\ \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_i \\ \vdots \\ w_n \\ w_{n+1} \\ w_{n+2} \end{bmatrix} - \left[\begin{array}{c|cc} & 0 & 0 \\ & 0 & 0 \\ -I_n & \vdots & \vdots \\ & \vdots & \vdots \\ & 0 & 0 \\ \hline e_n^T & -1 & 0 \\ -e_n^T & 0 & -1 \end{array}\right] \begin{bmatrix} z_1 \\ \vdots \\ z_i \\ \vdots \\ z_n \\ z_{n+1} \\ z_{n+2} \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \\ -b \\ b \end{bmatrix}, \tag{2a}$$

$$w_i \geq 0, \qquad z_i \geq 0, \qquad w_iz_i = 0, \qquad i = 1 \text{ to } n+2, \tag{2b}$$

where, for each k, $e_k = (1, \ldots, 1)^T$ is the k-dimensional column vector whose components are all 1's. This problem is the LCP$(\tilde{q}, \tilde{M})$.

Claim 2.1. Problem FKP has a solution if and only if the corresponding instance of KLCP has a solution.

Neccessity. Let $\tilde{x}$ be a solution of FKP. Define

$$\tilde{w}_{n+1} = \tilde{w}_{n+2} = \tilde{z}_{n+1} = \tilde{z}_{n+2} = 0;$$

and, for $i = 1$ to n, define

$$\tilde{w}_i = a_i(1 - \tilde{x}_i), \qquad \tilde{z}_i = a_i\tilde{x}_i.$$

Then, clearly,

$$\tilde{w}_i \geq 0, \qquad \tilde{z}_i \geq 0, \qquad \tilde{w}_i\tilde{z}_i = 0, \qquad i = 1 \text{ to } n+2.$$

Also, it is easy to see that

$$\tilde{w}_i + \tilde{z}_i = a_i, \qquad i = 1 \text{ to } n,$$

$$\tilde{w}_{n+1} - \tilde{z}_1 - \cdots - \tilde{z}_n + \tilde{z}_{n+1} = -b,$$

$$\tilde{w}_{n+2} + \tilde{z}_1 + \cdots + \tilde{z}_n + \tilde{z}_{n+2} = b.$$

Hence, $(\tilde{w}, \tilde{z})$ is a solution of the LCP $(\tilde{q}, \tilde{M})$.

Sufficiency. Let $(\hat{w}, \hat{z})$ be a solution of KLCP. Define

$$\hat{x}_i = \hat{z}_i / a_i, \qquad i = 1 \text{ to } n.$$

Since

$$\hat{w}_i\hat{z}_i = 0, \qquad \hat{w}_i + \hat{z}_i = a_i, \qquad i = 1 \text{ to } n,$$

$\hat{z}_i$ is either 0 or a_i. This implies that $\hat{x}_i = 0$ or 1. After adding the last two rows in (2), we get

$$\hat{w}_{n+1} + \hat{w}_{n+2} + \hat{z}_{n+1} + \hat{z}_{n+2} = 0.$$

This and nonnegativity imply that

$$\hat{w}_{n+1} = \hat{w}_{n+2} = \hat{z}_{n+1} = \hat{z}_{n+2} = 0.$$

Thus,

$$\hat{z}_1 + \cdots + \hat{z}_n = b, \qquad \text{if and only if } a_1\hat{x}_1 + \cdots + a_n\hat{x}_n = b.$$

Hence, $\hat{x}$ is a solution of the corresponding FKP. This completes the proof of our claim. □

From Claim 2.1, it is clear that FKP reduces to KLCP in polynomial time. Therefore, Theorem 2.1 follows from the fact that KLCP is a special case of GLCP. □

Now, let $\bar{M}$ be the matrix obtained from $\tilde{M}$, by setting

$$\tilde{m}_{ii} = -(n/2), \qquad i = n+1 \text{ and } n+2,$$

but leaving everything else unchanged. Then, it can be verified that $\bar{M}$ is a negative-semidefinite matrix. Using the same proof as that of Theorem 2.1, it can be verified that the LCP $(\tilde{q}, \bar{M})$ has a solution if and only if the FKP defined in Theorem 2.1 has a solution. Again, let $\bar{\bar{M}}$ be the matrix obtained from $\tilde{M}$ by setting

$$\tilde{m}_{n+1,n+1} = \tilde{m}_{n+2,n+2} = \text{an integer} < -n/2,$$

but leaving everything else unchanged. It can be verified that $\bar{\bar{M}}$ is negative definite. Using the same proof as that of Theorem 2.1, it can be verified that the LCP $(\tilde{q}, \bar{\bar{M}})$ has a solution if and only if the FKP defined in Theorem 2.1 has a solution. From this, we are led to the following corollary.

Corollary 2.1. The decision problem of checking whether the LCP (q, M) has a solution, is NP-complete, even if M is restricted to be negative definite or negative semidefinite.

3. NP-Completeness of GLCP in the Strong Sense

Garey and Johnson (Ref. 6) introduced the concept of pseudopolynomial time algorithm for decision problems. An algorithm for a decision problem is said to be a pseudopolynomial time algorithm if the computational effort required by it is bounded above by a polynomial function of the maximum among the absolute values of the data elements in the problem and the number of such data elements. The concept of strong NP-completeness, introduced by Garey and Johnson (Ref. 6), is related to the possibility of the existence of a pseudopolynomial time algorithm for the problem. If a decision problem is NP-complete in the strong sense, no pseudopolynomial time algorithm can exist for it unless all NP-complete problems can be solved in polynomial time.

Problem FKP has an algorithm whose worst-case computational complexity is $O(nb)$, which is therefore a pseudopolynomial time algorithm for that problem. So, our proof of Theorem 2.1 does not establish that GLCP is NP-complete in the strong sense. However, we now show that in fact GLCP is NP-complete in the strong sense.

Theorem 3.1. Problem GLCP is NP-complete in the strong sense.

Proof. Consider FIP, a decision problem of checking the feasibility of a 0-1 equality constrained integer programming problem defined as follows:

Problem FIP

Input: An $m \times n$ integer matrix A and an m-dimensional integer column vector $d = (d_i)$.

Question: Whether there exists a 0-1 solution $x = (x_j)$ satisfying

$$Ax = d,$$

$$x_j = 0 \text{ or } 1, \qquad j = 1 \text{ to } n.$$

Problem FIP is known to be NP-complete in the strong sense (Ref. 4). For any given instance of FIP, we can construct a corresponding instance of LCP $(\hat{q}, \hat{M})$, where $\hat{M}$ is a square matrix of order $(n+m+1) \times (n+m+1)$ and $\hat{q}$ is $(n+m+1)$-dimensional column vector, defined as follows:

$$\hat{M} = \left[\begin{array}{c|cc} -I_n & 0 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline A & -I_m & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline -e_m^T A & 0 \cdots 0 & -1 \end{array}\right], \qquad \hat{q} = \left[\begin{array}{c} e_n \\ \hline -d \\ \hline e_m^T d \end{array}\right].$$

We denote the LCP $(\hat{q}, \hat{M})$ by IPLCP. Then, by using the same method of proof as in Theorem 2.1, we can verify that IPLCP has a solution if and only if FIP has a solution. Since FIP is NP-complete in the strong sense, this shows that GLCP is NP-complete in the strong sense. □

Corollary 3.1. The decision problem of checking whether an LCP (q, M) has a solution is NP-complete in the strong sense even if M is restricted to be negative definite or negative semidefinite.

Proof. The proof of this corollary is essentially similar to the proof of Corollary 2.1. □

Corollary 3.2. The problem of finding a 0-1 solution to a system of linear equations can be transformed into an LCP.

Proof. The proof of Theorem 3.1 provides the necessary transformation, thus establishing this corollary. □

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