Optimal Bayesian Kalman Filtering with Prior Update

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Overview

What's this Paper?

- Introduce a new Kalman Filter: Optimal Bayesian Kalman Filter(OBKF)[Dehghannasiri et al., 2018]
- OBKF is a advanced Kalman Filter of IBR Kalman Filter[Dehghannasiri et al., 2017]
- OBKF exploit the measured data to estimate the true unknown value

Kalman Filter: Basic idea

- Outputs the optimal estimation of the state of a dynamic system
- Exploits both the prediction based on the model and the measurement
- Application Example: Robot Localization

Kalman Filter: Algorithm I

Kalman Filter works for linear dynamic systems:

Linear Dynamic System Example

$$\mathbf{x}_{k+1} = \mathbf{\Phi}_k \mathbf{x}_k + \mathbf{\Gamma}_k \mathbf{u}_k$$
$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k$$

- $\mathbf{x}_k$: *state vector*
- $\mathbf{y}_k$: *observation vector*
- $\mathbf{u}_k$, $\mathbf{v}_k$: *zero-mean noise vector*
- $\mathbf{\Phi}_k$, $\mathbf{\Gamma}_k$, $\mathbf{H}_k$: *transition matrix*

Kalman Filter: Algorithm II

Algorithm 1 Classic Kalman Filter

Input: $\hat{\mathbf{x}}_k$, ${\mathbf{P}^{\mathbf{x}}}_k$, $\mathbf{y}_k$

1: $\tilde{\mathbf{z}}_k = \mathbf{y}_k - \mathbf{H}_k \hat{\mathbf{x}}_k$

2: $\mathbf{K}_k = {\mathbf{P}^{\mathbf{x}}}_k \mathbf{H}_k^T (\mathbf{H}_k {\mathbf{P}^{\mathbf{x}}}_k \mathbf{H}_k^T + \mathbf{R})^{-1}$

3: $\hat{\mathbf{x}}_{k+1} = \boldsymbol{\Phi}_k \hat{\mathbf{x}}_k + \boldsymbol{\Phi}_k \mathbf{K}_k \tilde{\mathbf{z}}_k$

4: ${\mathbf{P}^{\mathbf{x}}}_{k+1} = \boldsymbol{\Phi}_k (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) {\mathbf{P}^{\mathbf{x}}}_k \boldsymbol{\Phi}_k^T + \boldsymbol{\Gamma}_k \mathbf{Q} \boldsymbol{\Gamma}_k^T$

Output: $\hat{\mathbf{x}}_{k+1}$, ${\mathbf{P}^{\mathbf{x}}}_{k+1}$

- $\hat{\mathbf{x}}_k$: *estimated mean*
- ${\mathbf{P}^{\mathbf{x}}}_k$: *estimated covariance*
- $\mathbf{R}$: *Noise covariance of* $\mathbf{u}_k$
- $\mathbf{Q}$: *Noise covariance of* $\mathbf{v}_k$

Problems of Kalman Filter

- The performance is sensitive to the accuracy of the noise covariance matrixes: $\mathbf{Q}$ and $\mathbf{R}$[Sangsuk-Iam and Bullock, 1990]

Uncertainty of the noise covariances

Assume covariances are parameterized by $\theta = [\theta_1, \theta_2]$

- The covariance matrix of $\mathbf{u}$ is $\mathbf{Q}^{\theta_1}$ (e.g. $\mathbf{Q}^{\theta_1} = \theta_1 \mathbf{I}$)
- The covariance matrix of $\mathbf{v}$ is $\mathbf{R}^{\theta_2}$ (e.g. $\mathbf{R}^{\theta_2} = \theta_2 \mathbf{I}$)
- Unknown prameter: $\theta = [\theta_1, \theta_2]$

If the θ used in the Algorithm is much different from the true θ, Kalman Filter provides poor estimation

Solutions of the Problem

- Non-Bayesian Approach: Adaptive Kalman Filter
  - Doesn't require any prior knowledge (θ is not a R.V.)
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  - Require many measured data
- Bayesian Approach: IBR Kalman Filter
  - Require prior knowledge (θ is a R.V.)
  - Robust: Guarantee the best average performance relative to the prior distribution
- Bayesian Approach: Optimal Bayesian Kalman Filter
  - Require prior knowledge (θ is a R.V.)
  - Utilize measured data to estimate unknown parameters
  - Optimal over the posterior distribution obtained from measured data

Basic Idea

- Assume the prior distribution $\pi(\theta)$ for the uncertainty of the covariance matrices
- Best estimation relative to $\pi(\theta)$:
  $\arg\min_{\hat{\mathbf{x}}_\theta(k)} E_\theta[E[(\mathbf{x}_\theta(k) - \hat{\mathbf{x}}_\theta(k))^T \times (\mathbf{x}_\theta(k) - \hat{\mathbf{x}}_\theta(k))]]$

Algorithm: IBR Kalman Filter

- IBR Kalman Filter is obtained just replacing $\mathbf{Q}$ and $\mathbf{R}$ in Calssic Kalman Filter with $E_{\theta_1}[\mathbf{R}^{\theta_1}]$ and $E_{\theta_2}[\mathbf{Q}^{\theta_2}]$ respectively

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Algorithm 2 IBR Kalman Filter

Input: $\hat{\mathbf{x}}_k^\theta$, $E_\theta[\mathbf{P}^{\mathbf{x},\theta}{}_k]$, $\mathbf{y}_k^\theta$

1: $\tilde{\mathbf{z}}_k^\theta = \mathbf{y}_k^\theta - \mathbf{H}_k \hat{\mathbf{x}}_k^\theta$

2: $\mathbf{K}_k^{\mathbf{\Theta}} = E_\theta[\mathbf{P}^{\mathbf{x},\theta}{}_k]\mathbf{H}_k^T E_\theta^{-1}[\mathbf{H}_k \mathbf{P}^{\mathbf{x},\theta}{}_k \mathbf{H}_k^T + \mathbf{R}^{\theta_2}]$

3: $\hat{\mathbf{x}}_{k+1}^\theta = \mathbf{\Phi}_k \hat{\mathbf{x}}_k^\theta + \mathbf{\Phi}_k \mathbf{K}_k^{\mathbf{\Theta}} \tilde{\mathbf{z}}_k^\theta$

4: $E_\theta[\mathbf{P}^{\mathbf{x},\theta}{}_{k+1}] = \mathbf{\Phi}_k(\mathbf{I} - \mathbf{K}_k^{\mathbf{\Theta}} \mathbf{H}_k) E_\theta[\mathbf{P}^{\mathbf{x},\theta}{}_k]\mathbf{\Phi}_k^T + \mathbf{\Gamma}_k E_{\theta_1}[\mathbf{Q}^{\theta_1}{}_k]\mathbf{\Gamma}_k^T$

Output: $\hat{\mathbf{x}}_{k+1}^\theta$, $E_\theta[\mathbf{P}^{\mathbf{x},\theta}{}_{k+1}]$

Basic Idea: OBKF

- Based on the IBR Kalman Filter
- Utilize measured data $\mathcal{Y}_k = \{\mathbf{y}_0, ..., \mathbf{y}_k\}$to obtain the posterior distribution $\pi(\theta|\mathcal{Y}_k)$
- Best estimation relative to $\pi(\theta|\mathcal{Y}_{k-1})$:
$\arg\min_{\hat{\mathbf{x}}_\theta(k)} E_\theta[E[(\mathbf{x}_\theta(k) - \hat{\mathbf{x}}_\theta(k))^T \times (\mathbf{x}_\theta(k) - \hat{\mathbf{x}}_\theta(k))]|\mathcal{Y}_{k-1}]$

Algorithm: OBKF

- OBKF is obtained just replacing **Q** and **R** in Calssic Kalman Filter with $E_{\theta_1}[\mathbf{Q}^{\theta_1}|\mathcal{Y}_k]$ and $E_{\theta_1}[\mathbf{R}^{\theta_2}|\mathcal{Y}_k]$ respectively

Algorithm 3 OBKF

Input: $\hat{\mathbf{x}}_k^{\theta}$, $E_{\theta}[\mathbf{P}^{\mathbf{x},\theta}{}_k|\mathcal{Y}_{k-1}]$, $\mathcal{Y}_k$

1: $\tilde{\mathbf{z}}_k^{\theta} = \mathbf{y}_k^{\theta} - \mathbf{H}_k\hat{\mathbf{x}}_k^{\theta}$

2: $\mathbf{K}_k^{\mathbf{\Theta}^*} = E_{\theta}[\mathbf{P}^{\mathbf{x},\theta}{}_k|\mathcal{Y}_{k-1}]\mathbf{H}_k^T E_{\theta}^{-1}[\mathbf{H}_k\mathbf{P}^{\mathbf{x},\theta}{}_k\mathbf{H}_k^T + \mathbf{R}^{\theta_2}|\mathcal{Y}_{k-1}]$

3: $\hat{\mathbf{x}}_{k+1}^{\theta} = \mathbf{\Phi}_k\hat{\mathbf{x}}_k^{\theta} + \mathbf{\Phi}_k\mathbf{K}_k^{\mathbf{\Theta}}\tilde{\mathbf{z}}_k^{\theta}$

4: $E_{\theta}[\mathbf{P}^{\mathbf{x},\theta}{}_{k+1}|\mathcal{Y}_k] = \mathbf{\Phi}_k(\mathbf{I} - \mathbf{K}_k^{\mathbf{\Theta}^*}\mathbf{H}_k)E_{\theta}[\mathbf{P}^{\mathbf{x},\theta}{}_k|\mathcal{Y}_k]\mathbf{\Phi}_k^T + \mathbf{\Gamma}_k E_{\theta_1}[\mathbf{Q}^{\theta_1}|\mathcal{Y}_k]\mathbf{\Gamma}_k^T$

Output: $\hat{\mathbf{x}}_{k+1}^{\theta}$, $E_{\theta}[\mathbf{P}^{\mathbf{x},\theta}{}_{k+1}|\mathcal{Y}_k]$

Algorithm: OBKF

- OBKF is obtained just replacing $\mathbf{Q}$ and $\mathbf{R}$ in Calssic Kalman Filter with $E_{\theta_1}[\mathbf{Q}^{\theta_1}|\mathcal{Y}_k]$ and $E_{\theta_1}[\mathbf{R}^{\theta_2}|\mathcal{Y}_k]$ respectively

Algorithm 4 OBKF

Input: $\hat{\mathbf{x}}_k^\theta$, $E_\theta[\mathbf{P}^{\mathbf{x},\theta}{}_k|\mathcal{Y}_{k-1}]$, $\mathcal{Y}_k$

1: $\tilde{\mathbf{z}}_k^\theta = \mathbf{y}_k^\theta - \mathbf{H}_k\hat{\mathbf{x}}_k^\theta$

2: $\mathbf{K}_k^{\mathbf{\Theta}^*} = E_\theta[\mathbf{P}^{\mathbf{x},\theta}{}_k|\mathcal{Y}_{k-1}]\mathbf{H}_k^T E_\theta^{-1}[\mathbf{H}_k\mathbf{P}^{\mathbf{x},\theta}{}_k\mathbf{H}_k^T + \mathbf{R}^{\theta_2}|\mathcal{Y}_{k-1}]$

3: $\hat{\mathbf{x}}_{k+1}^\theta = \mathbf{\Phi}_k\hat{\mathbf{x}}_k^\theta + \mathbf{\Phi}_k\mathbf{K}_k^{\mathbf{\Theta}}\tilde{\mathbf{z}}_k^\theta$

4: $E_\theta[\mathbf{P}^{\mathbf{x},\theta}{}_{k+1}|\mathcal{Y}_k] = \mathbf{\Phi}_k(\mathbf{I} - \mathbf{K}_k^{\mathbf{\Theta}^*}\mathbf{H}_k)E_\theta[\mathbf{P}^{\mathbf{x},\theta}{}_k|\mathcal{Y}_k]\mathbf{\Phi}_k^T + \mathbf{\Gamma}_k E_{\theta_1}[\mathbf{Q}^{\theta_1}|\mathcal{Y}_k]\mathbf{\Gamma}_k^T$

Output: $\hat{\mathbf{x}}_{k+1}^\theta$, $E_\theta[\mathbf{P}^{\mathbf{x},\theta}{}_{k+1}|\mathcal{Y}_k]$

How to find the posterior expectations: $E_{\theta_1}[\mathbf{Q}^{\theta_1}|\mathcal{Y}_k]$ and $E_{\theta_2}[\mathbf{R}^{\theta_2}|\mathcal{Y}_k]$?

Find Posterior Expectatoins: $E_{\theta_1}[\mathbf{Q}^{\theta_1}|\mathcal{Y}_k]$ and $E_{\theta_2}[\mathbf{R}^{\theta_2}|\mathcal{Y}_k]$

- Approximate $E_{\theta_1}[\mathbf{Q}^{\theta_1}|\mathcal{Y}_k]$ and $E_{\theta_1}[\mathbf{R}^{\theta_2}|\mathcal{Y}_k]$ using Metropolis Hastings MCMC
- MCMC requires the likelihood function $f(\mathcal{Y}_k|\theta)$
- $f(\mathcal{Y}_k|\theta)$ is calculated by:
  - Marginalize $f(\mathcal{Y}_k, \mathcal{X}_k|\theta)$ over $\mathcal{X}_k$
  - Factorize $f(\mathcal{Y}_k, \mathcal{X}_k|\theta)$ by Markov assumption
  - Using factor-graph, this step can be simplified
  - $f(\mathcal{Y}_k|\theta)$ can be calculated by recursive algorithm

Algorithm: Likelihood Function Calculation

Algorithm 5 Factor-Graph-Based Likelihood Function Calculation

Input: θ, $\mathcal{Y}_k$

$\mathbf{M}_0 \leftarrow E[\mathbf{x}_0],\ S_0 \leftarrow 1,\ \mathbf{\Sigma}_0 \leftarrow cov[\mathbf{x}_0],\ i \leftarrow 0$

while $i \leq k-1$ **do**

$\quad \mathbf{W}_i \leftarrow \mathbf{H}_i^T(\mathbf{R}^{\theta_2})^{-1}\mathbf{y}_i + \mathbf{\Sigma}_i^{-1}\mathbf{M}_i$

$\quad \mathbf{\Lambda}_i^{-1} \leftarrow \mathbf{\Phi}_i^T(\tilde{\mathbf{Q}}_i^{\theta_1})^{-1}\mathbf{\Phi}_i + \mathbf{H}_i^T(\mathbf{R}^{\theta_2})^{-1}\mathbf{H}_i + \mathbf{\Sigma}_i^{-1}$

$\quad \mathbf{\Sigma}_{i+1}^{-1} \leftarrow (\tilde{\mathbf{Q}}_i^{\theta_1})^{-1} - (\tilde{\mathbf{Q}}_i^{\theta_1})^{-1}\mathbf{\Phi}_i\mathbf{\Lambda}_i\mathbf{\Phi}_i^T(\tilde{\mathbf{Q}}_i^{\theta_1})^{-1}$

$\quad \mathbf{M}_{i+1} \leftarrow \mathbf{\Sigma}_{i+1}(\tilde{\mathbf{Q}}_i^{\theta_1})^{-1}\mathbf{\Phi_i}\mathbf{\Lambda}_i(\mathbf{H}_i^T(\mathbf{R}^{\theta_2})^{-1}\mathbf{y}_i + \mathbf{\Sigma}_i^{-1}\mathbf{M}_i)$

$\quad S_{i+1} \leftarrow$ using Eq.(29) in the paper

$\quad i \leftarrow i+1$

end while

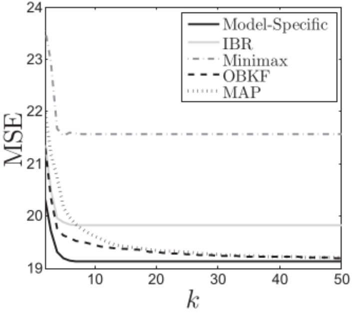
$\mathbf{\Delta}_k^{-1} \leftarrow \mathbf{H}_k^T(\mathbf{R}^{\theta_2})^{-1}\mathbf{H}_k + \mathbf{\Sigma}_k^{-1}$

$\mathbf{G}_k \leftarrow \mathbf{\Delta}_k(\mathbf{H}_k^T(\mathbf{R}^{\theta_2})^{-1}\mathbf{y}_k + \mathbf{\Sigma}_k^{-1}\mathbf{M}_k)$

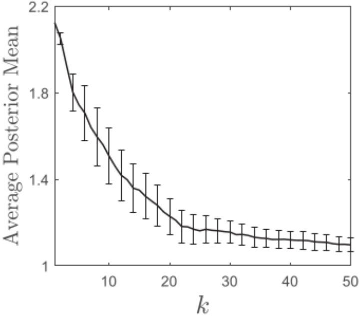
$f(\mathcal{Y}_k|\theta) \leftarrow$ using Eq.(34) in the paper

Output: $f(\mathcal{Y}_k|\theta)$

Performance: Accuracy



(a)



(b)

Figure: Performance analysis for specific $\theta_1 = 1$ and known θ_2
(a) OBKF achieves the lowest MSE
(b) Empirical average and variance of $E[\theta_1|\mathcal{Y}_k]$
(cited from [Dehghannasiri et al., 2018])

Performance: Data Efficiency

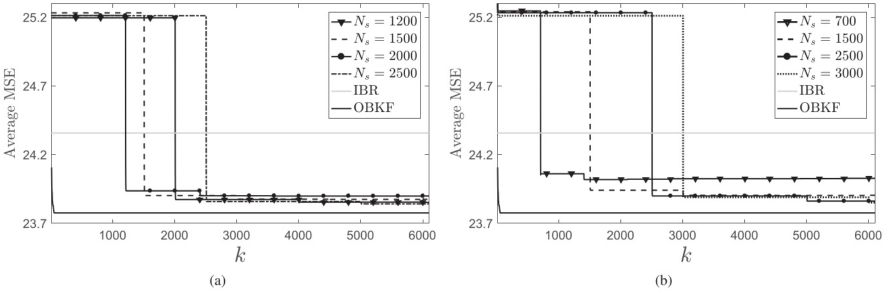


Figure: Unknown θ_1 and comparison with two adaptive Kalman Filter methods
(a) Myers method
(b) Mehra method
(cited from [Dehghannasiri et al., 2018])

Problems and Future Works

- If the prior distribution doesn't include the true value, the estimation won't converge
- Factor-graph and MCMC are computationally expensive. Finding efficient approach is a future work.

Summery

- OBKF converges to the optimal estimation as long as the prior distribution includes the true value
- OBKF requires much less data compared to Non-Bayesian methods
- The computational cost and deciding prior distribution are problems

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