

Project Report EEC264

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March 17, 2019

I. INTRODUCTION

This report explains about the paper "Optimal Bayesian Kalman Filtering with Prior Update"[1]. The optimal bayesian Kalman Filter(OBKF) is an advanced version of the intrinsically bayesian robust Kalman filter(IBRKF)[2], which is a Kalman filter exploiting the prior knowledge for the model.

The Kalman filter(classic KF)[3] is a widely used technique to estimate the state vector of a linear dynamics system from its previous estimation and the measurement. Although it provides the best estimation in some condition, it has a problem. That is, the classic KF is highly sensitive to the noise covariance of the target linear dynamic system[4]. To manage this problem, mainly two robust approaches, bayesian approach and non-bayesian approach, have been studied.

The adaptive Kalman filter[5][6] is a non-bayesian approach to achieve the robustness towards the uncertain system. It achieves the robustness by estimating the covariance matrixes during the state estimation. Although it doesn't require any prior knowledge, it needs a lot of observation data to tune the parameter. For a certain problem like gene regulatory network, the cost to obtain the data is expensive, so algorithms which doesn't require a lot of data are preferred.

On the other hand, because of its prior knowledge, the bayesian approach doesn't require so many data compared with non-bayesian approaches. The bayesian approach Kalman filter, IBRKF, is robust in the sense that it minimizes the average MSE relative to the prior distribution.

The IBRKF achieves the robustness in bayesian sense, but it doesn't utilizes the any information obtained from the observation. Exploiting the both prior knowledge and the observation data, the OBKF achieves the optimal estimation relative to the posterior distribution.

The rest of this paper is organized as follows. Section II explains the overview of the classic Kalman filter and its problem. In section III, a bayesian approach, IBR Kalman filter is explained. Section IV describes the OBKF, and

section V shows the simulated results and the performance of the OBKF. Finally, section VI summerizes the OBKF.

The results of the simulation on my laptop are shown in Fig. 1 and Fig. 2. The simulation will take about 3 or 4 hours.

II. KALMAN FILTER

The classic KF is the optimal linear estimator for linear dynamic systems. A linear dynamic system is written as the following equations.

$$\mathbf{x}_{k+1} = \mathbf{\Phi}_k \mathbf{x}_k + \mathbf{\Gamma}_k \mathbf{u}_k \quad (1)$$

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k \quad (2)$$

Where,

- \mathbf{x}_k : state vector
- \mathbf{y}_k : observation vector
- $\mathbf{u}_k, \mathbf{v}_k$: zero-mean gaussian noise vector
- $\mathbf{\Phi}_k, \mathbf{\Gamma}_k, \mathbf{H}_k$: transition matrix

And the noise statistics are as follows.

$$\mathbb{E}[\mathbf{u}_k \mathbf{u}_l^T] = \mathbf{Q} \delta_{kl}, \quad \mathbb{E}[\mathbf{v}_k \mathbf{v}_l^T] = \mathbf{R} \delta_{kl}, \quad \forall k, l = 0, 1, \dots \quad (3)$$

$$\mathbb{E}[\mathbf{v}_k \mathbf{x}_l^T] = \mathbf{0}, \quad \mathbb{E}[\mathbf{u}_k \mathbf{v}_l^T] = \mathbf{0} \quad \forall k, l = 0, 1, \dots \quad (4)$$

$$\mathbb{E}[\mathbf{u}_k \mathbf{y}_l^T] = \mathbf{0}, \quad 0 \leq l \leq k, \quad (5)$$

For such linear dynamic systems, the classic KF provides the linear estimation $\hat{\mathbf{x}}_k$ minimizing the mean squared error based on the observations $\mathbf{y}_l, l \leq k-1$. In other words, the estimated state vector is given as the following equation.

$$\hat{\mathbf{x}}_k = \arg \min_{\hat{\mathbf{x}}_k} \mathbb{E}[|\mathbf{x}_k - \hat{\mathbf{x}}_k|^2] \quad (6)$$

Let the mean of the estimated $\hat{\mathbf{x}}_k$ be itself $\hat{\mathbf{x}}_k$ and the covariance of it be as $\mathbf{P}_k^{\mathbf{x}}$. Then, the classic KF algorithm is obtained by the Algorithm 1.

Algorithm 1 Classic Kalman Filter

Input: $\hat{\mathbf{x}}_k, \mathbf{P}_k^{\mathbf{x}}, \mathbf{y}_k$
1: $\tilde{\mathbf{z}}_k = \mathbf{y}_k - \mathbf{H}_k \hat{\mathbf{x}}_k$
2: $\mathbf{K}_k = \mathbf{P}_k^{\mathbf{x}} \mathbf{H}_k^T (\mathbf{H}_k \mathbf{P}_k^{\mathbf{x}} \mathbf{H}_k^T + \mathbf{R})^{-1}$
3: $\hat{\mathbf{x}}_{k+1} = \Phi_k \hat{\mathbf{x}}_k + \Phi_k \mathbf{K}_k \tilde{\mathbf{z}}_k$
4: $\mathbf{P}_{k+1}^{\mathbf{x}} = \Phi_k (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_k^{\mathbf{x}} \Phi_k^T + \Gamma_k \mathbf{Q} \Gamma_k^T$
Output: $\hat{\mathbf{x}}_{k+1}, \mathbf{P}_{k+1}^{\mathbf{x}}$

This algorithm provides the best estimation which minimizes the cost function (6) and provides the optimal estimation. However, the algorithm estimates poorly if the covariance matrix \mathbf{Q} or \mathbf{R} are very different from the exact value. This drawback is proved in [4].

III. IBR KALMAN FILTER

The IBRKf is a robust Kalman filter exploiting the prior knowledge. To begin with, suppose the noise covariance are expressed by unknown parameter vector $\theta = [\theta_1, \theta_2]$. It is governed by the prior distribution $\pi(\theta)$, and the noise statistics is written as

$$\mathbb{E}[\mathbf{u}_k^{\theta_1} (\mathbf{u}_k^{\theta_1})^T] = \mathbf{Q}^{\theta_1} \delta_{kl} \quad (7)$$

$$\mathbb{E}[\mathbf{v}_k^{\theta_2} (\mathbf{v}_k^{\theta_2})^T] = \mathbf{R}^{\theta_2} \delta_{kl} \quad (8)$$

The IBRKf is robust in the sense of that it minimizes the cost function (6) relative to the prior distribution. That means, the IBRKf produces the $\hat{\mathbf{x}}_k$ given as

$$\hat{\mathbf{x}}_k = \arg \min_{\hat{\mathbf{x}}_k} \mathbb{E}_{\theta} [\mathbb{E} [|\mathbf{x}_k - \hat{\mathbf{x}}_k|^2]] \quad (9)$$

The IBRKf algorithm which provides (9) is similar to the one of classic KF. It's obtained by the Algorithm 2.

Algorithm 2 IBR Kalman Filter

Input: $\hat{\mathbf{x}}_k^{\theta}, \mathbb{E}_{\theta}[\mathbf{P}_k^{\mathbf{x}, \theta}], \mathbf{y}_k^{\theta}$
1: $\tilde{\mathbf{z}}_k^{\theta} = \mathbf{y}_k^{\theta} - \mathbf{H}_k \hat{\mathbf{x}}_k^{\theta}$
2: $\mathbf{K}_k^{\theta} = \mathbb{E}_{\theta}[\mathbf{P}_k^{\mathbf{x}, \theta}] \mathbf{H}_k^T \mathbb{E}_{\theta}^{-1} [\mathbf{H}_k \mathbf{P}_k^{\mathbf{x}, \theta} \mathbf{H}_k^T + \mathbf{R}^{\theta_2}]$
3: $\hat{\mathbf{x}}_{k+1}^{\theta} = \Phi_k \hat{\mathbf{x}}_k^{\theta} + \Phi_k \mathbf{K}_k^{\theta} \tilde{\mathbf{z}}_k^{\theta}$
4: $\mathbb{E}_{\theta}[\mathbf{P}_{k+1}^{\mathbf{x}, \theta}] = \Phi_k (\mathbf{I} - \mathbf{K}_k^{\theta} \mathbf{H}_k) \mathbb{E}_{\theta}[\mathbf{P}_k^{\mathbf{x}, \theta}] \Phi_k^T + \Gamma_k \mathbb{E}_{\theta_1}[\mathbf{Q}_k^{\theta_1}] \Gamma_k^T$
Output: $\hat{\mathbf{x}}_{k+1}^{\theta}, \mathbb{E}_{\theta}[\mathbf{P}_{k+1}^{\mathbf{x}, \theta}]$

It's worth nothing that the IBR Kalman Filter is obtained just replacing \mathbf{Q} and \mathbf{R} in Classic Kalman Filter with $\mathbb{E}_{\theta_1}[\mathbf{R}^{\theta_1}]$ and $\mathbb{E}_{\theta_2}[\mathbf{Q}^{\theta_2}]$ respectively. Since the covariance matrixes are same during the algorithm, the computational cost is almost same as the classic KF.

IV. OPTIMAL BAYESIAN KALMAN FILTER

Although IBRKf achieves the robustness relative to its prior knowledge, it doesn't exploit the whole information provided by the observation. The OBKF utilizes both the prior and posterior knowledge to achieve a better estimation. Compared with the IBRKf, it minimizes the cost relative to the posterior distribution, i.e.,

$$\hat{\mathbf{x}}_k = \arg \min_{\hat{\mathbf{x}}_k} \mathbb{E}_{\theta} [\mathbb{E} [|\mathbf{x}_k - \hat{\mathbf{x}}_k|^2 | \mathcal{Y}_k]] \quad (10)$$

where \mathcal{Y}_k is $\mathcal{Y}_k = \{\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_k\}$

Same as the IBRKf, the algorithm satisfying (10) is obtained just by replacing \mathbf{Q} and \mathbf{R} in classic KF with $\mathbb{E}_{\theta_1}[\mathbf{R}^{\theta_1} | \mathcal{Y}_k]$ and $\mathbb{E}_{\theta_2}[\mathbf{Q}^{\theta_2} | \mathcal{Y}_k]$ respectively. The algorithm is shown as Algorithm 3.

Algorithm 3 OBKF

Input: $\hat{\mathbf{x}}_k^{\theta}, \mathbb{E}_{\theta}[\mathbf{P}_k^{\mathbf{x}, \theta} | \mathcal{Y}_{k-1}], \mathcal{Y}_k$
1: $\tilde{\mathbf{z}}_k^{\theta} = \mathbf{y}_k^{\theta} - \mathbf{H}_k \hat{\mathbf{x}}_k^{\theta}$
2: $\mathbf{K}_k^{\theta*} = \mathbb{E}_{\theta}[\mathbf{P}_k^{\mathbf{x}, \theta} | \mathcal{Y}_{k-1}] \mathbf{H}_k^T \mathbb{E}_{\theta}^{-1} [\mathbf{H}_k \mathbf{P}_k^{\mathbf{x}, \theta} \mathbf{H}_k^T + \mathbf{R}^{\theta_2} | \mathcal{Y}_{k-1}]$
3: $\hat{\mathbf{x}}_{k+1}^{\theta} = \Phi_k \hat{\mathbf{x}}_k^{\theta} + \Phi_k \mathbf{K}_k^{\theta*} \tilde{\mathbf{z}}_k^{\theta}$
4: $\mathbb{E}_{\theta}[\mathbf{P}_{k+1}^{\mathbf{x}, \theta} | \mathcal{Y}_k] = \Phi_k (\mathbf{I} - \mathbf{K}_k^{\theta*} \mathbf{H}_k) \mathbb{E}_{\theta}[\mathbf{P}_k^{\mathbf{x}, \theta} | \mathcal{Y}_k] \Phi_k^T + \Gamma_k \mathbb{E}_{\theta_1}[\mathbf{Q}^{\theta_1} | \mathcal{Y}_k] \Gamma_k^T$
Output: $\hat{\mathbf{x}}_{k+1}^{\theta}, \mathbb{E}_{\theta}[\mathbf{P}_{k+1}^{\mathbf{x}, \theta} | \mathcal{Y}_k]$

Now the only remaining problem is, how to find the posterior expectations $\mathbb{E}_{\theta_1}[\mathbf{Q}^{\theta_1} | \mathcal{Y}_k]$ and $\mathbb{E}_{\theta_2}[\mathbf{R}^{\theta_2} | \mathcal{Y}_k]$. The posterior distribution $\pi(\theta | \mathcal{Y}_k)$ is necessary to calculate them. Since there is no closed-form solution for $\pi(\theta | \mathcal{Y}_k)$, the author of this paper employs MCMC to approximate $\mathbb{E}_{\theta_1}[\mathbf{Q}^{\theta_1} | \mathcal{Y}_k]$ and $\mathbb{E}_{\theta_2}[\mathbf{R}^{\theta_2} | \mathcal{Y}_k]$. From the Bayes theorem, the posterior distribution is proportional to the product of its likelihood and prior distribution, i.e., $\pi(\theta | \mathcal{Y}_k) \propto f(\mathcal{Y}_k | \theta) \pi(\theta)$. Then, once the likelihood function $f(\mathcal{Y}_k | \theta)$ is obtained, we can run the MCMC. From the Markov assumption of the system, it can be assumed that \mathbf{y}_k depends only on \mathbf{x}_k and θ . Then, the likelihood function is factorized as,

$$\begin{aligned}
f(\mathcal{Y}_k|\theta) &= \int \cdots \int f(\mathcal{Y}_k, \mathcal{X}_k|\theta) dx_0 \dots dx_k \\
&= \int \cdots \int \prod_{i=0}^k f(y_i|x_i, \theta) \\
&\quad \prod_{i=1}^k f(x_i|x_{i-1}, \theta) f(x_0) dx_0 \dots dx_k \quad (11)
\end{aligned}$$

Since this equation (11) is a factorization of a global function, it can be expressed as a factor-graph, and we can use sum-product algorithm[7] to compute $f(\mathcal{Y}_k|\theta)$. Finally, the likelihood function is obtained by the Algorithm 4.

Algorithm 4 Factor-Graph-Based Likelihood Function Calculation

Input: θ, \mathcal{Y}_k

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1:  $M_0 \leftarrow \mathbb{E}[x_0]$ 
2:  $S_0 \leftarrow 1$ 
3:  $\Sigma_0 \leftarrow \text{cov}[x_0]$ 
4:  $i \leftarrow 0$ 
5: while  $i \leq k-1$  do
6:    $W_i \leftarrow H_i^T (R^{\theta_2})^{-1} y_i + \Sigma_i^{-1} M_i$ 
7:    $\Lambda_i^{-1} \leftarrow \Phi_i^T (\tilde{Q}_i^{\theta_1})^{-1} \Phi_i + H_i^T (R^{\theta_2})^{-1} H_i + \Sigma_i^{-1}$ 
8:    $\Sigma_{i+1}^{-1} \leftarrow (\tilde{Q}_i^{\theta_1})^{-1} - (\tilde{Q}_i^{\theta_1})^{-1} \Phi_i \Lambda_i \Phi_i^T (\tilde{Q}_i^{\theta_1})^{-1}$ 
9:    $M_{i+1} \leftarrow \Sigma_{i+1} (\tilde{Q}_i^{\theta_1})^{-1} \Phi_i \Lambda_i (H_i^T (R^{\theta_2})^{-1} y_i + \Sigma_i^{-1} M_i)$ 
10:   $S_{i+1} \leftarrow S_i \sqrt{\frac{|\Lambda_i| |\Sigma_{i+1}|}{|\tilde{Q}_i^{\theta_1}| |\Sigma_i|}} \mathcal{N}(y_i; \mathbf{0}_{m \times 1}, R^{\theta_2}) \times$ 
     $\exp(\frac{M_{i+1}^T \Sigma_{i+1}^{-1} M_{i+1} + W_i^T \Lambda_i W_i - M_i^T \Sigma_i^{-1} M_i}{2})$ 
11:   $i \leftarrow i + 1$ 
12: end while
13:  $\Delta_k^{-1} \leftarrow H_k^T (R^{\theta_2})^{-1} H_k + \Sigma_k^{-1}$ 
14:  $G_k \leftarrow \Delta_k (H_k^T (R^{\theta_2})^{-1} y_k + \Sigma_k^{-1} M_k)$ 
15:  $f(\mathcal{Y}_k|\theta) \leftarrow S_k \sqrt{\frac{|\Delta_k|}{|\Sigma_k|}} \mathcal{N}(y_k; \mathbf{0}_{m \times 1}, R^{\theta_2}) \times$ 
     $\exp(\frac{G_k^T \Delta_k^{-1} G_k - M_k^T \Sigma_k^{-1} M_k}{2})$ 
Output:  $f(\mathcal{Y}_k|\theta)$ 

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Although the OBKF provides the best estimation relative to the posterior distribution, it has some drawbacks. Due to its MCMC and the factor-graph algorithm, the OBKF is computationally expensive. The computational cost of the factor-graph algorithm is proportional to k due to its while loop, and the computational cost of MCMC is depends on its number of sampling.

However, in some situation, the algorithm 4 and the MCMC can be omitted. For example, if the environment is stationary, it's not necessary to run the MCMC once the $\mathbb{E}_{\theta_1}[Q^{\theta_1}|\mathcal{Y}_k]$ and $\mathbb{E}_{\theta_2}[R^{\theta_2}|\mathcal{Y}_k]$ are obtained. Moreover, even if the covariance $\mathbb{E}_{\theta_1}[Q^{\theta_1}|\mathcal{Y}_k]$ and $\mathbb{E}_{\theta_2}[R^{\theta_2}|\mathcal{Y}_k]$ change, we can assume that the change of them are small between a short interval. In that case, it's not necessary to run the MCMC every measurement k , and the computational cost will be much smaller. Also, when implementing the OBKF, it is easy to accelerate the computation. The MCMC can be easily parallelized, and the speed of MCMC is proportional to the number of CPUs.

V. SIMULATION AND PERFORMANCE

The author compares the OBKF with the model specific Kalman filter, IBRKF, minimax Kalman filter, and MAP Kalman filter. The model specific Kalman filter is the classic KF with the true unknown parameter. The minimax Kalman filter is a robust Kalman filter, which provides the best worst case performance. The MAP Kalman filter is the classic KF with the MAP estimates of the unknown parameter. The trace of the covariance of the estimation error is used as a MSE metric to evaluate the accuracy of the estimate. The covariance of the error $x_k - \hat{x}_k$ is given as $\mathbf{P}_{k+1}^x(\theta; \theta')$ in the equation (12).

$$\begin{aligned}
\mathbf{P}_{k+1}^x(\theta; \theta') &= \Phi_k (\mathbf{I} - \mathbf{K}_k^{\theta'} \mathbf{H}_k) \mathbf{P}_k^x(\theta; \theta') (\mathbf{I} - \mathbf{K}_k^{\theta'} \mathbf{H}_k)^T \Phi_k^T \\
&\quad + \Gamma_k Q^{\theta_1} \Gamma_k^T + \Phi_k K_k^{\theta'} R^{\theta_2} (K_k^{\theta'})^T \Phi_k^T \quad (12)
\end{aligned}$$

Then, $\text{trace}(\mathbf{P}_{k+1}^x(\theta; \theta'))$ is used as MSE of the estimation.

To evaluate the performance of the OBKF, consider a 2-D tracking problem with an unknown parameter. In the problem, the state vector is $x_k = [p_x \ v_x \ p_y \ v_y]^T$, where p_x, v_x, p_y, v_y are the position and velocity of the x and y dimensions, respectively. The parameters in the (1) and (2) are set as follows.

$$\begin{aligned}
\Phi_k &= \begin{bmatrix} 1 & \tau & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \tau \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad H_k = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \Gamma_k = I \\
Q &= q \times \begin{bmatrix} \tau^3/3 & \tau^2/2 & 0 & 0 \\ \tau^2/2 & \tau & 0 & 0 \\ 0 & \tau & 0 & 0 \\ 0 & 0 & \tau^2/2 & \tau \end{bmatrix}, \quad R = r \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

Here, τ is the measurement interval. q and r are the

covariance noise intensity.

In the simulation, the author uses $\tau = 1$ second. The initial conditions are set as $\mathbb{E}[x_0] = [100 \ 10 \ 30 \ -10]^T$ and $\text{cov}[x_0] = \text{diag}([25 \ 2 \ 25 \ 2])$ where $\text{diag}(v)$ is a diagonal matrix whose diagonal elements are v . The parameter q is set to 2 and r is defined as a random variable, uniformly distributed over $[0.25, 4]$.

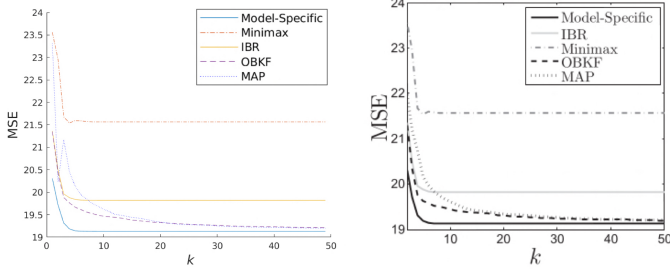


Figure 1: MSE for $r = 1$. The left figure shows the result simulated on my laptop. Right figure is cited from the OBKF paper[1]. The simulation is taken about 3 or 4 hours.

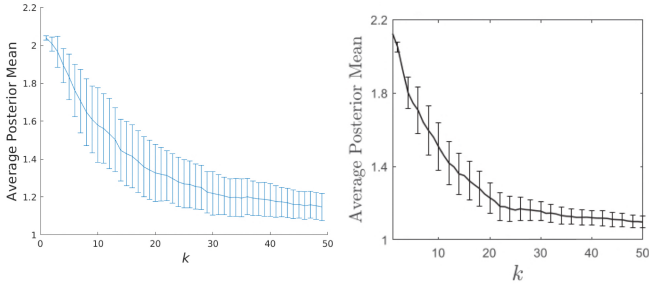


Figure 2: Empirical average and variance of $\mathbb{E}[r|Y_k]$. The left figure shows the result simulated on my laptop. Right figure is cited from the OBKF paper[1]. The simulation is taken about 3 or 4 hours.

As long as the prior distribution is accurate, the OBKF achieves the best performance from the view of the accuracy and the speed of convergence. Fig.1 compares the simulated MSE on my laptop with the figure from the paper. The results is averaged over 200 different sets of observations. You can see that in the both images, the OBKF outperforms the other Kalman filters. The OBKF converges the fastest and reaches almost the model-specific KF as k increases. Fig.2 compares the estimated r generated in the simulation of the Fig. 1. In the both image

of the Fig. 2, the estimated r converges to the true value $r = 1$ as k increases. That means that the estimation is getting close to the model-specific KF as the k increases.

Compared with the non-bayesian approach, the OBKF achieves much efficient estimation in the sense of the number of consumed data. Fig.4 compares the OBKF and the adaptive Kalman filter with unknown parameter r uniformly distributed over $[0.25, 4]$. In the Fig.4, the two adaptive methods, the Myers method[5] and the Mehra method[6] are used. The Myers method estimates the noise covariance empirically, on the other hand, the Mehra method estimates the Kalman gain matrix. The both methods depend on the parameter N_s , which is the number of the observations to tune the filter. If the N_s is small, the adaptive Kalman filter estimates poorly and can be unstable in the long run. From Fig.4, you can see that the OBKF achieves better estimation and converges much faster than the adaptive methods.

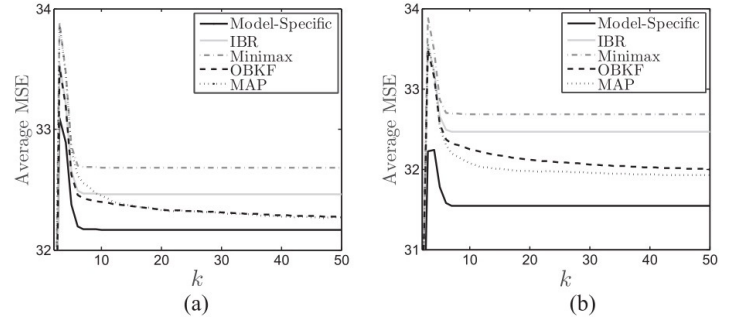


Figure 3: Average performance with unknown r . The assumed prior distribution is uniform distribution over the interval $[3, 6]$. (a) True uncertainty interval $[2, 7]$ (b) True uncertainty interval $[0.5, 8.5]$. Cited from [1]

Although the OBKF is accurate and efficient, the accuracy of the OBKF is sensitive to the prior knowledge. If the true prior distribution is different from the prior distribution in the OBKF, the OBKF doesn't converge to the optimal performance. Fig.3 shows the MSE of the Kalman filters with incorrect prior distribution. Since the prior distribution doesn't include the true value in some trial, the OBKF may not find the true value and doesn't converge to the optimal performance.

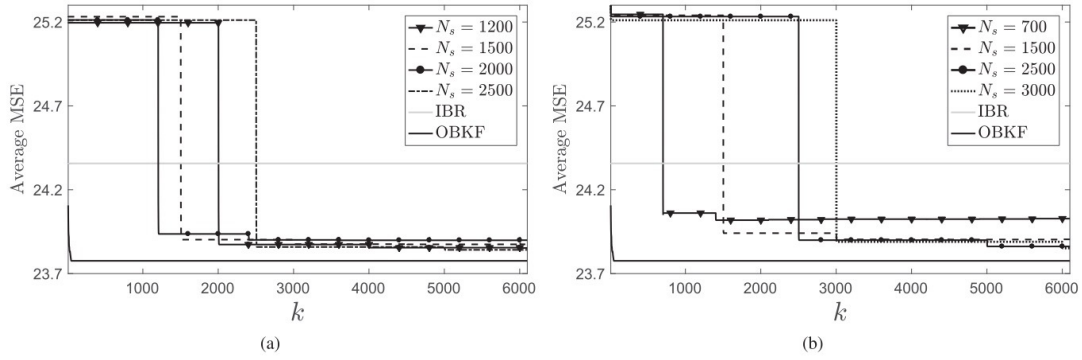


Figure 4: Comparison with the adaptive Kalman filters. (a) Unknown r and comparison with the Myers method[5]. (b) Unknown r and comparison with the Mehra method[6]. Cited from [1]

VI. CONCLUSION

The OBKF provides the optimal estimation relative to the posterior distribution. The algorithm is obtained just by replacing the covariance matrixes in the classic KF to the posterior ones. The posterior matrixes are estimated by MCMC, and the likelihood function is computed by the sum-prodduct algorithm.

Since it utilizes the prior knowledge, it doesn't require a lot of data compared with the adaptive filter. For the tracking problem, the OBKF achieves the better estimation than the adaptive method, even though the adaptive method uses much more data.

However, the OBKF is computationally expensive. This problem can be solved in some situation, such as when the environment is static or when the change of the unknown parameter is small. Developing more efficient algorithm is a future work.

Also, the performance of the OBKF is sensitive to the prior distribution. If the prior distribution of the OBKF is very different from the true prior distribution, the OBKF doesn't converge to the optimal estimation.

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