Optimal Bayesian Kalman Filtering with Prior Update

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March 1, 2019

Overview

What's this Paper?

- Introduce a new Kalman Filter: Optimal Bayesian Kalman Filter(OBKF)[Dehghannasiri et al., 2018]
- OBKF is a advanced Kalman Filter of IBR Kalman Filter[Dehghannasiri et al., 2017]
- OBKF exploit the measured data to estimate the true unknown value

Kalman Filter: Basic idea

- Outputs the optimal estimation of the state of a dynamic system
- Exploits both the prediction based on the model and the measurement
- Application Example: Robot Localization

Kalman Filter: Algorithm I

Kalman Filter works for linear dynamic systems:

Linear Dynamic System Example

$$\mathbf{x}_{k+1} = \mathbf{\Phi}_k \mathbf{x}_k + \mathbf{\Gamma}_k \mathbf{u}_k$$
$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k$$

- $\mathbf{x}_k$: *state vector*
- $\mathbf{z}_k$: *observation vector*
- $\mathbf{u}_k$, $\mathbf{v}_k$: *zero-mean noise vector*
- $\mathbf{\Phi}_k$, $\mathbf{\Gamma}_k$, $\mathbf{H}_k$: *transition matrix*

Kalman Filter: Algorithm II

Algorithm 1 Classic Kalman Filter

Input: $\hat{\mathbf{x}}_k$, ${\mathbf{P}^{\mathbf{x}}}_k$, $\mathbf{y}_k$

1: $\tilde{\mathbf{z}}_k = \mathbf{y}_k - \mathbf{H}_k \hat{\mathbf{x}}_k$

2: $\mathbf{K}_k = {\mathbf{P}^{\mathbf{x}}}_k \mathbf{H}_k^T (\mathbf{H}_k {\mathbf{P}^{\mathbf{x}}}_k \mathbf{H}_k^T + \mathbf{R})^{-1}$

3: $\hat{\mathbf{x}}_{k+1} = \mathbf{\Phi}_k \hat{\mathbf{x}}_k + \mathbf{\Phi}_k \mathbf{K}_k \tilde{\mathbf{z}}_k$

4: ${\mathbf{P}^{\mathbf{x}}}_{k+1} = \mathbf{\Phi}_k (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) {\mathbf{P}^{\mathbf{x}}}_k \mathbf{\Phi}_k^T + \mathbf{\Gamma}_k \mathbf{Q} \mathbf{\Gamma}_k^T$

Output: $\hat{\mathbf{x}}_{k+1}$, ${\mathbf{P}^{\mathbf{x}}}_{k+1}$

- $\hat{\mathbf{x}}_k$: *estimated mean*
- ${\mathbf{P}^{\mathbf{x}}}_k$: *estimated covariance*
- $\mathbf{R}$: *Noise covariance of* $\mathbf{u}_k$
- $\mathbf{Q}$: *Noise covariance of* $\mathbf{v}_k$

Problems of Kalman Filter

- The performance is sensitive to the accuracy of the noise covariance matrixes: $\mathbf{Q}$ and $\mathbf{R}$[Sangsuk-Iam and Bullock, 1990]

Solutions of the Problem

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  - Require many data
- Bayesian Approach: IBR Kalman Filter
  - Require prior knowledge
  - Robust: Guarantee the best average performance relative to the prior distribution
- Bayesian Approach: Optimal Bayesian Kalman Filter
  - Require prior knowledge
  - Utilize measured data to estimate unknown parameters
  - Optimal over the posterior distribution obtained from measured data

Basic Idea

- Assume the prior distribution for the uncertainty of the covariance matrices
- Guarantee the best average performance relative to the prior distribution

Bayesian Setup

Using $\theta = [\theta_1, \theta_2]$, assume the covariance matrices of the noise are parameterized as:

$$E[\mathbf{u}_k^{\theta_1}(\mathbf{u}_l^{\theta_1})^T] = \mathbf{Q}^{\theta_1}\delta_{kl}$$
$$E[\mathbf{v}_k^{\theta_2}(\mathbf{v}_l^{\theta_2})^T] = \mathbf{R}^{\theta_2}\delta_{kl}$$

And assume the prior distribution $\pi(\theta)$

Algorithm: IBR Kalman Filter

IBR Kalman Filter is obtained just replacing **Q** and **R** in Calssic Kalman Filter with $E_{\theta_1}[\mathbf{R}^{\theta_1}]$ and $E_{\theta_2}[\mathbf{Q}^{\theta_2}]$ respectively

Algorithm 2 IBR Kalman Filter

Input: $\hat{\mathbf{x}}_k^\theta$, $E_\theta[\mathbf{P}^{\mathbf{x},\theta}{}_k]$, $\mathbf{y}_k^\theta$

1: $\tilde{\mathbf{z}}_k^\theta = \mathbf{y}_k^\theta - \mathbf{H}_k \hat{\mathbf{x}}_k^\theta$

2: $\mathbf{K}_k^{\mathbf{\Theta}} = E_\theta[\mathbf{P}^{\mathbf{x},\theta}{}_k]\mathbf{H}_k^T E_\theta^{-1}[\mathbf{H}_k \mathbf{P}^{\mathbf{x},\theta}{}_k \mathbf{H}_k^T + \mathbf{R}^{\theta_2}]$

3: $\hat{\mathbf{x}}_{k+1}^\theta = \mathbf{\Phi}_k \hat{\mathbf{x}}_k^\theta + \mathbf{\Phi}_k \mathbf{K}_k^{\mathbf{\Theta}} \tilde{\mathbf{z}}_k^\theta$

4: $E_\theta[\mathbf{P}^{\mathbf{x},\theta}{}_{k+1}] = \mathbf{\Phi}_k(\mathbf{I} - \mathbf{K}_k^{\mathbf{\Theta}} \mathbf{H}_k) E_\theta[\mathbf{P}^{\mathbf{x},\theta}{}_k]\mathbf{\Phi}_k^T + \mathbf{\Gamma}_k E_{\theta_1}[\mathbf{Q}^{\theta_1}{}_k]\mathbf{\Gamma}_k^T$

Output: $\hat{\mathbf{x}}_{k+1}^\theta$, $E_\theta[\mathbf{P}^{\mathbf{x},\theta}{}_{k+1}]$

Basic Idea: OBKF

- Based on the IBR Kalman Filter
- Utilize measured data $\mathcal{Y}_k = \{\mathbf{y}_0, \ldots, \mathbf{y}_k\}$to obtain the posterior distribution $\pi(\theta|\mathcal{Y}_k)$
- Optimize the estimation over the posterior distribution

Algorithm: OBKF

OBKF is obtained just replacing **Q** and **R** in Calssic Kalman Filter with $E_{\theta_1}[\mathbf{Q}^{\theta_1}|\mathcal{Y}_k]$ and $E_{\theta_1}[\mathbf{R}^{\theta_2}|\mathcal{Y}_k]$ respectively

Algorithm 3 OBKF

Input: $\hat{\mathbf{x}}_k^{\theta}$, $E_\theta[\mathbf{P}^{\mathbf{x},\theta}{}_k|\mathcal{Y}_{k-1}]$, $\mathcal{Y}_k$

1: $\tilde{\mathbf{z}}_k^{\theta} = \mathbf{y}_k^{\theta} - \mathbf{H}_k\hat{\mathbf{x}}_k^{\theta}$

2: $\mathbf{K}_k^{\mathbf{\Theta}^*} = E_\theta[\mathbf{P}^{\mathbf{x},\theta}{}_k|\mathcal{Y}_{k-1}]\mathbf{H}_k^T E_\theta^{-1}[\mathbf{H}_k\mathbf{P}^{\mathbf{x},\theta}{}_k\mathbf{H}_k^T + \mathbf{R}^{\theta_2}|\mathcal{Y}_{k-1}]$

3: $\hat{\mathbf{x}}_{k+1}^{\theta} = \mathbf{\Phi}_k\hat{\mathbf{x}}_k^{\theta} + \mathbf{\Phi}_k\mathbf{K}_k^{\mathbf{\Theta}}\tilde{\mathbf{z}}_k^{\theta}$

4: $E_\theta[\mathbf{P}^{\mathbf{x},\theta}{}_{k+1}|\mathcal{Y}_k] = \mathbf{\Phi}_k(\mathbf{I} - \mathbf{K}_k^{\mathbf{\Theta}^*}\mathbf{H}_k)E_\theta[\mathbf{P}^{\mathbf{x},\theta}{}_k|\mathcal{Y}_k]\mathbf{\Phi}_k^T + \mathbf{\Gamma}_k E_{\theta_1}[\mathbf{Q}^{\theta_1}|\mathcal{Y}_k]\mathbf{\Gamma}_k^T$

Output: $\hat{\mathbf{x}}_{k+1}^{\theta}$, $E_\theta[\mathbf{P}^{\mathbf{x},\theta}{}_{k+1}|\mathcal{Y}_k]$

Algorithm: OBKF

OBKF is obtained just replacing **Q** and **R** in Calssic Kalman Filter with $E_{\theta_1}[\mathbf{Q}^{\theta_1}|\mathcal{Y}_k]$ and $E_{\theta_1}[\mathbf{R}^{\theta_2}|\mathcal{Y}_k]$ respectively

Algorithm 4 OBKF

Input: $\hat{\mathbf{x}}_k^{\theta}$, $E_\theta[\mathbf{P^{x,\theta}}_k|\mathcal{Y}_{k-1}]$, $\mathcal{Y}_k$

1: $\tilde{\mathbf{z}}_k^{\theta} = \mathbf{y}_k^{\theta} - \mathbf{H}_k\hat{\mathbf{x}}_k^{\theta}$

2: $\mathbf{K}_k^{\mathbf{\Theta}^*} = E_\theta[\mathbf{P^{x,\theta}}_k|\mathcal{Y}_{k-1}]\mathbf{H}_k^T E_\theta^{-1}[\mathbf{H}_k\mathbf{P^{x,\theta}}_k\mathbf{H}_k^T + \mathbf{R}^{\theta_2}|\mathcal{Y}_{k-1}]$

3: $\hat{\mathbf{x}}_{k+1}^{\theta} = \mathbf{\Phi}_k\hat{\mathbf{x}}_k^{\theta} + \mathbf{\Phi}_k\mathbf{K}_k^{\mathbf{\Theta}}\tilde{\mathbf{z}}_k^{\theta}$

4: $E_\theta[\mathbf{P^{x,\theta}}_{k+1}|\mathcal{Y}_k] = \mathbf{\Phi}_k(\mathbf{I} - \mathbf{K}_k^{\mathbf{\Theta}^*}\mathbf{H}_k)E_\theta[\mathbf{P^{x,\theta}}_k|\mathcal{Y}_k]\mathbf{\Phi}_k^T + \mathbf{\Gamma}_k E_{\theta_1}[\mathbf{Q}^{\theta_1}|\mathcal{Y}_k]\mathbf{\Gamma}_k^T$

Output: $\hat{\mathbf{x}}_{k+1}^{\theta}$, $E_\theta[\mathbf{P^{x,\theta}}_{k+1}|\mathcal{Y}_k]$

How to find the posterior expectations: $E_{\theta_1}[\mathbf{Q}^{\theta_1}|\mathcal{Y}_k]$ and $E_{\theta_1}[\mathbf{R}^{\theta_2}|\mathcal{Y}_k]$?

Find Posterior Expectatoins: $E_{\theta_1}[\mathbf{Q}^{\theta_1}|\mathcal{Y}_k]$ and $E_{\theta_1}[\mathbf{R}^{\theta_2}|\mathcal{Y}_k]$

- Approximate $E_{\theta_1}[\mathbf{Q}^{\theta_1}|\mathcal{Y}_k]$ and $E_{\theta_1}[\mathbf{R}^{\theta_2}|\mathcal{Y}_k]$ using Metropolis Hastings MCMC
- MCMC requires the likelihood function $f(\mathcal{Y}_k|\theta)$
- $f(\mathcal{Y}_k|\theta)$ is calculated by:
    - Marginalize $f(\mathcal{Y}_k, \mathcal{X}_k|\theta)$ over $\mathcal{X}_k$
    - Taking advantage of Markov Assumption, $f(\mathcal{Y}_k, \mathcal{X}_k|\theta)$ can be factorized
    - Using factor-graph, this step can simplify and become easy to understand

Performance: Accuracy

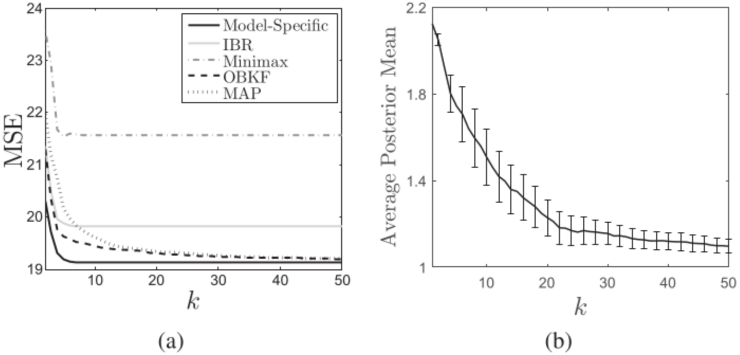

Figure: Performance analysis for specific $R = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ cited from [Dehghannasiri et al., 2018]
(a) OBKF achieves the lowest MSE
(b) Empirical average and variance of $E[r|\mathcal{Y}_k]$

Performance: Data Efficiency

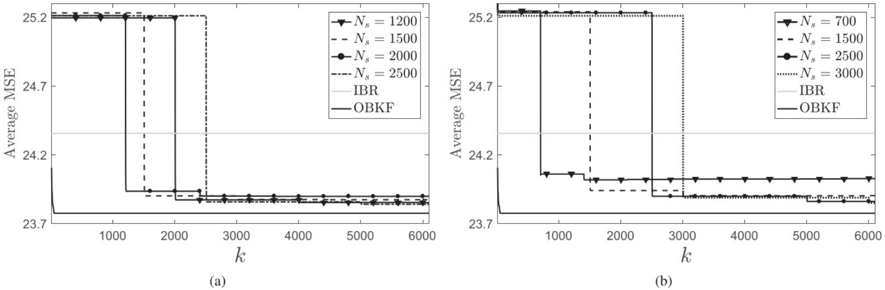

Figure: (a) Unknown r and comparison with the Myers method. (b) Unknown r and comparison with the Mehra method. cited from [Dehghannasiri et al., 2018]

Problems and Future Works

- If the prior distribution doesn't include the true value, the estimation won't converge
- Factor-graph and MCMC are computationally expensive. Finding efficient approach is a future work.

Summery

- OBKF converges to the optimal estimation as long as the prior distribution includes the true value
- OBKF requires much less data compared to Non-Bayesian methods
- The computational cost and deciding prior distribution are problems

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