Project Report EEC264

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I. Introduction

This report explains about a paper "Optimal Bayesian Kalman Filtering with Prior Update"[1]. The optimal bayesian Kalman Filter(OBKF) is an advanced version of the intrinsically bayesian robust Kalman filter(IBRKF)[2], which is a Kalman filter exploiting the prior knowledge for the model.

The Kalman filter(classic KF)[3] is a widely used technique to estimate the state vector of a linear dynamics system from its previous estimation and the measurement. Although it provides the best estimation in some condition, it has a problem. That is, the classic KF is highly sensitive to the noise covariance of the target linear dynamic system[4]. To manage this problem, mainly two robust approaches, bayesian approach and non-bayesian approach, have been studied.

The adaptive Kalman filter[5][6] is a non-bayesian approach to achieve the robustness towards the uncertain system. It estimates the covariance matrixes during the state estimation. Although it doesn't require any prior knowledge, the adaptive KF has a problem. That is, it needs a lot of observation data to tune the parameter. On the other hand, the bayesian approach doesn't require so many data comared with non-bayesian approaches. The bayesian approach Kalman filter, IBRKF, is robust in the sense of that it minimizes the average MSE relative to the prior distribution.

The IBRKF achieves the robustness in bayesian sense, but it doesn't utilizes the any information obtained from the observation. The OBKF exploits the both prior knowledge and the observation data, and it achieves the optimal estimation relative to the posterior distribution.

The rest of this paper is organized as follows. Section II shows the overview of the classic Kalman filter and its problem. In section III, a bayesian approach, IBR Kalman filter is explained. Section IV describes the OBKF, which is the main algorithm in this paper.

II. Kalman filter

The classic KF is the optimal linear estimator for linear dynamic systems. A linear dynamic system is written as

$$\mathbf{x}_{k+1} = \mathbf{\Phi}_k \mathbf{x}_k + \mathbf{\Gamma}_k \mathbf{u}_k \tag{1}$$

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k \tag{2}$$

Where,

- $\mathbf{x}_k$: *state vector*
- $\mathbf{y}_k$: *observation vector*
- $\mathbf{u}_k$, $\mathbf{v}_k$: *zero-mean gaussian noise vector*
- $\mathbf{\Phi}_k$, $\mathbf{\Gamma}_k$, $\boldsymbol{H}_k$: *transition matrix*

And the noise statistics are:

$$\mathbb{E}[\mathbf{u}_k \mathbf{u}_l^T] = \mathbf{Q}\delta_{kl},\ \mathbb{E}[\mathbf{v}_k \mathbf{v}_l^T] = \mathbf{R}\delta_{kl},\ \forall k, l = 0, 1, ... \tag{3}$$

$$\mathbb{E}[\mathbf{v}_k \mathbf{x}_l^T] = \mathbf{0}, \qquad \mathbb{E}[\mathbf{u}_k \mathbf{v}_l^T] = \mathbf{0}\ \forall k, l = 0, 1, ... \tag{4}$$

$$\mathbb{E}[\mathbf{u}_k \mathbf{y}_l^T] = \mathbf{0},\ \ 0 \le l \le k, \tag{5}$$

For such linear dynamic systems, the classic KF provides the linear estimation $\hat{\boldsymbol{x}}_k$ minimizing the mean squared error based on the observations $\boldsymbol{y}_l$, $l \le k-1$. The estimated state vector is given as,

$$\hat{\boldsymbol{x}}_k = \arg\min_{\hat{\boldsymbol{x}}_k} \mathbb{E}[|\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k|^2] \tag{6}$$

Let the mean of the estimated $\hat{\boldsymbol{x}}_k$ be itself $\hat{\boldsymbol{x}}_k$ and the covariance of it be as $\mathbf{P^x}_k$. Then, the classic KF algorithm is obtained by the Algorithm 1.

Algorithm 1 Classic Kalman Filter

Input: $\hat{\mathbf{x}}_k$, $\mathbf{P^x}_k$, $\mathbf{y}_k$

1: $\tilde{\mathbf{z}}_k = \mathbf{y}_k - \mathbf{H}_k \hat{\mathbf{x}}_k$
2: $\mathbf{K}_k = \mathbf{P^x}_k \mathbf{H}_k^T (\mathbf{H}_k \mathbf{P^x}_k \mathbf{H}_k^T + \mathbf{R})^{-1}$
3: $\hat{\mathbf{x}}_{k+1} = \mathbf{\Phi}_k \hat{\mathbf{x}}_k + \mathbf{\Phi}_k \mathbf{K}_k \tilde{\mathbf{z}}_k$
4: $\mathbf{P^x}_{k+1} = \mathbf{\Phi}_k (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{P^x}_k \mathbf{\Phi}_k^T + \mathbf{\Gamma}_k \mathbf{Q} \mathbf{\Gamma}_k^T$

Output: $\hat{\mathbf{x}}_{k+1}$, $\mathbf{P^x}_{k+1}$

This algorithm minimizes the (6), but it has a problem. This algorithm provides poor estimation if the covariance matrix $\mathbf{Q}$ and $\mathbf{R}$ is very different from the exact value. Therefore, a robust method is required to manage this problem.

III. IBR Kalman Filter

The IBRKF is a robust Kalman filter relative to the prior knowledge. It exploits the prior knowledge of the unknown parameters, and provides the robust estimation relative to the prior distribution.

To begin with, suppose the noise covariance are expressed by unknown parameter vector $\boldsymbol{\theta} = [\theta_1, \theta_2]$. It is governed by the prior distribution $\pi(\boldsymbol{\theta})$, and the noise statistics is written as

$$\mathbb{E}[\mathbf{u}_k^{\theta_1}(\mathbf{u}_l^{\theta_1})^T] = \mathbf{Q}^{\theta_1}\delta_{kl} \tag{7}$$

$$\mathbb{E}[\mathbf{v}_k^{\theta_2}(\mathbf{v}_l^{\theta_2})^T] = \mathbf{R}^{\theta_2}\delta_{kl} \tag{8}$$

The IBKF is robust in the sense of that it minimizes the cost function relative to the prior distribution. That means, the IBRKF produces the $\hat{\boldsymbol{x}}_k$ as

$$\hat{\boldsymbol{x}}_k = \arg\min_{\hat{\boldsymbol{x}}_k} \mathbb{E}_{\boldsymbol{\theta}}[\mathbb{E}[|\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k|^2]] \tag{9}$$

The algorithm which provides (9) is pretty similar to the one of classic KF. It's obtained by the Algorithm 2.

Algorithm 2 IBR Kalman Filter

Input: $\hat{\mathbf{x}}_k^{\boldsymbol{\theta}}$, $\mathbb{E}_{\boldsymbol{\theta}}[\mathbf{P}_k^{x,\boldsymbol{\theta}}]$, $\mathbf{y}_k^{\boldsymbol{\theta}}$

1: $\tilde{\mathbf{z}}_k^{\boldsymbol{\theta}} = \mathbf{y}_k^{\boldsymbol{\theta}} - \mathbf{H}_k\hat{\mathbf{x}}_k^{\boldsymbol{\theta}}$

2: $\mathbf{K}_k^{\Theta} = \mathbb{E}_{\boldsymbol{\theta}}[\mathbf{P}_k^{x,\boldsymbol{\theta}}]\mathbf{H}_k^T\mathbb{E}_{\boldsymbol{\theta}}^{-1}[\mathbf{H}_k\mathbf{P}_k^{x,\boldsymbol{\theta}}\mathbf{H}_k^T + \mathbf{R}^{\boldsymbol{\theta}_2}]$

3: $\hat{\mathbf{x}}_{k+1}^{\boldsymbol{\theta}} = \Phi_k\hat{\mathbf{x}}_k^{\boldsymbol{\theta}} + \Phi_k\mathbf{K}_k^{\boldsymbol{\Theta}}\tilde{\mathbf{z}}_k^{\boldsymbol{\theta}}$

4: $\mathbb{E}_{\boldsymbol{\theta}}[\mathbf{P}_{k+1}^{x,\boldsymbol{\theta}}] = \Phi_k(\mathbf{I} - \mathbf{K}_k^{\Theta}\mathbf{H}_k)\mathbb{E}_{\boldsymbol{\theta}}[\mathbf{P}_k^{x,\boldsymbol{\theta}}]\Phi_k^T + \Gamma_k\mathbb{E}_{\theta_1}[\mathbf{Q}_k^{\theta_1}]\Gamma_k^T$

Output: $\hat{\mathbf{x}}_{k+1}^{\boldsymbol{\theta}}$, $\mathbb{E}_{\boldsymbol{\theta}}[\mathbf{P}_{k+1}^{x,\boldsymbol{\theta}}]$

It's worth nothing that the IBR Kalman Filter is obtained just replacing $\mathbf{Q}$ and $\mathbf{R}$ in Classic Kalman Filter with $\mathbb{E}_{\theta_1}[\mathbf{R}^{\theta_1}]$ and $\mathbb{E}_{\theta_2}[\mathbf{Q}^{\theta_2}]$ respectively.

IV. Optimal Bayesian Kalman Filter

Although IBRKF achieves the robustness relative to its prior knowledge, it doesn't exploit the whole information provided by the observation. The OBKF utilizes both the prior and posterior knowledge to achieve a better estimation. Compared with the IBRKF, it minimizes the cost relative to the posterior distribution, i.e.,

$$\hat{\boldsymbol{x}}_k = \arg\min_{\hat{\boldsymbol{x}}_k} \mathbb{E}_{\boldsymbol{\theta}}[\mathbb{E}[|\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k|^2]|\mathcal{Y}_k] \tag{10}$$

where $\mathcal{Y}_k$ is $\mathcal{Y}_k = \{\boldsymbol{y}_0, \boldsymbol{y}_1, ..., \boldsymbol{y}_k\}$

Same as the IBRKF, the algorithm providing the $\hat{\boldsymbol{x}}_k$ satisfying (10) is obtained just by replacing $\mathbf{Q}$ and $\mathbf{R}$ in classic KF with $\mathbb{E}_{\theta_1}[\mathbf{R}^{\theta_1}|\mathcal{Y}_k]$ and $\mathbb{E}_{\theta_2}[\mathbf{Q}^{\theta_2}|\mathcal{Y}_k]$ respectively, shown as Algorithm 3.

Algorithm 3 OBKF

Input: $\hat{\mathbf{x}}_k^{\theta}$, $\mathbb{E}_{\theta}[\boldsymbol{P}_k^{x,\boldsymbol{\theta}}|\mathcal{Y}_{k-1}]$, $\mathcal{Y}_k$

1: $\tilde{\mathbf{z}}_k^{\boldsymbol{\theta}} = \mathbf{y}_k^{\boldsymbol{\theta}} - \mathbf{H}_k\hat{\mathbf{x}}_k^{\boldsymbol{\theta}}$

2: $\mathbf{K}_k^{\boldsymbol{\Theta}^*} = \mathbb{E}_{\theta}[\boldsymbol{P}_k^{x,\boldsymbol{\theta}}|\mathcal{Y}_{k-1}]\mathbf{H}_k^T\mathbb{E}_{\theta}^{-1}[\mathbf{H}_k\boldsymbol{P}_k^{x,\boldsymbol{\theta}}\mathbf{H}_k^T + \mathbf{R}^{\grave{}2}|\mathcal{Y}_{k-1}]$

3: $\hat{\mathbf{x}}_{k+1}^{\theta} = \boldsymbol{\Phi}_k\hat{\mathbf{x}}_k^{\theta} + \boldsymbol{\Phi}_k\mathbf{K}_k^{\boldsymbol{\Theta}}\tilde{\mathbf{z}}_k^{\theta}$

4: $\mathbb{E}_{\theta}[\boldsymbol{P}_{k+1}^{x,\boldsymbol{\theta}}|\mathcal{Y}_k] = \boldsymbol{\Phi}_k(\mathbf{I} - \mathbf{K}_k^{\boldsymbol{\Theta}^*}\mathbf{H}_k)\mathbb{E}_{\theta}[\boldsymbol{P}_k^{x,\boldsymbol{\theta}}|\mathcal{Y}_k]\boldsymbol{\Phi}_k^T + \boldsymbol{\Gamma}_k\mathbb{E}_{\theta_1}[\boldsymbol{Q}^{\boldsymbol{\theta}_1}|\mathcal{Y}_k]\boldsymbol{\Gamma}_k^T$

Output: $\hat{\mathbf{x}}_{k+1}^{\theta}$, $\mathbb{E}_{\theta}[\boldsymbol{P}_{k+1}^{x,\boldsymbol{\theta}}|\mathcal{Y}_k]$

Now the only remaining problem is how to find the posterior expectations: $\mathbb{E}_{\theta_1}[\boldsymbol{Q}^{\boldsymbol{\theta}_1}|\mathcal{Y}_k]$ and $\mathbb{E}_{\theta_2}[\boldsymbol{R}^{\boldsymbol{\theta}_2}|\mathcal{Y}_k]$.

To calculate them, the posterior distribution $\pi(\boldsymbol{\theta}|\mathcal{Y}_k)$ is necessary. Since there is no closed-form solution for $\pi(\boldsymbol{\theta}|\mathcal{Y}_k)$, the author of this paper employs MCMC to approximate $\mathbb{E}_{\theta_1}[\boldsymbol{Q}^{\boldsymbol{\theta}_1}|\mathcal{Y}_k]$ and $\mathbb{E}_{\theta_2}[\boldsymbol{R}^{\boldsymbol{\theta}_2}|\mathcal{Y}_k]$.

The posterior distribution is proportinal to the product of its likelihood and prior distribution, i.e., $\pi(\boldsymbol{\theta}|\mathcal{Y}_k) \propto f(\mathcal{Y}_k|\boldsymbol{\theta})\pi(\boldsymbol{\theta})$. Then, once the likelihood function $f(\mathcal{Y}_k|\boldsymbol{\theta})$ is obtained, we can run the MCMC.

From the Markov assumption of the system, $\boldsymbol{y}_k$ depends only on $\boldsymbol{x}_k$ and $\boldsymbol{\theta}$. Then, the likelihood function is transformed as,

$$\begin{aligned} & f(\mathcal{Y}_k|\boldsymbol{\theta}) \\ &= \int \cdots \int f(\mathcal{Y}_k, \mathcal{X}_k|\boldsymbol{\theta})dx_0...dx_k \\ &= \int \cdots \int \prod_{i=0}^{k} f(\boldsymbol{y}_i|\boldsymbol{x}_i, \boldsymbol{\theta}) \\ &\qquad \prod_{i=1}^{k} f(\boldsymbol{x}_i|\boldsymbol{x}_{i-1}, \boldsymbol{\theta})f(\boldsymbol{x}_0)dx_0...dx_k \end{aligned} \tag{11}$$

Since this equation (11) is a factorization, it can be expressed as a factor-graph, and we can use sum-product algorithm[7] to compute $f(\mathcal{Y}_k|\boldsymbol{\theta})$. Then, the likelihood function is obtained as the Algorithm 4.

Algorithm 4 Factor-Graph-Based Likelihood Function Calculation

Input: $\boldsymbol{\theta}$, $\mathcal{Y}_k$

$\boldsymbol{M}_0 \leftarrow \mathbb{E}[\boldsymbol{x}_0]$

$S_0 \leftarrow 1$

$\boldsymbol{\Sigma}_0 \leftarrow \text{cov}[\boldsymbol{x}_0]$

$i \leftarrow 0$

while $i \leq k-1$ **do**

$\quad \boldsymbol{W}_i \leftarrow \boldsymbol{H}_i^T(\boldsymbol{R}^{\theta_2})^{-1}\boldsymbol{y}_i + \boldsymbol{\Sigma}_i^{-1}\boldsymbol{M}_i$

$\quad \boldsymbol{\Lambda}_i^{-1} \leftarrow \boldsymbol{\Phi}_i^T(\tilde{\boldsymbol{Q}}_i^{\theta_1})^{-1}\boldsymbol{\Phi}_i + \boldsymbol{H}_i^T(\boldsymbol{R}^{\theta_2})^{-1}\boldsymbol{H}_i + \boldsymbol{\Sigma}_i^{-1}$

$\quad \boldsymbol{\Sigma}_{i+1}^{-1} \leftarrow (\tilde{\boldsymbol{Q}}_i^{\theta_1})^{-1} - (\tilde{\boldsymbol{Q}}_i^{\theta_1})^{-1}\boldsymbol{\Phi}_i\boldsymbol{\Lambda}_i\boldsymbol{\Phi}_i^T(\tilde{\boldsymbol{Q}}_i^{\theta_1})^{-1}$

$\quad \boldsymbol{M}_{i+1} \leftarrow \boldsymbol{\Sigma}_{i+1}(\tilde{\boldsymbol{Q}}_i^{\theta_1})^{-1}\boldsymbol{\Phi}_{\boldsymbol{i}}\boldsymbol{\Lambda}_i(\boldsymbol{H}_i^T(\boldsymbol{R}^{\theta_2})^{-1}\boldsymbol{y}_i + \boldsymbol{\Sigma}_i^{-1}\boldsymbol{M}_i)$

$\quad S_{i+1} \leftarrow \text{usingEq.}(29)\text{inthepaper}$

$\quad i \leftarrow i+1$

end while

$\boldsymbol{\Delta}_k^{-1} \leftarrow \boldsymbol{H}_k^T(\boldsymbol{R}^{\theta_2})^{-1}\boldsymbol{H}_k + \boldsymbol{\Sigma}_k^{-1}$

$\boldsymbol{G}_k \leftarrow \boldsymbol{\Delta}_k(\boldsymbol{H}_k^T(\boldsymbol{R}^{\theta_2})^{-1}\boldsymbol{y}_k + \boldsymbol{\Sigma}_k^{-1}\boldsymbol{M}_k)$

$f(\mathcal{Y}_k|\theta) \leftarrow \text{usingEq.}(34)\text{inthepaper}$

Output: $f(\mathcal{Y}_k|\theta)$

V. Performance

VI. Conclusion

References

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