Chapter
7

Chapter 7: Mathematical Background

Much of the text in this chapter comes directly from the book: Numerical Methods for Engineers.

Matrix Notation

A matrix consists of a rectangular array of elements represented by a single symbol. As depicted in the following picture, [A] is the shorthand notation for the matrix and a_{ij} designates an individual element of the matrix.

$$[A] = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & a_{m3} & & a_{mn} \end{pmatrix}$$

A horizontal set of elements is called a row and a vertical set is called a column. The first subscript *i* always designates the number of the row in which the element lies. The second subscript *j* designates the column. For example, element a_{23} is in row 2 and column 3.

The matrix in the figure above has m rows and n columns and is said to have a dimension of m by n (or m x n). It is referred to as an m-by-n matrix.

Matrices with row dimension m = 1, such as

$$[B] = [b_1\ b_2\ \ldots\ b_n]$$

are called *row vectors*. Note that for simplicity, the first subscript of each element is dropped. Also, it should be mentioned that there are times when it is desirable to employ a special shorthand notation to distinguish a row matrix from other types of matrices. One way to accomplish this is to employ special open-topped brackets.

Matrices with column dimension n = 1, such as

$$[C] = \begin{pmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_m \end{pmatrix}$$

are referred to as *column vectors*. For simplicity, the second subscript is dropped. As with the row vector, there are occasions when it is desirable to employ a special shorthand notation to distinguish a column matrix from other types of matrices. One way to accomplish this is to employ special brackets as in {B}.

Matrices where m = n are called *square matrices*. Fore example, a 4-by-4 matrix is

$$[A] = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

The diagonal consisting of the elements a_{11}, a_{22}, a_{33}, and a_{44} is termed the *principal* or *main diagonal* of the matrix.

Square matrices are particularly important when solving sets of simultaneous linear equations. For such systems, the number of equations (corresponding to rows) and the number of unknowns (corresponding to columns) must be equal in order for a unique solution to be possible. Consequently, square matrices of coefficients are encountered when dealing with such systems. Some special types of square matrices will now be described.

A *symmetrical matrix* is one where $a_{ij} = a_{ji}$ for all *i*'s and *j*'s. For example,

$$[A] = \begin{pmatrix} 5 & 1 & 2 \\ 1 & 3 & 7 \\ 2 & 7 & 8 \end{pmatrix}$$

is a 3-by-3 symmetrical matrix.

Matrix Operations

There are many operations that can be performed on matrices. One such operation is matrix multiplication.

(Matrix Product) Let A = [a_{nm}] be a matrix of size m x n and B = [b_{mp}] be a matrix of size n x p; (that is the number of columns of A equals the number of rows of B). Then AB is the m x p matrix C = [c_{np}] whose (n,p)-th element is defined by the formula

$$C_{np} \sum_{m=1}^{n} a_{nm} b_{mp} = a_{n1} b_{1p} + \ldots + a_{nn} b_{np}$$

Writing out the product explicitly,

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{np} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{pmatrix}$$

$$\begin{array}{cccc} b_{11} & b_{21} & \ldots & b_{m1} \\ \downarrow & \downarrow & & \downarrow \\ a_{11} & a_{12} & \ldots & a_{1m} \end{array}$$

Multiply each row in a_{nm} by each column in b_{mp} . The result being,

$c_{11} = a_{11}\ b_{11} + a_{12}\ b_{21} + \ldots + a_{1m} b_{m1}$

$c_{12} = a_{11} b_{12} + a_{12} b_{22} + \ldots + a_{1m} b_{m2}$

$c_{1p} = a11 b_{1p} + a_{12} b_{2p} + \ldots + a_{1m} b_{mp}$

$c_{21} = a_{21} b_{11} + a_{22} b_{22} + \ldots + a_{2m} b_{m1}$

$c_{22} = a_{21} b_{12} + a_{22} b_{22} + \ldots + a_{2m} b_{m2}$

$c_{2p} = a_{21} b_{1p} + a_{22} b_{2p} + \ldots + a_{2m} b_{mp}$

$c_{n1} = a_{n1} b_{11} + a_{n2} b_{21} + \ldots + a_{nm} b_{m1}$

$c_{n2} = a_{n1} b_{12} + a_{n2} b_{22} + \ldots + a_{nm} b_{m2}$

$c_{np} = a_{n1} b_{1p} + a_{n2} b_{2p} + \ldots + a_{nm} b_{mp}$