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MAT 220 Barbara Herzog
Analytic Geometry and Calculus I
Quick Paper Section 5.1

SECTION 5.1 Differential Equations: Growth and Decay

Differential Equations • Growth and Decay Models

Differential Equations

Up to now, we have learned to solve only two types of differential equations- those of the forms

$Y'=f(x)$ and $y''=f(x)$

In this section, we will learns how to solve a more general type of differential equation. The strategy is to rewrite the equation so that each variable occurs on only one side of the equation. This strategy is called **separation of variables.**

Growth and Decay Models

In many applications, the rate of change of a variable *y* is proportional to the value of *y*. If *y* is a function of time *t*, the proportionality can be written as follows.

Rate of change of y

is

$$\frac{dy}{dt} = ky$$

Proportional to y

Theorem 5.1 Exponential Growth and Decay Model

If *y* is a differentiable function of *t* such that $y>0$ and $y'=ky$, for some constant *K*, then

$$y = Ce^{kt}$$

C is the **initial value** of *y*, and *k* is the **proportionality constant. Exponential Growth** occurs when $k>0$, and **exponential decay** occurs when $k<0$.

Note: The statement of Theorem 4.12 doesn't tell how to distinguish between f(g(x)) and g'(x) in the integrand. As you become more experienced at integration, your skill in doing this will increase. Of course, part of the key is familiarity with derivatives.

Study Tip: There are several techniques for applying substitution, each differing slightly from others. However, you should remember that the goal is the same with every technique-you are trying to find and antiderivative of the integrand.

Examples 1 and 2 show how to apply the theorem *directly*, by recognizing the presence of *f(g(x)* and *g'(x)*. Note that the composite function in the integrand has an *outside function f* and an *inside function g*. Moreover, the derivative *g'(x)* is present as a factor of the integrand.

$$\int \overbrace{f}^{\text{Outside Function}}(\underbrace{g(x)}_{\text{Inside Function}})\overbrace{g'(x)}^{\text{Derivative of inside Function}}dx = F(g(x)) + C$$

Both of the integrands in Examples 1 and 2 fit the *f(g(x))g'(x)* pattern exactly-you only had to recognize the pattern. You can extend this technique considerably with the Constant Multiple Rule.

$$\int kf(x)dx = k\int f(x)dx.$$

Many integrands contain the essential part (the variable part) of *g'(x),* but are missing a constant multiple. In such cases you can multiply and divide by the necessary constant multiple.

Note: Be sure you see that the Constant Multiple Rule applies only to constants. You cannot multiply and divide by a variable and then move the variable outside the integral sign. For instance,

$$\int (x^2+1)^2 dx =$$

$$\neq \frac{1}{2x}\int (x^2+1)^2 \left(\frac{1}{2}\right)(2x)dx$$

After all, if it were legitimate to move variable quantities outside the integral sign, you could move the entire integrand out and simplify the whole process. But the result would be incorrect.

Change of variables

With a formal **change of variables**, you completely rewrite the integral in terms of *u* and *du* (or any other convenient variable). Although this procedure can involve more written steps than the pattern recognition illustrated in Examples 1 to 4, it is useful for complicated integrands. The change of variable technique uses the Leibniz notation for the differential. That is, if $u=g(x)dx$, then $du = g'(x)dx$, and the integral in Theorem 4.12 takes the form

Change of variables continued…

$$\int f(g(x))g'(x)dx = \int f(u)du = F(u)+C.$$

Study Tips: Because integration is usually more difficult than differentiation, you should always check your answer to an integration problem by differentiating.

When making a change of variables, be sure that your answer is written using the same variables as in the original integrand. For instance, in Example 7, you should not leave your answer as

$$\frac{1}{9}u^3+C$$

but rather replace u by sin 3x.

Guidelines for Making a Change of Variables

1) Choose a substitution $u=g(x)$. Usually, it is best to choose the *inner* part of a composite function, such as a quantity raised to a power.
2) Compute $du=g'(x)dx$.
3) Rewrite the integral in terms of the variable u.
4) Evaluate the resulting integral in terms of u.
5) Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6) Check your answer by differentiating.

The General Power Rule for Integration

One of the most common *u-substitutions* involves quantities in the integrand that are raised to a power. Because of the importance of this type of substitution, it is given a special name – the **General Power Rule.**

Theorem 4.13 The General Power Rule for Integration
If g is a differentiable function of x, then

$$\int [g(x)]^{n} g'(x)dx = \frac{[G(x)]^{n+1}}{n+1} + C \qquad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^{n} du = \frac{u^{n+1}}{n+1} + C \qquad n \neq -1$$

Change of Variable for Definite Integrals

When using *u-substitution* with a definite integral, it is often convenient to determine the limits of integration for the variable u rather that to convert the antiderivative back to the variable x and evaluate at the original limits. This change of variables is stated explicitly in the next theorem. The proof follows from Theorem 4.12 combined with the Fundamental Theorem of Calculus.

Theorem 4.14 Change of Variables for Definite Integrals

If the function $u = g(x)$ has a continuous derivative on the closed interval [a,b] and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Integration of Even and Odd Functions

Even with a change of variables, integration can be difficult. Occasionally, you can simplify the evaluation of a definite integral (over an interval that is symmetric about the y-axis or about the origin) by recognizing the integrand to be an even or odd function.

Theorem 4.15 Integration of Even and Odd Functions

Let f be integrable on the closed interval $[-a,a]$.

1) If f is an *even* function, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx.$
2) If f is an *odd* function, then $\int_{-a}^{a} f(x)dx = 0.$