MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.436J/15.085J — Fall 2020

Problem Set 1 — due Wednesday 9/9/2020

Readings:

(a) Notes from Lecture 1.
(b) Handout on background material on sets and real analysis (Recitation 1).

Supplementary readings:

[C], Sections 1.1-1.4.
[GS], Sections 1.1-1.3.
[W], Sections 1.0-1.5, 1.9.

Exercise 1. Let $\mathbb{N}$ be the set of positive integers.

(a) A function $f : \mathbb{N} \to \{0, 1\}$ is said to be *eventually zero* if there exists some N such that $f(n) = 0$, for all $n > N$. Show that the set of eventually zero functions is countable.

(b) Does the result from part (a) remain valid if we consider rational-valued eventually zero functions $f : \mathbb{N} \to \mathbb{Q}$?

(c) Does the result from part (a) remain valid if we consider real-valued eventually zero functions $f : \mathbb{N} \to \mathbb{R}$?

(d) A function $f : \mathbb{N} \to \mathbb{Q}$ is said to be *eventually constant* if there exists some $N \in \mathbb{N}$ and $q \in \mathbb{Q}$ such that $f(n) = q$, for all $n > N$. Are the eventually constant functions also countable?

Exercise 2. Let $\{x_n\}$ and $\{y_n\}$ be real sequences that converge to x and y, respectively. Provide a formal proof of the fact that $x_n + y_n$ converges to $x + y$.

Exercise 3. We are given a function $f : A \times B \to \mathbb{R}$, where A and B are nonempty sets.

(a) Assuming that the sets A and B are finite, show that

$$\max_{x\in A} \min_{y\in B} f(x, y) \leq \min_{y\in B} \max_{x\in A} f(x, y).$$

(b) For general nonempty sets (not necessarily finite), show that

$$\sup_{x\in A} \inf_{y\in B} f(x, y) \leq \inf_{y\in B} \sup_{x\in A} f(x, y).$$

Exercise 4. A probabilistic experiment involves an infinite sequence of trials. For $k = 1, 2, \ldots$, let A_k be the event that the kth trial was a success. Write down a set-theoretic expression that describes the following event:

B: For every k there exists an ℓ such that trials $k\ell$ and $k\ell^2$ were both successes.

Note: A "set theoretic expression" is an expression like $\bigcup_{k>5} \bigcap_{\ell<k} A_{k+\ell}$.

Exercise 5. Let $f_n, f, g : [0,1] \to [0,1]$ and $a, b, c, d \in [0,1]$. Derive the following set theoretic expressions:

(a) Show that

$$\{x \in [0,1] \mid \sup_n f_n(x) \leq a\} = \bigcap_n \{x \in [0,1] \mid f_n(x) \leq a\},$$

and use this to express $\{x \in [0,1] \mid \sup_n f_n(x) < a\}$ as a countable combination (countable unions, countable intersections and complements) of sets of the form $\{x \in [0,1] \mid f_n(x) \leq b\}$.

(b) Express $\{x \in [0,1] \mid f(x) > g(x)\}$ as a countable combination of sets of the form $\{x \in [0,1] \mid f(x) > c\}$ and $\{x \in [0,1] \mid g(x) < d\}$.

(c) Express $\{x \in [0,1] \mid \limsup_n f_n(x) \leq c\}$ as a countable combination of sets of the form $\{x \in [0,1] \mid f_n(x) \leq c\}$.

(d) Express $\{x \in [0,1] \mid \lim_n f_n(x) \text{ exists}\}$ as a countable combination of sets of the form $\{x \in [0,1] \mid f_n(x) < c\}$, $\{x \in [0,1] \mid f_n(x) > c\}$, etc. (Hint: think of $\{x \in [0,1] \mid \limsup_n f_n(x) > \liminf_n f_n(x)\}$).

Exercise 6. (a) Give an example of set Ω with two σ-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ such that $\mathcal{F}_1 \cup \mathcal{F}_2$ is NOT a σ-algebra. *Hint:* Consider $\Omega = \{a, b, c\}$.

(b) Let $\Omega \neq \emptyset$ be a set. Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ be an increasing sequence of σ-algebras. Show that $\mathcal{F} = \cup_{i=1}^{\infty} \mathcal{F}_i$ is an algebra.

(c) Consider the same situation as in part (b). Give an example where $\mathcal{F}$ is not a σ-algebra. *Hint:* Consider $\Omega = \mathbb{N}$ and $\mathcal{F}_n = \sigma(\{\{1\}, \{2\}, \cdots, \{n\}\})$. (Recall that $\sigma(\mathcal{C})$ is the smallest sigma algebra containing the collection of sets $\mathcal{C}$.) Then consider the set of even numbers for the union.

Remark: In fact a stronger statement is true: If $\mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \ldots$ is a *strictly increasing* sequence of σ-algebras then $\cup_{i=1}^{\infty} \mathcal{F}_i$ is NOT a σ algebra.

Exercise 7. Optional — not to be graded
This exercise develops an example that is meant to illustrate the following: if we work with fields instead of σ-fields, and if we only require finite additivity, then countable additivity will not be an automatic consequence, and the model may not correspond to any intuitive notion of probabilities.

Let $\Omega = \mathbb{N}$ (the positive integers), and let $\mathcal{F}_0$ be the collection of subsets of Ω that either have finite cardinality or their complement has finite cardinality. For any $A \in \mathcal{F}_0$, let $\mathbb{P}(A) = 0$ if A is finite, and $\mathbb{P}(A) = 1$ if A^C is finite.

(a) Show that $\mathcal{F}_0$ is a field but not a σ-field.

(b) Show that $\mathbb{P}$ is finitely additive on $\mathcal{F}_0$; that is, if $A, B \in \mathcal{F}_0$, and A, B are disjoint, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

(c) Show that $\mathbb{P}$ is not countably additive on $\mathcal{F}_0$; that is, construct a sequence of disjoint sets $A_i \in \mathcal{F}_0$ such that $\cup_{i=1}^{\infty} A_i \in \mathcal{F}_0$ and $\mathbb{P}\left(\cup_{i=1}^{\infty} A_i\right) \neq \sum_{i=1}^{\infty} \mathbb{P}\left(A_i\right)$.

(d) Construct a decreasing sequence of sets $A_i \in \mathcal{F}_0$ such that $\cap_{i=1}^{\infty} A_i = \emptyset$ for which $\lim_{i \to \infty} \mathbb{P}(A_i) \neq 0$.