MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Recitation 2 — 9/11/2020

Outline:

- Borel sets (See below)
- Lebesgue measure (From Lecture 2)
- Limits of sets (From Recitation 1)
- Monotone-class Theorem (From Lecture 1)
- An Example (see Section 2 below)

1 Borel Sets on $\mathbb{R}$

Recall that we defined the Borel σ-algebra over $\mathbb{R}$ as the σ-algebra generated by the intervals of the form (a, b) with $a < b$.

Definition 1. *A subset S of $\mathbb{R}$ is said to be open if for every $x \in S$, there exists an open interval (a, b) which is contained in S and which contains x.*

Theorem 1. *The following are Borel sets:*

1. *$\{a\}$ for any $a \in \mathbb{R}$.*
2. *intervals $(a, b], [a, b), [a, b]$ for $a \leq b$.*
3. *The set of rational numbers $\mathbb{Q}$.*
4. *Every open set in $\mathbb{R}$.*

Proof. **1)** Note that $A := (a-1, a+1)$, $B := (a-1, a)$, and $C := (a, a+1)$ are all Borel sets by definition. Recall that the Borel σ-algebra is closed under countable union, intersection, and complements, and hence $A \setminus (B \cup C)$ is also a Borel set. It is straightforward to verify that $A \setminus (B \cup C)$ is in fact equal to $\{a\}$.

2) Note that (a, b) is a Borel set by definition. Also, by part 1, $\{a\}$ and $\{b\}$ are both Borel sets. Given that Borel σ-field is closed under union, we obtain the desired result.

3) Note that by part 1, the singleton set $\{q\}$ is a Borel set for every $q \in \mathbb{Q}$. Since there are countable rational numbers, and the Borel σ-field is closed under countable union, the set of rational numbers is also a Borel set.

4) Let S be an open set in $\mathbb{R}$. By assumption on S, every $x \in S$ is contained in some interval (a, b) which is contained in S. Using the fact that rational numbers are dense in the reals, we can pick rational numbers q_x and r_x such that $a < q_x < x < r_x < b$ We see that any $x \in S$ is contained in one of the above constructed intervals with rational endpoints. Therefore we can write S as the following union of (possibly uncountably many) open intervals:

$$S = \bigcup_{x \in S} (q_x, r_x)$$

since there are countably many rationals, the number of such intervals is countable. We conclude that S is a union of countably many intervals (which are Borel sets), and is therefore a Borel set. □

2 Example: Translation Invariance of the Lebesgue Measure

Let $\Omega = [0, 1]$, and consider the Borel σ-algebra $\mathcal{B}$ on it with uniform (Lebesgue) probability measure $\mathbb{P}$. Our goal is to show that Lebesgue probability measure is translation invariant. More formally, we want to show the following result holds:

Theorem 2. *For any $A \subset \Omega$ and $x \in \Omega$, we define $A + x$ as*

$$A + x := \{a + x - \mathbb{1}\{a + x > 1\} : a \in A\}.$$

Then, for any $A \in \mathcal{B}$, $A + x$ is also a Borel set, and furthermore, $\mathbb{P}(A + x) = \mathbb{P}(A)$.

Proof. Fix $x \in \Omega$. Define $\mathcal{L}$ as the collection of Borel sets A such that $A + x$ is a Borel set, and its probability is equal to A, i.e.,

$$\mathcal{L} := \{A : A + x \in \mathcal{B}, \mathbb{P}(A + x) = \mathbb{P}(A)\}.$$

Note that $\mathcal{L}$ includes all intervals, as one can easily verify that every interval after being shifted by x can be cast as union of at most two intervals, and also its Lebesgue measure does not change. Hence, the monotone class generated by $\mathcal{L}$ will include Borel sets.

Therefore, by Monotone Class Theorem, it just suffices to show $\mathcal{L}$ is itself a monotone class. To show this, assume $A_n \uparrow A$ where $A_n \in \mathcal{L}$ for every n, and we need to prove $A \in \mathcal{L}$ (the same reasoning can be used for the case $A_n \downarrow A$).

Note that $A_n \uparrow A$ implies $A_n + x \uparrow A + x$. Also, since $A_n \in \mathcal{L}$, $A_n + x \in \mathcal{B}$. Since every σ-algebra is a monotone class itself, this implies that $A + x \in \mathcal{B}$. Also, by continuity of probability, $\mathbb{P}(A_n + x) \uparrow \mathbb{P}(A + x)$. However, recall that $A_n \in \mathcal{L}$, and since $\mathbb{P}(A_n + x) = \mathbb{P}(A_n)$. Thus, $\mathbb{P}(A_n) \uparrow \mathbb{P}(A + x)$. But, from the assumption $A_n \uparrow A$ along with the continuity of probability, we already know $\mathbb{P}(A_n) \uparrow \mathbb{P}(A)$. Thus $\mathbb{P}(A + x) = \mathbb{P}(A)$, and therefore, the proof is complete. $\square$