MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.436J/15.085J — Fall 2020

Problem Set 1 — due Wednesday 9/9/2020

Readings:
(a) Notes from Lecture 1.
(b) Handout on background material on sets and real analysis (Recitation 1).

Supplementary readings:
[C], Sections 1.1-1.4.
[GS], Sections 1.1-1.3.
[W], Sections 1.0-1.5, 1.9.

Exercise 1. Let $\mathbb{N}$ be the set of positive integers.

(a) A function $f : \mathbb{N} \to \{0, 1\}$ is said to be *eventually zero* if there exists some N such that $f(n) = 0$, for all $n > N$. Show that the set of eventually zero functions is countable.

(b) Does the result from part (a) remain valid if we consider rational-valued eventually zero functions $f : \mathbb{N} \to \mathbb{Q}$?

(c) Does the result from part (a) remain valid if we consider real-valued eventually zero functions $f : \mathbb{N} \to \mathbb{R}$?

(d) A function $f : \mathbb{N} \to \mathbb{Q}$ is said to be *eventually constant* if there exists some $N \in \mathbb{N}$ and $q \in \mathbb{Q}$ such that $f(n) = q$, for all $n > N$. Are the eventually constant functions also countable?

Solution:

(a) For a given positive integer N, let $A_N = \{f : \mathbb{N} \to \{0, 1\} : f(n) = 0, \forall n > N\}$. For a given N, each of $f(1), \cdots, f(N)$ can take one of two possible values 0 or 1. Hence the cardinality of A_N is given by $|A_N| = 2^N$. For example, for $N = 2$, there are four functions in the set A_2:

$$f(1)f(2)f(3)f(4)\cdots = 0000\cdots; \quad 1100\cdots; \quad 0100\cdots; \quad 1000\cdots.$$

The set of eventually zero from $\mathbb{N}$ to $\{0, 1\}$, denoted A, can be written as,

$$A = \bigcup_{N=1}^{\infty} A_N.$$

Since the union of countably many finite sets is countable, we conclude that the set of eventually zero functions from $\mathbb{N}$ to $\{0, 1\}$ is countable.

(b) Still, for a given positive integer N, let $A_N = \{f : \mathbb{N} \to \mathbb{Q} : f(n) = 0, \forall n > N\}$. For a given N and $i \le N$, $f(i)$ can take any value in $\mathbb{Q}$. Hence we conclude that A_N has the same cardinality as $\mathbb{Q}^N$ (the Cartesian product of N sets of rational numbers). Since $\mathbb{Q}$ is countable, and the Cartesian product of finitely many countable sets is countable, we know that A_N is countable, for any given N. Since the set of eventually zero functions from $\mathbb{N}$ to $\mathbb{Q}$ is the union of $A_1, A_2, \cdots$, it is countable, because the union of countably many countable sets is countable.

(c) Let A be the set of real values eventually zero functions on $\mathbb{N}$. We will show that this set is uncountable. Let $A_N = \{f : \mathbb{N} \to \mathbb{R} : f(n) = 0, \forall n > N\}$. As before the cardinality of A_N is same as that of $\mathbb{R}^N$ which is uncountable. Hence $A \supset A_N$ is also uncountable.

(d) For $N \in \mathbb{N}$ and $q \in \mathbb{Q}$, let $A_{N,q} = \{f : \mathbb{N} \to \mathbb{Q} : \ f(n) = q, \forall n > N\}$. Let A be the set of eventually constant functions from $\mathbb{N}$ to $\mathbb{Q}$. By definition, we have

$$A = \bigcup_{q \in \mathbb{Q}} \bigcup_{N \in \mathbb{N}} A_{N,q}$$

Similar to part b, $A_{N,q}$ has same cardinality as $\mathbb{Q}^N$. Since countable union of countable sets is countable, A is countable.

Exercise 2. Let $\{x_n\}$ and $\{y_n\}$ be real sequences that converge to x and y, respectively. Provide a formal proof of the fact that $x_n + y_n$ converges to $x + y$.

Solution: Fix some $\epsilon > 0$. Let n_1 be such that $|x_n - x| < \epsilon/2$, for all $n > n_1$. Let n_2 be such that $|y_n - y| < \epsilon/2$, for all $n > n_2$. Let $n_0 = \max\{n_1, n_2\}$. Then, for all $n > n_0$, we have

$$|(x_n + y_n) - (x + y)| \le |x_n - x| + |y_n - y| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which proves the desired result.

Exercise 3. We are given a function $f : A \times B \to \mathbb{R}$, where A and B are nonempty sets.

(a) Assuming that the sets A and B are finite, show that

$$\max_{x \in A} \min_{y \in B} f(x, y) \le \min_{y \in B} \max_{x \in A} f(x, y).$$

(b) For general nonempty sets (not necessarily finite), show that

$$\sup_{x\in A}\inf_{y\in B} f(x,y) \le \inf_{y\in B}\sup_{x\in A} f(x,y).$$

Solution:

(a) The proof rests on the application of the following simple fact: if $h(z) \le g(z)$ for all z in some finite set Z, then

$$\min_{z\in Z}\ h(z) \le \min_{z\in Z}\ g(z) \tag{1}$$

$$\max_{z\in Z}\ h(z) \le \max_{z\in Z}\ g(z). \tag{2}$$

Observe that for all x, y,

$$f(x,y) \le \max_{x\in A} f(x,y),$$

and Eq. (1) implies that for each x,

$$\min_{y\in B} f(x,y) \le \min_{y\in B}\ \max_{x\in A}\ f(x,y).$$

Now applying Eq. (2), let's take a maximum of both sides with respect to $x \in A$. Since the right-hand side is a number, it remains unchanged:

$$\max_{x\in A}\ \min_{y\in B}\ f(x,y) \le \min_{y\in B}\ \max_{x\in A}\ f(x,y),$$

which is what we needed to show.

(b) Along the same lines, we have the fact that if $h(z) \le g(z)$ for all $z \in Z$,

$$\inf_{z\in Z}\ h(z) \le \inf_{z\in Z}\ g(z) \tag{3}$$

$$\sup_{z\in Z}\ h(z) \le \sup_{z\in Z}\ g(z). \tag{4}$$

These follow immediately from the definitions of $\sup$ and $\inf$.

As before, we begin with

$$f(x,y) \le \sup_{x\in A}\ f(x,y),$$

for all x, y. By Eq. (3), for each x,

$$\inf_{y\in B}\ f(x,y) \le \inf_{y\in B}\ \sup_{x\in A}\ f(x,y),$$

and using Eq. (4),

$$\sup_{x\in A}\ \inf_{y\in B}\ f(x,y) \le \inf_{y\in B}\ \sup_{x\in A}\ f(x,y).$$

Exercise 4. A probabilistic experiment involves an infinite sequence of trials. For $k = 1, 2, \ldots$, let A_k be the event that the kth trial was a success. Write down a set-theoretic expression that describes the following event:

B: For every k there exists an ℓ such that trials $k\ell$ and $k\ell^2$ were both successes.

Note: A "set theoretic expression" is an expression like $\bigcup_{k>5} \bigcap_{\ell<k} A_{k+\ell}$.

Solution: $B = \bigcap_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} (A_{k\ell} \cap A_{k\ell^2})$.

Exercise 5. Let $f_n, f, g : [0,1] \to [0,1]$ and $a, b, c, d \in [0,1]$. Derive the following set theoretic expressions:

(a) Show that

$$\{x \in [0,1] \mid \sup_n f_n(x) \leq a\} = \bigcap_n \{x \in [0,1] \mid f_n(x) \leq a\},$$

and use this to express $\{x \in [0,1] \mid \sup_n f_n(x) < a\}$ as a countable combination (countable unions, countable intersections and complements) of sets of the form $\{x \in [0,1] \mid f_n(x) \leq b\}$.

(b) Express $\{x \in [0,1] \mid f(x) > g(x)\}$ as a countable combination of sets of the form $\{x \in [0,1] \mid f(x) > c\}$ and $\{x \in [0,1] \mid g(x) < d\}$.

(c) Express $\{x \in [0,1] \mid \limsup_n f_n(x) \leq c\}$ as a countable combination of sets of the form $\{x \in [0,1] \mid f_n(x) \leq c\}$.

(d) Express $\{x \in [0,1] \mid \lim_n f_n(x) \text{ exists}\}$ as a countable combination of sets of the form $\{x \in [0,1] \mid f_n(x) < c\}$, $\{x \in [0,1] \mid f_n(x) > c\}$, etc. (Hint: think of $\{x \in [0,1] \mid \limsup_n f_n(x) > \liminf_n f_n(x)\}$).

Solution: First observe the following set relations

$$[0,c) = \bigcup_{n=1}^{\infty} [0, c - \tfrac{1}{n}] \quad [0,c] = \bigcap_{n=1}^{\infty} [0, c + \tfrac{1}{n})$$
$$(c,1] = \bigcup_{n=1}^{\infty} [c + \tfrac{1}{n}, 1] \quad [c,1] = \bigcap_{n=1}^{\infty} (c - \tfrac{1}{n}, 1].$$

All conversions between strict and non-strict inequalities following from these relations and properties of the inverse image, i.e. homomorphism of arbitrary set operations. We will use the shorthand notation

$$\{f < a\} := \{x \in [0,1] \mid f(x) < a\}.$$

(a) Let $x \in \bigcap_n \{f_n \leq a\}$. Then, $f_n(x) \leq a$ for all $n \Longrightarrow \sup_n f_n(x) \leq a$, by definition of $\sup$ as a is an upper bound for $\{f_n(x)\}$. Therefore, as x was arbitrary,

$$\bigcap_{n=1}^{\infty} \{f_n \leq a\} \subset \{\sup_n f_n \leq a\}.$$

Let $x \in \{\sup_n f_n \leq a\}$. Then $\sup_n f_n(x) \leq a$ and for all n $f_n(x) \leq \sup_n(x) \leq a$. Therefore, as x was arbitrary,

$$\{\sup_n f_n \leq a\} \subset \bigcap_{n=1}^{\infty} \{f_n \leq a\}.$$

Hence $\{\sup_n f_n \leq a\} = \bigcap_n \{f_n \leq a\}$. By De Morgan's this relation also implies

$$\{\sup_n f_n > a\} = \bigcup_{n=1}^{\infty} \{f_n > a\}.$$

Similar results hold for $\inf$.

Let $f = \sup_n f_n$. Using the above comment

$$\begin{aligned}
\{\sup_n f_n < a\} &= \{f < a\} \\
&= f^{-1}([0, a)) \\
&= \bigcup_{k=1}^{\infty} f^{-1}\left([0, a - \tfrac{1}{k}]\right) \\
&= \bigcup_{k=1}^{\infty} \{\sup_n f_n \leq a - \tfrac{1}{k}\} \\
&= \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \{f_n \leq a - \tfrac{1}{k}\}.
\end{aligned}$$

(b) Using countability and density of the rationals

$$\begin{aligned}
\{f > g\} &= \bigcup_{q \in \mathbb{Q}} \{f > q\} \cap \{q > g\} \\
&= \bigcup_{q \in \mathbb{Q}} \{f > q\} \cap \{q \geq g\} \\
&= \bigcup_{q \in \mathbb{Q}} \{f \geq q\} \cap \{q > g\}.
\end{aligned}$$

(c)

$$
\begin{aligned}
\{\limsup_{n\to\infty} f_n \le c\} &= \{\inf_{n\ge 1}\sup_{k\ge n} f_k \le c\} \\
&= \bigcap_{m=1}^{\infty}\{\inf_{n\ge 1}\sup_{k\ge n} f_k < c + \tfrac{1}{m}\} \\
&= \bigcap_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\{\sup_{k\ge n} f_k < c + \tfrac{1}{m}\} \\
&= \bigcap_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcup_{\ell=1}^{\infty}\{\sup_{k\ge n} f_k \le c + \tfrac{1}{m} - \tfrac{1}{\ell}\} \\
&= \bigcap_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcup_{\ell=1}^{\infty}\bigcap_{k=n}^{\infty}\{f_k \le c + \tfrac{1}{m} - \tfrac{1}{\ell}\}.
\end{aligned}
$$

(d)

$$
\begin{aligned}
\{\lim_{n\to\infty} f_n \ \text{ exists}\} &= \{\liminf_{n\to\infty} f_n = \limsup_{n\to\infty} f_n\} \\
&= \{\liminf_{n\to\infty} f_n < \limsup_{n\to\infty} f_n\}^c \quad (\liminf_{n\to\infty} f_n(x) \le \limsup_{n\to\infty} f_n(x)) \\
&= \left(\bigcup_{q\in\mathbb{Q}}\{\liminf_{n\to\infty} f_n < q\} \cap \{q < \limsup_{n\to\infty} f_n\}\right)^c \qquad (\text{part } b) \\
&= \bigcap_{q\in\mathbb{Q}}\{\liminf_{n\to\infty} f_n \ge q\} \cup \{\limsup_{n\to\infty} f_n \le q\}.
\end{aligned}
$$

The sets $\{\liminf_{n\to\infty} f_n \ge q\}$ and $\{\limsup_{n\to\infty} f_n \le q\}$ can be expressed as countable combinations using part (c) and the fact that

$$
\begin{aligned}
-\limsup_{n\to\infty} f_n(x) &= -\inf_{n\ge 1}\sup_{k\ge n} f_k(x) \\
&= \sup_{n\ge 1}\inf_{k\ge n}(-f_k(x)) \\
&= \liminf_{n\to\infty}(-f_n(x)),
\end{aligned}
$$

i.e. $\{\liminf_{n\to\infty} f_n \ge q\} = \{\limsup_{n\to\infty}(-f_n) \le -q\}$. More specifically,

$$
\bigcap_{q\in\mathbb{Q}}\left[\left(\bigcap_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcup_{\ell=1}^{\infty}\bigcap_{k=n}^{\infty}\{f_k \ge c - \tfrac{1}{m} + \tfrac{1}{\ell}\}\right) \cup \left(\bigcap_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcup_{\ell=1}^{\infty}\bigcap_{k=n}^{\infty}\{f_k \le c + \tfrac{1}{m} - \tfrac{1}{\ell}\}\right)\right].
$$

Using one of the later two expressions of part (b), we can drop one of the outer intersections

$$\bigcap_{q\in\mathbb{Q}}\left[\left(\bigcup_{n=1}^{\infty}\bigcup_{\ell=1}^{\infty}\bigcap_{k=n}^{\infty}\{f_k \geq c+\tfrac{1}{\ell}\}\right)\cup\left(\bigcap_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcup_{\ell=1}^{\infty}\bigcap_{k=n}^{\infty}\{f_k \leq c+\tfrac{1}{m}-\tfrac{1}{\ell}\}\right)\right]$$

or

$$\bigcap_{q\in\mathbb{Q}}\left[\left(\bigcap_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcup_{\ell=1}^{\infty}\bigcap_{k=n}^{\infty}\{f_k \geq c-\tfrac{1}{m}+\tfrac{1}{\ell}\}\right)\cup\left(\bigcup_{n=1}^{\infty}\bigcup_{\ell=1}^{\infty}\bigcap_{k=n}^{\infty}\{f_k \leq c-\tfrac{1}{\ell}\}\right)\right].$$

Exercise 6. (a) Give an example of set Ω with two σ-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ such that $\mathcal{F}_1 \cup \mathcal{F}_2$ is NOT a σ-algebra. *Hint:* Consider $\Omega = \{a, b, c\}$.

(b) Let $\Omega \neq \emptyset$ be a set. Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset ...$ be an increasing sequence of σ-algebras. Show that $\mathcal{F} = \cup_{i=1}^{\infty}\mathcal{F}_i$ is an algebra.

(c) Consider the same situation as in part (b). Give an example where $\mathcal{F}$ is not a σ-algebra. *Hint:* Consider $\Omega = \mathbb{N}$ and $\mathcal{F}_n = \sigma\left(\{\{1\}, \{2\}, \cdots, \{n\}\}\right)$. (Recall that $\sigma(\mathcal{C})$ is the smallest sigma algebra containing the collection of sets $\mathcal{C}$.) Then consider the set of even numbers for the union.

Remark: In fact a stronger statement is true: If $\mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq ...$ is a *strictly increasing* sequence of σ-algebras then $\cup_{i=1}^{\infty}\mathcal{F}_i$ is NOT a σ algebra.

Solution:

(a) Let $\Omega = \{a, b, c\}$. Let $\mathcal{F}_1 = \{\emptyset, \{a\}, \{b, c\}, \Omega\}$ and $\mathcal{F}_2 = \{\emptyset, \{b\}, \{a, c\}, \Omega\}$. We can easily check that both $\mathcal{F}_1$ and $\mathcal{F}_2$ are σ-algebras. Indeed, consider $\mathcal{F}_1$. We have

- $\emptyset \in \mathcal{F}_1$
- Trivially for all $A \in \mathcal{F}_1$, we have $A^c \in \mathcal{F}_1$ (there are four choices for A; consider $A = \{b, c\}$ for example)
- For countable union, note that there are only four **distinct** choices of $A \in \mathcal{F}_1$. Hence it is enough to check finite union of four sets. Formally, let $A_i \in \mathcal{F}_1$, $i \in \{1, 2, 3, 4\}$. Consider $A = \cup_{i=1}^{4}A_i$. If any of the A_i is Ω, then $A = \Omega \in \mathcal{F}_1$. If say $A_1 = \emptyset$, then $A = \cup_{i=2}^{4}A_i$. Hence it is enough to verify the union property for $A_i \in \{\{a\}, \{b, c\}\}$. But trivially, $\{a\} \cup \{b, c\} = \Omega \in \mathcal{F}_1$. Hence $A \in \mathcal{F}_1$.

Now let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 = \{\emptyset, \{a\}, \{b\}, \{b, c\}, \{a, c\}, \Omega\}$. Let $A_1 = \{a\}$ and $A_2 = \{b\}$. Note that $A_1 \cup A_2 = \{a, b\} \notin \mathcal{F}$. Hence $\mathcal{F}$ is not a σ-algebra.

(b) Let $\mathcal{F} = \cup_i \mathcal{F}_i$. Let us verify the three properties of an algebra:

- Trivially $\emptyset \in \mathcal{F}$ since the empty set is in $\mathcal{F}_1 \subset \mathcal{F}$.
- Let $A \in \mathcal{F}$. Hence there is an i such that $A \in \mathcal{F}_i$ by the definition of union of sets. Since $\mathcal{F}_i$ is a σ-algebra, this means $A^c \in \mathcal{F}_1 \subset \mathcal{F}$. Hence $A^c \in \mathcal{F}$.
- Let $A, B \in \mathcal{F}$. Hence there are i, j (not necessarily distinct) such that $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$. Let $k = \max(i, j)$. Then $\mathcal{F}_i, \mathcal{F}_j \subset \mathcal{F}_k$. Hence $A, B \in \mathcal{F}_k$. Therefore $A \cup B \in \mathcal{F}_k \subset \mathcal{F}$. So $A \cup B \in \mathcal{F}$.

(c) Let $\Omega = \mathbb{N}$. Let $[n] = \{1, 2, \cdots, n\}$. Let

$$\mathcal{F}_n = \{A : A \subset [n]\} \cup \{A : \Omega \setminus A \subset [n]\}$$

that is, $\mathcal{F}_n$ consists of all subsets of $[n]$ and their complements in Ω. First we will show that $\mathcal{F}_n$ is a σ-algebra on Ω. Trivially, $\mathcal{F}_n$ is closed under complements. The only non-trivial property to check is the countable union property. But note that $\mathcal{F}_n$ is a finite collection. Hence enough to check that finite unions are closed. This again follows from noting that for A_1 and A_2 such that $A_1 \subset [n]$ and $\Omega \setminus A_2 \subset [n]$, we have that $(A_1 \cup A_2)^c = A_1^c \cap A_2^c \subset [n]$. Hence $A_1 \cup A_2 \in [n]$. (This is pretty much the only case we need to check).

Now we will show that

$$\mathcal{F}_n = \sigma(\{\{1\}, \{2\}, \cdots, \{n\}\}) \equiv \mathcal{G}_n$$

Since $\{i\} \in \mathcal{F}_n$ for all $i \in [n]$, clearly $\mathcal{G}_n \subset \mathcal{F}_n$. If $A \in \mathcal{F}_n$, then either $A = \cup_{i \in A}\{i\} \in \mathcal{G}_n$ or $A^c = \cup_{i \in A^c}\{i\} \subset \mathcal{G}_n$. Hence $\mathcal{F}_n \subset \mathcal{G}_n$. Therefore $\mathcal{F}_n = \mathcal{G}_n$.

Clearly $\mathcal{G}_n \subset \mathcal{G}_{n+1}$.

Now we can see that

$$\mathcal{F} = \cup_n \mathcal{F}_n = \{A \subset \mathbb{N} : |A| < \infty \text{ or } |\Omega \setminus A| < \infty\}$$

.

Let $A_n = \{2n\}$, for $n \in \mathbb{N}$. Note that $A_n \in \mathcal{F}_{2n}$. Let $A = \cup_n A_n = \{2n : n \in \mathbb{N}\}$. That is, A is just the set of even positive integers. Since none of A and $\Omega \setminus A$ are finite sets, $A \notin \mathcal{F}$. Hence $\mathcal{F}$ is not a σ-algebra.

Exercise 7. Optional — not to be graded

This exercise develops an example that is meant to illustrate the following: if we work with fields instead of σ-fields, and if we only require finite additivity, then countable additivity will not be an automatic consequence, and the model may not correspond to any intuitive notion of probabilities.

Let $\Omega = \mathbb{N}$ (the positive integers), and let $\mathcal{F}_0$ be the collection of subsets of Ω that either have finite cardinality or their complement has finite cardinality. For any $A \in \mathcal{F}_0$, let $\mathbb{P}(A) = 0$ if A is finite, and $\mathbb{P}(A) = 1$ if A^C is finite.

(a) Show that $\mathcal{F}_0$ is a field but not a σ-field.

(b) Show that $\mathbb{P}$ is finitely additive on $\mathcal{F}_0$; that is, if $A, B \in \mathcal{F}_0$, and A, B are disjoint, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

(c) Show that $\mathbb{P}$ is not countably additive on $\mathcal{F}_0$; that is, construct a sequence of disjoint sets $A_i \in \mathcal{F}_0$ such that $\cup_{i=1}^{\infty} A_i \in \mathcal{F}_0$ and $\mathbb{P}\left(\cup_{i=1}^{\infty} A_i\right) \neq \sum_{i=1}^{\infty} \mathbb{P}\left(A_i\right)$.

(d) Construct a decreasing sequence of sets $A_i \in \mathcal{F}_0$ such that $\cap_{i=1}^{\infty} A_i = \emptyset$ for which $\lim_{i\to\infty} \mathbb{P}(A_i) \neq 0$.

Solution:

(a) The empty set has zero cardinality, and therefore belongs to $\mathcal{F}_0$. Furthermore, if $A \in \mathcal{F}_0$, then either A or A^c has finite cardinality. It follows that either A^c or $(A^c)^c$ has finite cardinality, so that $A^c \in \mathcal{F}_0$.

Suppose that $A, B \in \mathcal{F}_0$. If both A and B are finite, then $A \cup B$ is also finite and belongs to $\mathcal{F}_0$. Suppose now that at least one of A or B is infinite. We have $A \cup B = (A^c \cap B^c)^c$. Since $A^c \cap B^c$ is finite, it follows that $A \cup B \in \mathcal{F}_0$. This shows that $\mathcal{F}_0$ is a field.

To see that $\mathcal{F}_0$ is not a σ-field, note that $\{2n\} \in \mathcal{F}_0$ for every $n \in \mathbb{N}$, but the set $\bigcup_{n=0}^{\infty}\{2n\}$, the set of even natural numbers, is not in $\mathcal{F}_0$.

(b) Let $A, B \in \mathcal{F}_0$ be disjoint. If both A and B are finite, then $\mathbb{P}\left(A \cup B\right) = 0 = \mathbb{P}(A) + \mathbb{P}(B)$. Suppose that either A or B (or both) is infinite. Since A and B are disjoint, we have $A \subset B^c$ and $B \subset A^c$. It follows that A and B cannot both be infinite. Therefore, $\mathbb{P}(A \cup B) = 1 = \mathbb{P}(A) + \mathbb{P}(B)$, and $\mathbb{P}$ is finitely additive.

(c) Note that $\{n\} \in \mathcal{F}_0$ and $\bigcup_{n\geq 1}\{n\} = \Omega$. However, $\mathbb{P}(\{n\}) = 0$ while $\mathbb{P}(\Omega) = 1$, hence $\mathbb{P}$ is not countably additive.

(d) Let $A_n = \{n, n+1, \ldots\}$. Then $(A_n)_{n\geq 1}$ forms a decreasing sequence of sets with $\bigcap_n A_n = \emptyset$. But $\mathbb{P}(A_n) = 1$ for all n, hence $\lim_{n\to\infty} \mathbb{P}(A_n) = 1$.