6.S088 Problem Set 2

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Problem 3

$$K(\mathbf{x}, \mathbf{x}') = e^{-L\|\mathbf{x}-\mathbf{x}'\|^2} = e^{-L\|\mathbf{x}\|^2} e^{-L\|\mathbf{x}'\|^2} e^{2L\langle \mathbf{x}, \mathbf{x}' \rangle} \tag{1}$$

For the term $e^{2L\langle \mathbf{x}, \mathbf{x}' \rangle}$, we use Taylor expansion of e^z around $z = 0$ and geometry

$$e^{2L\langle \mathbf{x}, \mathbf{x}' \rangle} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(2L \langle \mathbf{x}, \mathbf{x}' \rangle\right)^k \tag{2}$$

Then we have

$$\langle \mathbf{x}, \mathbf{x}' \rangle^k = \left(\sum_{i=1}^{d} \mathbf{x}_i \mathbf{x}'_i \right)^k = \sum_{j \in [d]^k} \left(\prod_{i=1}^{k} \mathbf{x}_{j_i} \right) \left(\prod_{i=1}^{k} \mathbf{x}'_{j_i} \right) \tag{3}$$

where j enumerates over all selections of k coordinates of $\mathbf{x}$. Then by plugging in (2), (3) to (1) results in

$$K(\mathbf{x}, \mathbf{x}') = \langle \phi(\mathbf{x}), \phi(\mathbf{x}') \rangle = \prod_{k=0}^{\infty} \prod_{j \in [d]^k} \phi_{k,j}(\mathbf{x}) \phi_{k,j}(\mathbf{x}')$$

$$\text{where } \phi_{k,j}(\mathbf{x}) = e^{-L\|\mathbf{x}\|^2} \frac{2L^{\frac{K}{2}}}{\sqrt{k!}} \prod_{i=0}^{k} \mathbf{x}_{j_i}$$

Problem 4

$$\begin{aligned} \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j K\left(x^{(i)}, x^{(j)}\right) &= \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j \langle \psi(x^{(i)}), \psi(x^{(j)}) \rangle_{\mathcal{H}} \\ &= \left(\sum_{i=1}^{n} c_i \psi(x^{(i)}) \right)^T \left(\sum_{i=1}^{n} c_i \psi(x^{(i)}) \right) \geq 0 \end{aligned}$$

$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j K\left(x^{(i)}, x^{(j)}\right) \geq 0$ implies K is *positive semi-definite*.