Problem 1.

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$A \in \mathbb{R}^{n \times n}$

(a) If $\{\lambda_i\}_{i=1}^{n} \subset (-1,1]$,

Compute $(I+A)^{-1}$

$$A = Q \Lambda Q^{-1}$$

$$I = Q Q^{-1}$$

$$\begin{aligned} I + A &= QQ^{-1} + Q\Lambda Q^{-1} \\ &= \underset{\text{invertible since } A \text{ is diagonalizable.}}{Q} (I + \Lambda) Q^{-1}. \end{aligned}$$

$$\underset{\sim}{(I+A)^{-1}} = Q^{-1} \underbrace{(I+\Lambda)^{-1}}_{\text{diagonal elements are eigenvalues of } (I+A)^{-1}.} Q$$

$$\boxed{\Rightarrow \begin{pmatrix} \frac{1}{1+\lambda_1} & & & \\ & \frac{1}{1+\lambda_2} & & \\ & & \ddots & \\ & & & \frac{1}{1+\lambda_n} \end{pmatrix}}$$

(b) $$A^k = \underbrace{Q \Lambda \cancel{Q^{-1}} \cancel{Q} \Lambda \cancel{Q^{-1}} \cdots \cancel{Q} \Lambda Q^{-1}}_{k \text{ times}}$$

$$= Q \Lambda^k Q^{-1}.$$

$$= Q \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} Q^{-1}$$

$$\boxed{\lambda_1^k, \ldots \lambda_n^k \text{ are eigenvalues of } A^k.}$$

Solution2

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(a)

$\Sigma \in \mathbb{R}^{n \times m}$

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}$$

Reorder $\sigma_1 \cdots \sigma_n$ by decending order

$\bar{\sigma}_1 \cdots \bar{\sigma}_n$ and let

$\bar{\sigma}_{k+1}, \cdots, \bar{\sigma}_n = 0$ wlog if exists

such zero eigenvalues.

Then

$$\boxed{\Sigma^{+} = \begin{bmatrix} \bar{\sigma}_1 & & & & 0 \\ & \ddots & & & \\ & & \bar{\sigma}_k & & \\ & & & 0 & \\ 0 & & & & \ddots \; 0 \end{bmatrix}}$$

(b) $A = U \Sigma V^T \in \mathbb{R}^{n \times m}$

$$A^{+} = V \Sigma^{+} U^T$$

① $A A^{+} A = U \Sigma V^T V \Sigma^{+} U^T U \Sigma V^T$

$$= U \underbrace{\Sigma \Sigma^{+} \Sigma}_{} V^T$$
$$= U \Sigma V^T$$
$$= A$$

② $A^{+} A A^{+} = V \Sigma^{+} U^T U \Sigma V^T V \Sigma^{+} U^T$

$$= V \Sigma^{+} \Sigma \Sigma^{+} U^T$$
$$= V \Sigma^{+} U^T$$
$$= A^{+}$$

③ $(A A^{+})^T = (U \Sigma V^T V \Sigma^{+} U^T)^T$

$$= (U \Sigma \Sigma^{+} U^T)^T$$
$$= U (\Sigma \Sigma^{+})^{T} U^T$$
$$= U \Sigma \Sigma^{+} U^T$$
$$= U \Sigma V^T V \Sigma^{+} U^T.$$

④ $(A^{+} A)^T = (V \Sigma^{+} U^T U \Sigma V^T)^T$

$$= (V \Sigma^{+} \Sigma V^T)^T$$
$$= V (\Sigma^{+} \Sigma)^T V^T$$
$$= V \Sigma^{+} \Sigma V^T$$
$$= V \Sigma^{+} U^T U \Sigma V^T$$
$$= A^{+} A.$$

Since $A^{+} = V \Sigma^{+} U^T$ satisfies

$\boxed{4 \text{ conditions}, \quad A^{+} = V \Sigma^{+} U^T}$.

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$$\mathcal{L}(w) = \| wX - y \|_2^2$$

$$wX - y = [w_1\ w_2 \cdots w_d] \begin{bmatrix} x_1^{(1)} & x_1^{(2)} & \cdots & x_1^{(n)} \\ \vdots & \vdots & & \vdots \\ x_d^{(1)} & x_d^{(2)} & & x_d^{(n)} \end{bmatrix}$$

$$- [y^{(1)} \cdots y^{(n)}]$$

$$= [w_1 x_1^{(1)} + w_2 x_2^{(1)} + \cdots w_d x_d^{(1)}, \cdots,$$

$$w_1 x_1^{(n)} + w_2 x_2^{(n)} + \cdots w_d x_d^{(n)}]$$

$$- [y^{(1)}, \cdots, y^{(n)}]$$

$1 \times n$

$$= \left[\sum_{j=1}^{d} w_j x_j^{(1)} - y^{(1)}, \cdots \sum_{j=1}^{d} w_j x_j^{(n)} - y^{(n)} \right] \in \mathbb{R}^{1 \times n}$$

$$\| wX - y \|_2^2 = \left(\sum w_j x_j^{(1)} - y^{(1)} \right)^2 + \cdots + \left(\sum w_j x_j^{(n)} - y^{(n)} \right)^2$$

$$= \mathcal{L}(w)$$

$$\frac{\partial L}{\partial w_i} = 2 \left(\sum_{j=1}^{d} w_j x_j^{(1)} - y^{(1)} \right) \cdot x_i^{(1)} + \cdots$$

$$+ 2 \left(\sum_{j=1}^{d} w_j x_j^{(n)} - y^{(n)} \right) \cdot x_i^{(n)}$$

$\therefore$

$$\boxed{\nabla L = 2 \underbrace{(wX - y)}_{\substack{\uparrow \\ \underbrace{1\times d\ d\times n - 1\times n}_{\substack{\uparrow \\ 1\times n}}}} \underbrace{X^T}_{n\times d} \in \mathbb{R}^{1\times d}}$$

Problem 6.

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① $f(x) = \max(x, 0)$.

$$= \begin{cases} x & \text{if } x \geq 0. \\ 0 & \text{if } x < 0 \end{cases}$$

$$\| f \|_{L^2(\gamma)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|^2 e^{-\frac{x^2}{2}} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \underbrace{|x|^2}_{x^2} e^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{2} \cdot \mathbb{E}_{X \sim N(0,1)}[X^2] = \boxed{\frac{1}{2}}$$

Since $Var(X) = \mathbb{E}[X^2] - \{\mathbb{E}[X]\}^2$,

$$\mathbb{E}[X^2] = Var(X) + \{\mathbb{E}[X]\}^2$$

$$= 1 + 0$$

$$= 1$$

② $\|g\|_{L^2(\gamma)}$ when $g(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

$$\|g\|_{L^2(\gamma)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |g(x)|^2 e^{-\frac{x^2}{2}} dx$$

$$= \underline{\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{x^2}{2}} dx}$$

half of the entire probability.

$$\boxed{= \frac{1}{2}}$$

Problem 8

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$$\int_{-\infty}^{\infty} e^{-x^2+4x}\,dx$$

since $-x^2+4x = -(x-2)^2+4$

$$= \int_{-\infty}^{\infty} e^{-(x-2)^2} \cdot e^4\,dx$$

$$= e^4 \int_{-\infty}^{\infty} e^{-(x-2)^2}\,dx$$

$$\left(\begin{array}{l} \text{since} \\ x-2 = \dfrac{u}{\sqrt{2}} \rightarrow (x-2)^2 = \dfrac{u^2}{2} \\ \\ dx = \dfrac{1}{\sqrt{2}}\,du \end{array} \right)$$

$$= e^4 \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} \cdot \frac{1}{\sqrt{2}}\,du$$

$$= e^4 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} \cdot e^{-\frac{u^2}{2}}\,du$$

$$= e^4 \cdot \sqrt{\pi} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi}} \cdot e^{-\frac{u^2}{2}}\,du$$

$$\boxed{= e^4 \cdot \sqrt{\pi}.}$$

Problem 9

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(a)

$w \in \mathbb{R}^{1\times n}$

$w^T \in \mathbb{R}^{n\times 1}$

$ww^T \in \mathbb{R}^{1\times 1}$.

$$[w_1 \ w_2 \cdots w_n]\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

$$= w_1^2 + w_2^2 + \cdots w_n^2$$

$$\mathbb{E}[ww^T] = \mathbb{E}[w_1^2] + \mathbb{E}[w_2^2] + \cdots [w_n^2]$$

$$= \mathbb{E}[\|w\|^2]$$

since $Var(w_1) = \mathbb{E}[w_1^2] - [\mathbb{E}[w_1]]^2$,

$$\Rightarrow Var(w_1) + [\mathbb{E}[w_1]]^2 = \mathbb{E}[w_1^2] = 1.$$

$$= 1 + 1 + \cdots + 1$$

$$\boxed{= n.}$$

$$\boxed{\mathbb{E}[ww^T] \text{ and } \mathbb{E}[\|w\|^2] = n.}$$

(b) $\mathbb{E}[\langle w, v\rangle]$

$$= \mathbb{E}[w_1 v_1] + \mathbb{E}[w_2 v_2] + \cdots \quad \mathbb{E}[w_n v_n]$$

since $Cov(w_1, v_1) = \mathbb{E}[w_1 v_1] - \mathbb{E}[w_1]\mathbb{E}[v_1]$

$$0 = \underbrace{\mathbb{E}[w_1 v_1]}_{\parallel \atop 0.} - 0.$$

$$\boxed{= 0.}$$