6.S088 Problem Set 2

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Problem 2

(a)

$$
\begin{aligned}
w^{(t+1)} &= w^{(t)} - \eta \nabla_w \mathcal{L}\left(w^{(t)}\right) \\
&= w^{(t)} + \eta (y - wX) X^T
\end{aligned}
$$

(b)

Let $S = XX^T$ and $S' = yX^T$. Then we can express the equation we get from (a) as follows:

$$
\begin{aligned}
w^{(t+1)} &= w^{(t)} + \eta (y - wX) X^T \\
&= w^{(t)} (I - \eta S) + \eta S'
\end{aligned}
$$

Then we solve the recurrence relation and get

$$
w^{(t)} = \eta S' \left[(I - \eta S)^{t-1} + (I - \eta S)^{t-2} + \ldots + (I - \eta S)^{1} + I \right]
$$

(... I fail to solve this problem 2 until the end but I solved all other problems except problem 6.)

Problem 3

$$
K(\mathbf{x}, \mathbf{x}') = e^{-L\|\mathbf{x}-\mathbf{x}'\|^2} = e^{-L\|\mathbf{x}\|^2} e^{-L\|\mathbf{x}'\|^2} e^{2L\langle \mathbf{x}, \mathbf{x}' \rangle} \tag{1}
$$

For the term $e^{2L\langle \mathbf{x}, \mathbf{x}' \rangle}$, we use Taylor expansion of e^z around $z = 0$ and geometry

$$
e^{2L\langle \mathbf{x}, \mathbf{x}' \rangle} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(2L \langle \mathbf{x}, \mathbf{x}' \rangle\right)^k \tag{2}
$$

Then we have

$$
\langle \mathbf{x}, \mathbf{x}' \rangle^k = \left(\sum_{i=1}^{d} \mathbf{x}_i \mathbf{x}'_i \right)^k = \sum_{j \in [d]^k} \left(\prod_{i=1}^{k} \mathbf{x}_{j_i} \right) \left(\prod_{i=1}^{k} \mathbf{x}'_{j_i} \right) \tag{3}
$$

where j enumerates over all selections of k coordinates of $\mathbf{x}$. Then by plugging in (2), (3) to (1) results in

$$
\begin{gathered}
K(\mathbf{x}, \mathbf{x}') = \langle \phi(\mathbf{x}), \phi(\mathbf{x}') \rangle = \prod_{k=0}^{\infty} \prod_{j \in [d]^k} \phi_{k,j}(\mathbf{x}) \phi_{k,j}(\mathbf{x}') \\
\text{where } \phi_{k,j}(\mathbf{x}) = e^{-L\|\mathbf{x}\|^2} \frac{2L^{\frac{K}{2}}}{\sqrt{k!}} \prod_{i=0}^{k} \mathbf{x}_{j_i}
\end{gathered}
$$

Problem 4

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K\left(x^{(i)}, x^{(j)}\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \langle \psi(x^{(i)}), \psi(x^{(j)})\rangle_{\mathcal{H}}$$
$$= \left(\sum_{i=1}^{n} c_i \psi(x^{(i)})\right)^T \left(\sum_{i=1}^{n} c_i \psi(x^{(i)})\right) \geq 0$$

$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K\left(x^{(i)}, x^{(j)}\right) \geq 0$ implies K is *positive semi-definite*.