6.S091 Problem Set 1

Suyeol Yun
6.S091: Causality

January 16, 2023

Preliminaries

(a)

$$\begin{aligned}
\mathbb{P}_{\mathcal{X}} &= P(U)P(A|U)P(M|A)P(Y|M,U) \\
&= \mathrm{Ber}(0.5)\,\mathrm{Ber}(U/4)\,\mathrm{Ber}(0.5+0.1A)\,\mathrm{Ber}(M/2+U/4)
\end{aligned}$$

(b)

$$\begin{aligned}
P(Y|M=0,A=0) &= \mathbb{1}_{\{\varepsilon_y+U/4\geq 1\}} \sim \mathrm{Ber}(U/4) \\
P(Y=1|M=0,A=0) &= P(Y=1|M=0,A=0,U=0)P(U=0) \\
&\quad + P(Y=1|M=0,A=0,U=1)P(U=1) \\
&= 0\cdot 1/2 + 1/4\cdot 1/2 \\
&= 1/8 \\
P(Y=1|M=0,A=1) &= P(Y=1|M=0,A=0) \\
&= 1/8
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(A=1) &= \sum_{U\in\{0,1\}} \mathbb{P}(A=1|U)P(U) && \text{(Law of Total Probability)} \\
&= \sum_{U\in\{0,1\}} \mathbb{P}(\epsilon_a + U/4 \geq 1)P(U) \\
&= \sum_{U\in\{0,1\}} \mathbb{P}(\epsilon_a \geq 1 - U/4)P(U) \\
&= \mathbb{P}(\epsilon_a \geq 3/4)P(U=1) + \mathbb{P}(\epsilon_a \geq 1)P(U=0) \\
&= 1/4\cdot 1/2 + 0\cdot 1/2 \\
&= 1/8
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(M=1) &= \sum_{A\in\{0,1\}} \mathbb{P}(M=1|A)P(A) && \text{(Law of Total Probability)} \\
&= \sum_{A\in\{0,1\}} \mathbb{P}(\varepsilon_m + 10(1-A) \leq 60)P(A) \\
&= \sum_{A\in\{0,1\}} \mathbb{P}(\varepsilon_m \leq 60 - 10(1-A))P(A) \\
&= \mathbb{P}(\varepsilon_m \leq 60)P(A=1) + \mathbb{P}(\varepsilon_m \leq 50)P(A=0) \\
&= 3/5\cdot 1/8 + 1/2\cdot 7/8 \\
&= 41/80
\end{aligned}$$

$$
\begin{aligned}
\mathbb{P}(Y=1) &= \sum_{M\times U\in\{0,1\}\times\{0,1\}} \mathbb{P}(Y=1|M,U)P(M,U) && \text{(Law of Total Probability)}\\
&= \sum_{M\times U\in\{0,1\}\times\{0,1\}} \mathbb{P}(Y=1|M,U)P(M|U)P(U)\\
&= \sum_{M\in\{0,1\}} \mathbb{P}(Y=1|M,U=0)P(M|U=0)P(U=0)\\
&+ \sum_{M\in\{0,1\}} \mathbb{P}(Y=1|M,U=1)P(M|U=1)P(U=1)\\
&= \mathbb{P}(Y=1|M=0,U=0)P(M=0|U=0)\cdot 1/2\\
&+ \mathbb{P}(Y=1|M=1,U=0)P(M=1|U=0)\cdot 1/2\\
&+ \mathbb{P}(Y=1|M=0,U=1)P(M=0|U=1)\cdot 1/2\\
&+ \mathbb{P}(Y=1|M=1,U=1)P(M=1|U=1)\cdot 1/2\\
&= 0\cdot 41/80\cdot 1/2\\
&+ 1/2\cdot 39/80\cdot 1/2\\
&+ 1/4\cdot 39/80\cdot 1/2\\
&+ 3/4\cdot 41/80\cdot 1/2\\
&= 3/8
\end{aligned}
$$

In the above calculation, it's worth noting that U nad M are correlated via A, i.e. $U \to A \to M$. M and U are not independent and thus, $P(M|U)$ is computed as follows.

$$
\begin{aligned}
P(M=0|U=0) &= P(M=0|U=0,A=0)P(A=0) + P(M=0|U=0,A=1)P(A=1)\\
&= 1/2\cdot 7/8 + 3/5\cdot 1/8\\
&= 41/80\\
P(M=1|U=0) &= 1-41/80 = 39/80\\
P(M=0|U=1) &= P(M=0|U=1,A=0)P(A=0) + P(M=0|U=1,A=1)P(A=1)\\
&= 1/2\cdot 7/8 + 2/5\cdot 1/8\\
&= 39/80\\
P(M=1|U=1) &= 41/80
\end{aligned}
$$

Interventional

(c)

$$
\begin{aligned}
\mathbb{P}_{\mathcal{X}}(U,A,M,Y \mid \mathrm{do}(A=1)) &= P(U)P(A^I=1)P(M|A=1)P(Y|M,U)\\
&= \mathrm{Ber}(0.5)\cdot 1\cdot \mathrm{Ber}(0.6)\cdot \mathrm{Ber}(M/2+U/4)\\
&= \mathrm{Ber}(0.5)\cdot \mathrm{Ber}(0.6)\cdot \mathrm{Ber}(M/2+U/4)
\end{aligned}
$$

(d)

$$
\begin{aligned}
\mathbb{P}_{\mathcal{X}}(Y=1 \mid \text{ do } (A=1)) &= \sum_{M\times U\in\{0,1\}\times\{0,1\}} \mathbb{P}(Y=1|\text{ do } (A=1), M, U)P(M,U|\text{ do } (A=1)) \\
&= \sum_{M\times U\in\{0,1\}\times\{0,1\}} \mathbb{P}(Y=1|\text{ do } (A=1), M, U)P(M|\text{ do } (A=1))P(U|\text{ do } (A=1)) \\
&= \mathbb{P}(Y=1|\text{ do } (A=1), M=0, U=0)P(M=0|\text{ do } (A=1))P(M|\text{ do } (A=1))P(U=0|\text{ do } (A=1)) \\
&+ \mathbb{P}(Y=1|\text{ do } (A=1), M=1, U=0)P(M=1|\text{ do } (A=1))P(M|\text{ do } (A=1))P(U=0|\text{ do } (A=1)) \\
&+ \mathbb{P}(Y=1|\text{ do } (A=1), M=0, U=1)P(M=0|\text{ do } (A=1))P(M|\text{ do } (A=1))P(U=1|\text{ do } (A=1)) \\
&+ \mathbb{P}(Y=1|\text{ do } (A=1), M=1, U=1)P(M=1|\text{ do } (A=1))P(M|\text{ do } (A=1))P(U=1|\text{ do } (A=1)) \\
&= 0\cdot 2/5\cdot 0 + 1/2\cdot 3/5\cdot 0 + 1/4\cdot 2/5\cdot 1 + 0.75\cdot 3/5\cdot 1 \\
&= 11/20 \\
&= 0.55 \\
\mathbb{P}_{\mathcal{X}}(Y=0 \mid \text{ do } (A=1)) &= 1-0.55 = 0.45
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}_{\mathcal{X}}(Y=1 \mid \text{ do } (A=0)) &= \sum_{M\times U\in\{0,1\}\times\{0,1\}} \mathbb{P}(Y=1|\text{ do } (A=0), M, U)P(M,U|\text{ do } (A=0)) \\
&= \sum_{M\times U\in\{0,1\}\times\{0,1\}} \mathbb{P}(Y=1|\text{ do } (A=0), M, U)P(M|\text{ do } (A=0))P(U|\text{ do } (A=0)) \\
&= \mathbb{P}(Y=1|\text{ do } (A=0), M=0, U=0)P(M=0|\text{ do } (A=0))\cdot P(U=0|\text{ do } (A=0)) \\
&+ \mathbb{P}(Y=1|\text{ do } (A=0), M=1, U=0)P(M=1|\text{ do } (A=0))\cdot P(U=0|\text{ do } (A=0)) \\
&+ \mathbb{P}(Y=1|\text{ do } (A=0), M=0, U=1)P(M=0|\text{ do } (A=0))\cdot P(U=1|\text{ do } (A=0)) \\
&+ \mathbb{P}(Y=1|\text{ do } (A=0), M=1, U=1)P(M=1|\text{ do } (A=0))\cdot P(U=1|\text{ do } (A=0)) \\
&= 0\cdot 1/2\cdot 4/7 + 1/2\cdot 1/2\cdot 4/7 + 1/4\cdot 1/2\cdot 3/7 + 0.75\cdot 1/2\cdot 3/7 \\
&= 5/14 \\
\mathbb{P}_{\mathcal{X}}(Y=0 \mid \text{ do } (A=0)) &= 1-5/14 = 9/14
\end{aligned}
$$