6.S091 Problem Set 1

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6.S091: Causality

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Preliminaries

(a)

$$\begin{aligned}\mathbb{P}_{\mathcal{X}}(U, A, M, Y) &= P(U)P(A|U)P(M|A)P(Y|M, U)\\ &= \mathrm{Ber}(0.5)\,\mathrm{Ber}(U/4)\,\mathrm{Ber}(0.5 + 0.1A)\,\mathrm{Ber}(M/2 + U/4)\end{aligned}$$

(b)

$$\begin{aligned}&\mathbb{P}_{\mathcal{X}}(Y = 1) = 0.38125\\ &\mathbb{P}_{\mathcal{X}}(Y = 1 \mid M = 0, A = 0) = 0.10714285714285714\\ &\mathbb{P}_{\mathcal{X}}(Y = 1 \mid M = 0, A = 1) = 0.25\end{aligned}$$

* I computed the above values by computing the distribution $\mathbb{P}_{\mathcal{X}}(U, A, M, Y)$ as dictionary in python and marginalized. See https://github.com/syyunn/6.S091/blob/main/pset1/problem_1b.py

Interventional

(c)

$$\begin{aligned}\mathbb{P}_{\mathcal{X}}(U, A = 1, M, Y \mid \mathrm{do}(A = 1)) &= P(U)P(A = 1|A = 1)P(M|A = 1)P(Y|M, U, A = 1)\\ &= \mathrm{Ber}(0.5) \cdot 1 \cdot \mathrm{Ber}(0.6) \cdot \mathrm{Ber}(M/2 + U/4)\\ &= \mathrm{Ber}(0.5) \cdot \mathrm{Ber}(0.6) \cdot \mathrm{Ber}(M/2 + U/4)\end{aligned}$$

In the similar vein, we can compute

$$\begin{aligned}\mathbb{P}_{\mathcal{X}}(U, A = 1, M, Y \mid \mathrm{do}(A = 0)) &= P(U)P(A = 0|A = 0)P(M|A = 0)P(Y|M, U, A = 1)\\ &= \mathrm{Ber}(0.5) \cdot 1 \cdot \mathrm{Ber}(0.5) \cdot \mathrm{Ber}(M/2 + U/4)\\ &= \mathrm{Ber}(0.5) \cdot \mathrm{Ber}(0.5) \cdot \mathrm{Ber}(M/2 + U/4)\end{aligned}$$

(d)

$$\mathbb{P}_{\mathcal{X}}(Y \mid \mathrm{do}(A = 1)) = \begin{cases} 0.425 & \text{if } Y = 1\\ 0.575 & \text{if } Y = 0\end{cases} \tag{1}$$

$$\mathbb{P}_{\mathcal{X}}(Y \mid \mathrm{do}(A = 0)) = \begin{cases} 0.375 & \text{if } Y = 1\\ 0.625 & \text{if } Y = 0\end{cases} \tag{2}$$

* I computed the above values by computing the distribution $\mathbb{P}_{\mathcal{X}}(U, A, M, Y)$ as dictionary in python and marginalized. See https://github.com/syyunn/6.S091/blob/main/pset1/problem_1d_doA0.py and https://github.com/syyunn/6.S091/blob/main/pset1/problem_1d_doA1.py

(e)

In case of $do(A=0)$, we have

$$\mathbb{P}_{\mathcal{X}}(Y=1 \mid \mathrm{do}(A=0)) = 0.375 \quad \text{from (d)}$$

$$\begin{aligned}\sum_{u\in\{0,1\}} \mathbb{P}_{\mathcal{X}}(Y=1 \mid A=0, U=u)\mathbb{P}_{\mathcal{X}}(U=u) &= \mathbb{P}_{\mathcal{X}}(Y=1 \mid A=0, U=1)\mathbb{P}_{\mathcal{X}}(U=1) + \mathbb{P}_{\mathcal{X}}(Y=1 \mid A=0, U=0)\mathbb{P}_{\mathcal{X}}(U=0) \\ &= 0.5 * 1/2 + 0.25 * 1/2 \\ &= 0.375 \quad \text{(match)}\end{aligned}$$

$$\mathbb{P}_{\mathcal{X}}(Y=0 \mid \mathrm{do}(A=0)) = 0.625 \quad \text{from (d)}$$

$$\begin{aligned}\sum_{u\in\{0,1\}} \mathbb{P}_{\mathcal{X}}(Y=0 \mid A=0, U=u)\mathbb{P}_{\mathcal{X}}(U=u) &= \mathbb{P}_{\mathcal{X}}(Y=0 \mid A=0, U=1)\mathbb{P}_{\mathcal{X}}(U=1) + \mathbb{P}_{\mathcal{X}}(Y=0 \mid A=0, U=0)\mathbb{P}_{\mathcal{X}}(U=0) \\ &= 0.5 * 1/2 + 0.75 * 1/2 \\ &= 0.625 \quad \text{(match)}\end{aligned}$$

In case of $do(A=1)$, we have

$$\mathbb{P}_{\mathcal{X}}(Y=1 \mid \mathrm{do}(A=1)) = 0.425 \quad \text{from (d)}$$

$$\begin{aligned}\sum_{u\in\{0,1\}} \mathbb{P}_{\mathcal{X}}(Y=1 \mid A=1, U=u)\mathbb{P}_{\mathcal{X}}(U=u) &= \mathbb{P}_{\mathcal{X}}(Y=1 \mid A=1, U=1)\mathbb{P}_{\mathcal{X}}(U=1) + \mathbb{P}_{\mathcal{X}}(Y=1 \mid A=1, U=0)\mathbb{P}_{\mathcal{X}}(U=0) \\ &= 0.55 * 1/2 + \text{undefined} \\ &= \text{undefined}\end{aligned}$$

$$\mathbb{P}_{\mathcal{X}}(Y=0 \mid \mathrm{do}(A=1)) = 0.625 \quad \text{from (d)}$$

$$\begin{aligned}\sum_{u\in\{0,1\}} \mathbb{P}_{\mathcal{X}}(Y=0 \mid A=1, U=u)\mathbb{P}_{\mathcal{X}}(U=u) &= \mathbb{P}_{\mathcal{X}}(Y=0 \mid A=1, U=1)\mathbb{P}_{\mathcal{X}}(U=1) + \mathbb{P}_{\mathcal{X}}(Y=0 \mid A=1, U=0)\mathbb{P}_{\mathcal{X}}(U=0) \\ &= 0.45 * 1/2 + \text{undefined} \\ &= \text{undefined}\end{aligned}$$

(f)

In case of $do(A=0)$, we have

$$\begin{aligned}&\mathbb{P}_{\mathcal{X}}(Y=1 \mid \mathrm{do}(A=0)) = 0.375 \quad \text{from (d)} \\ &\sum_{m\in\{0,1\}} \mathbb{P}_{\mathcal{X}}(M=m \mid A=0) \sum_{a'\in\{0,1\}} \left(\mathbb{P}_{\mathcal{X}}\left(Y=1 \mid M=m, A=a'\right)\mathbb{P}_{\mathcal{X}}\left(A=a'\right)\right) \\ &= \mathbb{P}_{\mathcal{X}}(M=0 \mid A=0) \cdot \sum_{a'\in\{0,1\}} \left(\mathbb{P}_{\mathcal{X}}\left(Y=1 \mid M=0, A=a'\right)\mathbb{P}_{\mathcal{X}}\left(A=a'\right)\right) \\ &+ \mathbb{P}_{\mathcal{X}}(M=1 \mid A=0) \cdot \sum_{a'\in\{0,1\}} \left(\mathbb{P}_{\mathcal{X}}\left(Y=1 \mid M=1, A=a'\right)\mathbb{P}_{\mathcal{X}}\left(A=a'\right)\right) \\ &= 1/2 \cdot (0.10714285714285714 \cdot 0.875 + 0.25 \cdot 0.125) + 1/2 \cdot (0.6071428571428571 \cdot 0.875 + 0.75 \cdot 0.125) \\ &= 0.375 \quad \text{(match)}\end{aligned}$$

$$\begin{aligned}&\mathbb{P}_{\mathcal{X}}(Y=0 \mid \mathrm{do}(A=0)) = 0.625 \quad \text{from (c)} \\ &\sum_{m\in\{0,1\}} \mathbb{P}_{\mathcal{X}}(M=m \mid A=0) \sum_{a'\in\{0,1\}} \left(\mathbb{P}_{\mathcal{X}}\left(Y=0 \mid M=m, A=a'\right)\mathbb{P}_{\mathcal{X}}\left(A=a'\right)\right) \\ &= \mathbb{P}_{\mathcal{X}}(M=0 \mid A=0) \cdot \sum_{a'\in\{0,1\}} \left(\mathbb{P}_{\mathcal{X}}\left(Y=0 \mid M=0, A=a'\right)\mathbb{P}_{\mathcal{X}}\left(A=a'\right)\right) \\ &+ \mathbb{P}_{\mathcal{X}}(M=1 \mid A=0) \cdot \sum_{a'\in\{0,1\}} \left(\mathbb{P}_{\mathcal{X}}\left(Y=0 \mid M=1, A=a'\right)\mathbb{P}_{\mathcal{X}}\left(A=a'\right)\right) \\ &= 1/2 \cdot (0.8928571428571429 \cdot 0.875 + 0.7500000000000001 \cdot 0.125) + 1/2 \cdot (0.39285714285714285 \cdot 0.875 + 0.25 \cdot 0.125) \\ &= 0.6250000000000000218750 \quad \text{(match)}\end{aligned}$$

In case of $do(A = 1)$, we have

$$
\begin{aligned}
&\mathbb{P}_{\mathcal{X}}(Y = 1 \mid \mathrm{do}(A = 1)) = 0.425 \quad \text{from (d)} \\
&\sum_{m\in\{0,1\}} \mathbb{P}_{\mathcal{X}}(M = m \mid A = 1) \sum_{a'\in\{0,1\}} \left(\mathbb{P}_{\mathcal{X}}\left(Y = 1 \mid M = m, A = a'\right)\mathbb{P}_{\mathcal{X}}\left(A = a'\right)\right) \\
&= \mathbb{P}_{\mathcal{X}}(M = 0 \mid A = 1) \cdot \sum_{a'\in\{0,1\}} \left(\mathbb{P}_{\mathcal{X}}\left(Y = 1 \mid M = 0, A = a'\right)\mathbb{P}_{\mathcal{X}}\left(A = a'\right)\right) \\
&+ \mathbb{P}_{\mathcal{X}}(M = 1 \mid A = 1) \cdot \sum_{a'\in\{0,1\}} \left(\mathbb{P}_{\mathcal{X}}\left(Y = 1 \mid M = 1, A = a'\right)\mathbb{P}_{\mathcal{X}}\left(A = a'\right)\right) \\
&= 0.4 \cdot (0.10714285714285714 \cdot 0.875 + 0.25 \cdot 0.125) + 0.6 \cdot (0.6071428571428571 \cdot 0.875 + 0.75 \cdot 0.125) \\
&= 0.42499999999999997650 \quad \text{(match)}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{P}_{\mathcal{X}}(Y = 0 \mid \mathrm{do}(A = 1)) = 0.575 \quad \text{from (c)} \\
&\sum_{m\in\{0,1\}} \mathbb{P}_{\mathcal{X}}(M = m \mid A = 1) \sum_{a'\in\{0,1\}} \left(\mathbb{P}_{\mathcal{X}}\left(Y = 0 \mid M = m, A = a'\right)\mathbb{P}_{\mathcal{X}}\left(A = a'\right)\right) \\
&= \mathbb{P}_{\mathcal{X}}(M = 0 \mid A = 1) \cdot \sum_{a'\in\{0,1\}} \left(\mathbb{P}_{\mathcal{X}}\left(Y = 0 \mid M = 0, A = a'\right)\mathbb{P}_{\mathcal{X}}\left(A = a'\right)\right) \\
&+ \mathbb{P}_{\mathcal{X}}(M = 1 \mid A = 1) \cdot \sum_{a'\in\{0,1\}} \left(\mathbb{P}_{\mathcal{X}}\left(Y = 0 \mid M = 1, A = a'\right)\mathbb{P}_{\mathcal{X}}\left(A = a'\right)\right) \\
&= 0.4 \cdot (0.8928571428571429 \cdot 0.875 + 0.7500000000000001 \cdot 0.125) + 0.6 \cdot (0.39285714285714285 \cdot 0.875 + 0.25 \cdot 0.125) \\
&= 0.57500000000000001625 \quad \text{(match)}
\end{aligned}
$$

Problem 2

(a)

Let's take a arbitrary node $\gamma_m \in \gamma$. Then γ_m should be unblocked since γ is d-connecting path and it means all nodes in γ are unblocked. Then since γ_m is unblocked, we can think of two cases where γ_m is collider or γ_m is non-collider. First, in case of γ_m is collider, then $\gamma_m \in S$ or its descendent $\mathrm{de}_{\mathcal{G}}(\gamma_m) \in S$. If $\gamma_m \in S$, $\gamma_m \in V$ since V include S itself. If $\mathrm{de}_{\mathcal{G}}(\gamma_m) \in S$, $\gamma_m \in \mathrm{an}_{\mathcal{G}}(S)$, and it implies $\gamma_m \in V$. Second, in case of γ_m is non-collider, $\gamma_m \notin S$. In this case, let's think about the sub-path $\gamma_m \to \gamma_{m+1}, \gamma_{m+2}, \cdots, b$ where b is the destination of the path γ. Among the nodes in this sub-path, if there exists a node that is collider, then this node is in S or ancestor of S as we've shown in the first scenario. Then γ_m becomes ancestor of S and it implies $\gamma_m \in V$. In other case, if there is no collider in the sub-path, then γ_m is an ancestor of b and it implies $\gamma_m \in V$. Therefore, in any cases, $\gamma_m \in V$.

(b)

From (a), we know that $\gamma_m \in S$ only in case that γ_m is collider. By the definition of moral graph, if γ_m is collider, we can marry γ_{m-1} and γ_{m+1} which are left and right nodes of γ_m. Therefore, with given path γ, we can construct a new path γ' by visiting each node in γ and marry left and right nodes when the node we're visiting is collider. Then this new path γ' doesn't pass through S and γ' is a valid path in moral graph $\overline{\mathcal{G}[\mathbf{V}]}$ since its all nodes and edges are included in $\overline{\mathcal{G}[\mathbf{V}]}$.

Problem 3

(a)

$$
\begin{aligned}
\mathbb{P}_{\mathcal{X}}(Y = 1 \mid A = N) &= \sum_{S\in\{L,R\}} \mathbb{P}_{\mathcal{X}}(Y = 1 \mid S, A = N)\mathbb{P}_{\mathcal{X}}(S) \\
&= 0.73 \cdot 0.49 + 0.93 \cdot 0.51 \\
&= 0.832 \\
\mathbb{P}_{\mathcal{X}}(Y = 1 \mid A = O) &= \sum_{S\in\{L,R\}} \mathbb{P}_{\mathcal{X}}(Y = 1 \mid S, A = O)\mathbb{P}_{\mathcal{X}}(S) \\
&= 0.69 \cdot 0.49 + 0.87 \cdot 0.51 \\
&= 0.7818
\end{aligned}
$$

(b)

Since S satisfies backdoor criterion, $\mathbb{P}_{\mathcal{X}}(Y = 1 \mid \mathrm{do}(A = N)) = \sum_{\mathbf{s}} \mathbb{P}_{\mathcal{X}}(Y \mid A = N, \mathbf{S} = \mathbf{s})\mathbb{P}_{\mathcal{X}}(\mathbf{S} = \mathbf{s}) = 0.832$ and $\mathbb{P}_{\mathcal{X}}(Y = 1 \mid \mathrm{do}(A = O)) = \sum_{\mathbf{s}} \mathbb{P}_{\mathcal{X}}(Y \mid A = O, \mathbf{S} = \mathbf{s})\mathbb{P}_{\mathcal{X}}(\mathbf{S} = \mathbf{s}) = 0.7818$.

Problem 4

β_{aw}	$\hat{\beta}_{aw}$	$\hat{\beta}_{yw}$	$\widehat{\beta_{yw}}/\widehat{\beta_{aw}}$	$\lvert\mathbb{E}[Y\|A] - \widehat{\beta_{yw}}/\widehat{\beta_{aw}}\rvert$
5	4.52	33.84	7.47	0.03
0.5	0.72	5.54	7.60	0.1
0.05	0.32	2.49	7.62	0.12

As shown in the table, the ratio well estimates the true $\mathbb{E}[Y|A] = 0.75$. Also we can notice that the absolute value of bias $|\mathbb{E}[Y|A] - \widehat{\beta_{yw}}/\widehat{\beta_{aw}}|$ increases as β_{aw} decreases.

* I computed the above values by computing the distribution $\mathbb{P}_{\mathcal{X}}(U, A, M, Y)$ as dictionary in python and marginalized. See https://github.com/syyunn/6.S091/blob/main/pset1/problem4.py