6.S091 Problem Set 1

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6.S091: Causality

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Preliminaries

(a)

$$\begin{aligned}\mathbb{P}_{\mathcal{X}} &= P(U)P(A|U)P(M|A)P(Y|M,U)\\ &= \mathrm{Ber}(0.5)\,\mathrm{Ber}(U/4)\,\mathrm{Ber}(0.5+0.1A)\,\mathrm{Ber}(M/2+U/4)\end{aligned}$$

(b)

$$\begin{aligned}&\mathbb{P}_{\mathcal{X}}(Y=1) = 0.38125\\ &\mathbb{P}_{\mathcal{X}}(Y=1 \mid M=0, A=0) = 0.10714285714285714\\ &\mathbb{P}_{\mathcal{X}}(Y=1 \mid M=0, A=1) = 0.25\end{aligned}$$

Interventional

(c)

$$\begin{aligned}\mathbb{P}_{\mathcal{X}}(U, A^I=1, M, Y \mid \mathrm{do}(A=1)) &= P(U)P(M|A=1)P(Y|M,U,A=1)\\ &= \mathrm{Ber}(0.5)\cdot\mathrm{Ber}(0.6)\cdot\mathrm{Ber}(M/2+U/4)\\ &= \mathrm{Ber}(0.5)\cdot\mathrm{Ber}(0.6)\cdot\mathrm{Ber}(M/2+U/4)\end{aligned}$$

$$\begin{aligned}\mathbb{P}_{\mathcal{X}}(U, A^I=1, M, Y \mid \mathrm{do}(A=0)) &= P(U)P(M|A=0)P(Y|M,U,A=0)\\ &= \mathrm{Ber}(0.5)\cdot\mathrm{Ber}(0.5)\cdot\mathrm{Ber}(M/2+U/4)\\ &= \mathrm{Ber}(0.5)\cdot\mathrm{Ber}(0.5)\cdot\mathrm{Ber}(M/2+U/4)\end{aligned}$$

(d)

$$\mathbb{P}_{\mathcal{X}}(Y \mid \mathrm{do}(A=1)) = \begin{cases}0.425 & \text{if } Y=1\\ 0.575 & \text{if } Y=0\end{cases} \tag{1}$$

$$\mathbb{P}_{\mathcal{X}}(Y \mid \mathrm{do}(A=0)) = \begin{cases}0.375 & \text{if } Y=1\\ 0.625 & \text{if } Y=0\end{cases} \tag{2}$$

(e)

If $A=1, U=0$, $P(A=1, U=0) = 0$. Therefore, $P(Y|A=1, U=0)$ is undefined. So I check only for the case of $do(A=0)$ which doesn't involve $P(Y|A=1, U=0)$ as a component in its process of computation.

$$
\begin{aligned}
\mathbb{P}_{\mathcal{X}}(Y=1 \mid \mathrm{do}(A=0)) &= 0.375 \\
\sum_{u\in\{0,1\}} \mathbb{P}_{\mathcal{X}}(Y=1 \mid A=0, U=u)\mathbb{P}_{\mathcal{X}}(U=u) &= \mathbb{P}_{\mathcal{X}}(Y=1 \mid A=0, U=1)\mathbb{P}_{\mathcal{X}}(U=1) + \mathbb{P}_{\mathcal{X}}(Y=1 \mid A=0, U=0)\mathbb{P}_{\mathcal{X}}(U=0) \\
&= 0.5 * 1/2 + 0.25 * 1/2 \\
&= 0.375 \quad \text{(match)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}_{\mathcal{X}}(Y=0 \mid \mathrm{do}(A=0)) &= 0.625 \\
\sum_{u\in\{0,1\}} \mathbb{P}_{\mathcal{X}}(Y=0 \mid A=0, U=u)\mathbb{P}_{\mathcal{X}}(U=u) &= \mathbb{P}_{\mathcal{X}}(Y=0 \mid A=0, U=1)\mathbb{P}_{\mathcal{X}}(U=1) + \mathbb{P}_{\mathcal{X}}(Y=0 \mid A=0, U=0)\mathbb{P}_{\mathcal{X}}(U=0) \\
&= 0.5 * 1/2 + 0.75 * 1/2 \\
&= 0.625 \quad \text{(match)}
\end{aligned}
$$

(f)

$$
\begin{aligned}
&\sum_{m\in\{0,1\}} \mathbb{P}_{\mathcal{X}}(M=m \mid A=0) \sum_{a'\in\{0,1\}} \left(\mathbb{P}_{\mathcal{X}}\left(Y \mid M=m, A=a'\right)\mathbb{P}_{\mathcal{X}}\left(A=a'\right)\right) \\
&=
\end{aligned}
$$

Problem 2

(a)

(Ask whether we start from $\mathbf{V} = \overline{\mathrm{an}}_{\mathcal{G}}(\mathbf{A} \cup \mathbf{B} \cup \mathbf{S})$) By the setting, $\mathbf{V} = \overline{\mathrm{an}}_{\mathcal{G}}(\mathbf{A} \cup \mathbf{B} \cup \mathbf{S})$. Since γ is d-connecting path, all nodes in γ are unblocked. For a node to be unblocked, either 1) it is not collider and not in S or 2) it is collider but it is not in S and so is its decendents. In both cases, the node is not in $\mathbf{S}$. In addition, since the definition of path doesn't include starting and end point which are in A and B respectively, we can exlcude the self $A \cup B \cup S$ from V, i.e. $\mathbf{V} = \mathrm{an}_{\mathcal{G}}(\mathbf{A} \cup \mathbf{B} \cup \mathbf{S})_{\blacksquare}$

(b)

Since there exists a d-connecting path γ, $(\mathbf{A}, \mathbf{B}, \mathbf{S}) \notin \mathcal{I}_{\perp\!\!\!\perp}(\mathcal{G})$. Since all nodes in γ doesn't include any node in S as shown in (a), we can use γ as a path in $\overline{\mathcal{G}[\mathbf{V}]}$ from A to B which doesn't pass through S by replacing the directed edges to undirected ones.

Problem 3

(a)

$$
\begin{aligned}
\mathbb{P}_{\mathcal{X}}(Y=1 \mid A=N) &= \sum_{S\in\{L,R\}} \mathbb{P}_{\mathcal{X}}(Y=1 \mid S, A=N)\mathbb{P}_{\mathcal{X}}(S) \\
&= 0.73 \cdot 0.49 + 0.93 \cdot 0.51 \\
&= 0.832 \\
\mathbb{P}_{\mathcal{X}}(Y=1 \mid A=O) &= \sum_{S\in\{L,R\}} \mathbb{P}_{\mathcal{X}}(Y=1 \mid S, A=O)\mathbb{P}_{\mathcal{X}}(S) \\
&= 0.69 \cdot 0.49 + 0.87 \cdot 0.51 \\
&= 0.7818
\end{aligned}
$$

(b)

(Ask how to draw ζ^I) Since S satisfies backdoor criterion, we can say
$\mathbb{P}_{\mathcal{X}}(Y=1 \mid \mathrm{do}(A=N)) = \sum_{\mathbf{s}} \mathbb{P}_{\mathcal{X}}(Y \mid A=N, \mathbf{S}=\mathbf{s})\mathbb{P}_{\mathcal{X}}(\mathbf{S}=\mathbf{s}) = 0.832$ and $\mathbb{P}_{\mathcal{X}}(Y=1 \mid \mathrm{do}(A=O)) = \sum_{\mathbf{s}} \mathbb{P}_{\mathcal{X}}(Y \mid A=O, \mathbf{S}=\mathbf{s})\mathbb{P}_{\mathcal{X}}(\mathbf{S}=\mathbf{s}) = 0.7818$.

Problem 4