Differientiable Manifold

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1 Manifold: the Local Trivialization

Def 1.1 (*C^r Manifold*). $\{M, \mathscr{S}\}$ called a $C^{r\in\mathbb{N}}$ *Manifold* consists of:

(i) A A_2, T_2 *Topological Space* M.

(ii) A *differential structure* $\mathscr{S}$ $i.e.$ A collection of *coordinate systems* $\{(U_\alpha, \varphi_\alpha)\}$:

- An *open cover* of M $\{U_\alpha\}_\alpha$: $\bigcup_\alpha U_\alpha = M$.
- An *atlas* $\{(U_\alpha, \varphi_\alpha)\} = \{(U, x^1, \cdots, x^n)\}$:
  the *coordinate chart* φ_α is a *homeomorphism* into: $U_\alpha \to \varphi_\alpha(U_\alpha) \subset R^n$,
  and the *coordinate funtion* x^i is the ith *Euclidean coordinate* of φ_α.
- Transition mappings $\varphi_\beta \circ \varphi_\alpha^{-1}$ is C^r, for all α, β.

Remark. The construction of $\mathscr{S}$ implies the fundamental idea of the topics relate to *manifolds*, namely:

the LOCAL TRIVALIZATION.

We will see the idea embodied as the so-called duality *intrinsic defintions* and *local coordinate expressions* in the next section, albeit the word *intrinsic* means "regardless of coordinates" but not always *global*.

Attdef 1.1.1 (*morphism*). It's a natural idea from the perspective of Category Theory as a corollary of the Remark above:

Any definition relate to C^r *differentiability* denpends on the C^r *differentiability* of coordinate chart transformation, like C^r *mapping*, C^r *diffeomorhism*, and C^r *compatible*.$Especially$ we suppose *diffeomorphic maniflods* are same, and it's an unsovlved core problem to find all the different *differantial structures* on S^n.

Attdef 1.1.2. Unfortunately, you still have to learn some information from the definition:

- According to the fact that $R^n \not\cong R^m$ if $n \neq m$, we denotes n as the dimension of M if M is connected.
- Notice that exists a unique $\mathscr{S}_{max} \supset \mathscr{S}, \forall \mathscr{S}$. Thus we identity $\mathscr{S}$ with $\mathscr{S}_{max}$ WLOG(pf. by *AC*).
- Besides, for any $C^{r \geqslant 1}$ *Manifold*, there exists a *compatible* C^∞ *differential structure*. Thus we often ignore "C^∞" for the sake of convinience when comes to *diffeomorphic maniflods*.

Def 1.2 (*Lie Group*). G is a C^r *Lie Group* if G is a C^r *manifold* and a *group*, while the group operation $\mu : G \times G \to G$ is C^r.

Def 1.3 (*Fibre Bundle*). the 5-tuple (E, M, π, F, G) is the *fibre bundle* over *M*, if:

(i) M, E, F are *manifolds*, $\pi : E \twoheadrightarrow M$ is a projection.

(ii) G is a *lie group* and has a (right) C^r *action* on F.

(iii) $\forall U_\alpha$, exists a *local trivalization* ψ_α is a *homeomorphism*:

$$\pi^{-1}(U_\alpha) \to U_\alpha \times F,$$
$$\pi^{-1}(p) \to \{p\} \times F.$$

(iv) Exists a C^∞ connection function $g_{\beta\alpha} : U_\alpha \cap U_\beta \to G$:

$$(p, v) \mapsto \psi_\beta \circ \psi_\alpha^{-1}(p, v) = (p, g_{\beta\alpha}(p)v), \forall v \in \{p\} \times F.$$

We notes π as the *fibre projection*, and F as the *fiber*.Besides,the space B is called the *base space*, E is the *total space*, G is the *structue group*.

To be mentioned, readers should be aware that another definition of *fibre bundles* on some books does not include the *group structure* G.

Attdef 1.3.1 (*Pirincipal Bundle* and *Vector Bundle*). *Especially,*

- (P, M, π) is called the *Pirincipal Bundle* if G has a (right) C^r *action* on P, and each *fiber* $\pi^{-1}(p)$ is a $G-$ *torsor*/$G-$ *principal homogeneous spcae*(that G acts *freely* and *transitively*).

- (E, M, π) is called the k-*Vector Bundle* if F is a k-module of *dimension* r and $G = GL(r, k)$, and we will write *Real Vector Bundle* as *Bundle* for short in the following text.

Attdef 1.3.2 (*morphism*)**.** Simliar with the remark on the def of *Manifold*, *Bundle Mapping* is just the C^r *mapping* pair (F, f) satisfying a natural commute diagram and F is linear pointwisely, and (F, f) is a *Bundle Equivalence* iff (F, f) and (G, g) are inverses to each other. $Especially$, called a *Bundle Morphism* and a *Bundle Isomorhism* if $f = id_M$.

Attdef 1.3.3. We can also define *Manifold* and *Vector Bundle* in the perspective of *Quotient Topology* $i.e.$ "*locally* gluing *trival Manifolds* or *Bundlles* together":

- A *Manifold* $M \triangleq \bigsqcup_\alpha U_\alpha / \sim$.
- A *Vector Bundle* $E \triangleq \bigsqcup_\alpha (U_\alpha \times\ k^r)/\sim$, where $g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(r, k)$ statisfying (to provide *local linearlization*):

$$g_{\alpha\alpha} = id,$$
$$\text{the } \textit{cocycle} \text{ condition:}\ \ g_{\beta\alpha} \cdot g_{\alpha\gamma} \cdot g_{\gamma\beta} = id.$$

Attdef 1.3.4. There are some basic *Algebraic Operations* over *Bundles* defined pointwisely:

Whitney Sum (Direct Sum) $\oplus$, *Dual Bundles* *, *Tensor Product* $\otimes$, and *wedge product* $\bigwedge$.

Lemma 1.1 (Paracompact)**.** *A locally compact, Lindelof space X is paracompact. Especially if we consider a manifold M, a secondly countable LCH, then each open cover of M has a countable, locally finite refinement consisting of open sets with compact closures (i.e. compact closure).[cf.Warner]*

Proof. (hint) Consider the "exhaution(穷竭)" construction $\{G_i\}_i \in \mathbb{N}$:

$$\bar{G}_i \subset G_{i+1},\ \bar{G}_i\ compact;\ \bigcup_i G_i = M$$

Q.E.D.

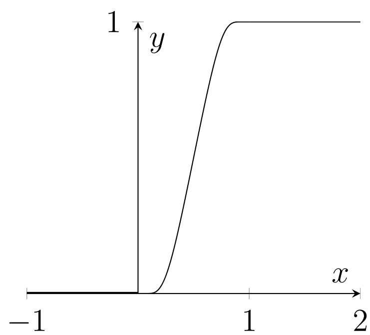
Lemma 1.2 (bulge/truncation function)**.** *There exists several smooth functions with special properties:*

(i) *A truncation function ϕ exists, if $\phi : \mathbb{R} \to [0,1]$ is smooth s.t.:*

$$\phi(x) = \begin{cases} 1, x \geqslant 1 \\ 0, x \leqslant 0 \end{cases}$$

Proof. Consider a C^∞ function $\varphi(x) = \begin{cases} e^{-\frac{1}{x}}, x > 0 \\ 0, x \leqslant 0 \end{cases}$, where $\varphi : \mathbb{R}^n \to [0,1]$. Then let

$$\phi(x) = \frac{\varphi(x)}{\varphi(x) + \varphi(1-x)}.$$



Q.E.D.

(ii) *A bulge function ψ exists, if $\psi : \mathbb{R}^n \to [0,1]$ is smooth s.t.:*

$$\psi(x) = \begin{cases} 1, x \in D_{r_1} \\ 0, x \notin D_{r_2} \end{cases}$$

Proof.

$$\psi(x) = \phi(\frac{r_2 - \|x\|}{r_2 - r_1})$$

Q.E.D.

(ii+) *A ultra-bulge function $\widetilde{\psi}$ exists for any open (coordinate) neighborhood U, if exists V with its compact closure $\bar{V} \subset U$ s.t.:*

$$\psi(x) = \begin{cases} 1, x \in V \\ 0, x \notin U \end{cases}$$

Proof. (hint) To combine a set of bulge functions $\{\psi_i\}$, consider the function:

$$\psi = 1 - \prod_i (1 - \psi_i)$$

. Q.E.D.

Thm-Lem 1.3 (P.O.U.). *For every (countable) open cover $\{U_i\}$ of a manifold M, there exists $\{g_\alpha : 0 \leqslant g_\alpha \leqslant 1\}$ s.t.:*

(i) $\exists i$, *supp*$(g_\alpha) \subset U_i(\alpha)$ *compact for each α;*

(i') *supp*$(g_\alpha) \subset\subset U_\alpha$ *for each α;*

(ii) $\sum_\alpha g_\alpha = 1$, *and for each $x \in M$, cofinite $\{g_\alpha\}$ are zero.*

For the i' version, we sometimes call $\{g_\alpha\}$ as the P.O.U. subordinate to $\{U_\alpha\}$. [1]

2 From (Co-)Tangent to Differentiation, finally Calculus

Def 2.1 (*Section*, as a generalization of (*vector-valued) function*). s is a C^r *section* of a *bundle* E, if $s : M \to E$ satisfying $\pi \circ s = id_M$. We noted $\Omega(E)$ as the collection of all sections of E.

Remark. A section s is a $C^\infty(M)$-module, where $C^\infty(M)$ is the *vector space* of C^∞ function from M to $\mathbb{R}$.

Def 2.2 (*Tangent, Cotangent*). Let M be a *manifold* and p be a point at M, then:

- $T_p^* M \triangleq F_p/F_p^2$ is called the *Cotangent Space* to M at p, where F_p is all finite linear combinations of k-fold products of elements of $\tilde{F}_p$, the C^∞ function germ at p, that vanishes at p.

[1] *A hint for the proof*: Combine the two lemmas above, while using a property of *LCH*:

$$\forall V \text{ with compact closure } \bar{V} \subset U, \exists V' \text{ with compact closure, s.t. } \bar{V} \subset V' \subset \bar{V}' \subset U$$

- A *tangent vector* $v_p : C^\infty(U_p) \to \mathbb{R}$ is a special *linear functional* satisfying the so-called *Leibniz's Rule*.

  Then T_pM denotes the set of *tangent vectors* to M at p and is called the *Tangent Space* to M at p.

The *tangent mapping* $\mathrm{d}f_p$ (or f_{*p}) and the *cotangent mapping* f^{*p} are induced by nutural diagrams of f. Besides, it's clear that T_pM, T_p^*M are both *vector spaces*, and $T_p^*M = (T_pM)^*$. Let:

$$TM \triangleq \bigcup_p T_pM,\ T^*M \triangleq \bigcup_p T_p^*M.$$

They are both *real vector bundles* and are respectively called the *Tangent Bundle* and the *Cotangent Bundle*.

Remark (intrinsic def). let

$$\begin{aligned} < \cdot, \cdot >: T_pM \times T_p^*M \to \mathbb{R}, \\ (x, u) \mapsto < x, u >|_p = x_p(u). \end{aligned}$$

It's clear that the definitions above does not depend on the coordinate system, which is called *intrinsic*.

Def-Thm 2.3 (*tangent vector fields* and *1-parameter transformation groups*). There exists a 1-1 correspondence to M between:

- the collection of *tangent vector fields* $TM = \{X_i\}_i$,
- a collection of C^k *additive group actions* $\{\phi : I_{\epsilon>0} \times U_{p\in M} \to M\}$ where $\phi(t \in I_\epsilon, \cdot) \triangleq \phi_t$ is a *diffeomorphism*.

If $\phi(\cdot, q)$ is the unique *integral curve* passing through q, $\forall q \in U$:

$$\begin{aligned} &\phi'(\cdot, q) = X_{\phi(\cdot,q)}, \\ &\phi(0, q) = q. \end{aligned}$$

Specifically, ϕ is called a *1-parameter transformation groups* if there exists a continuation $\phi : \mathbb{R} \times M \to M$, and its corresponding X is then *complete*.

Prop-Thm 2.3.1 (Local expressions for (co-)tangent vectors).

Lemma 2.1 (inverse mapping). *$f : M^n \to N^n$ is a $C^{r\geqslant 1}$ mapping and df_p is invertible, then exists $U \ni x$ s.t. $f\ |_U : U \to V$ is a C^r diffeomorhism.*

Remark. df_p need be a *topological morphism* if $r = 1$.

Def 2.4 (*immersion, embedding, submersion* and *submanifold*). Let $f : M^m \to N^n$ be a $C^{r\geqslant 1}$ mapping, then:

(i) f is *immersed/submersed*, if df is *injective/surjective*;

(ii) f is *emebedded*, if f is *homeomorprhic* from M to $f(M)$ and *immersed*, i.e. f is *open* and *injective* and df is *injective*;

(iii) And if f is an inclusion, we say M is an *immersed/regular manifold* of N if f is *immersed/embedded*.

Remark. We give the structue of a *regular manifold* as its "coordinate" definition:

$\forall p \in M, \exists \{U_p, x^1, \cdots, x^n\},\ s.t.\ M \cap U = \{q \in U : x^i(q) = 0,\ m+1 \leqslant i \leqslant n\}.$

It's clear that *immersions/submersions* also have their local coordinate structure, albeit this note is too small to contain them.
Besides, this remark immediately lead to the following theorem 2.2.

Thm 2.2. *Let $f : M \to N$ be a $C^{r\geqslant 1}$ mapping and $q \in f(M)$, then $f^{-1}(q)$ is a regular submanifold of M, if:* $\mathrm{rank}(df_p) \equiv l$, $\forall p \in f^{-1}(q)$. *And* $\mathrm{codim}(f^{-1}(q)) = l$.[2]
Especially, this thm is always presented as the following two useful versions:

(i) Regular Level Set Thm: if df_p is epimorphic;

(ii) Constant Rank Thm: if $\mathrm{rank}(df_p) \equiv l$, $\forall p \in M$.

Def 2.5 (transversal intersection).

Thm 2.3 (transversal intersection).

Thm 2.4 (Sard).

Def 2.6 (*Derivation* on an Algebra). Let A be a k-*algebra*, then a *derivation* of A is a linear endomorphism X of A that satisfys the *Leibniz's Rule*:

$$X(fg) = fXg + gXf$$

And the collection on *derivations* of A is denoted by $Der(A)$.

[2] *A hint for the proof*: *rank* is unchanging under perturbation.

Remark. Notice that the *commutator* of arbitrary two derivations $[X, Y] := XY - YX$ is still a *commutator*, immediately we find that $Der(C^\infty(M)) = TM$ actually is a $\mathbb{R}$-*lie algebra*. (BUT NOT *homogeneous* over $C^\infty(M)$!) [3]

Def 2.8 (*Differential form* and *Exterior Differentiation*). Let $\bigwedge(T^*M)$ be the *Grassmann algebra* (one kind of *graded algebra*) of a *tangent bunlde* T^*M over $C^\infty(M)$, then:

- $\omega \in \bigwedge^k(T^*M)$ is called a k-*(differential) form*.
- A linear endomorphism $d : \bigwedge(T^*M) \to \bigwedge(T^*M)$ $(k \in \mathbb{N}^*)$ is called the *exterior differentiation operator* over $\bigwedge(T^*M)$ if:
  (i) $d(f) = df$, if $f \in C^\infty(M)$.
  (ii) *graded Leibniz's rule*: $d(\omega \wedge \nu) = d\omega \wedge \nu + (-1)^{deg(\omega)}\omega \wedge d\nu$.
  (iii) $d^2 = 0$.

Remark. For $d : \bigwedge^{k\in\mathbb{N}}(T^*M) \to \bigwedge^{k+1}(T^*M)$ and $d \circ d = 0$, we then have a *chain*, called *De Rham cohomology* and denoted by $H^k_{dR}(M)$:

$$0 \xrightarrow{d} \bigwedge^{1}(T^*M) \xrightarrow{d} \cdots \xrightarrow{d} \bigwedge^{n-1}(T^*M) \xrightarrow{d} \bigwedge^{n}(T^*M) \xrightarrow{d} 0$$

Prop 2.8.1 (the 2rd Def of Exterior Differentiation).

$$\begin{aligned} d\omega(X_0, \cdots, X_k) = &\sum_{i=0}^{k}(-1)^i X_i\omega(X_0, \cdots, \hat{X}_i, \cdots, X_k) \\ &+ \sum_{0\leqslant i<j\leqslant k}(-1)^{i+j}\omega([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_k) \end{aligned}$$

Def 2.9 (*Distribution* and its *annihilator*). $\mathscr{D}$ is a k-*dimensional Distribution* on a manifold M^n, if there exists a collection of *non-zero vector fields* $\{X_i\}_{1\leqslant i\leqslant k}$, s.t. $\mathscr{D}(p) = span\{X_i(p)\}_{1\leqslant i\leqslant k} \subset T_pM$.

And we denoted the *ideal* in $\bigwedge(T^*M)$ generated by $n-k$ independent annihilators of $\mathscr{D}$ as $\mathscr{I}(\mathscr{D}) :=< anni(\mathscr{D}) >$.

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[3]

Def 2.7 (*Lie algebra* and *Lie bracket*). $\mathfrak{g}$ is a k-*Lie algebra*, if $\mathfrak{g}$ is a k-module together with a *Lie bracket* $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which is a *bilinear*, *anti-communtative* operator satisfying the *Jocobian identity*:

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0, \forall X, Y, Z \in \mathfrak{g}$$

Lemma 2.5. *Let X be a vector field and $p \in M$, then X has a local expression $\{U_p, y^1, \cdots, y^n\}$ if p is not a sigular point of X:*

$$X\mid_{U_p} = \frac{\partial}{\partial y_1}.$$

Even further, Let $\{X_i\}_{i \leqslant k}$ be a set of vector fields doesn't vanash at $p \in M$, then $\{X_i\}$ still has a local expression $\{U_p, y^1, \cdots, y^n\}$, if X_i linear independent pointwisely and $[X_i, X_j] = 0$ for each i, j:

$$X_i\mid_{U_p} = \frac{\partial}{\partial y_i},\ 1 \leqslant i \leqslant k$$

Def-Thm 2.10 (*Frobenius: completely integrable conditions*)**.** Let a *k-dimensional Distribution* $\mathscr{D}$ and its *ideal* $\mathscr{I}(\mathscr{D})$ be defined on M^n as above. We say $\mathscr{D}$ is *completely integrable* and F is the *integral (sub)manifold* of $\mathscr{D}$, if(TFAE):

- There exists a unique *k-dimensional maximal integral (sub)manifold* F of $\mathscr{D}$ "passing through" any point $q \in M$, if exists an *injective immersion* $i : F \to M$ s.t. $i_{*p}(T_pF) = \mathscr{D}(q = i(p))$.

  Specifically, we could have an *atlas* $\{(V, y^1, \cdots, y^k, y^{k+1}(q), \cdots, y^n(q))\}$ glued by coordinate slices locally which passes through q.

- The *distribution* $\mathscr{D}$ is *involutive*, if the *commutator* $[\cdot, \cdot]$ is well-defined (close) on $\mathscr{D}$.

- Equivalently, let N be a manifold and τ be an *injective immersion* $\tau : N \to M$, then N satisfys the so-called *Pfaff's equations*:

  $$\omega^\alpha\mid_N = 0.\ (\omega^\alpha \in T^*M,\ \alpha = 1, 2, \cdots, n-k)$$

  then the equations ought to be *completely integrable*, i.e. exists the above-mentioned F that "passes through" q and an *injective immersion* $i : F \to M$ s.t. $i^*\omega^\alpha \equiv 0$.

- The *ideal* $\mathscr{I}(\mathscr{D})$ is a *differential ideal*, if $d(\mathscr{I}) \subset \mathscr{I}$, i.e.

  $$d\omega^\alpha \equiv 0\ (mod\,\omega^1, \cdots, \omega^{n-k}),\ 1 \leqslant \alpha \leqslant n-k.$$

Def 2.11 (*Lie Derivate*)**.** For each $X \in TM$, there exists a *Lie derivate* $\mathcal{L}_X \in End(\Omega(E))$ defined by:

$$(\mathcal{L}_X S)(p) = \lim_{t \to 0} \frac{(\phi_{-t})_* S_{\phi(t)} - S_p}{t}$$

Especially, we have $\mathcal{L}_X f = < X, f >$ and $\mathcal{L}_X Y = [X, Y]$.

Prop 2.11.1 (Cartan formulas).

$$\begin{aligned} d\mathcal{L}_X - \mathcal{L}_X d &= 0, \\ \mathcal{L}_X i(Y) - i(Y)\mathcal{L}_X &= i_{[X,Y]}, \\ \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X &= \mathcal{L}_{[X,Y]}, \\ di_X + i_X d &= \mathcal{L}_X. \end{aligned}$$

Def 2.12 (Boundary operator).

Def 2.13 (*Orientation*). M^m is *orientable*, if exists a *nowhere-vanishing* m-form ω. We denote $[\omega] \triangleq \{f\omega : f > 0\}$ as an *orientation* on M, then it's clear that every connected manifold (only) has 2 *orientatations* $\mathcal{O}^+$ and $\mathcal{O}^-$.

Thm 2.6 (Newton-Leibniz).

$$\int_D d\omega = \int_{\partial D} \omega$$

3 Further formal constructions

Def 3.1 (Pricipal bundle and Associate Bundle).

Def 3.2 (*Connection*).