AN INTRODUCTION TO ÉTALE COHOMOLOGY

ELLIOT HAO (SHIZE HAO)

Contents

Convention and Notation

We denote Sch_k as the category of $k-$schemes (schemes over k), where k is some base field, base ring or even some base scheme. We usually omit k if it doesn't matter, and we stress it only when we care about bases.

Most of the time, we do not distinguish between rings and their associated affine schemes, or between modules over a ring A and the quasi-coherent sheaves over $\operatorname{Spec} A$, as they are essentially equivalent. However, we may use $\widetilde{M}$ to explicitly denote the quasi-coherent sheaf associated with a module M.

We also use the somewhat informal term "local". By affine-local, we mean working within affine open neighborhoods. This term is particularly relevant when we "reduce to affine cases," a process that involves the following steps::

(1) Verifying our assertions in an arbitrary affine open neighborhood;
(2) Ensuring compatibility of these assertions across overlaps of different affine neighborhoods.

To illustrate this explicitly, suppose we are working with affine neighborhoods $U \cong \operatorname{Spec} A$, $V \cong \operatorname{Spec} B$, along with their intersection $U \cap V$. The intersection $U \cap V$ can be covered by a family of distinguished open subsets $\{W_i\}$, where $W_i \cong \operatorname{Spec} A_{f_i}$, for some $f_i \in A$ (alternatively, $W_i \cong \operatorname{Spec} B_{g_i}$, for $g_i \in B$, depending on your choice of localization). Here, the localized rings $A_{f_i} = \mathcal{A}[T]/(Tf_i - 1)$ correspond to the distinguished open set $D(f_i) = \operatorname{Spec} A \setminus V(f)$, which satisfies $\operatorname{Spec} A_{f_i} \cong D(f_i) \subset \operatorname{Spec} A$.

In this context, verifying a claim involves:

(1) Establishing the claim on any affine neighborhoods $\operatorname{Spec} A$;
(2) Confirming that the restriction of the situation on $\operatorname{Spec} A$ to $D(f)$, for any $f \in A$, is consistent with the analogous situation on $\operatorname{Spec} A_f$

1. Étale Morphisms: Equivalent Definitions and Examples

The term "étale morphism" is significant in algebraic geometry. On one hand, it encodes the condition of local isomorphism, serving as an algebraic analogue of a covering map; on the other hand, it is a broad generalization of finite separable extensions, carrying strong number-theoretic implications. The following theorem provides a formal interpretation of these ideas, presenting the equivalent definitions of an étale morphism:

Theorem 1.1. *Assume $f : X \to S$. We call f étale if it satisfies any of the following equivalent conditions:*

(1) f is smooth with relative dimension 0;
(2) f is both smooth and unramified;
(3) f is flat and unramified;
(4) f is formally étale and locally of finite presentation;
(5) Affine locally, f can be represented by a standard étale algebra $R[T]/(F)_G$, where F is monic and F' is invertible.

Proof. (1) $\Leftrightarrow$ (2): Apply Propositions 1.12 and 1.8; this is Corollary 1.13.
(2) $\Leftrightarrow$ (4): Apply Propositions 1.16 and 1.19; this is Proposition 1.20.
(2) $\Rightarrow$ (3): See Lemma 1.11.
(2) $\Leftarrow$ (3): To be continued. This requires Theorem 1.21, or see [Mil13, Proposition 2.9].
(1) $\Leftarrow$ (5): This is Example 1.3.
(1) $\Rightarrow$ (5): To be continued. □

Warning. We define a smooth morphism as one where the Jacobian matrix is injective. Some sources define a smooth morphism as flat, locally of finite presentation, and with geometrically regular fibers. This approach requires a lot of efforts dealing with flatness and regularity, whereas the Jacobian criterion is more intuitive and straightforward, which is why we adopt it here.

In the following sections, we will unravel each equivalent definition of an étale morphism step by step. Definitions (1) and (2) are sometimes referred to as the "Jacobian Criterion," while (4) is often called the "functorial approach." On the other hand, (3) is the most technical condition, bearing a more commutative-algebraic flavor, so we will address it at the end. For definition (5), I am not sure whether I will prove it fully at this stage...

1.1. Motivation: The Jacobian Criterion and Beyond.

We start with a classical "linear algebra" approach, using the Jacobian criterion to discuss smooth and étale morphisms, concluding with an interpretation of étale morphisms as "local isomorphisms."

In this subsection, we assume that $f : X \to S$ in Sch_k is locally of finite presentation. Thus, X has affine local coordinates: for any point $x \in X$, there exists an affine open neighborhood U of x such that $f(U) \cong \operatorname{Spec} R$ and $U \cong \operatorname{Spec} A$, where $A \cong R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ with $X = \{x_1, \ldots, x_n\}$ and $I = (f_1, \ldots, f_m)$.

For simplicity, we may sometimes further assume $S = \operatorname{Spec} R$ and even $R \cong k[t_1, \ldots, t_s]$ when necessary. Hence, f (affine locally) factors as follows:

$$X \xrightarrow{\tau} \mathbb{A}^n_S \xrightarrow{\pi} S$$

In these affine local coordinates, we can define the Jacobian matrix $J(x)$ with entries $\frac{\partial f_j}{\partial x_i}(x)$, whose values lie in $\kappa(x)$. This leads to the following definition:

Definition 1.2. In the above setting, the map $f : X \to S$ is called smooth of relative dimension d at a point $x \in X$ if the Jacobian matrix $J(x)$ is injective and of rank $r = n - d$. In particular, we say $X \to S$ is étale at the point x if the Jacobian matrix $J(x)$ is non-singular.

If f is smooth (resp. étale) at every point, we say that f is smooth (resp. étale).

Now, we list some important examples. Note that these examples assume some results that have not yet been proven.

Example 1.3. A standard étale R-algebra is $A := R[T]/(F)_G$, where F is monic and the Jacobian F' is invertible. Thus, $\operatorname{Spec} A \to \operatorname{Spec} R$ is étale. In particular, if $A = k[T]/F$ with F' invertible, this is a finite separable extension.

Example 1.4. The degree n covering $\pi_n : \mathbb{P}^1 \to \mathbb{P}^1$ is defined by sending x to x^n. π_n is ramified at 0 and ∞; see Corollary 1.13:

Algebraically, if we localize at 0 or ∞ and reduce modulo maximal ideals, the field extension $k \to k[x]/x^n$ is ramified.

However, if we invert the uniformizers at 0 and ∞, we obtain the multiplicative group $\mathbb{G}_m$ and an étale map of degree n: $\mathbb{G}_m \to \mathbb{G}_m$.

Example 1.5. Open immersions are étale morphisms, meaning that a Zariski covering is an étale covering.

Example 1.6. By Lemma 1.11, étale morphisms are flat morphisms of finite presentation. In other words, an étale covering is an fppf covering.

This definition should convince readers that étale morphisms are essentially local isomorphisms. However, it is somewhat indirect, so we introduce a more straightforward interpretation involving Kähler differentials. First, we recall some basic facts about Kähler differentials:

Lemma 1.7. *Let $f : X \to Y$ be a morphism of S-schemes. The following two sequences are exact:*

$$\mathcal{C}_f \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to 0$$

$$f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

where f is an immersion in the first sequence and $\mathcal{C}_f$ is the conormal sheaf of f.

Proof. The proof requires two equivalent definitions of Kähler differentials. First, utilizing the concrete construction of differentials, we verify the sequences are exact in the affine case. Then, applying universal properties, we check exactness after applying $Hom_{\mathcal{O}_x}(-, M)$ for any $\mathcal{O}_X$-module M. The remaining details follow by unpacking definitions:

(To be continued) $\square$

The following proposition formalizes the local-isormorphism interpretation involving Kähler differentials within the affine setting of this subsection, where $S = \operatorname{Spec} R$, i.e., $f : X \to S$ in Sch_k factors as:

$X \xrightarrow{\tau} \mathbb{A}^n_S \xrightarrow{\pi} S$ and $\mathcal{C}_\tau \cong I/I^2$:

Proposition 1.8. *The following are equivalent:*

(1) f is smooth of relative dimension d at x, i.e., the Jacobian $J : (I/I^2)_x \otimes_{\mathcal{O}_{X,x}} \kappa(x) \to \Omega_{\mathbb{A}^n_S/S} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ is injective.

(2) In a neighborhood of x, the following sequence is split-exact:

$$0 \to \mathcal{C}_\tau \to \tau^*\Omega_{\mathbb{A}^n_S/S} \to \Omega_{X/S} \to 0$$

The above two implies $\Omega_{X/S}$ is locally free of rank d at x and the following:

(3) In a neighborhood of x, the following sequence is also split-exact:

$$0 \to f^*\Omega_{S/k} \to \Omega_{X/k} \to \Omega_{X/S} \to 0$$

Furthermore, if X is smooth over k at the point x, the above three conditions are equivalent.

Proof. The implication (1) $\Rightarrow$ (2) is trivial. For (2) $\Rightarrow$ (1), note that $\Omega_{\mathbb{A}^n_S/S}$ is a free $\mathcal{O}_S$-module of rank n.

Thus, we can fix a basis $s_1, \cdots, s_n$ such that $s_1, \cdots, s_{m\leqslant n}$ generates $I/I^2 \otimes_R \kappa(x)$. By Nakayama's Lemma, this implies that $s_1, \cdots, s_{m\leqslant n}$ generates $(I/I^2)_x$. Consequently, we can define a retraction $r : M_x \to (I/I^2)_x$ of $j : (I/I^2)_x \to M_x$, mapping $s_{i\leqslant m}$ to s_i and $s_{i>m}$ to 0 so that $jr = \mathrm{id}$.

Thus, by Lemma 1.7, M_x splits as a direct sum of two free modules, $(I/I^2)_x$ and $(\Omega_{X/S})_x$. Again by Nakayama's Lemma, in the finitely generated setting, splitting at the stalks extends to a neighborhood of x. This completes the proof of (2) $\Rightarrow$ (1).

Now, to prove (3): we use the factorization $f = \pi\tau$ and the fact that τ^* preserves split-exact sequences. Thus, we reduce to the case where $\tau = \mathrm{id}$ and $f = \pi$, which can be verified by following the proof of Lemma 1.7 (2).

For $(3) \Rightarrow (1)+(2)$, we consider the following diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{C}_\tau \arrow[r, equal] \arrow[d] & \mathcal{C}_\tau \arrow[d] & \\
0 \arrow[r] & \tau^*\pi^*\Omega_{S/k} \arrow[r] \arrow[d, equal] & \tau^*\Omega_{\mathbb{A}^n_S/k} \arrow[r] \arrow[d] & \tau^*\Omega_{\mathbb{A}^n_S/S} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f^*\Omega_{S/k} \arrow[r] & \Omega_{X/k} \arrow[r] \arrow[d] & \Omega_{X/S} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since X is smooth over k, the bottom row and left column are split-exact at the point x.

Since $\Omega_{\mathbb{A}^n_S/k}$ is locally free at the point x and, by $(1) \Rightarrow (2)$, the top row is exact at x when $\tau = \mathrm{id}$, we conclude that the top sequence is also split-exact at x.

Therefore, it is straightforward to show that the remaining right column is split-exact. □

Remark 1.9. Note that this proposition still holds when S is not affine. The assumption of S being affine is for simplicity of notation.

We immediately deduce the case of étale morphisms:

Corollary 1.10. *Assume that X is a smooth k-scheme, then $f : X \to S$ is étale if and only if $f^*\Omega_{S/k} \cong \Omega_{X/k}$. In particular, we have $\Omega_{X/S} = 0$.*

One might feel that the above definition suffices for characterizing étale morphisms. However, we have a simplified criterion involving flatness: when f is unramified, f is smooth if and only if it is flat. To show that this is indeed a "simpler" or "weaker" condition, we provide an interesting proof directly deducing flatness from formal smoothness ([CL16]):

Lemma 1.11. *If $f : X \to S$ is smooth, then f is flat.*

Proof. ([CL16]) Since f is locally of finite presentation and flatness is a local property, we may assume $f : X \to S$ is an affine scheme map $\varphi : A \to B$. Let R be a localization of a polynomial A-algebra such that there is a surjective ring map $\psi : R \twoheadrightarrow B$ with a finitely generated kernel I.

Then $\hat{R}/\hat{I} = R/I = B$, so $I = \ker(\hat{\psi} : \hat{R} \to B)$. We aim to show there exists a section $B \to \hat{R}$. By induction on $i \geqslant 2$, we apply the formal smoothness of f (Proposition 1.19) to construct A-algebra homomorphisms $\psi_i : B \to \hat{R}/\hat{I}^i$:

$$\begin{tikzcd}
 & & & B \arrow[dl, dashed, "\psi_{i-1}"'] \arrow[d, "\psi_i"] & \\
0 \arrow[r] & \hat{I}^{i-1}/\hat{I}^i \arrow[r] & \hat{R}/\hat{I}^i \arrow[r] & \hat{R}/\hat{I}^{i-1} \arrow[r] & 0
\end{tikzcd}$$

making the triangles commute. Since $\hat{R}$ is complete in the $\hat{I}$-adic topology, we have a limit homomorphism $f = \lim_i \psi_i : B \to \lim_i \hat{R}/\hat{I}^i = \hat{R}$ such that $\phi\psi = \mathrm{id}_B$. Thus, B is a direct summand of the flat A-module $\hat{R}$ and hence also flat.

□

Definition-Proposition 1.12. Let $f : X \to S$ be a locally of finite type morphism, mapping x to $s = f(x)$. Then f is called unramified at x, if it satisfies the following equivalent assertions:

(1) $(\Omega_{X/S})_x = 0$.
(2) $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ is a finite separable extension of $\kappa(s)$.

(2+) There is a K-algebra isomorphism $(\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}) \bigotimes_{\kappa(s)} K \cong K^n$ for some algebraically closed extension K of $\kappa(s)$ and some positive integer n.

(3) The diagonal $\Delta_{X/S} : X \to X \times_S X$ is an open immersion in a neighborhood of x.

We say that f is unramified if it is unramified at every point. In particular, $X_{\kappa(s)} := X \times_S \operatorname{Spec} \kappa(s)$ is called unramified over $\kappa(s)$, if $X_{\kappa(s)}$ is geometrically reduced $\kappa(s)$-scheme and $\dim X_s = 0$.

Proof. The proof for the equivalence of these conditions proceeds as follows:

(2) $\Leftrightarrow$ (2+): This follows from the primitive element theorem and the Chinese Remainder Theorem (details omitted).

(2) $\Rightarrow$ (1): By Nakayama's Lemma, it suffices to show that $\Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x) \cong \Omega_{\kappa(x)/\kappa(s)} = 0$. Thus, it remains to show that $\kappa(x) \otimes_{\kappa(s)} \kappa(x) \to \kappa(x)$ is injective, where the kernel of this map is generated by $dx := 1 \otimes x - x \otimes 1$, with $dx = \frac{d(f(x))}{f'(x)} = 0$.

(1) $\Rightarrow$ (2+): We may assume $\mathcal{O}_{S,s} = \kappa(s) = K$ is algebraically closed. Then the differential of the K-local algebra $\mathcal{O}_{X,x}$ vanishes if and only if $\mathcal{O}_{X,x} \cong K^n$ for some n.

(1) $\Rightarrow$ (3): By Nakayama's Lemma, the sheaf $\mathcal{I}$ associated with $\Delta_{X/S}$ vanishes in a neighborhood of x, so $\Delta_{X/S}$ is an open immersion around x.

For (3) $\Rightarrow$ (1), we show this in the next section. $\square$

We immediately obtain the following important corollary:

Corollary 1.13. *$f : X \to S$ is etale if and only if f is smooth and unramified*

Proof. Apply Proposition 1.8 and 1.12. $\square$

1.2. **Functorial Approach.** This approach is sometimes called the Nilpotent Thickening or Infinitesimal Lifting Approach.

Definition 1.14. A closed immersion $i : T \to T_0$ of S-**affine** schemes is called a *thickening of $X \to S$ of order at most* n (also referred to as an infinitesimal extension), if the associated quasi-coherent sheaf $\mathcal{N}$ satisfies $\mathcal{N}^{n+1} = 0$. If the thickening is of order at most 1, we call $i : T \to T_0$ a square-zero thickening.

We refer to the following diagram 1.2 as the thickening square of order at most n, if it is a commutative diagram with i a thickening of order at most n:

$$\begin{tikzcd} T_0 \arrow[r, "\eta_0"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r] & S \end{tikzcd}$$

Definition 1.15. A morphism $f : X \to S$ of schemes is called*formally etale*(respectively, *formally smooth*, *formally unramified*), if, for every square-zero thickening square, the lifting problem admits exactly one (respectively,at least one, at most one) solution. That is, there exists unique (respectively,at least one, at most one) morphism $\eta : T \to X$ that factors through the thickening square:

$$\begin{tikzcd} T_0 \arrow[r, "\eta_0"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed, "\eta"] & S \end{tikzcd}$$

These definitions are local in essence as the following:

Proposition 1.16. *$f : X \to S$ is formally unramified if and only if $\Omega_{X/S} = 0$. Particularly, "formally unramified" is a local property.*

Proof. We first define an inclusion of immersion of $X \times_S X$ as $X \hookrightarrow X^{(1)} \hookrightarrow X \times_S X$. Here $X \hookrightarrow X \times_S X$ is defined by $\mathcal{I}$ and $X^{(1)} \hookrightarrow X \times_S X$ is defined by $\mathcal{I}^2$. Then the key is to utilize the following lifting problem:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i"'] & X \times_S X \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] \\ X^{(1)} \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

To be continued. □

Proposition 1.17. *The properties "formally smooth", "formally unramified", and "formally etale" are local on the source and the target. Specifically, suppose we have a morphism of schemes $f : X \to S$, then the following assertions are equivalent:*

(1) $f : X \to S$ is formally smooth (respectively, unramified, etale);
(2) for every point $x \in X$, there exist affine opens $x \in U \subset X$ and $f(x) \in V \subset Y$ with $f(U) \subset V$ such that $f|_U : U \to V$ is formally smooth (respectively, unramified, etale);

Particularly, in this proposition, we will assume that f is locally of finte presentation. For a general proof, it is necessary to show that the projectivity of $\Omega^1_{X/S}$ is a local property. This involves the highly nontrivial fact that projective modules admit faithfully flat descent; see [sta24, 05A9].

Proof. The "formally unramfied" case follows directly by unpacking the definition, while the formally étale case follows from the unramfied case and smooth case. Thus, we only need to prove the formally smooth case. For formally smooth case, (1) ⇒ (2) is also literally by defnition. Thus we only need to show (2) ⇒ (1) for formally smooth case [sta24, $0D0F$]:

We first reduce to the case $S \cong \operatorname{Spec} R$ by examining the gluing conditions. We also wish to show that this reduction holds true for the source X.

Define $\mathcal{L}_{\eta_0}$ (abbreviated as $\mathcal{L}_\eta$ when unambiguous) as the sheaf of lifts of η_0 on T, which is also defined on T_0 since T_0 and T share the same topological space. For each $U \subset T$, $\mathcal{L}_\eta(U)$ is the set of morphisms $\eta|_U$ that solve the lifting problem in the thickening square 1.2, restricted to U:

$$\begin{tikzcd} T_0|_U \arrow[r, "\eta_0|_U"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T|_U \arrow[r] \arrow[ur, dashed, "\eta|_U"] & S \end{tikzcd}$$

Next, define a Hom-sheaf of groups on X: $\mathcal{G} := \mathcal{D}er_S(X, \eta_{0,*}i^*\mathcal{N}) \cong \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/S}, \eta_{0,*}i^*\mathcal{N})$. We prove that $\mathcal{G}$ acts simply and transitively on $\mathcal{L}_\eta$ by reducing to the affine cases. On some affine opens $\operatorname{Spec} C/N \cong U \subset T_0$ and $\operatorname{Spec} A \cong V \subset X$ as follows:

$$\begin{tikzcd} C/N & A \arrow[l, "\varphi_0"'] \arrow[dl, dashed, "\varphi"] \\ C \arrow[u] & R \arrow[l] \arrow[u] \end{tikzcd}$$

we have $\mathcal{G}|_V \cong Der_R(A, (N \bigotimes_C C/N)_A) = Der_R(A, N_A)$. Thus, fixing an R-algebra morphism $\psi \in \mathcal{L}_{\eta_0}$, we can show that the map $(\cdot - \psi) : \mathcal{L}_{\eta|_U} \rightleftarrows \mathcal{G}|_U : (\cdot + \psi)$ is a well-defined bijection:

$$(\varphi - \psi)(ab) = (\varphi - \psi)(a)\varphi(b) + \psi(a)(\varphi - \psi)(b) = (\varphi - \psi)(a)\varphi_0(b) + \varphi_0(a)(\varphi - \psi)(b).$$
$$(D + \psi)(ab) = \varphi_0(a)D(b) + \varphi_0(b)D(a) + \psi(a)\psi(b) = (D + \psi)(a)(D + \psi)(b).$$

From now on, we assume f is locally of finite presentation to avoid the discussion involving local projectivity. For any A-module M, we use $\widetilde{M}$ to explicitly denote the quasi-coherent sheaf associated with it. For any $a \in A$, we denote by M_a the localized A_a-module $M \otimes_A A_a$, where $A_a = A[T]/(Ta - 1)$. As topological spaces, we have $\operatorname{Spec} A_a \cong D(a) = \operatorname{Spec} A \setminus V((a))$.

We now show that $\mathcal{G}$ is quasi-coherent. Specifically, for some affine open $\operatorname{Spec} A \cong V \subset X$ and any $a \in A$, we aim to prove that $\mathcal{G}(V)_a \cong \mathcal{G}(D(a))$. Following the notation above, for certain A-modules M and M', we have $\Omega^1_{X/S}|_V \cong \widetilde{M}$ and $\eta_{0,*}i^*\mathcal{N}|_V \cong \widetilde{M'}$, noting that $\eta_{0,*}i^*\mathcal{N}$ is quasi-coherent.

Consider a natural map $\psi : \mathcal{G}(V)_a \cong \operatorname{Hom}_A(M, M')_a \to \operatorname{Hom}_A(M, M'_a) \cong \operatorname{Hom}_{A_a}(M_a, M'_a) \cong \mathcal{G}(D(a))$. Since f being locally of finite presentation implies that $\Omega^1_{X/S}$ is locally of finite presentation, we may assume that there exists an exact sequence $A^m \to A^n \to M \to 0$ for some $m, n \in \mathbb{N}$. Using this presentation, we can show that the map ψ is an isomorphism, leveraging the fact that the Hom functor preserves kernels. Consequently, $\mathcal{G}$ is quasi-coherent.

It follows that $H^1(T_0, \mathcal{G}) = 0$ because the cohomology of a quasi-coherent sheaf on an affine scheme vanishes. Now, assuming (2) (i.e., $\mathcal{L}_\eta$ is not a zero sheaf), we conclude that $\mathcal{L}_\eta$ is a trivial $\mathcal{G}$-torsor since $H^1(T_0, \mathcal{G}) = 0$. In other words, local solutions to the lifting problem on affine opens can be glued up to form a global solution.

Thus, we reduce the proposition to the case where X is also affine, essentially completing the proof of (1) $\Leftrightarrow$ (2). □

Remark 1.18. Formally smooth share certain results with Proposition 1.8; see [sta24, 06B7] and [sta24, 06B6].

Proposition 1.19. *Assume that $f : X \to S$ is locally of finite presentation. Then f is formally smooth if and only if f is smooth of some relative dimension d.*

Proof. To be continued; Utilizing the proposition 1.8. □

Corollary 1.20. *$f : X \to S$ is étale if and only if f is formally étale and locally of finite presentation.*

1.3. More on Smooth morphisms.

Slogan: smooth = flat + geometrically regular +locally of finite presentation.

Theorem 1.21. *$X \to S$ is smooth at x if and only if f is flat and locally of finite presentation at x, and the $\kappa(f(x))$-scheme $f^{-1}(f(x))$ is geometrically regular.*

Proof. To be continued. □

Theorem 1.22. *Criterion for Noetherian schemes: Completions...*

2. Étale Fundamental Group

In this section, we study the étale fundamental group.

Let $\text{FÉT}_{/X}$ be the category of finite étale morphisms $Y \to X$. For simplicity, we assume X is connected throughout and may fix a geometric point $\bar{x} \in X$. As discussed in Section 1, a finite étale morphism $Y \to X$ serves as the algebraic analogue of a covering space and the geometric generalization of a finite separable field extension. Such morphisms are often called finite étale covers of X.

A natural question arises following this analogue:

Question 2.1. can we define the fundamental group $\pi_1(X, \bar{x})$ and establish a correspondence between finite étale covers of X and $\pi_1(X, x)$-sets, similar to constructions in algebraic topology and Galois theory?

The answer is yes, and we will first provide an informal explanation. A rigorous treatment will follow later, after introducing Theorem 2.4 and its basic applications.

Let $\mathbb{F}_{\pi_1(X,\bar{x})}$ denote the category of finite $\pi_1(X, \bar{x})$-sets. It has a subcategory $\text{Orb}_{\pi_1(X,\bar{x})}$, consisting of finite transitive $\pi_1(X, \bar{x})$-sets (i.e., $\pi_1(X, \bar{x})$-orbits).

First, we assert the existence of a universal covering space $\widetilde{X}$ of X (in a formal sense). Intuitively, $\widetilde{X}$ is the "biggest" finite étale cover of X, or equivalently, the initial object in $\text{FÉT}_{/X}$.

This implies that, formally, $\widetilde{X}$ should be viewed as the projective limit of all finite étale covers $\{X_i\}$, expressed as $\widetilde{X} =$" $\varprojlim_i$" X_i. The étale fundamental group of $(X, \bar{x})$ is then defined as the (opposite) automorphism groups of the universal covering $\pi_1(X, \bar{x}) := \mathrm{Aut}_X(\widetilde{X})^{\mathrm{opp}}$, where $(\cdot)^{\mathrm{opp}}$ denotes the opposite group. This def

Next, we claim that the existence of the universal covering gives rise to an equivalence of categories, realized through the so-called fibre functor: $F_{\bar{x}} : \mathrm{F\acute{E}T}_{/X} \to \mathbb{F}_{\pi_1(X,\bar{x})}$, defined by $F_{\bar{x}} = \mathrm{Hom}(\widetilde{X}, \cdot) := \varinjlim_i \mathrm{Hom}_{\mathrm{F\acute{E}T}_{/X}}(X_i, \cdot)$. Here, $\pi_1(X, \bar{x}) = \mathrm{Aut}_X(\widetilde{X})^{\mathrm{opp}} \cong \mathrm{Aut}(F_{\bar{x}})$ acts naturally on $F_{\bar{x}}(Y)$, justifying the use of the opposite group $(\cdot)^{\mathrm{opp}}$ in the definition of $\pi_1(X, \bar{x})$.

Remark 2.2. Readers should note that although the fibre functor is always constructible, while the universal covering may not exist, even when considering the limit " $\varprojlim_i$" X_i in $\mathrm{Sch}_{/X}$. However, this poses no real obstacle: in practice, computations rely entirely on the fibre functor. Furthermore, if you are those who are devoted to the cult of the Yoneda lemma, the fibre functor and the universal covering can even be regarded as one and the same. We will revisit this point later.

Now, a finite etale cover $[\pi : Y \to X]$ is called Galois if $\deg \pi = \# \mathrm{Aut}_X(Y)$. Assuming the existence of Galois closures, the projective system $\{X_i\}$ of the universal covering can be replaced by its subsystem consisting of Galois covers. This new system remains projective, leading the relation:

$$\pi_1(X, \bar{x}) = \mathrm{Aut}_X(\widetilde{X})^{\mathrm{opp}} = \varprojlim_i \mathrm{Aut}_X(X_i)^{\mathrm{opp}}.$$

In particular, this relation demonstrates that $\pi_1(X, \bar{x})$ is a profinite group.

Remark 2.3. From this perspective, one can define $\Pi(X)$, the étale fundamental groupoid of X, as the full subcategory of $\mathrm{Fun}(\mathrm{F\acute{E}T}_{/X}, \mathrm{Sets})$ generated by fibre functors $F_{\bar{x}}$ up to isomorphisms. In particular, the étale fundamental group has the relation $\pi(X, \bar{x}) = \mathrm{Aut}(F_{\bar{x}})$.

Eventually, we arrive at the Fundamental Theorem of Galois Theory, which synthesizes these definitions and concepts:

Theorem 2.4. $F_{\bar{x}} : \mathrm{F\acute{E}T}_{/(X,\bar{x})} \to \mathbb{F}_{\Pi(X,\bar{x})}$ *is an isomorphism.*

In particular, a connected étale cover $Y \in \mathrm{F\acute{E}T}_{/(X,\bar{x})}$ *is Galois if and only if the corresponding stabilizer of the* $\pi(X, \bar{x})$*-orbit* $F_{\bar{x}}(Y)$ *is a normal subgroup of* $\mathrm{Orb}_{\pi(X,\bar{x})}$.

The proof involves technical lemmas that extend Galois theory to the setting of schemes. While the details are omitted here, this theorem provides a powerful correspondence, enabling us to study $\pi_1(X, \bar{x})$ through its action on finite étale covers. Let us explore some illustrative examples.

Example 2.5. Let X be $\mathrm{Spec}\, k$, where K is a field. Then $\mathrm{F\acute{E}T}_k$ consists of finite étale k-algebra, which are,by Proposition 1.12, finite products of finite separable extensions of k. The universal covering is the separable algebraic closure k^{sep}, and $\pi_1(\mathrm{Spec}\, k) = \mathrm{Aut}_{\mathrm{Spec}\, k}(\mathrm{Spec}\, k^{\mathrm{sep}})^{\mathrm{opp}} = \mathrm{Gal}(k^{\mathrm{sep}}/k)$. This recovers the usual Galois Theory.

Remark 2.6. You might note that this example also gives us a hint of the this "opposite" construction for π_1, since affine schemes have the opposite morphisms of commutative ring; See Example 2.5. However, people (including this notes) feel free to discard $(\cdot)^{\mathrm{opp}}$ whenever they want, since we have natural isomorphisms $G \cong G^{\mathrm{opp}}$ for all groups.

Example 2.7. Let X be a normal complex projective variety. Then by the Riemann Existence Theorem, $\pi_1(X) = \varprojlim_i \mathrm{Aut}_X(X_i) = \varprojlim_i \mathrm{Aut}_{X(\mathbb{C})}(X_i(\mathbb{C})) = \pi_1^{\mathrm{top}}(X(\mathbb{C}))^{\wedge}$, the pro-completion of $\pi_1^{\mathrm{top}}(X)$.

Example 2.8. Let $X = \mathbb{P}^1$, the complex projective line. For any cover $\pi : Y \to \mathbb{P}^1$, the Riemann-Hurwitz formula $2g(Y) - 2 = -2\deg(\pi)$ implies that $\deg(\pi) = g(Y) = 1$ and hence π is an isomorphism. This shows that $\pi_1(\mathbb{P}^1) = 0 = \pi_1^{\mathrm{top}}(\mathbb{P}^1)$.

We can now address the unresolved issues we didn't deal with earlier. Readers might note that we left three main questions that required further elaboration:

(1) Whether a "good enough" construction of the universal covering and the fibre functor exists.
(2) How the terminology of Galois covers provides a computable formula for the fundamental group.
(3) A naturaly way to understand the following observation: in order to compute the étale fundamental group, the term projective limit appears several times. Thus, it seems that everything we meet is somehow "projective".

For the first question, the Yoneda lemma, ensures that constructing the fibre functor suffices to address this question. Upon closer examination, the answer actually already lies in the name "fibre functor". Specifically, we define the fibre functor $F_{\bar{x}} : \mathrm{F\acute{E}T}_{/X} \to \mathrm{Sets}$ by sending $[Y \to X]$ to $\mathrm{Hom}_X(\bar{x}, Y)$, where $\bar{x} \in X$ is a chosen geometric point. Here, a geometric point $\bar{x}$ of the point $\mathrm{Spec}\,\kappa(x) \to X$ is a lifting $\mathrm{Spec}\,\kappa(x) \to \mathrm{Spec}\,\Omega \to X$, where Ω is a separablly closure of $\kappa(x)$. This construction mirrors the approach used for defining the topological fundamental group, providing a satisfying answer.

Furthermore, using this new construction of $F_{\bar{x}}$, we can also provide a more illustrative definition of Galois covers. A Galois cover $Y \to X$ is a finite étale cover such that $\mathrm{Aut}_X(Y)$ acts transitively on $F_{\bar{x}}(Y)$. In fact, $F_{\bar{x}}(Y) = \mathrm{Aut}_X(Y)$, and $F_{\bar{x}}(Y)$ is an $\mathrm{Aut}_X(Y)$-torsor. Denoting the category of pointed (connected) galois cover over (X, x) by $\mathcal{G}_{\bar{x}}$, and the category of pointed, connected étale covers $(Y, \bar{y}) \to (X, \bar{x})$ by $I_{\bar{x}}$. Unsurprisingly, $\mathcal{G}_{\bar{x}}$ is a cofinal subcategory of $I_{\bar{x}}$, indicating the existence of the Galois closure. For a rigorous proof of all the claims in this paragraph, we refer to [GW23, §20.16] or [Fu15, Proposition 3.2.10].

Turning to the third issue, the observation is correct and suggests an abstract-nonsense slogan: when studying étale covers (and by extension, étale sites ane étale cohomology), it is more natarual to consider not $\mathrm{F\acute{E}T}_{/X}$ itself, but rather its procompletion $\mathrm{Pro}\,\mathrm{F\acute{E}T}_{/X}$. Following this principle, some people have extended étale cohomology to pro-étale cohomology, which is a more natural framework.

Formally, the Pro-completion is described as follows. For any category C, we have the Yoneda embedding $y : C^{\mathrm{op}} \hookrightarrow C^\vee = \mathrm{Fun}(C, Set)$, satisfying $y(x) := \mathrm{Hom}_C(x, -)$. We define $(\mathrm{Pro}\,C)^{\mathrm{op}}$ as a full subcategory of $C^\vee$ generated by all objects $\lim_{i\in I} x_i$, where I is cofiltered. In fact, $C^\vee$ can be regarded as the free cocompletion of C^{op}, and $\mathrm{Pro}\,C$ should be thought of as "the projective limit part" of C.

Finally, we state a lemma that ties these observations together, demonstrating that the two constructions of the étale fundamental groupoid are equivalent and that this groupoid is indeed "Pro-Finite":

Lemma 2.1. [GW23, Theorem 20.86, Proposition 20.93, Remark 20.98]

Let $\bar{x}$ be a geometric point of a connected scheme X. Then, $\Pi(X)$ is a groupoid and also a full subcategroy of $(\mathrm{Pro}\,\mathrm{F\acute{E}T}_{/X})^{\mathrm{op}}$:

$$\Pi(X) \subset (\mathrm{Pro}\,\mathrm{F\acute{E}T}_{/X})^{\mathrm{op}} \subset (\mathrm{F\acute{E}T}_{/X})^\vee$$

Concretely:

(1) The fibre functor $F_{\bar{x}}$ of $\bar{x}$ is pro-representable by finite étale cover, i.e., $F_{\bar{x}} \in \mathrm{Pro}\,\mathrm{F\acute{E}T}_{/X}$ is a profinite étale cover. Specifically, $\mathrm{colim}_{i\in I_{\bar{x}}} \mathrm{Hom}(Y_i, Y) \cong F_{\bar{x}}(Y)$ with $I_{\bar{x}}$ cofilterd.
(2) $\Pi(X)$ is indeed a groupoid, i.e., any morphism between fibre functors in $(\mathrm{F\acute{E}T}_{/X})^\vee$ is an isomorphism. Furthermore, $\pi_0(\Pi(X)) = \pi_0(X) = 1$.
(3) The fundamental group $\pi(X, \bar{x})$ of $\bar{x}$ is profinite. Particularly, $\pi(X, \bar{x}) = \varprojlim_{X_i \in \mathcal{G}_{\bar{x}}} \mathrm{Aut}_X(X_i)$.

Proof. For (1), fix $Y \in \text{FÉT}_{/X}$. For any $(Z, \bar{z}) \in \text{FÉT}_{/(X,\bar{x})}$, there is a natural map $\text{Hom}(Z, Y) \to F_{\bar{x}}(Y)$ sending f to $f(\bar{z})$. This is injective, according to [Mil13, Corrollary 2.16] . This also implies that there is at most one map between any $(Y_1, \bar{y_1}), (Y_2, \bar{y_2}) \in \text{FÉT}_{/(X,\bar{x})}$. Notice that the connected component of $Y_1 \times_X Y_2$ containg $(\bar{y_1}, \bar{y_2})$ is the cofiltered object of $(Y_1, \bar{y_1})$ and $(Y_2, \bar{y_2})$, showing that $I_{\bar{x}}$ is cofiltered.

The natural map $\text{colim}_{i \in I_{\bar{x}}} \text{Hom}(Y_i, Y) \cong F_{\bar{x}}(Y)$ is a filtered colimit of injective map, hence still injective. Besides, this natural map is also surjective: for any $\bar{y} \in F_{\bar{x}}(Y)$, the connected component Y_0 of Y containg $\bar{y}$ essentially lying in $\text{Hom}(Y_0, Y)$.

For (2), we use the confinality of $\mathcal{G}_{\bar{x}} \hookleftarrow I_{\bar{x}}$. To be continued.

For(3), we have $\pi(X, \bar{x}) =$ [1] $\text{Hom}(F_{\bar{x}}, F_{\bar{x}}) = \varprojlim_{X_i \in \mathcal{I}_{\bar{x}}} F_{\bar{x}}(X_i) = \varprojlim_{X_i \in \mathcal{G}_{\bar{x}}} F_{\bar{x}}(X_i) = \varprojlim_{X_i \in \mathcal{G}_{\bar{x}}} \text{Aut}_X(X_i)$, here $\pi(X, \bar{x})$ is profinite sin $\# \text{Aut}_X(X_i)$ is finite. $\square$

3. Faithfully Flat Descent

3.1. fpqc Morphisms are Effective Epimorphisms.

Definition 3.1. [sta24, 00WP] Let $\mathcal{C}$ be a category. A morphism $f : U \to X$ is called an effective epimorphism if the fiber product $V \times_U V$ exists and the diagram $U \times_X U \underset{\text{pr}_2}{\overset{\text{pr}_1}{\rightrightarrows}} U \xrightarrow{f} X$ is a coequalizer.

We say that $U \to X$ is an universal effective epimorphism if for any morphism $U \to X$ the base change $V \times_X U \to V$ is an effective epimorphism.

Remark 3.2. For simplicity, we use a slightly modified version of the terminology found in [sta24, 00WP]. However, these two terminologies are largely equivalent. Assuming that arbitrary coproducts exist in $\mathcal{C}$ and letting $U = \bigsqcup_i U_i$ (and $X = U$), we can use the fact that the Hom functor preserves limits to show that these definitions are essentially equivalent.

This definition might initially seem a little weird at first. However, it becomes quite intuitive in the context of homotopy colimits. Specifically, U can be interpreted as the geometric realization of the Čech nerve of the morphism $U \to X$; see [Lur24, Corollary 04WP].

Furthermore, this perspective will become even more natural if you read [Lur24, Subsection 04WG] more. In this work, Lurie shows that the colimit of the projection of the sieve generated by f onto $\mathcal{C}$ is preciesely U itself! In fact, Lurie refers to effective epimorphisms as quotient morphisms, which is a more intuitive term. The change in terminology likely arises from the fact that quotient morphisms may no longer be epimorphisms in the context of $\infty-$categories.

Example 3.3. Quotient morphisms are surjective maps in Sets.

This example shows that if we furhter more truncating 1-categories to 0-categories, i.e. Sets, then effective epimorphism are just epimorphisms.

From now on we let the $(1-)$ category $\mathcal{C}$ be Sch_S, and we aim to charaterize quotient morphisms in Sch_S. The answer is given by *fpqc* morphisms, which are faithfully flat and quasi-compact morphisms. Specifically, faithfully flat ring maps in Ring are effective epimorphisms.

To prove this claim, we first review an interesting property of faithfully flat morphisms. Recall that a morphism $f : X \to Y$ in schemes is flat if $f^* : \text{Qcoh}_Y \to \text{Qcoh}_X$ is an exact functor:

Definition-Proposition 3.4. [Fu15, Proposition 1.2.1.] For any flat morphism $f : X \to Y$ in schemes, f is called faithfully flat, if the following equivalent conditions hold:

(1) $f^* : \text{Qcoh}_Y \to \text{Qcoh}_X$ is faithful;
(2) $f^*\mathcal{F} = 0$ if and only if $\mathcal{F} = 0$;
(2+) $f : X \to Y$ is surjective;
(3) $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence if and only if it remains exact along pullback f^*.

[1] This is only a inclusion if we do not assume (2)

In particular, f^* is conservative if it satisfies all the above conditions.

Proof. Conservativity and (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) is a general result that holds for any exact functor between Abelian categories, omit;

For (2) $\Leftrightarrow$ (2+), this can be reduced to affine case, where we may assume $f^* = B\otimes_A : \mathrm{Mod}_A \to \mathrm{Mod}_B$.

Then, $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is surjective if and only if for any non-zero prime $A/p \neq 0$, it follows that $B \otimes_A A/p \neq 0$. This is equivalent to saying that if $B \otimes_A M = 0$, then $M = 0$. $\square$

Remark 3.5. Condition (2+) suggests that faithfully flat morphisms are indeed a generalization of surjective maps, as are effective epimorphisms.

From Proposition 3.4, we immediately obtain the following important fact:

Theorem 3.6. *Faithfully flat ring maps in* Ring *are effective epimorphisms.*

Proof. We want to show: $A \longrightarrow B \underset{p_2}{\overset{p_1}{\rightrightarrows}} B \otimes_A B$ is an equalizer diagram, where $p_1 : b \mapsto b \otimes 1$ and $p_2 : b \mapsto 1 \otimes b$.

Note that $A \to B$ is injective from Proposition 3.4 (2) and $A \to B$ factor through $A \to \ker(p_1 - p_2) \to B$. Thus, we need to show $A \to \ker(p_1 - p_2)$ is surjective. By Proposition 3.4 (3), it is equivalently to show that $A \to \ker(p_1 - p_2)$ remains surjective after tensoring with $B\otimes_A$:

$$B \xrightarrow{s} B \otimes_A \ker(p_1 - p_2) \longrightarrow B \otimes_A B \rightrightarrows B \otimes_A B \otimes_A B$$

This is straightforward: for any $x \in B$ and $y \in \ker(p_1 - p_2) \subset B$, we have $x \otimes y = xy \otimes 1 + x \otimes 1 \cdot (1 \otimes y - y \otimes 1) = xy \otimes 1 \in \operatorname{im}(s)$, indicating that $\operatorname{im}(s) = B \otimes_A \ker(p_1 - p_2)$, which shows that s is surjective. $\square$

We can generalize this theorem to the global case Sch_S, if we utilize the following criterion:

Proposition 3.7. *[Mil13, Proposition 6.6][Mil80, II 1.5] Let $\mathcal{F}$ be a presheaf for the flat (resp. etale) site on a scheme X. Then $\mathcal{F}$ is a sheaf if and only if $\mathcal{F}$ satisfies the sheaf condition for Zariski open coverings and for flat(resp. etale) coverings $U \to X$ with U and X both affine.*

The proof of this proposition involves some diagram chasing, which we omit here; see the reference for details. Now, combining all of the above facts, we can deduce the following main theorem:

Theorem 3.8. *fpqc morphisms in* Sch_S *are effective epimorphisms. Equivalently,* $\mathrm{Hom}_S(-, Y)$ *is an fpqc sheaf for any* $Y \in \mathrm{Sch}_S$

Proof. The case $S = \operatorname{Spec}\mathbb{Z}$ is straightforward since $\mathrm{Hom}_{\operatorname{Spec}\mathbb{Z}}(-, Y)$ preserving limit implies Proposition 3.7 (1) and Theorem 3.6 implies Proposition 3.7 (2).

For $S \neq \operatorname{Spec}\mathbb{Z}$, we only need to show that a coequalizer diagram $U \times_X U \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} U \xrightarrow{f} X$ in schemes is also a coequalizer diagram in $S-$schemes. Using the fact that $U \to X$ is an epimorphism, the rest of the proof involves only easy diagram chasing, $\square$

Beyond the fact that $fpqc$ morphisms are effective epimorphisms, we can further show that $fpqc$ morphisms are effective for descending objects, morphisms, and many properties; see [sta24, Section 08WE] for a comprehensive treatment. In particular, an non-trivial fact is that $fpqc$ morphisms are effective descent morphisms for modules(for quasi-coherent sheaves)([sta24, Definition 08WY]), which is our main goal in the following subsection.

3.2. fpqc Morphisms are Effective Descent Morphisms.

Slogan: Effective descent for modules along faithfully flat ring maps.[sta24, 023N]

—Bhargav Bhatt

Definition 3.9. [sta24, Definition 023A] For any morphism $f : U \to X$ of schemes, we can define its asscoiated (2-truncated) Čech nerve, denoted as $\check{C}(U/X)$, as follows:

$$\begin{tikzcd} U \times_X U \times_X U \arrow[r, shift left=2, "\mathrm{pr}_{01}"] \arrow[r, "\mathrm{pr}_{02}" description] \arrow[r, shift right=2, "\mathrm{pr}_{12}"'] & U \times_X U \arrow[r, shift left=2, "\mathrm{pr}_0"] \arrow[r, shift right=2, "\mathrm{pr}_1"'] & U \arrow[l, "\Delta" description] \arrow[r, "f"] & X \end{tikzcd}$$

A descent datum for quasi-coherent sheaves with respect to a morphism $f : U \to X$ is a pair $(\mathcal{F}, \varphi)$, where $\mathcal{F}$ is a quasi-coherent sheaf on U, and $\varphi : \mathrm{pr}_0^* \mathcal{F} \to \mathrm{pr}_1^* \mathcal{F}$ is an isomorphism of modules over $U \times_X U$. This isomorphism must satisfy the cocycle condition on $U \times_X U \times_X U$, which states that $\mathrm{pr}_{02}^* \varphi = \mathrm{pr}_{12}^* \varphi \circ \mathrm{pr}_{01}^* \varphi$.

Furthermore, a descent datum is called effective if there exists a quasi-coherent sheaf $\widetilde{\mathcal{F}}$ on X along with an isomorphism $h : f^* \widetilde{\mathcal{F}} \cong \mathcal{F}$ such that $\varphi = \mathrm{pr}_1^*(h) \circ \mathrm{pr}_0^*(h)^{-1}$.

We denote the category of descent datum for quasi-coherent sheaves with respect to a morphism $f : U \to X$ as $\mathrm{Tot}(\mathrm{Qcoh}(\check{C}(U/X)))$ and $\mathrm{Tot}(\mathrm{Qcoh}(\check{C}(U/X)))_{\mathrm{eff}}$ for the effective part. Morphisms in this category are self-explanatory: they are morphisms between quasi-coherent sheaves that are compatible with the additional isomorphisms φ.[2]

Example 3.10. Note that there naturally exists a functor $H : \mathrm{Qcoh}(X) \to \mathrm{Tot}(\mathrm{Qcoh}(\check{C}(U/X)))$. Specifically, for any quasi-coherent sheaf $\mathcal{F}$ on X, we naturally and functorially obtain $(f^*\mathcal{F}, \tau^*)$ in $\mathrm{Tot}(\mathrm{Qcoh}(\check{C}(U/X)))$, where $\tau : U \times_X U \to U \times_X U$ is the swap of coordinate. You may check that this construction will corresponds to the functor $\widetilde{L}$ in Example 3.15 via Lemma 3.16 in affine cases.

Remark 3.11. H is indeed has a natural construction. Note that the Čech nerve $\mathrm{Qcoh}(\check{C}(U/X))$ can be naturally extended to $\check{C} : \Delta^{\triangleleft}_{\leqslant 2} \to \mathrm{Cat}$ via the permutation of coordinates, such as τ in the above example. Hence, H can be also constructed by the universal property of the 2-limit totalization.

Our main goal in this subsection is to show that the above functor is, in fact, an equivalence. There are various proofs of this result, such as the one in [sta24, Section 08WE], all of which are essentially based on the "fundamental" properties of fpqc morphisms, as presented in Proposition 3.4.

My approach here involves the concept of monadicity, as it encapsulates the technical details within the coherent comodule structure, making the proof clearer. This approach provides a transparent view of how Proposition 3.4 with an extrac condition implies the equivalence $\mathrm{Qcoh}(X) \simeq \mathrm{Tot}(\mathrm{Qcoh}(\check{C}(U/X)))$.

To see the power of monads, we first show how to encode descent datum into a comodule structure over a comonad.

Definition 3.12. A (co-)monad T in D is a (co-)monoid object in $\mathrm{End}(D)$, and a T-(co-)module M in D is a (co-)module over T satisfying certain compatibility conditions.

Example 3.13. An Adjoint pair $(L \dashv R, \eta : \mathrm{id}_C \to RL, \epsilon : LR \to \mathrm{id}_D)$ between categories C and D induces a monad (RL, η) and a comonad (LR, ϵ). Furthermore, any object $c \in C$ induces a comodule Lc over the comonad LR. We denote this assignment as $\widetilde{L} : C \to \mathrm{coMod}_{LR}(D)$. Similarly. Rd is RL-module for any $d \in D$.

Example 3.14. There is always a natural free-forgetful adjoint pairs $F \dashv U$ between Sets and some algebraic catgory $\mathcal{A}$, where $\mathcal{A}$ could be Mon, the category of monoids, or Mod_R, the category of left $R-$modules for some ring R. Then, we have $\mathrm{Mod}_{UF}(\mathrm{Sets}) \simeq \mathcal{A}$.

[2]This notation may seem confusing. In some reference, it could be denoted as $\mathrm{Desc}_{U \to X}$, which may be more intuitive. However, we use this notation here to maintain consistency with derived algebraic geometry, as descent data are indeed the totalization (homotopy limit) of quasi-coherent sheaves of the Čech nerve associated to $u \to X$. Note that this limit is essentially 2-categorical, as we work in the 2-category Cat.

Example 3.15. In Example 3.13, we can let $A \to B$ be a ring morphism, R be the restriction of scalars functor $\mathcal{R} : \mathrm{Mod}_B$ to Mod_A and L be the extension of scalars $B \otimes_A : \mathrm{Mod}_A \to \mathrm{Mod}_B$. We can then define a comonad $T := LR \in \mathrm{End}(\mathrm{Mod}_B)$, which will also denote as $B \otimes_A$ if there is no ambiguity.

Globally, we can take a morphism $f : U \to X$ in schemes and define $T = f^* f_*$ as a comonad over Qcoh_U, the category of quasi-coherent sheaves over $\mathcal{O}_U$. Then, we have the following proposition:

Lemma 3.16. *For any $f : X \to Y$ between schemes, if either X and Y are affine or f is flat and quasi-compact and quasi-separated (qcqs). Then, a comodule over the comonad $T = f^* f_*$ is precisely a descent datum for quasi-coherent sheaves with respect to f, i.e.* $\mathrm{coMod}_T(\mathrm{Qcoh}_U) \simeq \mathrm{Tot}(\mathrm{Qcoh}(\check{C}(U/X)))$.

The first is a direct computation that establishes a bijection between T-comodules and descent data, though proving it as a categorical equivalence would require careful and detailed verification. The second proof, which is more streamlined, essentially requires Theorem 3.22. We will sketch this later due to space constraints; see Proof 3.2. The first proof is as follows:

Proof. [Li24, Lemma 7.9.5] A T-comodule $\mathcal{F}$ is equipped with an $\mathcal{O}_U$-module map $\delta : \mathcal{F} \to T\mathcal{F}$. First, observe the tautological/flat base-change formula, $T = f^* f_* \cong \mathrm{pr}_{0,*} \mathrm{pr}_1^*$, which, affine locally, maps $a \otimes m$ to $a \otimes 1 \otimes m$. This formula is the only key element needed to establish this proposition and will also feature in our second proof.

Consequently, we find that $\mathrm{Hom}(\mathcal{F}, T\mathcal{F}) \cong \mathrm{Hom}(\mathcal{F}, \mathrm{pr}_{0,*} \mathrm{pr}_1^* \mathcal{F}) \cong \mathrm{Hom}(\mathrm{pr}_0^* \mathcal{F}, \mathrm{pr}_1^* \mathcal{F})$, showing that assigning $\delta : \mathcal{F} \to T\mathcal{F}$ is equivalent to assigning $\varphi : \mathrm{pr}_0^* \mathcal{F} \to \mathrm{pr}_1^* \mathcal{F}$.[3]

Examining this definition in affine local terms, we see that the cocycle condition translates into the coassociativity law, given the symmetric monoidal nature of the tensor product.

Note that $\Delta_{U/X,*}\varphi = \epsilon\delta$. Thus, the counit law $\epsilon\delta = \mathrm{id}_{\mathcal{F}}$ corresponds to $\Delta_{U/X,*}\varphi = \mathrm{id}_{\mathcal{F}}$. The crucial observation here is that coassociativity implies $(\Delta_{U/X,*}\varphi)^2 = \Delta_{U/X,*}\varphi$. Therefore, by working affine locally and examining the associated coefficient matrices, we conclude that φ is an isomorphism if and only if $\Delta_{U/X,*}\varphi$ is an isomorphism, which is equivalent to $\Delta_{U/X,*}\varphi$ being the identity.

□

Remark 3.17. f_* may not represent push-forward if f is not qcqs, it serves instead as a formal adjoint according to Gabber's result; See disccussion on Math.StackExchange.

Remark 3.18. This Proof should not be too overwhelming if you are comfortable manipulating algebraic structures over categories. However, if these concepts are unfamiliar, feel free to bypass the monadic terminology by treating the adjoint pair $f^* \dashv f_*$ as a generic one and consider a $f^* f_*$-comodule is the same as a descent datum. For those less experienced in algebraic geometry, it is advisable to initially explore this section within the context of affine schemes, particularly when interpreting the statement and proof of the following comonadicity theorem.

Now, we introduce some necessary notation before introducing the core theorem, which is supposed to get pay back soon...

We denote the indexing category $[0] \underset{\delta_{10}}{\overset{\delta_{11}}{\rightrightarrows}} [1]$ as $\Delta_{\leqslant 1}$, which is part of the simplicial category.

Formally, $\Delta_{\leqslant 1}$ can be extended to a cone with itself as the base, denoted by $\Delta_{\leqslant 1}^{\triangleleft}$[4]. Additionally, in the diagram below, the cone $\Delta_{\leqslant 1}^{\triangleleft}$ (the solid part) extends further to form a "split cone"(the

[3]Interested readers can verify for themselves that this correspondence is functorial, i.e., compatible with the additional structures of $\mathrm{coMod}_T(\mathrm{Qcoh}_U)$ and $\mathrm{Tot}(\mathrm{Qcoh}(\check{\mathrm{C}}(U/X)))$. The author omits this part as writing out every detail may be too extensive.

[4]In some lieterature it denoted as $\Delta_{+,\leqslant 1}$

entire diagram), denoted by $\Delta_{-\infty,\leqslant 1}$:

$$\begin{tikzcd} {[-1]} \arrow[r, "\delta_0"'] & {[0]} \arrow[l, bend right, dashed, "\sigma_{-1}"'] \arrow[r, shift left, "\delta_{11}"] \arrow[r, shift right, "\delta_{10}"'] & {[1]} \arrow[l, bend right=60, dashed, "\sigma_0"'] \end{tikzcd}$$

Accordingly, we call the functor $\alpha : \Delta_{-\infty,\leqslant 1} \to C$ a split equalizer in C.

Furthermore, $\alpha : \Delta_{\leqslant 1} \to C$ is called split if it can be extended to a split equalizer diagram $\widetilde{\alpha} : \Delta_{-\infty,\leqslant 1} \to C$; for some functor $F : C \to D$, $\alpha : \Delta_{\leqslant 1} \to C$ is called F- split, if $F \circ \alpha$ admits a splitting $\widetilde{F \circ \alpha}$ in D. We then have the following:

Proposition 3.19. *The index category $\Delta_{-\infty,\leqslant 1}$ is an absolute equalizer diagram, i.e. for any functor $\alpha : \Delta_{-\infty,\leqslant 1} \to C$, the diagram $\alpha(\Delta^{\lhd}_{\leqslant 1})$ is an equalizer diagram.*

Remark 3.20. The notation here is somehow a little confusing. However, one can interpret $[-1] = \{-\infty\}$, $[0] = \{-\infty, 0\}$,$[1] = \{-\infty, 0, 1\}$; the maps among them are order preserving map sending $-\infty$ to $-\infty$; $\delta_{10}(0) = 0$, $\delta_{11}(0) = 1$, $\sigma_0(0) = -\infty$, and $\sigma_0(1) = 0$. This witnesses a splitting: the equalizer $[-1]$ is the "$\{-\infty\}$" part of $[0]$ and $[1]$, and $[0]$ is the "$\{-\infty, 1\}$" part of $[1]$.

Example 3.21. Any comodule M over comonad T produces a splitting equalizer:

$$\begin{tikzcd} M \arrow[r, "\delta"'] & TM \arrow[l, bend right, "\epsilon_M"'] \arrow[r, shift left, "\mu"] \arrow[r, shift right, "T\delta"'] & TTM \arrow[l, bend right=60, "\epsilon_{TM}"'] \end{tikzcd}$$

The following is the core theorem in this section:

Theorem 3.22. *(Barr-Beck comonadicity theorem)*

Given an adoint pair $L \dashv R$ between monodial categories C and D, we can define $T := LR \in$ $\mathrm{End}(C)$ *as follow:* $\begin{tikzcd} C \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & D \arrow[l, shift left, "R"] \arrow[loop right, "T"] \end{tikzcd}$.

Then the following are equivalent:

1. *(1) (L, R) is comonadic, i.e. $\widetilde{L} : C \xrightarrow{\simeq} \mathrm{coMod}_T(D)$, where $\widetilde{L}$ is defined in Example 3.13;*
2. *(2) L is conservative; any L-split diagram creates the corresponding equalizer: for any L-split diagram $\alpha : \Delta_{\leqslant 1} \to C$ with the associated split diagram $\beta : \Delta_{-\infty,\leqslant 1} \to D$ such that $\beta|_{\Delta_{\leqslant 1}} = L\alpha$, α can be extended to an equilzer diagram $\widetilde{\alpha} : \Delta^{\lhd}_{\leqslant 1} \to C$ such that $\widetilde{\alpha}|_{\Delta_{\leqslant 1}} = \alpha$ and $\beta|_{\Delta^{\lhd}_{\leqslant 1}} = L\widetilde{\alpha}$.*

In particular, the inverse of $\widetilde{L}$ is K, fitting into the following equalizer diagram:

$$K(M) \longrightarrow RTM \rightrightarrows RTM$$

Remark 3.23. For our 1-categorical case, this Theorem is still hold if we do not require L to be conservative in (2).

Remark 3.24. In our context, we let the adjoint pair be $\begin{tikzcd} \mathrm{Qcoh}_X \arrow[r, shift left, "f^*"] \arrow[r, phantom, "\perp"] & \mathrm{Qcoh}_U \arrow[l, shift left, "f_*"] \end{tikzcd}$. Then the inverse of $\widetilde{f^*}$ should be $K(\mathcal{F}) =?$. *To be continued.*

Corollary 3.25. *For f a faithfully flat morphism in schemes, we have* $\mathrm{Qcoh}_X \simeq \mathrm{coMod}_{f^*f_*}(\mathrm{Qcoh}_U)$.

Proof. We want to show (2) of Theorem 3.22: f^* is conservative, Qcoh_U admits arbitrary equalizers, and f^* preserves equalizers. This is just Proposition 3.4. $\square$

Theorem 3.26. *fpqc morphisms are effective for descent. In other words, we have* $\mathrm{Qcoh}_X \simeq \mathrm{Tot}(\mathrm{Qcoh}(\check{C}(U/X)))$. *Equivalently, the prestack* $\mathrm{Qcoh} : \mathrm{Sch}^{op} \to \mathrm{Cat}$ *satisfies fpqc descent.*

Proof. Combining Corollary 3.25 and Theorem 3.16 proves the statement for fp-qc-qs morphisms. For general fpqc cases, we need to reduce to the affine case. To be Continued. ☐

Corollary 3.27. *The prestack* $\mathrm{Qcoh} : \mathrm{Sch}^{op} \to \mathrm{Cat} \to \mathrm{Gpd}$ *satisfies fpqc descent*

Proof. $\mathrm{Cat} \to \mathrm{Gpd}$ is the right adjoint functor to the inclusion $\mathrm{Gpd} \to \mathrm{Cat}$ and hence preserves limits. ☐

Futhermore, since rings form a basis for schemes with respect to the Zariski topology (and hence with resoect to fpqc topology), we directly deduce that $\mathrm{Qcoh}_X \cong \lim_{\mathrm{Spec}\, R \to X} \mathrm{Mod}_R$ and hence Qcoh is the right Kan extenion of $\mathrm{Mod} : \mathrm{Ring} \to \mathrm{Gpd}$.

Now we present an alternate proof for Proposition 3.16 ([Lur17, Theorem 4.7.5.2]):

Proof. We want show that $L : \mathrm{Tot}(\mathrm{Qcoh}(\check{\mathrm{C}}(U/X))) \to \mathrm{Qcoh}_U$ is comonoadic, where L is defined by the universal property. By Barr-Beck Theorem 3.22, it is equivalent to the following:

(1) L admits a right adjoint R;
(2) L is is conservative;
(3) Every L-split cosimplicial diagram creates the limit.

Denote the Čech nerve $\check{\mathrm{C}}(U/X))$ as $\check{\mathrm{C}}. : \Delta_{\leqslant 2} \to \mathrm{Cat}$ and let $d^{0*} := \check{\mathrm{C}}(\delta_{*0} : [*] \to [* + 1])$ be the coface maps. Then, following the proof of [Lur17, Theorem 4.7.5.2.], it is sufficient to show that: For every map $\alpha : [m] \to [n]$ in $\Delta_{\leqslant 2}$ and the induced diagram

$$\begin{tikzcd} \check{\mathrm{C}}_m \arrow[r, "d^0"] \arrow[d, "\check{\mathrm{C}}(\alpha)"] & \check{\mathrm{C}}_{m+1} \arrow[d, "\check{\mathrm{C}}(\alpha \star \mathrm{id}_{[0]})"] \\ \check{\mathrm{C}}_n \arrow[r, "d^0"] & \check{\mathrm{C}}_{n+1} \end{tikzcd},$$

the horizontal arrows admit right adjoints $d^{0,R}$ which also commute with the vertical arrows.

In our setting, d^0 are given by pr_0^* and pr_{02}^* and $d^{0,R}$ are given by $\mathrm{pr}_{0,*}$ and $\mathrm{pr}_{02,*}$. Thus the diagram commutativity condition is essentially the same as the base change formula $f^* f_* \cong \mathrm{pr}_{0,*}\, \mathrm{pr}_1^*$, which is the only ingredient we need for the proof. ☐

Remark 3.28. In the proof of [Lur17, Theorem 4.7.5.2.], Lurie also show that the construction of R satisiefies $L \circ R = d^{00,R} \circ \check{\mathrm{C}}(\delta_{11}) = \mathrm{pr}_{0,*}\, \mathrm{pr}_1^*$.

If we apply the second proof to deduce the equivalence $\mathrm{Qcoh}_X \simeq \mathrm{coMod}_{f^* f_*}(\mathrm{Qcoh}_U)$ in Theorem 3.26, we can then provide a more transparent proof of it. Here a fun fact is that we use the Barr-Beck Theorem twice[Lur17, Corollary 4.7.3.16.]:

Proof. The key is to utilize the following diagram:

$$\begin{tikzcd} \mathrm{Qcoh}_X \arrow[rr, "H"] \arrow[dr, dashed] \arrow[ddr, bend right, "f^*"'] & & \mathrm{Tot}(\mathrm{Qcoh}(\check{\mathrm{C}}(U/X))) \arrow[dl, dashed] \arrow[ddl, bend left, "L"] \\ & \mathrm{coMod}_{f^* f_*}(\mathrm{Qcoh}_U) \arrow[d] & \\ & \mathrm{Qcoh}_U & \end{tikzcd}$$

The outer triangular is commutative, where H is constructed in Example 3.10 and Remark 3.11. Then, H is an equivalence, if $f^* \dashv f_*$ and $L \dashv R$ are both comonadic and the corresponding comonad are the same, i.e. $f^* f_* \cong \mathrm{pr}_{0,*}\, \mathrm{pr}_1^* \cong LR$. ☐

Remark 3.29. In summary, we can collect all the sufficient conditions required to establish this equivalence H as follows:

(1) f^* is conservative;
(2) Every f^*-split cosimplicial diagram creates the limit.

(3) f^* satisfies the certain Beck-Chevalley condition $f^* f_* \cong \mathrm{pr}_{0,*} \mathrm{pr}_1^*$.

(1) and (2) is for Corollary 3.25, and (3) if for Lemma 3.16. Notably, if f^* is not conservative, then $\mathrm{coMod}_{f^* f_*}(\mathrm{Qcoh}_U) \to \mathrm{Qcoh}_X$ is fully faithful.

These conditions are sometime referred to as the descent criterion; see [Lur17, Corollary 4.7.5.3] for the original reference with proof and [Kha23, Theorem 3.2.6] for the corrected statement.

4. A Bit of 6-Functor Formalism (Under Construction)

5. Cohomology of Curves (Under Construction)

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Email address: `shize_hao@college.harvard.edu`