# A GLIMPSE OF AFFINE SCHEMES

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Abstract. In the setting of ring theory and point set topology, we define the geometric version of a special kind of commutative ring, which is called the affine scheme. To be specific, we start with the concept of spectra of commutative rings and then define rings of functions on the spectrum of a Dedekind domain. Also, we prove the nontriviality of the ideal class group of $\mathbb{Z}[\sqrt{-6}]$ to exemplify the concept of affine schemes. In appendices, we discuss sheaves and ramification.

Contents



## 1. Introduction

We here give a short introduction to the background and motivation of scheme theory, the geometry version of ring theory. Most of it will not be covered in the actual presentation. The main references of the lecture notes are [Har77] and [Vak24].

In classical algebraic geometry, one studies *algebraic varieties*, which are (irreducible) zero-loci of polynomials over affine or projective space, typically over an algebraically closed field $k$. In this case, Hilbert's Nullstellensatz gives us a dictionary between algebra and geometry: *e.g.* radical ideals to subsets of affine space, prime ideals to varieties, maximal ideals to points. By arriving at a reasonable notion of functions on varieties (via *coordinate rings*), we obtain an equivalence of categories between affine varieties and finitely generated, reduced $k$-algebras.

Some natural questions arise: what if $k$ is not algebraically closed? What if we want to glue affine varieties together? What if we allow nilpotents in our rings? The correct geometric structures that generalize varieties and which address these three problems are called *schemes*. Amazingly, the category of affine schemes is isomorphic to the category of commutative rings, which enlarges the equivalence for affine varieties.

Nowadays, scheme theory has become the language of algebraic geometry and number theory. In a single sentence, *a scheme is a locally affine sheaf.* It seems confusing at first, but it turns out to be a statement of "abstract nonsense" (category theory) except for the word "affine." Indeed, the key point of understanding scheme theory is understanding how natural an affine scheme is. After that, a scheme, in general, is just obtained by gluing affine schemes: think of the latter as the role that open disks play in differential geometry, which topological manifolds are locally comprised of.

Then what is an affine scheme? In some sense, knowing about prime ideals in commutative rings *is* knowing about schemes. However, the definition of affine schemes provides new insight into ring theory. It tells us how to understand algebra geometrically, and vice versa, which is called the *duality between algebra and geometry*.

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## 2. Case Studies: Spectra of $\mathbb{C}[x]$ and $\mathbb{Z}$

In this section, we will define the spectra of commutative rings $\mathbb{C}[x]$ and $\mathbb{Z}$. We will compare the difference between the Zariski topology and the usual Euclidean topology on the complex line, then argue how reasonable the definition is.

First, we define the spectrum of a commutative ring $A$. We write

$$\operatorname{Spec} A = \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ is a prime ideal of A}\}$$

**Proposition 2.1.** *The set of all prime ideals in $\mathbb{C}[x]$ is $\{(x-a) : a \in \mathbb{C}\} \sqcup \{(0)\}$, denoted as $\underline{\operatorname{Spec}}\,\mathbb{C}[x]$; the set of all prime ideals in $\mathbb{Z}$ is $\{(p) : p$ prime in $\mathbb{Z}\} \sqcup \{(0)\}$, denoted as $\underline{\operatorname{Spec}}\,\mathbb{Z}$.*

*Proof.* We first claim that $\mathbb{C}[x]$ and $\mathbb{Z}$ are PIDs. Then we know their irreducible elements are, respectively, linear polynomials and the usual prime numbers:

- According to the fundamental theorem of algebra, any polynomial in $\mathbb{C}[x]$ can be uniquely factored into the product of (irreducible) linear monomials. Besides, any nonzero ideal in $\mathbb{C}[x]$ is generated by some polynomial of the smallest degree. If not, we can fix $f$ and $g$ in $\mathbb{C}[x]$, where $\deg f$ is the smallest in $\mathbb{C}[x]$. Since $\mathbb{C}[x]$ admits the Euclidean algorithm:
$$g(x) = f(x) \cdot d(x) + r(x), \text{ where } \deg r < \deg f \text{ or } r = 0,$$
it can only have $f \mid g$, which means $\mathbb{C}[x]$ is a PID.
- Similarly, according to the fundamental theorem of arithmetic, we can prove $\mathbb{Z}$ is a PID.

Next, since $\mathbb{C}[x]$ and $\mathbb{Z}$ are integral domains, the zero ideal is a prime ideal in $\mathbb{C}[x]$ and $\mathbb{Z}$.

Finally, notice that a nonzero ideal in a PID is a prime ideal if and only if it is generated by an irreducible element. ☐

Next, we will introduce the topology on $\underline{\operatorname{Spec}}\,\mathbb{C}[x]$ and $\underline{\operatorname{Spec}}\,\mathbb{Z}$. For those unfamilar with topology, we want to characterize the notion of "nearness" in some flexible way. The building blocks are *open sets*, and we will also define *closed subsets* as complements of open sets.

The following is the formal definition:

**Definition 2.2.** A *topology* on a set $X$ is a nonempty collection of subsets of $X$, called open sets, such that any union of open sets is open, any finite intersection of open sets is open, and both $X$ and the empty set are open. A set equipped with a topology is called a *topological space.* An open set is defined as the complement of a closed set.

Now that we have defined what a topology is, we can immediately define a somewhat weird topology of a commutative ring. First we discuss this in general, and then we will go back to our cases $\operatorname{Spec}\mathbb{C}[x]$ and $\operatorname{Spec}\mathbb{Z}$ to examine the definition via examples.

**Definition-Proposition 2.3.** *The Zariski topology on $\underline{\operatorname{Spec}}\,A$*

*Denote by $\underline{\operatorname{Spec}}\,A$ the set of all prime ideals in $A$, where $A$ is a commutative ring.*

*Then, for each ideal $I \lhd A$, we define $V(I) := \{\mathfrak{p} \supset I \mid \mathfrak{p}$ is a prime ideal of $A\}$. The collection of $V(I)$ for all $I$ define closed sets on $\underline{\operatorname{Spec}}\,A$. We call the above topology on* $\operatorname{Spec} A$ *the Zariski topology of $A$.*

*Proof.* We denote $J$ as an index set and check three axioms of closed sets one by one:

(1) $\varnothing = V(A)$; $\underline{\operatorname{Spec}}\, A = V((0))$.
(2) $\bigcap_{j\in J} V(I_j) = V(\sum_{j\in J} I_j)$.
(3) $V(I) \cup V(J) = V(IJ)$

Immediately, we observe that $V(-)$ is inclusion-reversing: if $I \subset J$, then $V(I) \supset V(J)$.

For (1), by definition, no prime ideal contains 1, so $V(1) = \varnothing$. Moreover, since 0 is in every ideal, $V(0) = \operatorname{Spec} A$.

For (2), if $I_j$ is any collection of ideals, the minimal ideal containing all of them is their sum, $\sum_j I_j$. We have that $V(\sum_j I_j) \subset V(I_j)$ for each $j$, so $V(\sum_j I_j) \subset \bigcap_j V(I_j)$. On the other hand, if $\mathfrak{p}$ is a prime ideal containing all $I_j$, then $\mathfrak{p}$ must contain $\sum_j I_j$ since $\sum_j I_j$ is the smallest ideal containing all $I_j$. It follows that $\bigcap V(I_j) \subset V(\sum_j I_j)$, so that $\bigcap V(I_j) = V(\sum I_j)$, as desired.

For (3) we have the inclusions $IJ \subseteq I, J$. Hence $V(I), V(J) \subseteq V(IJ)$, so $V(I) \cap V(J) \subseteq V(IJ)$. For the reverse inclusion, if $\mathfrak{p}$ is a prime ideal containing $IJ$, then $\mathfrak{p}$ contains either $I$ or $J$. Otherwise, we would have an element $ab \in IJ \subseteq \mathfrak{p}$ with $x, y \notin \mathfrak{p}$, which is impossible. $\square$

**Remark 2.4.** In general, any arbitrary ring homomorphism $\varphi : A \to B$ naturally induces a contravariant map $f :=$ "$\varphi^{-1}$"[1]:$\operatorname{Spec} B \to \operatorname{Spec} A$, which is defined as $f(p) := \varphi^{-1}(p)$. It also shows why we define the spectrum of a ring is the set of all prime ideals.

The usual topology on $\mathbb{C}$ is defined by open disks and an open set on $\mathbb{C}$ should be some union of open disks. However, the Zariski topology on $\underline{\operatorname{Spec}}\, \mathbb{C}[x] \overset{1:1}{\leftrightarrow} \mathbb{C} \sqcup \{(0)\}$ is totally different to the usual topology on $\mathbb{C}$ in the following ways:

(1) *Fuzzy Point.* $\operatorname{Spec} \mathbb{C}[x]$ contains an extra point (0) than $\mathbb{C}$, called the generic point:

**Proposition 2.5.** *The smallest closed set containing* (0) *is the whole complex line.*[2]

*Proof.* Property of integral domain. $\square$

Recall that the smallest closed set containing some point is called the closure of these points. It is a set consisting of these points and boundary points near them. Thus, the proposition indicates that every point in $\operatorname{Spec} \mathbb{C}[x]$ is a boundary point near the generic point. It seems weird in the first place, but what actually happens here is that we should take a generic point as *a higher dimension point existing nowhere but almost everywhere.* Intuitively, we also call it a fuzzy point, and we will also draw a picture in the presentation to show the meaning.

(2) *Coarse Topology.* $\operatorname{Spec} \mathbb{C}[x]$ is much coarser than the usual topology:

**Proposition 2.6.** *Closed sets of* $\operatorname{Spec} \mathbb{C}[x]$ *are finite sets or the whole complex line.*

*Proof.* By prime decomposition, $V(I) = V(\prod_i p_i) = \cup_i V(p_i) = \cup_i \{p_i\}$. $\square$

As the proposition shows, Zariski topology is extremely coarse, offering a relatively vague picture of the complex line compared to the Euclidean topology. It simply associates each maximal ideal with a closed point, unlike the Euclidean topology which contains open disks to represent neighborhoods around a point. In the Zariski topology on $\operatorname{Spec} \mathbb{C}$, there are no open disks or concept of proximity.[3] It even fails to be Hausdorff, thus having poor separation properties. However, it is a useful language for studying varieties and schemes.

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There is a similar proposition for the integers:

[1]"$\varphi^{-1}$" as we defined here is the right inverse of $\varphi$ but not a left inverse

[2]For those of you who've taken a topology course, this is unusual behavior! Most topological spaces outside of algebraic geometry are at least $T_0$, which is to say that its points are closed. Not so in this case; some points exhibit a "largeness"/"ever-presence".

[3]As open disks shrink towards a point in a complex plane, points within smaller disks are closer to the fixed point. In this sense, we convince ourselves that open neighborhoods encode the information about the proximity of one point to another.

**Proposition 2.7.** *The smallest closed set containing* $(0)$ *in* $\operatorname{Spec}\mathbb{Z}$ *is* $\operatorname{Spec}\mathbb{Z}$ *itself. Closed sets of* $\operatorname{Spec}\mathbb{Z}$ *are also finite sets or* $\operatorname{Spec}\mathbb{Z}$ *itself.*

To see that the only infinite closed set is the whole space, consider a closed set $V(I)$ with $I \subseteq \mathbb{Z}$ and suppose $V(I)$ is infinite. If $n \in I$ then $n \in (p)$ for infinitely many primes $p$. If $n \neq 0$, this means that $p \mid n$ for infinitely many primes, which is impossible. It follows that $n = 0$.

To round out this section of the talk, we'd like to give one further example of affine spaces that is especially simple.

**Example 2.8.** Let $k$ be a field. Then $\operatorname{Spec} k$ consists of just one point, corresponding to the unique maximal ideal $(0)$. It is a closed point.

Now we introduce a useful technique derived from commutative algebra called *localization.*

Informally, we would like to invert certain subsets of a ring (say, a domain) to obtain a new ring that is intermediate between our original one and its field of fractions. We'll do this during the presentation, and briefly describe the universal property of localization.

**Definition-Proposition 2.9.** *We say* $S$ *is a multiplicative subset of an arbitrary integral domain* $A$*, if* $1 \in S$ *and* $S$ *is closed under multiplication, which means:* $s, s' \in S$ *implies* $ss' \in S$*.*

*Now, the localization of* $A$ *with respect to* $S$ *is an integral domain* $S^{-1}A := \{(a, s) \in A \times S\}/\sim$*, where* $(a, s) \sim (a', s')$ *if and only if* $as' = a's$*. Sometimes we denote* $(a, s)$ *in a more intuitional way as* $\frac{a}{s}$*.*

*Proof of well-defined-ness.* The ring structure of $S^{-1}A$ is induced by the product ring $A \times A$. Also, the relationship $\sim$ defined above is indeed an equivalence relationship. □

**Remark 2.10.** Here we have a natural injective ring homomorphism $A \to S^{-1}A$ defined by sending $x$ to $\frac{x}{1}$.

**Remark 2.11.** In this talk, We will only need a special localization, called the localization ring at a prime ideal $p$, denoted as $A_p$, where the corresponding multiplicative subset $s = A \setminus p$. Particularly, the field of fractions of $A$ is just localization at zero ideal $K = A_0$. By the Remark 2.10, we then obtain a nice diagram of maps in the following sense:

$$\begin{tikzcd} & A_p \arrow[dr, "\text{inside } p"] & \\ A \arrow[ur, "\text{outside } p"] \arrow[rr, "\text{invert all}"'] & & K \end{tikzcd}$$

The left top arrow inverts elements outside $p$ and the right top arrow inverts elements inside $p$. Combining them is just inverting all elements at once.

## 3. Affine schemes of Dedekind domains

Imitating the proofs in the propositions 2.5 and 2.6, we have a straightforward generalization for an arbitrary Dedekind domain $A$:

**Proposition 3.1.** *For any Dedekind domain* $A$*, the smallest closed set containing* $(0)$ *in* $\operatorname{Spec} A$ *is* $\operatorname{Spec} A$ *itself. Closed sets of* $\operatorname{Spec} A$ *are also finite points or* $\operatorname{Spec} A$ *itself.*

So far, we've finally completed the *space* piece of the *affine scheme* puzzle. Next, we will talk about the "rings of functions" piece. We begin with the following question:

**Question 3.2.** What should an "algebraic" function[4] on the spectrum of a ring look like?

To give a correct definition of an algebraic function, we must first have the following structure theorem of the localization ring.[5]:

[4]The formal name for "algebraic function" will be "regular function".

[5]localizations of Dedekind domains at their primes end up being *discrete valuation rings*, or regular local rings of dimension one - in many respects, these are the nicest rings which are not fields, and enjoy wonderfully simple ideal structure

**Proposition 3.3.** *Suppose that $A$ is a Dedekind domain with a nonzero prime ideal $p$, then the localization ring $A_p$ is an integral domain. The only nonzero ideals in it are $(x^n)$ for some $x \in p$ and all non-negative integers $n$.*

*Proof.* We fix a nonzero ideal $I \lhd A_p$. Then, $I = \varphi(\text{“}\varphi^{-1}\text{”}(I)) = p^n \prod_i p^{n_i} A_p$, for a naturally defined map $\varphi : A \to A_p$. Notice that for any $i$ there exists some $y \in p_i \setminus p$ that is invertible in $A_p$. Hence $I = p^n A_p$. We will write $p^n A_p$ as $p^n$ for simplicity.

Now, we only need to show $pA_p = (x)$ for some $x$.

Choose $x \in p \setminus p^2$, which means $(x) \subset A_p$ and $(x) \neq p^k$ for $k \geqslant 2$. Also, we have proved that $(x) = p^n$ for some $n$, which it turns out to be that we must have $p = (x)$. □

**Corollary 3.4.** *For any nonzero prime ideal $p \lhd A$, the natural map $\tau : \operatorname{Spec} A_p \to \operatorname{Spec} A$ is the embedding from a pair of points to the whole space.*

Then its spectrum has two points, and we'll draw a picture - one of them is a closed point corresponding to $\mathfrak{m}$ and the other is a generic point for $(0)$.

**Corollary 3.5.** *Denote $K$ as the field of fractions of $A$ and also the field of fractions of $A_p$. We can define the (discrete) valuation $v_p : K \to \mathbb{Z} \sqcup \{\infty\}$. It satisfies:*

*(1) $v_p^{-1}(\mathbb{Z}_{\geqslant n}) = (x^n)$, where $(x^n)$ is a fractional ideal in $A_p$;*

*(2) $v_p^{-1}(\{n\}) = x^n A_p^*$, where $A_p^*$ is the set of invertible elements in $A_p$.*

*In particular, $v_p^{-1}(\mathbb{Z}_{\geqslant 0}) = A_p$ and $v_p^{-1}(\{0\}) = A_p \setminus (x) = A_p^*$. Also $v_p^{-1}(\{\infty\}) = \{0\}$*

*Proof.* Notice that for each $p$, $A_p$ is a subring of $K$. Then, for any $f, g \in A$, we define $v_p(f) := ord_p(f)$ and $v_p(\frac{f}{g}) := \frac{v_p(f)}{v_p(g)}$. We observe that $(1) \setminus (x) = A_p^*$. Also, notice that $v_p(f) = n$ if and only if $f \in (x^n) \setminus (x^{n+1})$. Thus, $v_p^{-1}(\mathbb{Z}_{\geqslant n}) = (x^n)$. □

This corollary shows that for each element we can define a valuation $v_p : K \to \mathbb{Z}$. Fix a prime ideal $p$ here. Then, for every element $u$ in $K$, we have a factorization $u = x^n \cdot v$ for some $v \in A_p^*$. This discussion provides some insight to give the first definition of algebraic functions in the following sense:

**Answer 3.6.** *the weaker*

An “algebraic” function f is an element in $K$, called a rational function. It is defined as $f(p) := v_p(f)$. Hence, sometimes We call $K$ the function field.

However, it contains relatively less information. We will illustrate our point by drawing the graphs of some rational functions in $\mathbb{Q}$ and $\mathbb{C}(x)$[6] in our presentation.

In some sense, we should take the factorization $u = x^n \cdot v$ as *the valuation version of the Laurent series.* For $n \geqslant 0$, $u$ is a well-defined Taylor series with the order of zero equals to $n$; For $n < 0$, $u$ is infinite at $p$ with the order of pole equals to $-n$. Hence, a localization $A_p$ can be taken as a ring of all Taylor's expansions nearby $p$, which is also called *germs* at $p$.

Also, notice that $f(p)$ should not be well-defined When $f \notin \mathbb{A}_p)$. Furthermore, the set of non-defined points is a finite set, according to the prime decomposition.

**Remark 3.7.** Recall that a spectrum is very coarse in general. It contains relatively less neighborhood information. On the contrary, a localization ring at $p$ describes the information about rings of functions in some neighborhood of $p$. Thus, the localization completes the neighborhood information algebraically, which we lost in the topology.

We eventually collected all the facts to answer the Question 3.2:

**Answer 3.8.** *the stronger*

An algebraic function is a rational function, and it is a partially defined function on some open subset. For each point, it takes values on the localization rings at this point.

[6] Try to find the analogue between localization ring $\mathbb{C}[x]_{(x)}$ and $C[[x]]$ by yourself! (Hint) They are: rational functions admit factorization $x^n \cdot v$ with non-negative $n$; analytic functions in the form of Taylor's expansion nearby point $0 \in \mathbb{C}$!

Combining $\operatorname{Spec} A$ and rings of functions on $\operatorname{Spec} A$, we obtain a beautiful structure for $A$, called an *affine scheme*. Here is the formal definition:

**Definition 3.9.** For a Dedekind domain $A$ and its field of fractions $K$, We define a ring of functions on an open set in $\operatorname{Spec} A$ to be $U \subset \operatorname{Spec} A$ to be $A(U) = \{f \in K \mid v_{p\in U}(f) \geqslant 0\}$.

Then, an affine scheme of $A$ is a tuple $(A, \operatorname{Spec} A, A(-))$. Sometimes we also call $K$ the function field of the affine scheme.

**Remark 3.10.** Notice that $A(U) = \{f \in K \mid v_{p\in U}(f) \geqslant 0\} = \{f \in K \mid f \in A_p\} = \cap_{p\in U} A_p$, so the definition of the affine scheme only requires the existence of the function field. It means we have a straightforward generalization of the definition of affine scheme for any arbitrary integral domain!

We will draw a nice picture to show how an algebraic function on $\operatorname{Spec} \mathbb{Z}$ takes different values at different points. The key point here is that codomains change with the change of points, and the valuation at a fixed point indicates the quantitative tendency somehow.

**Remark 3.11.** For any $p$ and $p'$, $f(p) = f(p')$ in the function field $K$ but should not be taken as the same value in their codomains, since their codomains are not the same!

Their codomains are the same up to an isomorphism $\sigma \in \operatorname{Gal}(K)$[7] in the following sense:

$$\begin{tikzcd} p \arrow[r, mapsto, "f"] & f(p) \in A_p \hookrightarrow K \arrow[d, "\sigma"] \\ p' \arrow[r, mapsto, "f"] & f(p') \in A_{p'} \hookrightarrow K \end{tikzcd}$$

**Historical Remark 3.12.** We tried to avoid any modern language, including category theory and sheaf theory, which is unavoidable when giving a formal definition of an affine scheme. However, we have clarified the key idea here.

*The very idea of scheme is of infantile simplicity — so simple, so humble, that no one before me thought of stooping so low. So childish, in short, that for years, despite all the evidence, for many of my erudite colleagues, it was really "not serious"! —Grothendieck*

## 4. Case studies: $\operatorname{Spec} \mathbb{Z}[\sqrt{-6}]$

In this section, we will reduce our definition of an affine scheme to a special case $\operatorname{Spec} \mathbb{Z}[\sqrt{-6}]$. We make some concrete computations to see what happens here.

We first introduce a proposition derived from [Mar18], but we reformulate it slightly differently. In the following proposition, we call a map $\pi : X \to Y$ a covering map is the map is surjective; furthermore, the preimage of $p \in Y$ $\pi^{-1}(p)$ is called the fiber at $p$.

**Proposition 4.1** (Theorem 25 in [Mar18] Chapter3)**.** *Ramification in* $\pi : \operatorname{Spec} \mathbb{Z}[\sqrt{-6}] \to \operatorname{Spec} \mathbb{Z}$

*We have a natural covering map* $\pi : \operatorname{Spec} \mathbb{Z}[\sqrt{-6}] \to \operatorname{Spec} \mathbb{Z}$*, which is induced by the ring homomorphism from* $\mathbb{Z}$ *to* $\mathbb{Z}[\sqrt{-6}]$*. Furthermore,* $\mid \pi^{-1}(p) \mid= 1$ *or* $2$*.*

*Proof.* According to Remark 2.4, $\mathbb{Z} \to \mathbb{Z}[\sqrt{-6}]$ induce a natural map $\pi : \operatorname{Spec} \mathbb{Z}[\sqrt{-6}] \to \operatorname{Spec} \mathbb{Z}$. Some prime $q$ lying over $p \in \operatorname{Spec} \mathbb{Z}$ is just the same as $q \in \pi^{-1}(p)$.

For some fixed quadratic field $K/\mathbb{Q}$, Theorem 25 in [Marcus Chapter 3] gives a general prime decomposition formula in $K$ for every prime in $\mathbb{Z}$. Here we fix $K = \mathbb{Q}(\sqrt{-6})$. According to Corollary 2 in [Marcus Chapter 2], the integer ring of $\mathbb{Q}(\sqrt{-6})$ is $\mathbb{Z}[\sqrt{-6}]$.

Then we immediately obtain $\mid \pi^{-1}(p) \mid= 1$ or $2$ as desired. What's more, we can compute the Legrende symbol for $p \geqslant 5$:

$$\left(\frac{-6}{p}\right) = (-1)^{\frac{p-1}{2}}(-1)^{\frac{p^2-1}{8}}(-1)^{\lfloor \frac{p+1}{6} \rfloor} = \begin{cases} -1, p \equiv 13, 15, 17, 21, 23 \mod 24 \\ 1, \text{else} \end{cases}$$

Then we can prove that:

[7]these isomorphisms satisfy the cocycle condition: the key condition makes a presheaf to be a sheaf.

- $p$ is (totally) ramified and $p\mathbb{Z}[\sqrt{-6}] = (p, \sqrt{-6})^2$, if $p$ is (2) or (3);
- $p$ is split and $p\mathbb{Z}[\sqrt{-6}] = q_p\bar{q}_p$, if $\left(\dfrac{-6}{p}\right) = 1$;
- $p$ is inert and $p\mathbb{Z}[\sqrt{-6}] = q_p$ prime, if $\left(\dfrac{-6}{p}\right) = -1$.

Also, we can compute that the discriminant $\Delta = -2^3 \cdot 3$. Then $\pi$ ramifies at $p$ if and only if $p = (2)$ or $(3)$. □

**Proposition 4.2.** $\mathrm{Cl}(\mathbb{Z}[\sqrt{-6}])$ *is nontrivial. Particularly,* $(2, \sqrt{-6}])$ *is not principal.*

*Proof.* We denote $A = \mathbb{Z}[\sqrt{-6}]$, $q_2 = (2, \sqrt{-6}])$ and $q_3 = (3, \sqrt{-6}])$. We prove that $(2, \sqrt{-6}])$ is not a principal ideal by contradiction.

Suppose $q_2 = (f)$ for some $f \in \mathbb{Q}[\sqrt{-6}]$. Then, $v_{q_2}(f) = v_{q_2}(q_2) = 1$ and $v_q(f) = v_q(q_2) = 0$ for any $q \neq q_2$.

This means that $f \in (q_2 \setminus q_2^2) \bigcap (q_3^0 \setminus q_3)$. Or equivalently, it means that $f = \sqrt{-6} \cdot s = t$, where $s \in (A \setminus q_3)$ and $t \in (A \setminus q_2)$. However, $t = \sqrt{-6} \cdot s \in (\sqrt{-6}) \subset q_3$. Contradiction!

We will prove it also by drawing a picture. The key point here is to see the value of $f$ at special point (that is ramified point) to show the contradiction. □

**Remark 4.3.** We can also prove $p_3$ in a parallel process. Notice that $q_2^2$, $q_2q_3$, $q_3^2$ are all principal, hence $q_2 \sim q_3$ in $\mathrm{Cl}(\mathbb{Z}[\sqrt{-6}])$. What's more, $q_2$ actually is the only nontrivial element, while we have not proved that $q_{p\geqslant 5}$ principal yet.

## 5. Appendix: Sheaves

We mentioned briefly that to formalize these notions of "rings of functions", we need the machinery of sheaves. *A priori* they are valued in the general category of abelian groups, but can take values in more restrictive categories: sheaves valued in modules over a ring, rings, etc. are all invaluable in algebraic geometry.

**Definition 5.1.** Let $X$ be a topological space. A *presheaf* $\mathscr{F}$ of abelian groups on $X$ consists of the following data:

(1) for every open set $U \subseteq X$, an abelian group $\mathscr{F}(U)$
(2) for every inclusion $V \subseteq U$ of open sets, a morphism of abelian groups $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$ such that (i) $\mathscr{F}(\varnothing) = 0$, (ii) $\rho_{UU} = \mathrm{id}_U$ for all open sets $U$, and (iii) such that for every inclusion $W \subseteq V \subseteq U$ of open sets we have $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

$$\begin{tikzcd} U \arrow[r, "\rho_{UV}"] \arrow[dr, "\rho_{UW}"'] & V \arrow[d, "\rho_{VW}"] \\ & W \end{tikzcd}$$

Elements $s \in \mathscr{F}(U)$ are called *sections* of $\mathscr{F}$ over $U$.

**Definition 5.2.** A presheaf is a *sheaf* if it satisfies the following additional condition: for every open cover $\{U_i\}$ of each open set $U$, if there are sections $s_i \in \mathscr{F}(U_i)$ for each $i$ such that $s_i|_{U_i \cap U_j} = s_i|_{U_j}$, there exists a unique $s \in \mathscr{F}(U)$ such that $s_{U_i} = s_i$ for all $i$.

### 5.1. **Constructions with sheaves.** 

First we consider restrictions of (pre-)sheaves.

**Definition 5.3.** Let $\mathcal{F}$ be a presheaf on a topological space $X$, and let $U$ be an open subset of $X$. Then we can define the *restriction presheaf* $\mathscr{F}|_U$ to satisfy $\mathscr{F}|_U(V) = \mathscr{F}(V)$ for all open $V \subseteq U$, and with the restriction maps inherited from $\mathscr{F}$. Moreover, if $\mathscr{F}$ is a sheaf, then so is $\mathscr{F}|_U$.

**Definition 5.4.** The *stalk* of a presheaf $\mathscr{F}$ at a point $x \in X$ is defined to be

$$\mathscr{F}_x = \varinjlim_{U \ni x} \mathscr{F}(U)$$

where the right-hand side denotes the *direct limit* of the groups $\mathscr{F}(U)$, as $U$ varies over the neighborhoods of $x$. In less fancy language, $\mathscr{F}_x$ can be described as the quotient of the disjoint union of all sections,

$$\bigsqcup_{U \ni x} \mathscr{F}(U)/\sim$$

where $s_U \in \mathscr{F}(U)$ and $s_V \in \mathscr{F}(V)$ are equivalent when there exists an open neighborhood $W \subseteq U \cap V$ of $x$ such that $s_U|_W = s_V|_W$. These equivalence classes are called *germs* at $x$, and comprise the stalk $\mathscr{F}_x$.

Oftentimes, properties of $\mathscr{F}$ can be deduced at the (typically simpler) level of stalks, exhibiting a local-global principle for (pre-)sheaves.

Recall that a *local ring* is a ring with a unique maximal ideal.

**Definition 5.5.** A *locally ringed space* $(X, \mathscr{O}_X)$ consists of a topological space $X$ equipped with a sheaf of rings $\mathscr{O}_X$ such that each stalk $\mathscr{O}_{X,x}$ is a local ring.

With this setup, an affine scheme will be a locally ringed space $(X, \mathscr{O}_X)$ that is isomorphic to the spectrum of a ring $A$, where sections over open sets $U$ of the latter are the regular functions defined on $U$. Then schemes are those locally ringed spaces which admit coverings by affine schemes, and morphisms of schemes are defined in the category of locally ringed spaces.

## 6. Appendix: Ramification

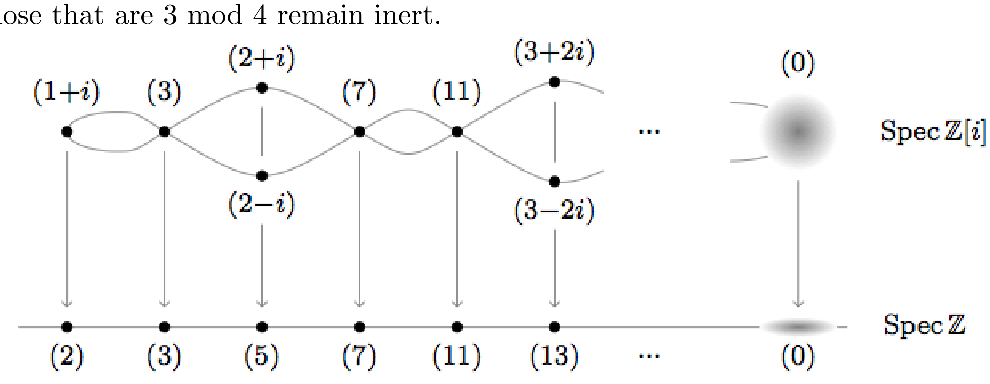
To elaborate on what we've talked about in our presentation, we discuss the phenomenon of ramification more generally.

Let $A$ be a Dedekind domain with field of fractions $K$. Let $L/K$ be a finite, separable extension, and $B$ the integral closure of $A$ in $L$. Let $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} B$. The inclusion $A \hookrightarrow B$ induces a morphism of spectra in the reverse direction, $f : X \to Y$. If $\mathfrak{p}$ is a prime (equivalently, maximal) ideal of $A$, then we can consider

$$\mathfrak{p}B = \prod_{i=1}^{r} \mathfrak{q}_i^{e_i}$$

where the right-hand side is the unique prime factorization of $\mathfrak{p}$ upstairs in the Dedekind domain $B$. Recall from class that the $e_i$ are the *ramification indices*: below we show why this name is justified, and how the phenomenon of lifting primes is related to branched coverings of curves.

In the below diagram we realize $\operatorname{Spec} \mathbb{Z}[i]$ as a twofold branched cover of $\operatorname{Spec} \mathbb{Z}$. The cover is only branched at $(2)$, since we have $(1+i)^2 = (2)$ in $\operatorname{Spec} \mathbb{Z}[i]$; primes that are 1 mod 4 split, and those that are 3 mod 4 remain inert.

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