Set theory

From the book: Naive Set Theory by Paul R. Halmos

This set of notes is almost a re-edited version of the original book, but in some places, a unique notation system was used, and there are also specific additions, such as exercise solutions. These notes were written exclusively for myself and may contain errors.

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Contents

1 Axiom of extension

On thing that the development will not include is a definition of sets. The situation is analogous to the familiar axiomatic approach to elementary geometry. That approach does not offer a definition of points and lines; instead it describes what it is that one can do with those objects. The semi-axiomatic point of view adopted here assumes that the reader has the ordinary, human, intuitive (and frequently erroneous) understanding of what sets are; the purpose of the exposition is to delineate some of the many things that one can correctly do with them.

Sets, as they are usually conceived, have *elements* or *members*. An element of a set may be a wolf, a grape, or a pigeon. It is important to know that a set itself may also be an element of some other set. Mathematics is full of examples of sets of sets. A line, for instance, is a set of points; the set of all lines in the plane is a natural example of a set of sets (of points). What may be surprising is not so much that sets may occur as elements, but that for mathematical purposes no other elements need ever be considered.

The principal concept of set theory, the one that in completely axiomatic studies is the principal primitive (undefined) concept, is that of *belonging*. If x belongs to A (x is an element of A, x is *contained* in A), we shall write

$$x \in A.$$

A possible relation between sets, more elementary than belonging, is *equality*. The equality of two sets A and B is universally denoted by the familiar symbol

$$A = B;$$

the fact that A and B are not equal is expressed by writing

$$A \neq B.$$

The most basic property of belonging is its relation to equality, which can be formulated as follows.

Axiom of extension. *Two sets are equal if and only if they have the same elements.*

With greater pretentiousness and less clarity: a set is determined by its extension.

It is valuable to understand that the axiom of extension is not just a logically necessary property of equality but a non-trivial statement about belonging. One way to come to understand the point is to consider a partially analogous situation in which the analogue of the axiom of extension does not hold. Suppose, for instance, that we consider human beings instead of sets, and that, if x and A are human beings, we write $x \in A$ whenever x is an ancestor of A. The analogue of the axiom of extension would say here that if two human beings are equal, then they have the same ancestors (this is the "only if" part, and it is true), and also that if two human beings have the same ancestors, then they are equal (this is the "if" part, and it is false).

If A and B are sets and if every element of A is an element of B, we say that A is a *subset* of B, or B *includes* A, and we write

$$A \subset B$$

or

$$B \supset A.$$

The wording of the definition implies that each set must be considered to be included in itself ($A \subset A$); this fact is described by saying that set inclusion is *reflexive*. (Note that, in the same sense of the word, equality also is reflexive.) If A and B are sets such that $A \subset B$ and $A \neq B$, the word *proper* is used (proper subset, proper inclusion). If A, B, and C are sets such that $A \subset B$ and $B \subset C$, then $A \subset C$; this fact is described by saying that set inclusion is *transitive*. (This property is also shared by equality.)

If A and B are sets such that $A \subset B$ and $B \subset A$, then A and B have the same elements and therefore, by the axiom of extension, $A = B$. This fact is described by saying that set inclusion is *antisymmetric*. (In this respect set inclusion behaves differently from equality. Equality is *symmetric*, in the sense that if $A = B$, then necessarily $B = A$.) The axiom of extension can, in fact, be reformulated in these terms: if A and B are sets, then a necessary and sufficient condition that $A = B$ is that both $A \subset B$ and $B \subset A$. Correspondingly, almost all proofs of equalities between two sets A and B are split into two parts; first show that $A \subset B$, and then show that $B \subset A$.

Observe that belonging ($\in$) and inclusion ($\subset$) are conceptually very different things indeed. One important difference has already manifested itself above: inclusion is always reflexive, whereas it is not at all clear that belonging is ever reflexive. That is: $A \subset A$ is always true; is $A \in A$ ever true? It is certainly not true of any reasonable set that anyone has ever seen. Observe, along the same lines, that inclusion is transitive, whereas belonging is not. Everyday examples, involving, for instance, super-organizations whose members are organizations, will readily occur to the interested reader.

2 Axiom of specification

All the basic principles of set theory, except only the axiom of extension, are de- signed to make new sets out of old ones. The first and most important of these basic principles of set manufacture says, roughly speaking, that anything intelligent one can assert about the elements of a set specifies a subset, namely, the subset of those elements about which the assertion is true.

Before formulating this principle in exact terms, we look at a heuristic example. Let A be the set of all men. The sentence "x is married" is true for some of the elements x of A and false for others. The principle we are illustrating is the one that justifies the passage from the given set A to the subset (namely, the set of all married men) specified by the given sentence. To indicate the generation of the subset, it is usually denoted by

$$\{x \in A : x \text{ is married}\}.$$

Similarly

$$\{x \in A : x \text{ is not married}\}$$

is the set of all bachelors;

$$\{x \in A : \text{ the father of } x \text{ is Adam}\}$$

is the set that contains Seth, Cain and Abel and nothing else, and

$$\{x \in A : x \text{ is the father of Abel}\}$$

is the set that contains Adam and nothing else. Warning: a box that contains a hat and nothing else is not the same thing as a hat, and, in the same way, the last set in this list of examples is not to be confused with Adam. The analogy between sets and boxes has many weak points, but sometimes it gives a helpful picture of the facts.

All that is lacking for the precise general formulation that underlies the examples above is a definition of *sentence*. Here is a quick and informal one. There are two basic types of sentences, namely, assertions of belonging,

$$x \in A$$

and assertions of equality,

$$A = B$$

all other sentences are obtained from such *atomic* sentences by repeated applications of the usual logical operators, subject only to the minimal courtesies of grammar and unambiguity. To make the definition more explicit (and longer) it is necessary to append to it a list of the "usual logical operators" and the rules of syntax. An adequate (and, in fact, redundant) list of former contains seven items:

1. *and*,
2. *or (in the sente of "either – or – or both")*,
3. *not*,
4. *if – then – (or implies)*,
5. *if and only if*,
6. *for some (or there exists)*,
7. *for all*.

As for the rules of sentence construction, they can be described as follows.

- Put "not" before a sentence and enclose the result between parentheses. (The reason for parentheses, here and below, is to guarantee unambiguity. Note, incidentally, that they make all other punctuation marks unnecessary. The complete parenthetical equipment that the definition of sentences calls for is rarely needed. We shall always omit as many parentheses as it seems safe to omit without leading to confusion. In normal mathematical practice, to be followed in this book, several different sizes and shapes of parentheses are used, but that is for visual convenience only.)
- Put "and" or "or" or "if and only if" between two sentences and enclose the result between parentheses.
- Replace the dashes in "if – then –" by sentences and enclose the result in parentheses.
- Replace the dash in "for some –" or in "for all –" by a letter, follow the result by a sentence, and enclose the whole in parentheses.

We are now ready to formulate the major principle of set theory, often referred to by its German name *Aussonderungsaxiom*.

> **Axiom of specification.** *To every set A and to every condition $S(x)$ corresponds a set B whose elements are exactly those elements x of A for which $S(x)$ holds.*

A "condition" here is just a sentence. The symbolism is intended to indicate the letter x if *free* in the sentence $S(x)$; that means that x occurs in $S(x)$ at least once without being introduced by one of the phrases "for some x" or "for all x". It is an immediate consequence of the axiom of extension that the axiom of specification determines the set B uniquely. To indicate the way B is obtained from A and from $S(x)$ it is customary to write

$$B = \{x \in A : S(x)\}.$$

To obtain an amusing and instructive application of the axiom of specification, consider, in the role of $S(x)$, the sentence

$$\text{not } (x \in x).$$

It will be convenient, here and throughout, to write "$x \notin A$" instead of "not $(x \in A)$"; in this notation, the role of $S(x)$ is now played by

$$x \notin x.$$

It follows that, whatever the set A may be, if $B = \{x \in A : x \notin x\}$, then, for all y,

$$y \in B \text{ if and only if } (y \in A \text{ and } y \notin y). \qquad (\star)$$

Can it be that $B \in A$? We proceed to prove that the answer is no. Indeed, if $B \in A$, then either $B \in B$ also (unlikely, but not obviously impossible), or else $B \notin B$. If $B \in B$, then, by $(\star)$, the assumption $B \in A$ yields $B \notin B$ – a contradiction. If $B \not inB$, then, by $(\star)$ again, the assumption $B \in A$ yields $B \in B$ – a contradiction. This completes the proof that is impossible, so that we must have $B \notin A$. The most interesting part of this conclusion is that there exists something (namely B) that does not belong to A. The set A in this argument was quite arbitrary. We have proved, in other words, that *nothing contains everything*,

or, more spectacularly

there is no universe.

"Universe" here is used in the sense of "universe of discourse", meaning, in any particular discussion, a set that contains all the objects that enter into that discussion.

In older (pre-axiomatic) approaches to set theory, the existence of universe was takes for granted, and the argument in the preceding paragraph was known as the *Russell's paradox.* The moral is that it is impossible, especially in mathematics, to get something for nothing. To specify a set, it is not enough to pronounce some magic words (which may form a sentence such as "$x \notin z$"); it is necessary also to have at hand a set to whose elements the magic words apply.

3 Unordered pairs

For all that has been said so far, we might have been operating in a vacuum. To give the discussion some substance, let us now officially assume that

there exists a set.

Since later on we shall formulate a deeper and more useful existential assumption, this assumption plays a temporary role only. One consequence of this innocuous seeming assumption is that there exists a set without any elements at all. Indeed, if A is a set, apply the axiom of specification to A with the sentence "$x \neq x$". The result is the set $\{x \in A : x \neq x\}$, and that set, clearly, has no elements. The axiom of extension implies that there can be only one set with no elements. The usual symbol for that set is

$$\emptyset;$$

the set is called *empty set*. The empty set is a subset of every set, or, in other words, $\emptyset \subset A$ for every A. To establish this, we might argue as follows. It is to be proved that every element in $\emptyset$ belongs to A; since there are no elements in $\emptyset$, the condition is automatically fulfilled. The reasoning is correct but perhaps unsatisfying. Since it is a typical example of frequent phenomenon, a condition holding in the "vacuous" sense, a word of advice to the inexperienced reader might be in order. To prove that something is true about the empty set, prove that it cannot be false. How, for instance, could it be false that $\emptyset \subset A$? It could be false only if $\emptyset$ had an element that did not belong to A. Since $\emptyset$ has no elements at all, this is absurd. Conclusion: $\emptyset \subset A$ is not false.

The set theory developed so far is still a pretty poor thing; for all we know there is only one set and that one is empty. Are there enough sets to ensure that every set is an element of some set? Is it true that for any two sets there is a third one that they both belong to? What about three sets, or four, or any number? We need a new principle of set construction to resolve such questions. The following principle is a good beginning.

Axiom of pairing. *For any two sets there exists a set that they both belong to.*

Note that this is just the affirmative answer to the second question above. To reassure worriers, let us hasten to observe that words such as "two", "three", and "four" used above, do not refer to the mathematical concepts bearing those names, which will be defined later; at present such words are merely the ordinary linguistic abbreviations for "something and then something else" repeated an appropriate number of times. Thus, for instance, the axiom of pairing, in unabbreviated form, says that if a and b are sets, then there exists a set A such that $a \in A$ and $b \in A$.

One consequence (in fact an equivalent formulation) of the axiom of pairing is that for any two sets there exists a set that contains both of them and nothing else. Indeed, if a and b are sets, and if A is a set such that $a \in A$ and $b \in A$, then we can apply the axiom of specification to A with the sentence "$x = a$ or $x = b$". The result is the set

$$\{x \in A : x = a \text{ or } x = b\},$$

and that set, clearly, contains just a and b. The axiom of extension implies that there can be only one set with this property. The usual symbol for that set is

$$\{a, b\};$$

the set is called the *pair* (or, by emphatic comparison with a subsequent concept, the *unordered pair*) formed by a and b.

If, temporarily, we refer to the sentence "$x = a$ or $x = b$" as $S(x)$, we may express the axiom of pairing by saying that there exists a set B such that

$$x \in B \text{ if and only if } S(x). \tag{$\star$}$$

The axiom of specification, applied to a set A, asserts the existence of a set B such that

$$x \in B \text{ if and only if } (x \in A \text{ and } S(x)). \tag{$\star\star$}$$

The relation between ($\star$) and ($\star\star$) typifies something that occurs quite frequently. All the remaining principles of set construction are pseudo-special cases of the axiom of specification in the sense which ($\star$) is a pseudo-special case of ($\star\star$). They all assert the existence of a set specified by a certain condition; if it were known in advance that there exists a set containing all t he specified elements, then the existence of a set containing just them would indeed follow as a special case of the axiom of specification.

If a is a set, we may form the unordered pair $\{a, a\}$. That unordered pair is denoted by

$$\{a\}$$

and is called the *singleton* of a; it is uniquely characterized by the statement that it has a as its only element. Thus, for instance, $\emptyset$ and $\{\emptyset\}$ are very different sets; the former has no elements, whereas the latter has the unique element $\emptyset$. To say that $a \in A$ is equivalent to saying that $\{a\} \subset A$.

The axiom of pairing ensures that every set is an element of some set and that any two sets are simultaneously elements of some one and the same set. (The corresponding questions for three and four and more sets will be answered later.) Another pertinent comment is that from the assumptions we have made so far we can infer the existence of very many sets indeed. For examples consider the sets $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, $\{\{\{\emptyset\}\}\}$, etc.; consider the pairs, such as $\{\emptyset, \{\emptyset\}\}$, formed by any two of them; consider the pairs formed by any two such pairs, or else the mixed pairs formed by any singleton and any pair; proceed so on ad infinitum.

Exercise. Are all the sets obtained in this way distinct from one another?

Before continuing our study of set theory, we pause for a moment to discuss a notational matter. It seems natural to denote the set B described in ($\star$) by $\{x : S(x)\}$ in the special case that was there considered

$$\{x : x = a \text{ or } x = b\} = \{a, b\}.$$

We shall use this symbolism whenever it is convenient and permissible to do so. If, that is, $S(x)$ is a condition on x such that the x's that $S(x)$ specifies constitute a set, then we may denote that set by

$$\{x : S(x)\}.$$

In case A is a set and $S(x)$ is $(x \in A)$, then it is permissible to form $\{x : S(x)\}$; in fact

$$\{x : x \in A\} = A.$$

If A is a set and $S(x)$ is an arbitrary sentence, it is permissible to form $\{x : xinA \text{ and } S(x)\}$; this set is the same as $\{x \in A : S(x)\}$. As further examples, we note that

$$\{x : x \neq x\} = \emptyset$$

and

$$\{x : x = a\} = \{a\}.$$

In case $S(x)$ is $(x \notin x)$, or in case $S(x)$ is $(x = x)$, the specified x's do not constitute a set.

Despite the maxim about never getting something for nothing, it seems a little harsh to be told that certain sets are not really sets and even their names must never be mentioned. Some approaches to set theory try to soften the blow by making systematic use of such illegal sets but just not calling them sets; the customary word is ”class”. A precise explanation of what classes really are nd how they are used is irrelevant in the present approach. Roughly speaking, a class may be identified with a condition (sentence), or, rather, with the ”extension” of a condition.

4 Unions and intersections

If A and B are sets, it is sometimes natural to wish to unite their elements into one comprehensive set. One way of describing such a comprehensive set is to require it to contain all the elements that belong to at least one of the two members of the pair $\{A, B\}$. This formulation suggests a sweeping generalization of itself; surely a similar construction should apply to arbitrary collections of sets and not just to pairs of them. What is wanted, in other words, is the following principle of set construction.

Axiom of unions. For every collection of sets there exists a set that contains all the elements that belong to at least one set of the given collection.

Here it is again: for every collection $\mathcal{C}$ there exists a set U such that if $x \in X$ for some $X \in \mathcal{C}$, then $x \in U$. (Note that "at least one" is the same as "some").

The comprehensive set U described above may be too comprehensive; it may contain elements that belong to none of the sets X in the collection $\mathcal{C}$. This is easy to remedy; just apply the axiom of specification to form the set

$$\{x \in U : x \in X \text{ for some } X \in \mathcal{C}\}.$$

(The condition here is a translation into idiomatic usage of the mathematically more acceptable "for some X ($x \in X$ and $X \in \mathcal{C}$)".) It follows that, for every x, a necessary and sufficient condition that x belong to this set is that x belong to X for some X in $\mathcal{C}$. If we change notation and call the new set U again, then

$$U = \{x : x \in X \text{ for some } X \in \mathcal{C}\}.$$

This set U is called the *union* of the collection $\mathcal{C}$ of sets; note that the axiom of extension guarantees its uniqueness. The simplest symbol for U that is in use at all is not very popular in mathematical circles; it is

$$\bigcup \mathcal{C}.$$

Most mathematicians prefer something like

$$\bigcup \{X : X \in \mathcal{C}\}$$

or

$$\bigcup_{X \in \mathcal{C}} X.$$

Further alternatives are available in certain important cases; they will be described in due course.

For the time begin we restrict our study of the theory of union to the simplest facts only. The simplest fact of all is that

$$\bigcup_{X \in \emptyset} X = \emptyset,$$

and the next simplest fact is that

$$\bigcup_{X \in \{A\}} = A.$$

The proofs are immediate from the definitions.

There is a little more substance in the union of pairs of sets (which is what started this whole discussion anyway). In that case of special notation is used:

$$\bigcup_{X \in \{A, B\}} X = A \cup B.$$

The general definition of unions implies in the special case that $x \in A \cup B$ if and only if x belongs to either A or B or both; it follows that

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

Here are some easily proved facts about the unions of pairs.

- $A \cup \emptyset = A$.

  Editor's proof. If $x \in A \cup \emptyset$ then $x \in A$ or $x \in \emptyset$. Since $x \in \emptyset$ is always false, we get $x \in A$, therefore $A \cup \emptyset \subset A$. If $x \in A$, then obviously $x \in A$ or $x \in \emptyset$, therefore $A \subset A \cup \emptyset$. We get $A \subset \emptyset = A$.

- $A \cup B = B \cup A$ (commutativity).

  Editor's proof.
  $$\{x : x \in A \text{ or } x \in B\} = \{x : x \in B \text{ or } x \in A\}.$$

- $A \cup (B \cup C) = (A \cup B) \cup C$ (associativity).

  Editor's proof.
  $$\{x : x \in A \text{ or } (x \in B \text{ or } x \in C)\} = \{x : (x \in A \text{ or } x \in B) \text{ or } x \in C\}.$$

- $A \cup A = A$ (idempotence).

  Editor's proof.
  $$\bigcup_{X \in \{A\}} = A.$$

- $A \subset B$ if and only if $A \cup B = B$.

  Editor's proof. Assume $A \subset B$. From the definition $B \subset A \cup B$. Propose that $A \cup B \subset B$ is false. Then there exists an element $x \in A \cup B$ so that $x \notin B$. This x must be in A, but due to our assumption for every $y \in A$, $y \in B$ is also true. Therefore $A \cup B = B$.

  Now assume that $A \cup B = B$. Propose that there exist an element $x \in A$, so that $x \notin B$. By definition $x \in A \cup B$. But $A \cup B \subset B \not\ni x$, so $A \cup B = B$ cannot be true.

Every student of mathematics should prove these things for himself at least once in his life. The proofs are based on the corresponding elementary properties of the logical operator *or*.

An equally simple but quite suggestive fact is that

$$\{a\} \cup \{b\} = \{a, b\}.$$

What this suggests is the way to generalize pairs. Specially, we write

$$\{a, b, c\} = \{a\} \cup \{b\} \cup \{c\}.$$

The equation defines its left side. The right side should by rights have at least one pair of parentheses in it, but, in view of the associative law, their omission can lead to no misunderstanding. Since it is easy to prove that

$$\{a, b, c\} = \{x : x = a \text{ or } x = b \text{ or } x = c\},$$

we know now that for every three sets there exits a set that contains them and nothing else; it is natural to call that uniquely determined set the (*unordered*) *triple* formed by them. The extension of the notation and terminology thus introduced to more terms (*quadruples*, etc.) is obvious.

The formation of unions has many points of similarity with another set-theoretic operation. If A and B are sets, the *intersection* of A and B is the set

$$A \cap B$$

defined by

$$A \cap B = \{x \in A : x \in B\}.$$

The definition is symmetric in A and B even if it looks otherwise; we have

$$A \cap B = \{x \in B : x \in A\},$$

and, in fact, since $x \in A \cap B$ if and only if x belongs to both A and B, it follows that

$$A \cap B = \{x : x \in A \text{ and } x \ \in B\}.$$

The basic facts about intersections, as well as their proofs, are similar to the basic facts about union:

- $A \cap \emptyset = \emptyset$,
- $A \cap B = B \cap A$,
- $A \cap (B \cap C) = (A \cap B) \cap C$,
- $A \cap A = A$,
- $A \subset B$ if and only if $A \cap B = A$.

Pairs of sets with an empty intersection occur frequently enough to justify the use of a special word: if $A \cap B = \emptyset$, the sets A and B are called *disjoint.* The same word is sometimes applied to a collection of sets to indicate that any two distinct sets of the collection are disjoint; alternatively we may speak in such a situation of a *pairwise disjoint* collection.

Two useful facts about unions and intersections involve both the operations at the same time:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

These identities are called the *distributive laws.* By way of a sample of a set-theoretic proof, we prove the second one. If x belongs to the left side, then x belongs either to A or to both B and C; if x is in A, then x is in both $A \cup B$ and $A \cup C$, and if x is in both B and C, then, again x is in both $A \cup B$ and $A \cup C$; it follows that, in any case, x belongs to the right side. This proves that the right side includes the left. To prove the reverse inclusion, just observe that if x belongs to both $A \cup B$ and $A \cup C$, then x belongs either to A or to both B and C.

The formation of the intersection of two sets A and B, or, we might as well say, the formation of the intersection of a pair $\{A, B\}$ of sets, is a special case of a much more general operation. (This is another respect in which the theory of intersections imitates that of unions.) The existence of the general operation of intersection depends on the fact that for each non-empty collection of sets there exists a set that contains exactly those elements that belong to every set of the given collection. In other words: for each collection $\mathcal{C}$, other than $\emptyset$, there exists a set V such that $x \in V$ if and only if $x \in X$ for every $X \in \mathcal{C}$. To prove this assertion, let A be any particular set in $\mathcal{C}$ (this step is justified by the fact that $\mathcal{C} \neq \emptyset$) and write

$$V = \{x \in A : x \in X \text{ for every } X \in \mathbb{C}\}.$$

(The condition means "for all X (if $X \in \mathcal{C}$, then $x \in X$)".) The dependence of V on the arbitrary choice of A is illusory; in fact

$$V = \{x : x \in X \text{ for every } X \in \mathcal{C}\}.$$

The set V is called the *intersection* of the collection $\mathcal{C}$ of sets; the axiom of extension guarantees its uniqueness. The customary notation is similar to the one for unions:

$$\bigcap\{X : X \in \mathcal{C}\}$$

or

$$\bigcap_{X \in \mathcal{C}} X.$$

Exercise. A necessary and sufficient condition that $(A \cap B) \cup C = A \cap (B \cup C)$ is that $C \subset A$. Observe that the condition has nothing to do with the set B.

Editor's proof. Assume that A, B, C are sets, and $C \subset A$ is true. Let x belong to the left side of the equation; that is, x belongs to both A and B or x belongs to C. Given our assumption, x is in A either way. Since, the set A contains the right side of the equation, we get $(A \cap B) \cup C \subset A \cap (B \cup C)$. Now, let x be contained in the right side of the equation: $x \in (A \cap B) \cup (A \cap C)$. If $x \in A \cup B$, clearly x belongs to the left side. Lastly, if $x \in A \cap C = C$, then again, x belongs to the left side.

Assume that the equation mentioned in the exercise is true. Since A, B, C are arbitrary sets, let B equal the empty set, so we get

$$(A \cap \emptyset) \cup C = A \cap (\emptyset \cup C),$$

or in an equivalent form

$$C = A \cap C.$$

5 Complements are powers

If A and B are sets, the *difference* between A and B, more often known as the *relative complement* of B in A, is the set $A - B$ defined by

$$A - B = \{x \in A : x \notin B\}.$$

Note that in this definition it is not necessary to assume that $B \subset A$. In order to record the basic facts about complementation as simply as possible, we assume nevertheless (in this section only) that all the sets to be mentioned are subsets of one and the same set E and that all complements (unless otherwise mentioned) are formed relative to that E. In such situations (and they are quite common) it is easier to remember the underlying set E than to keep writing it down, and this makes it possible to simplify the notation. An often used symbol for the temporarily absolute (as opposed to relative) complement of A is A'. In terms if this symbol the basic facts about complementation can be stated as follows:

- $(A')' = A$,
- $\emptyset' = E$, $E' = \emptyset$,
- $A \cap A' = \emptyset$, $A \cup A' = E$,
- $A \subset B$ if and only if $B' \subset A'$.

The most important statements about the complements are the so-called *De Morgan laws*:

$$(A \cup B)' = A' \cap B', \quad (A \cap B)' = A' \cup B'.$$

(We shall see presently that the De Morgan laws hold for the unions and intersections of larger collections of sets than just pairs.) These facts about complementation imply that the theorems of set theory usually come in pairs. If in an inclusion or equation involving unions, intersections, and complements of subsets of E we replace each set by its complement, interchange unions and intersections, and reverse all inclusions, the result is another theorem. This fact is sometimes referred to as the *principle of duality* for sets.

Here are some easy exercises on complementation.

- $A - B = A \cap B'$.

  Editor's proof.
  $$\{x \in A : x \notin B\} = \{x \in A : x \in B'\}.$$

- $A \subset B$ if and only if $A - B = \emptyset$.

  Editor's proof. If $A \subset B$, then $A - B = (A \cap B) - B = (A \cap B) \cap B' = A \cap (B \cap B') = A \cap \emptyset = \emptyset$.

  Assume that both $A - B = \emptyset$ and $A \not\subset B$ are true. Then, there exists an element $x \in A$ such that $x \notin B$. By definition
  $$A - B = \{x \in A : x \notin B\} = \emptyset,$$
  a contradiction.

- $A - (A - B) = A \cap B$.

  Editor's proof.

  $A - (A - B) = A - (A \cap B') = A \cap (A \cap B')' = A \cap (A' \cup B) = (A \cap A') \cup (A \cap B) = A \cap B.$

- $A \cap (B - C) = (A \cap B) - (A \cap C)$

  Editor's proof.

$$(A \cap B) - (A \cap C) = (A \cap B) \cap (A \cap C)' = (A \cap B) \cap (A' \cup C') =$$

$$(A \cap B \cap A') \cup (A \cap B \cap C') = A \cap B \cap C' = A \cap (B \cap C')\ A \cap (B - C).$$

- $A \cap B \subset (A \cap C) \cup (B \cap C')$.

  Editor's proof. According to the distributive laws we get

$$(A \cap C) \cup (B \cap C') =$$

$$\big((A \cap C) \cup B\big) \cap \big((A \cap C) \cup C'\big) =$$

$$\big((A \cap C) \cup B\big) \cap \big((C' \cup A) \cap (C' \cup C)\big) =$$

$$\underbrace{\big((A \cap C) \cup B\big)}_{=:P} \cap \underbrace{(C' \cup A)}_{=:Q}.$$

  Let $x \in A \cap B$. With the pre-defined P, Q sets, we get that $x \in P$ and $x \in Q$.

- $(A \cup C) \cap (B \cup C') \subset A \cup B$.

  Editor's proof. According to the distributive laws we get

$$(A \cup C) \cap (B \cup C') =$$

$$\big((A \cup C) \cap B\big) \cup \big((A \cup C) \cap C'\big) =$$

$$\big((A \cup C) \cap B\big) \cup \big((C' \cap A) \cup (C' \cap C)\big) =$$

$$\underbrace{\big((A \cup C) \cap B\big)}_{=:P} \cup \underbrace{(C' \cap A)}_{=:Q}.$$

  Let $x \in P \cup Q$. If $x \in P$, then $x \in B \subset A \cup B$. On the other way, if $x \in Q$, then $x \in A \subset A \cup B$. Either way, we get that $x \in A \cup B$.

If A and B are sets, the *symmetric difference* (or *Boolean sum*) of A and B is the set $A + B$ defined by

$$A + B = (A - B) \cup (B - A).$$

This operation is commutative ($A + B = B + A$) and associative ($A + (B + C) = (A + B) + C$), and is such that $A + \emptyset = A$ and $A + A = \emptyset$.

This may be the right time to straighten out a trivial but occasionally puzzling part of the theory of intersections. Recall, to begin with, that intersections were defined for non-empty collections only. The reason is that the same approach to the empty collection does not define a set. Which x's are specified by the sentence

$$x \in X \text{ for every } X \in \emptyset?$$

As usual for questions about $\emptyset$ the answer is easier to see for the corresponding negative question. Which x's do *not* satisfy the stated condition? If it is not true that $x \in X$ for every $X \in \emptyset$, then there must exist an X in $\emptyset$ such that $x \notin X$; since, however, there do not exist any X's in $\emptyset$ at all, this is absurd. Conclusion: no x fails to satisfy the stated condition, or, equivalently, every x does satisfy it. In other words, the x's that the condition specifies exhaust the (nonexistent) universe. There is no profound problem here; it is merely a nuisance to be forced always to be making qualifications and exceptions because some set somewhere along some construction might turn out to be empty. There is nothing to be done about this; it is just a fact of life.

If we restrict our attention to subsets of a particular set E, as we have temporarily agreed to do, then the unpleasantness described in the preceding paragraph appears to go away. The point is that in that case we can define the intersection of a collection $\mathcal{C}$ (of subsets of E) to be the set

$$\{x \in E : x \in X \text{ for every } X \in \mathcal{C}\}.$$

This is nothing revolutionary; for each non-empty collection, the new definition agrees with the old one. The difference is in the way the old and the new definitions treat the empty collection; according to the new definition

$$\bigcap_{X \in \emptyset} X = E.$$

(For which elements x of E can it be false that $x \in X$ for every $X \in \emptyset$?) The difference is just a matter of language. A little reflection reveals that the "new" definition offered for the intersection of a collection $\mathbb{C}$ of subsets E is really the same as he old definition of the intersection of the collection $\mathcal{C} \cup \{E\}$, and the latter is never empty.

We have been considering subsets of a set E; do those subsets themselves constitute a set? The following principle guarantees that the answer is yes.

Axiom of powers. *For each set there exists a collection of sets that contains among its elements all the subsets of the given set.*

In other words, if E is a set, then there exists a set (collection) $\mathcal{P}$ such that if $X \subset E$, then $X \in \mathcal{P}$.

The set $\mathcal{P}$ described above maybe larger than wanted; it may contain elements other than the subsets of E. This is easy to remedy; just apply the axiom of specification to form the set $\{X \in \mathcal{P} : X \subset E\}$. (Recall that "$X \subset E$" says the same thing as "for all x (if $x \in X$ then $x \in E$)".) Since, for every X, a necessary and sufficient condition that X belongs to this set is that X be a subset of E, it follows that if we change notation and call this set $\mathcal{P}$ again, then

$$\mathcal{P} = \{X : X \subset E\}.$$

The set $\mathcal{P}$ is called the *power set* of E; the axiom the extension guarantees its uniqueness. The dependence of $\mathcal{P}$ on E is denoted by writing $\mathcal{P}(E)$ instead of just $\mathcal{P}$.

Because the the set $\mathcal{P}(E)$ is very big in comparison with E; it is not easy to give examples. If $E = \emptyset$, the situation is clear enough; the set $\mathcal{P}(\emptyset)$ is the singleton $\{\emptyset\}$. The power sets of singletons and pairs are also easily describable; we have

$$\mathcal{P}(\{a\}) = \{\emptyset, \{a\}\}$$

and

$$\mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a\, b\}\}.$$

The power set of a triple has eight elements. The reader can probably guess (and is hereby challenged to prove) the generalization that includes all these statements: the power set of a finite set with, say, n elements has 2^n elements. (Of course concepts like "finite" and "2^n" have no official standing for us yet; this should not prevent them from being unofficially understood.)

Editor's proof. Let us have a number $n \in \mathbf{N}$, and a set A with length n. May we denote the sets in $\mathcal{P}(A)$ of length $i \in \{1, \ldots, n\}$ with the symbol A^i. Then

$$|\mathcal{P}(A)| = \sum_{i=0}^{n} |A^i| = \sum_{i=0}^{n} \binom{n}{i}.$$

According to the binomial theorem

$$\sum_{i=0}^{n} \binom{n}{i} = \sum_{i=0}^{n} \binom{n}{i} \cdot 1^{n-i} \cdot 1^i = (1+1)^n = 2^n.$$

The occurrence of n as an exponent (the n-th power of 2) has something to do with the reason why a power set bears its name.

If $\mathcal{C}$ is a collection of subsets of a set E (that is, $\mathcal{C}$ is a subcollection of $\mathcal{P}(E)$), then write

$$\mathcal{D} = \{X \in \mathcal{P}(E) : X' \in \mathcal{C}\}.$$

(To be certain that the condition used in the definition of $\mathcal{D}$ is a sentence in the precise technical sense, it must be rewritten in something like the form

$$\text{for some } Y\Big(Y \in \mathcal{C} \text{ and for all } x\big(x \in X \text{ if and only if } (x \in E \text{ and } x \notin Y)\big)\Big).$$

Similar comments often apply when we wish to use defined abbreviations instead of logical and set-theoretic primitives only. The translation rarely requires any ingenuity and we shall usually omit it.) It is customary to denote the union and the intersection of the collection $\mathcal{D}$ by the symbols

$$\bigcup_{X \in \mathcal{C}} X' \quad \text{and} \quad \bigcap_{X \in \mathcal{C}} X'.$$

In this notation the general forms of the De Morgan laws become

$$\left(\bigcup_{X \in \mathcal{C}} X\right)' = \bigcap_{X \in \mathcal{C}} X'$$

and

$$\left(\bigcap_{X \in \mathcal{C}} X\right)' = \bigcup_{X \in \mathcal{C}} X'.$$

The proofs of these equations are immediate consequences of the appropriate definitions.

Exercise. Prove that $\mathcal{P}(E) \cap \mathcal{P}(F) = \mathcal{P}(E \cap F)$.

Editor's proof. Let $X \in \mathcal{P}(E) \cap \mathcal{P}(F)$. Then $X \subset E$ and $X \subset F$, therefore $X \subset E \cap F$, thus $X \in \mathcal{P}(E \cap F)$. Now, let $X \in \mathcal{P}(E \cap F)$, we get $X \subset E \cap F$, that is $X \subset E$ and $X \subset F$, therefore $X \in \mathcal{P}(E) \cap \mathcal{P}(F)$.

Exercise. Prove that $\mathcal{P}(E) \cup \mathcal{P}(F) \subset \mathcal{P}(E \cup F)$.

Editor's proof. Let $X \in \mathcal{P}(E) \cup \mathcal{P}(F)$. That means either $X \subset E$ or $X \subset F$ is true. Since $X \subset E \cup F$, we get $X \in \mathcal{P}(E \cup F)$.

These assertions can be generalized to

$$\bigcap_{X\in\mathcal{C}} \mathcal{P}(X) = \mathcal{P}\left(\bigcap_{X\in\mathcal{C}} X\right)$$

and

$$\bigcup_{X\in\mathcal{C}} \mathcal{P}(X) \subset \mathcal{P}\left(\bigcup_{X\in\mathcal{C}} X\right).$$

Further elementary facts:

$$\bigcap_{X\in\mathcal{P}(E)} X = \emptyset,$$

and

$$\text{if } E \subset F, \text{ then } \mathcal{P}(E) \subset \mathcal{P}(F).$$

A curious question concerns the commutativity of the operators $\mathcal{P}$ and $\cup$.

> **Exercise.** Show that E is always equal to
>
> $$\bigcup_{X\in\mathcal{P}(E)} X$$
>
> (that is $E = \bigcup \mathcal{P}(E)$), but that the result of applying $\mathcal{P}$ and $\bigcup$ to E in the other order is a set that includes E as a subset, typically a proper subset.

Editor's proof. Let $x \in E$. Since $E \subset E$ ($E \in \mathcal{P}(E)$), we get

$$x \in \bigcup_{X\in\mathcal{P}(E)} X.$$

Now let

$$x \in \bigcup_{X\in\mathcal{P}(E)} X.$$

An equivalent formulation is the following: x belongs to the preceding set if and only if there exists a set $A \subset E$ so that $x \in A$. The first half of the exercise is proved.

Before proving the next statement, let us observe the following: given a set A, and one of its element $a \in A$, then $a \in \mathcal{P}(a)$; given another element $b \in A$, we again, get $b \in \mathcal{P}(b)$. The statements $a, b \in \mathcal{P}(a \cup b)$ are also true. This can be generalized the following way

$$A \subset \mathcal{P}\left(\bigcup_{x\in A} x\right).$$

In order to prove the last bit ("typically a proper subset"), first lets use the notation $\mathcal{X}$ instead of E. We may notice that there may be a set X in $\mathcal{X}$ such that X contains an element x whose singleton $\{x\}$ does not belong to the "big" set $\mathcal{X}$.

However when we apply the union operator to the set $\mathcal{X}$:

$$\bigcup_{X\in\mathcal{X}} X,$$

we get a new set that contains x. After using the power operator:

$$\mathcal{P}\left(\bigcup_{X\in\mathcal{X}} X\right),$$

the singleton $\{x\}$ will be part of the resulting set, since

$$\{x\} \subset \bigcup_{X \in \mathcal{X}} X.$$

6 Ordered pairs

What does it mean to arrange the elements of a set A in some order? Suppose, for instance, that the set A is the quadruple $\{a, b, c, d\}$ of distinct elements, and suppose that we want to consider its elements in the order

$$c\,b\,d\,a.$$

Even without a precise definition of what this means, we can do something set-theoretically intelligent with it. We can, namely, consider, for each particular spot in the ordering, the set of all those elements that occur at or before that spot; we obtain in this way the sets

$$\{c\}\,\{c, b\}\,\{c, b, d\}\,\{c, b, d, a\}.$$

We can go on then to consider the set (or collection, if that sounds better)

$$\mathcal{C} = \{\{c\}\,\{c, b\}\,\{c, b, d\}\,\{c, b, d, a\}\}$$

that has exactly those sets for its elements. In order to emphasize that the intuitively based and possibly unclear concept of order has succeeded producing something solid and simple, namely plain, unembellished set $\mathcal{C}$, the elements of $\mathcal{C}$, and *their* elements, are presented above in a scrambled manner. (The lexicographically inclined reader might be able to see a method in the manner of scrambling.)

Let us continue to pretend for a while that we do know what order means. Suppose that in a hasty glance at the preceding paragraph all we could catch is the set $\mathcal{C}$; can we use it to recapture the order that gave rise to it? The answer is easily seen to be yes. Examine the elements of $\mathcal{C}$ (they themselves are sets, of course) to find one that is included in all the others; since $\{c\}$ fills the bill (and nothing else does) we know that c must have been the first element. Look next for the next smallest element of $\mathcal{C}$, i.e., the one that is included in all the ones that remain after $\{c\}$ is removed; since $\{b, c\}$ fills the bills (and nothing else does), we know that b must have been the second element. Proceeding thus (only two more steps are needed) we pass from the set $\mathcal{C}$ to the given ordering of the given set A.

The mores is this: we may not know precisely what it means to order the elements of a set A, bu with each order we can associate a set $\mathcal{C}$ of subsets of A in such a way that the given order can be uniquely recaptured from $\mathcal{C}$. (Here is a non-trivial exercise: find an intrinsic characterization of those sets of subsets of A that correspond to some order in A. Since "order" has no official meaning for us yet, the whole problem is officially meaningless. Nothing that follows depends on the solution, but t he reader would learn something valuable trying to find it.) The passage from an order in A to the set $\mathcal{C}$, and back, we illustrated above for a quadruple; for a pair everything becomes at least twice as simple. If $A = \{a, b\}$ and if, in the desired order, a comes first, then $\mathcal{C} = \{\{a\}, \{a, b\}\}$; if, however, b comes first, then $\mathcal{C} = \{\{b\}, \{a, b\}\}$.

The *order pair* of a and b, with *first coordinate* a and *seconds coordinate* b, is the set (a, b) defined by

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

However convincing the motivation of this definition may be, we must still prove that the result has the main property that an ordered pair must have to deserve its name. We must show that if (a, b) and (x, y) are ordered pairs and if $(a, b) = (x, y)$, then $a = x$ and $b = y$. To prove this, we note first that if a and b happen to be equal, then the ordered pair (a, b) is the same as the singleton $\{\{a\}\}$. If, conversely, (a, b) is a singleton, then $\{a\} = \{a, b\}$, so that $b \in \{a\}$, and therefore $a = b$. Suppose now that $(a, b) = (x, y)$. If $a = b$, then both (a, b) and (x, y) are singletons, so that $x = y$; since $\{x\} \in (a, b)$ and $\{a\} \in (x, y)$, it follows that a, b, x, and y are all equal. If $a \neq b$, then both (a, b) and (x, y) contain exactly one singleton, namely $\{a\}$ and $\{x\}$

respectively, so that $a = x$. Since in this case it is also true that both (a, b) and (x, y) contain exactly one unordered pair that is not a singleton, namely $\{a, b\}$ and $\{x, y\}$ respectively, it follows that $\{a, b\} = \{x, y\}$, and therefore, in particular, $b \in \{x, y\}$. Since b cannot be x (for then we should have $a = x$ and $b = x$, and, therefore, $a = b$), we must have $b = y$, and the proof is complete.

If A and B are sets, does there exists a set that contains all the unordered pairs (a, b) with a in A and b in B? It is quite easy to see that the answer is yes. Indeed, if $a \in A$ and $b \in B$, then $\{a\} \subset A$ and $\{b\} \subset B$, and therefore $\{a, b\} \subset A \cup B$. Since also $\{a\} \subset A \cup B$, it follows that both $\{a\}$ and $\{a, b\}$ are elements of $\mathcal{P}(A \cup B)$. This implies that $\{\{a\}, \{a, b\}\}$ is a subset of $\mathcal{P}(A \cup B)$, and hence that it is and element of $\mathcal{P}(\mathcal{P}(A \cup B))$; in other words $(a, b) \in \mathcal{P}(\mathcal{P}(A \cup B))$ whenever $a \in A$ and $b \in B$. Once this is known, it is a routine matter to apply the axiom of specification and the axiom of extension to produce the unique set $A \times B$ that consists exactly of the ordered pairs (a, b) with a in A and b in B. This set is called the *Cartesian product* of A and B; it is characterized by the fact that

$$A \times B = \{x : x = (a, b) \text{ for some } a \text{ in } A \text{ and for some } b \text{ in } B\}.$$

The Cartesian product of two sets is a set of ordered pairs (that is, a set each of whose elements is an ordered pair), and the same is true for every subset of a Cartesian product. It is of technical importance to know that we can go in the converse direction also: every set of ordered pairs is a subset of the Cartesian product of two sets. In other words: if R is a set such that every element of R is an ordered pair, then there exists two sets A and B such that $R \subset A \times B$. The proof is elementary. Suppose indeed that $x \in R$, so that $x = \{\{a\}, \{a, b\}\}$ for some a and for some b. The problem is to dig out a and b from under the braces. Since the elements of R are sets, we can form the union of the sets ni R; since x is one of the sets in R, the elements of x belong to that union. Since $\{a, b\}$ is one of the elements of x, we may write, in what has been called the brutal notation above, $\{a, b\} \in \bigcup R$. One set of braces has disappeared; let us do the same thing again to make the other set go away. Form the union of the sets in $\bigcup R$. Since $\{a, b\}$ is one of those sets, it follows that the elements of $\{a, b\}$ belongs to that union, and hence both a and b belongs to $\bigcup\bigcup R$. This fulfills the promise made above; to exhibit R as a subset of some $A \times B$, we may take both A and B to be $\bigcup\bigcup R$. It is often desirable to take A and B as small as possible. To do so, just apply the axiom of specification to produce the sets

$$A = \{a : \text{ for some } b\,((a, b) \in R)\}$$

and

$$B = \{b : \text{ for some } a\,((a, b) \in R)\}.$$

These sets are called *projections* of R onto the first and second coordinates respectively.

However important set theory may be now, when it began some scholars considered it a disease from which, it was to be hoped, mathematics would soon recover. For this reason many set-theoretic considerations were called pathological, and the word lives on in mathematical usage; it often refers to something the speaker does not like. The explicit definition of an ordered pair ($(a, b) = \{\{a\}, \{a, b\}\}$) is frequently relegated to pathological set theory. For the benefit of those who think that in this case the name is deserved, we note that the definition has served its purpose by now and will never be used again. We need to know that ordered pairs are determined by and uniquely determine their first and second coordinates, the Cartesian products can be formed, and that every set of ordered pairs is a subset of some Cartesian products; which particular approach is used to achieve these ends is immaterial.

It is easy to locate the source of the mistrust and suspicion that many mathematicians feel toward the explicit definition of ordered pair given above. The trouble is not that there is anything wrong or anything missing; the relevant properties of the concept we defined are all correct (that is, in accord with the demands of intuition) and all the correct properties are present. The trouble is that the concept has some irrelevant properties that tare accidental and distracting. The theorem that $(a, b) = (x, y)$ if and only if $a = x$ and $b = y$ is the sort of thing we expect to learn about ordered pairs. The fact that $\{a, b\} \in (a, b)$, on the other hand, seems accidental; it is a freak property of the definition rather than an intrinsic property of the concept.

The charge of artificiality is true; but it is not too high a price to pay for conceptual economy. The concept of an ordered pair could have been introduced as an additional primitive, axiomatically endowed with just the right properties, no more and no less. In some theories this is done. The mathematician's choice is between having to remember a few more axioms and having to forget a few accidental facts; the choice is pretty clearly a matter of taste. Similar choices occur frequently in mathematics; in this book, for instance, we shall encounter them again in connection with the definition of numbers of various kinds.

Exercise. If A, B, X, and Y are sets, then

- $(A \cup B) \times X = (A \times X) \cup (B \times X)$,
- $(A \cap B) \times (X \cap Y) = (A \times X) \cap (B \times Y)$,
- $(A - B) \times X = (A \times X) - (B \times X)$.

If either $A = \emptyset$ or $B = \emptyset$, then $A \times B = \emptyset$, and conversely. If $A \subset X$ and $B \subset Y$, then $A \times B \subset X \times Y$, and (provided $A \times B \neq \emptyset$) conversely.

Editor's proof. Assume
$$x \in (A \cup B) \times X;$$
we may write $x = (a, b)$, such that, $a \in A$ or $a \in B$, and $b \in X$. If $a \in A$ and $b \in X$, then $(a, b) \in A \times X$, and secondly, $a \in B$ and $b \in X$ gives us $(a, b) \in B \times X$; we have $x \in (A \times X) \cup (B \times X)$. Now, using the same notations, let us say that $x = (a, b) \in (A \times X) \cup (B \times X)$. We get $a \in A$ or $a \in B$, $b \in X$. The first proof is complete.

Assume
$$x = (a, b) \in (A \cap B) \times (X \cap Y);$$
then a is in both A and B, b is in both X and Y. In order for x to be in the set on the right side of the equation, x has to be both in $A \times X$ and $B \times Y$. Since $a \in A$ and $b \in X$ we get $x \in A \times X$. The same is true for the remaining sets: $x \in B \times Y$. We get $x \in (A \times X) \cap (B \times Y)$. Now, let us see the reverse direction, that is, suppose $x = (a, b) \in (A \times X) \cap (B \times Y)$. That is $(a, b) \in A \times X$ and $(a, b) \in B \times Y$. We get $a \in A$ and $a \in B$, therefore $a \in A \cap B$, and analogously $b \in B \times Y$, so $x \in (A \cup B) \times (B \times Y)$.

Assume
$$x = (a, b) \in (A - B) \times X;$$
we have $a \in A$ but $a \notin B$ and $b \in X$. Assume the contrary: $x \notin (A \times X) - (B \times X)$. We get $x \notin (A \times X) \cap (B \times X)'$, therefore $x \in \big((A \times X) \cap (B \times X)'\big)' = (A \times X)' \cup (B \times X)$; that is, $x \notin A \times X$ or $x \in B \times X$. If $x \notin A \times X$, then $a \notin A$ or $b \notin X$, which cannot be. On the other side, if $x \in B \times X$, then $a \in B$, which is also false. Now let us assume that $x \in (A \times X) - (B \times X)$, that is $x \in (A \times X) \cap (B \times X)'$. We get $(a, b) \in A \times X$ and $(a, b) \notin B \times X$. That is $a \in A$ and $b \in X$ and either $a \notin B$ or $b \notin X$. Since $b \in X$, $b \notin X$ can never be true. This means x is involved in the set of $(A \times X) - (B \times X)$ if and only if $a \in A$ and $a \notin B$ and $b \in X$.

7 Relations

Using ordered pairs, we can formulate the mathematical theory of relations in set-theoretic language. By a relation we mean here something like marriage (between men and women) belonging (between elements and sets). More explicitly, what we shall call a relation is sometimes called a *binary* relation. An example of a ternary relation is parenthood for people (Adam and Eve are the parents of Cain). In this book we shall have no occasion to treat the theory of relations that are ternary, quaternary, or worse.

Looking at any specific relation, such as marriage for instance, we might be tempted to consider certain ordered pairs (x, y), namely just those for which x is man, y is a woman, and x is married to y. We have not yet seen the definition of the general concept of a relation, but it seems plausible that, just as in this marriage example, every relation should uniquely determine the set of all those ordered pairs for which the first coordinate does stand in that relation to the second. If we know the relation, we know the set, and, better yet, if we know the set, we know the relation. If, for instance, we were presented with the set of ordered pairs of people that corresponds to marriage, then, even if we forgot the definition of marriage, we could always tell when a man x is married to a woman y and when not; we would just have to see whether the ordered pair (x, y) does or does not belong to the set

We may not know what a relation is, but we do know what a set is, and the preceding considerations establish a close connection between relations and sets. The precise set-theoretic treatment of relations takes advantage of that heuristic connection; the simplest to do is to define a relation to be the corresponding set. This is what we do; we hereby define a *relation* as a set of ordered pairs. Explicitly: a set R is a relation if each element of R is an ordered pair; this means, of course, that if $z \in R$, then there exists x and y so that $z = (x, y)$. If R is a relation, it is sometimes convenient to express the fact that $(x, y) \in R$ by writing

$$xRy$$

and saying, as in everyday language, that x stands in the relation R to y.

The least exciting relation is the empty one. (To prove that $\emptyset$ is a set of ordered pairs, look for an element of $\emptyset$ that is not an ordered pair.) Another dull example is the Cartesian product of any two sets X and Y. Here is a slightly more interesting example: let X be any set, and let R be the set of all those pairs (x, y) in $X \times X$ for which $x = y$. The relation R is just the relation of equality between elements of X; if x and y are in X; then xRy mean the same as $x = y$. One more example will suffice for now: let X be any set, and let R be the set of all those pairs (x, A) in $X \times \mathcal{P}(X)$ for which $x \in A$. This relation R is just the relation of belonging between elements of X and subsets of X; if $x \in X$ and $A \in \mathcal{P}(X)$, then xRA means the same as $x \in A$.

In the preceding section we saw that associated with every set R of ordered pairs there are two sets called the projections of R onto the first and second coordinates. In the theory of relations these sets are known as the *domain* and the *range* of R (abbreviated dom R and ran R); we recall that they are defined by

$$\operatorname{dom} R = \{x : \text{ for some } y\,(xRy)\}$$

and

$$\operatorname{ran} R = \{y : \text{ for some } x\,(xRy)\}.$$

If R is the relation of marriage, so that xRy means that x is a man, y is a woman, and x and y are married to one another; then $\operatorname{dom} R$ is the set of married men and $\operatorname{ran} R$ is the set of married women. Both the domain and the range of $\emptyset$ are equal to $\emptyset$. If $R = X \times Y$, then $\operatorname{dom} R = X$ and $\operatorname{ran} R = Y$. If R is equality in X, then $\operatorname{dom} R = \operatorname{ran} R = X$. If R is belonging, between X and $\mathcal{P}(x)$, then $\operatorname{dom} R = X$ and $\operatorname{ran} R = \mathcal{P}(X) - \{\emptyset\}$.

If R is a relation included in a Cartesian product $X \times Y$ (so that $\operatorname{dom} R \subset X$ and $\operatorname{ran} R \subset Y$), it is sometimes convenient to say that R is a relation *from* X *to* Y; instead of a relation X to X we may speak of a relation *in* X. A relation R in X is *reflexive* if xRx for every x in X; it is *symmetric* if xRy implies that yRx; and it is *transitive* if xRy and yRx imply that xRz.

Exercise. For each of these three possible properties, find a relation that does not have that property but does have the other two.

If we have a set $X = \{a, b, c\}$, then

- R_1 is not reflexive, but is symmetric and transitive if
$$R_1 = \emptyset;$$
- R_2 is not symmetric, but is reflexive and transitive if
$$R_2 = \{(a, a), (b, b), (c, c), (a, b)\};$$
- R_3 is not transitive, but is reflexive and symmetric if
$$R_3 = \{(a, a), (b, b), (c, c), (a, b), (b, a), (b, c), (c, b)\}.$$

A relation in a set is an *equivalence relation* if it is reflexive, symmetric, and transitive. The smallest equivalence relation in a set X is the relation of equality in X; the largest equivalence relation in X is $X \times X$.

There is an intimate connection between equivalence relations in a set X and certain collections (called partitions) of subsets of X. A *partition* of X is a disjoint collection $\mathcal{C}$ of non-empty subsets of X whose union is X. If R is an equivalence relation in X, and if x is in X, the *equivalence class* of x with respect to R is the set of all those elements y in X for which xRy. (The weight of tradition makes the use of the word "class" at this point unavoidable.) Examples: if R is equality in X, then each equivalence class is a singleton; if $R = X \times X$, then the set X itself is the only equivalence class. There is no standard notation for the equivalence class of x with respect to R; we shall usually denote it by x/R, and we shall write X/R for the set of all equivalence classes. (Pronounce X/R as "X modulo R", or, in abbreviated form, "X mod R".)

Exercise. Show that X/R is indeed a set by exhibiting a condition that specifies exactly the subset X/R of the power set $\mathcal{P}(X)$.

Editor's proof. The condition S_1 guarantees that if $A \in \mathcal{P}(X)$, then all the elements of A are in relation:
$$S_1(x) = \text{for every } x\,(x \in A \text{ and (for every } y\,(y \in A \text{ and } xRy))).$$
The condition S_2 guarantees that if $x \in X$ and xRy for some $y \in A$ then $x \in A$:
$$S_2(x) = \text{for every } x\,(x \in X \text{ and (if for some } y\,(y \in A \text{ and } xRy) \text{ then } x \in A).$$
$$X/R = \{C \in \mathcal{P}(X) : C \neq \emptyset \text{ and for every } x\,(x \in C \text{ and } (S_1(x) \text{ and } S_2(x)))\}.$$

Now forget R for a moment and begin anew with partition $\mathcal{C}$ of X. A relation, which we shall call $X/\mathcal{C}$, is defined in X by writing
$$xX/\mathcal{C}y$$

just in case x and y belong to the same set of the collection $\mathcal{C}$. We shall call $X/\mathcal{C}$ the relation *induced* by the partition $\mathcal{C}$.

In the preceding paragraph we saw how to associate a set of subsets of X with every equivalence relation in X and how to associate a relation in X with every partition of X. The connection between equivalence relations and partitions can be described by saying that the passage from $\mathcal{C}$ to $X/\mathcal{C}$ is exactly the reverse of the passage from R to X/R. More explicitly: if R is an equivalence relation in X, then the set of equivalence classes is a partition of X that induces the relation R, and if $\mathcal{C}$ is a partition of X, then the induced relation is an equivalence relation whose set of equivalence classes is exactly $\mathcal{C}$.

For the proof, let us start with an equivalence relation R. Since each x belongs to some equivalence class (for instance $x \in x/R$), it is clear that the union of the equivalence classes is all X. If $z \in x/R \cap y/R$, then xRz and zRy, and therefore xRy. This implies that if two equivalence classes have an element in common, then they are identical, or, in other words, that two distinct equivalence classes are always disjoint. The set of equivalence classes is therefore a partition. To say that two elements belongs to the same set (equivalence class) of this this partition means, by definition, that stand in the relation R to one another. This proves the first half of our assertion.

The second half is easier. Start partition $\mathcal{C}$ and consider the induced relation. Since every element of X belongs to some set of $\mathcal{C}$, reflexivity just says that x and x are in the same set of $\mathcal{C}$. Symmetry says that if x and y are in the same set of $\mathcal{C}$, then y and x are in the same set of $\mathcal{C}$, and this is obviously true. Transitivity says that if x and y are in the same set of $\mathcal{C}$ and if y and z are in the same set of $\mathcal{C}$, then x and z are in the same set of $\mathcal{C}$, and this too is obvious. The equivalence class of each x in X is just the set of $\mathcal{C}$ which x belongs. This completes the proof of everything that was promised.