Vibrating string

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The entire derivation of the theory and more detailed documentation can be found in the following book, on which this note is also heavily based:

[1] Simon Péter Lajos (2012) *Introduction to Analysis 2*, Eötvös Loránd University, Budapest, ISBN: 9789633122570.

1 History and theory

The task is to determine the shape of a homogeneous vibrating string fixed at both ends. Let us assume that the string has a length $l \in \mathbf{R}^+$ and vibrates in the transverse plane. By selecting an appropriate coordinate system, the shape of the string can be expressed as:

$$u : [0,\, l] \times \mathbf{R}_0^+ \to \mathbf{R}$$

where $u(x,t) \in \mathbf{R}$ $\big((x,\, t) \in [0,l] \times \mathbf{R}_0^+\big)$ refers to the displacement of the string at point x at time t.

If $u \in D^2$ (u is twice differentiable), then (based on physical considerations) it satisfies the partial differential equation with the constant $q \in \mathbf{R}^+$:

$$\partial_{22} u = q \cdot \partial_{11} u,$$

which is a (special) partial differential equation. Given the initial shape of the string and its velocity $\big(u(x,0)$ and $\partial_2 u(x,0)$ for $x \in [0,l]\big)$, the function u can be determined.

The results obtained from ordinary differential equations can often be successfully applied when solving partial differential equations. Thus, the above equation can also model the motion of a vibrating string. Let $u \in C^2$ (u is twice continously differentiable) and given

$$f,\, g \in \mathbf{R} \to \mathbf{R}$$

functions, such that they satisfy the following boundary and initial conditions:

$$\begin{aligned}
&(\mathbf{1}) \quad u(0,\, t) = u(l,\, t) = 0 \quad && (t \in \mathbf{R}_0^+),\\
&(\mathbf{2}) \quad u(x,\, 0) = f(x) && (x \in [0,\, l]),\\
&(\mathbf{3}) \quad \partial_2 u(x,\, 0) = g(x) && (x \in [0,\, l]).
\end{aligned}$$

Based on the work of *Euler*, *Lagrange*, *D'Alembert*, *D. Bernoulli*, and finally *Fourier*, the following method was refined (applicable to many similar problems). The fundamental idea is to seek the solution u in the form:

$$u(x,t) = F(x) \cdot G(t) \quad \big((x,\, t) \in [0,\, l] \times \mathbf{R}_0^+\big)$$

(known as the *Fourier method*), where $F, G \in \mathbf{R} \to \mathbf{R}$ are (suitable) functions that are twice continuously differentiable. By omitting the details, this problem can be reduced to a homogeneous linear second-order differential equation with constant coefficients.

It can be easily verified that for any $n \in \mathbf{N}$ and parameters $a_n, b_n, d_n \in \mathbf{R}$, the functions, where $(x,\, t) \in [0,\, l] \times \mathbf{R}_0^+$:

$$u_n(x,t) := d_n \sin\left(\frac{n\pi x}{l}\right) \cdot \left(a_n \cos\left(\frac{\pi\sqrt{q}nt}{l}\right) + b_n \sin\left(\frac{\pi\sqrt{q}nt}{l}\right)\right)$$

are solutions (for the criteria ($\mathbf{1}$)). Let us note that under suitable conditions, the function

$$u(x,t) := \sum_{n=0}^{\infty} u_n(x,t) \quad \big((x,\, t) \in [0,\, l] \times \mathbf{R}_0^+\big)$$

(function series sum) also exists as a solution, and the initial conditions take the following forms:

$$u(x,0) = \sum_{n=0}^{\infty} a_n d_n \cdot \sin\left(\frac{n\pi x}{l}\right) = f(x) \qquad (x \in [0,l]),$$

$$\partial_2 u(x,0) = \frac{\pi\sqrt{q}}{l} \cdot \sum_{n=0}^{\infty} b_n d_n \cdot n \cdot \sin\left(\frac{n\pi x}{l}\right) = g(x) \quad (x \in [0,l]).$$

Thus, these equalities imply that f and g represent the Fourier series expansions (specifically sine series) of the functions f and g.

2 Applying the results

In my example, I used the results for a much simpler case. Let $l,\, h \in \mathbf{R}^+$ represent the length of the string and the *initial height of the string*, respectively. The initial shape and velocity (f and g in the previous section) of the string is determined by the functions that satisfy the following criterion:

$$f : [0,\, l] \to \mathbf{R}, \quad f(x) := ax^2 + bx + c \quad (x \in [0,\, l],\, a,\, b,\, c \in \mathbf{R})$$

and

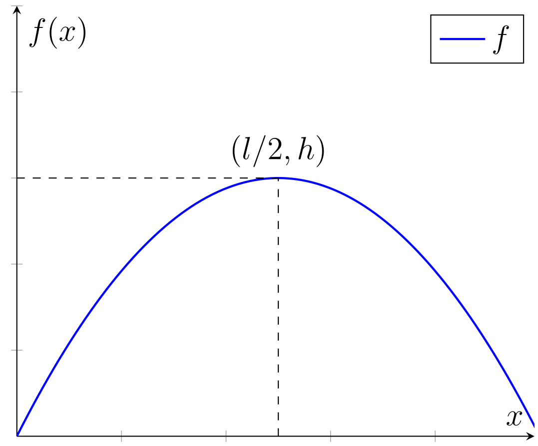
$$f(0) = 0,\; f(l) = 0,\; f(l/2) = h.$$

This problem is straightforward to solve using simple interpolation:

$$f(x) = -\frac{Q}{l}x^2 + Qx \quad (x \in [0,\, l]),$$

where $Q := \frac{4h}{l}$. Also, let the velocity function g be as follows

$$g : [0,\, l] \to \mathbf{R}, \quad g(x) := -f(x) \quad (x \in [0,\, l]).$$



The last task left is to – for every $n \in \mathbf{N}$ – determine the coefficients a_n, b_n and c_n from the previous section by expanding the functions f and g into sine series. Again, by omitting the details this can be done quite easily. For every $n \in \mathbf{N}^+$, let

$$a_n := -1,$$

$$b_n := \frac{l}{n\sqrt{q}\pi},$$

$$d_n := \frac{4lQ\big((-1)^n - 1\big)}{\pi^3 n^3}.$$

Thus,

$$f(x) = \sum_{n=1}^{\infty} a_n d_n \sin\left(\frac{\pi n x}{l}\right) \quad (x \in [0,\, l]),$$

and

$$g(x) = \frac{\pi\sqrt{q}}{l} \sum_{n=1}^{\infty} b_n d_n n \sin\left(\frac{\pi n x}{l}\right) \quad (x \in [0,\, l]).$$

Finally, the function

$$u : [0,\, l] \to \mathbf{R}_0^+ \to \mathbf{R}$$

takes the form

$$u(x,t) := \sum_{n=1}^{\infty} u_n(x,t) \quad \big((x,\, t) \in [0,\, l] \times \mathbf{R}_0^+\big),$$

where – for every $n \in \mathbf{N}^+$ and $(x,\, t) \in [0,\, l] \times \mathbf{R}_0^+$ – the u_n functions are defined the same as in the previous section:

$$u_n(x,\, t) :=$$

$$\underbrace{\frac{4lQ\big((-1)^n - 1\big)}{\pi^3 n^3}}_{d_n} \sin\left(\frac{n\pi x}{l}\right) \cdot \left(\underbrace{(-1)}_{a_n} \cos\left(\frac{\pi\sqrt{q}nt}{l}\right) + \underbrace{\frac{l}{n\sqrt{q}\pi}}_{b_n} \sin\left(\frac{\pi\sqrt{q}nt}{l}\right)\right).$$