Documentation

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In this PDF, I aim to describe concepts related to computer graphics, with a focus on the mathematical aspects. While this information may be useful to others, please note that it is primarily for my own purposes, and I do not assume responsibility for any errors.

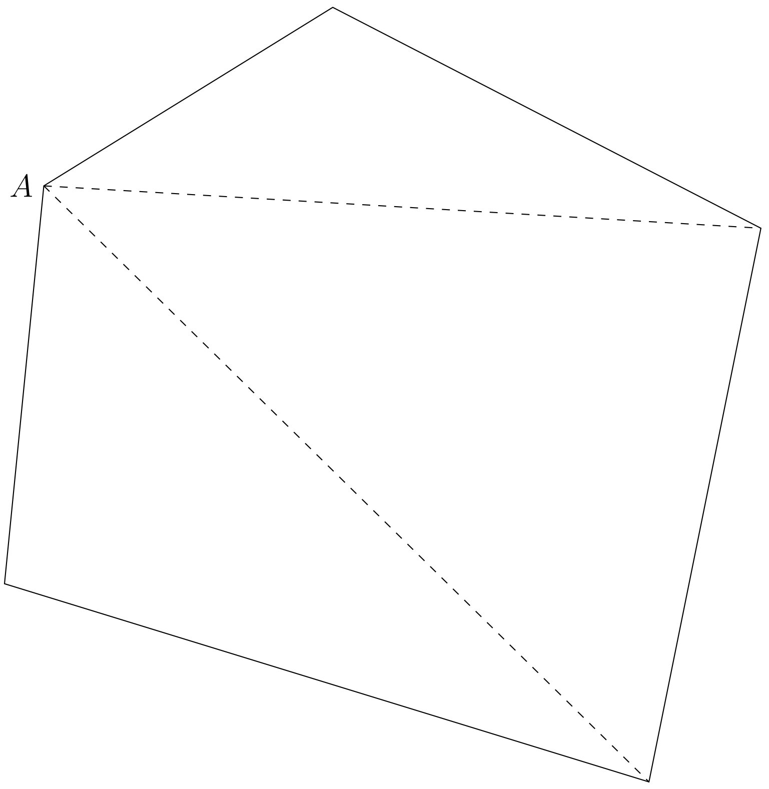
Contents

1 Convex Mesh Division

In this section, I will explain a simple yet effective method to divide any convex polyhedron into arbitrarily small tetrahedrons. This can be useful for creating destructible objects. Let us assume we have a convex polyhedron, where we only know the vertices of each face, and each face can be any polygon.

1.1 Polyhedron to Tetrahedrons

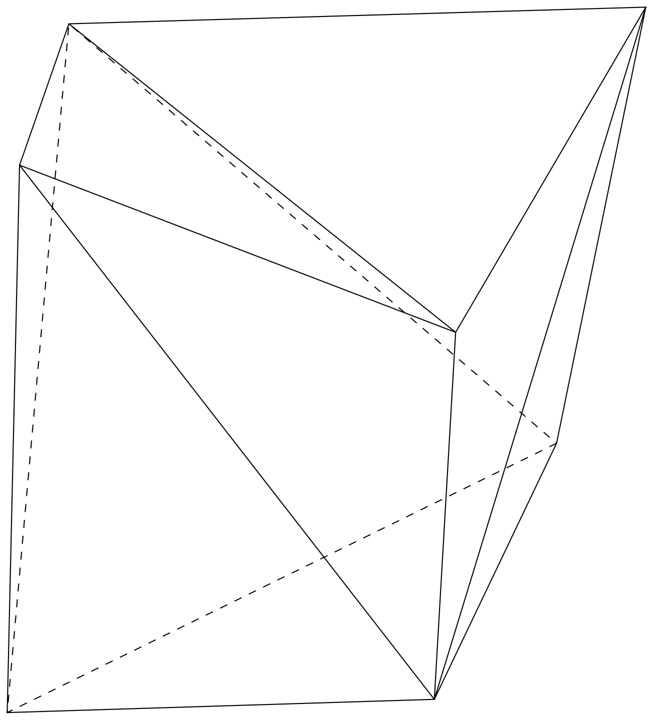
First, we aim to divide our polyhedron into smaller tetrahedrons and subsequently divide those tetrahedrons further. As you might have guessed, a tetrahedron is also a polyhedron. We will use a slightly different approach for dividing tetrahedrons. For simplicity, convert all faces into triangles, which is straightforward:



All you need to do is select any vertex A and connect it to every vertex in the polygon except its neighbors.

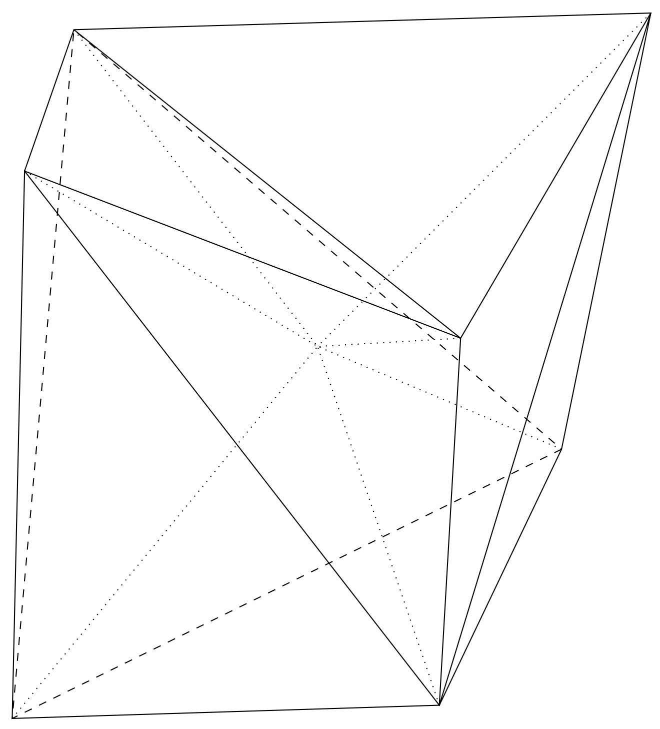
```
struct Triangle {
    glm::vec3 v1;
    glm::vec3 v2;
    glm::vec3 v3;
};

std::vector<Triangle> GetTrianglesFromPolygon(const std::vector<glm::vec3
    >& polygon) {
        std::vector<Triangle> result;

        for (int i = 2; i < polygon.size(); ++i) {
            result.push_back(Triangle{polygon[0], polygon[i-1], polygon[i]});
        }

        return result;
    }
```

Now we have any array of triangles that build up our convex polyhedron.



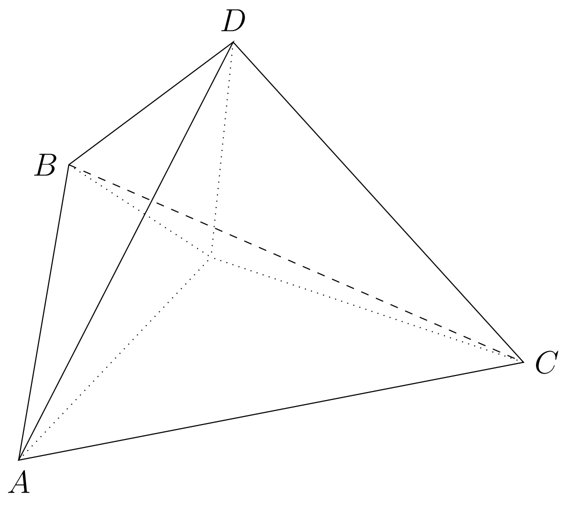
The simplest method for dividing a convex polyhedron into smaller tetrahedrons is to find the center of the polyhedron and then connect this center to each of the polyhedron's triangular faces.

```
    std::vector<Tetra> GetTetras(const std::vector<Triangle>& triangles) {
        glm::vec3 center{0.0f, 0.0f, 0.0f};
        for (const auto& t : triangles) {
            center += t.v1; center += t.v2; center += t.v3;
        }
        center /= (triangles.size() * 3.0f);
        std::vector<Tetra> tetras;
        for (const auto& t : triangles)
        {
            tetras.push_back({t.v1, t.v2, t.v3, center});
        }
        return tetras;
    }
```

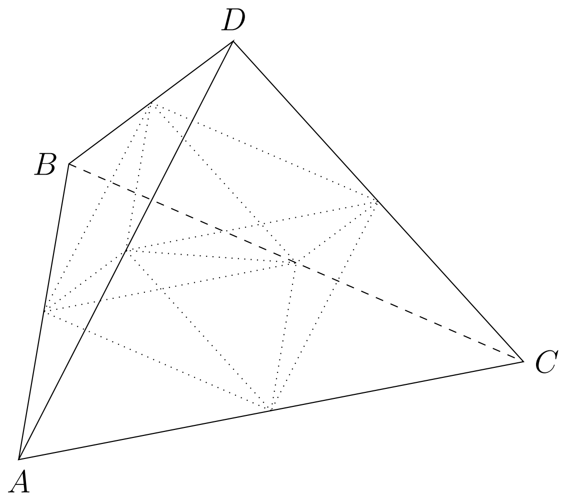


1.2 Tetrahedrons to tetrahedrons

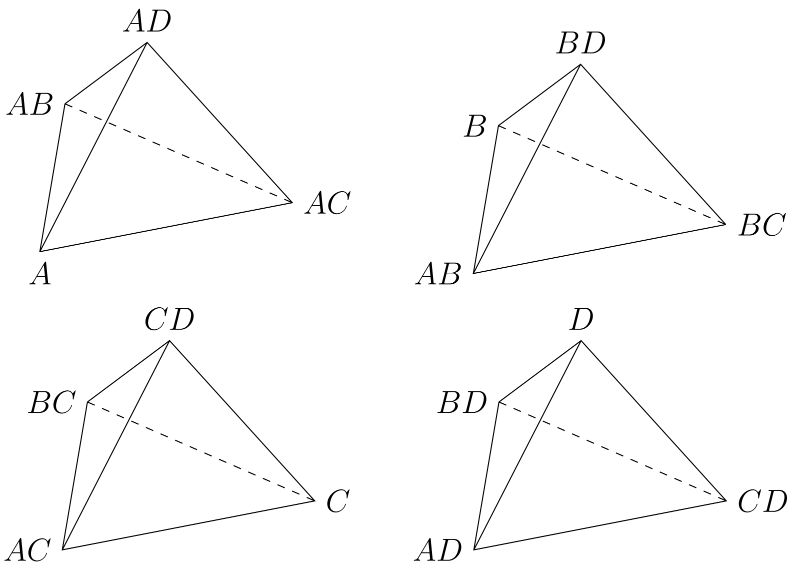
We have successfully divided our mesh into tetrahedrons. While further subdivision might be desirable, this method may not be the most effective for dividing the tetrahedrons themselves. Let's explore why this particular tetrahedron could be an example of our polyhedron:



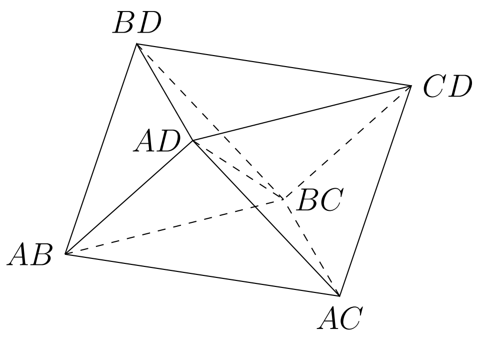
As illustrated, continuing this pattern would result in all the sub-tetrahedrons having identical faces. This would produce very thin tetrahedrons with wide faces, which might not be ideal for our purposes. Therefore, let's explore a different algorithm. Our overall goal is to calculate the midpoints of all the edges in each tetrahedron and use these midpoints to generate smaller tetrahedrons.



The illustration may be tricky, but there are actually eight smaller tetrahedrons within the larger one. Identifying the vertices is straightforward:



Within the tetrahedron, there is a hexahedron comprised of four tetrahedrons.



The algorithm is quite straightforward. You start with six vertices and need to select two that do not share a common vertex. Following this labeling approach, if you examine the labels of the chosen midpoints, they will encompass all the vertices. In this case, the appropriate pairs of vertices might be:

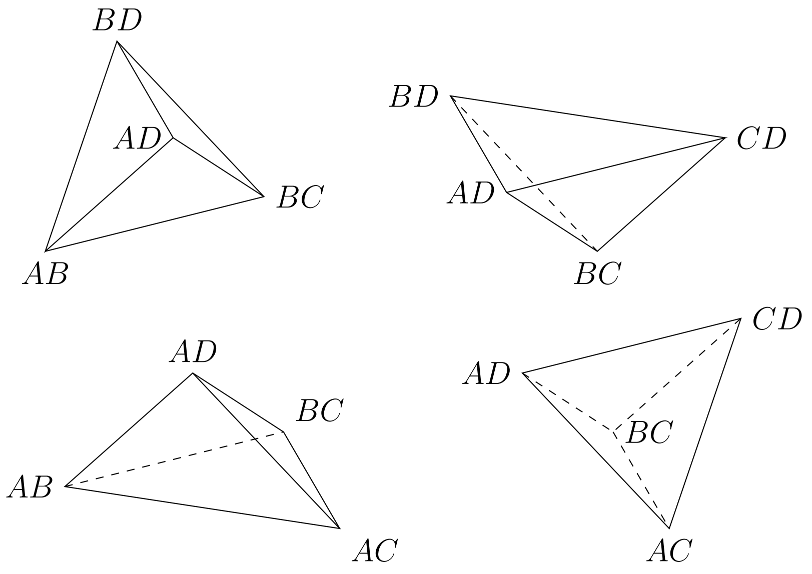
$$AD - BC,\ AB - CD,\ BD - AC$$

All four tetrahedrons will contain these two vertices. Next, we need to determine the order in which to assemble the remaining four vertices. For instance, if we choose AD and BC as the two vertices, the tetrahedrons should be assembled as follows:

1. AD, BC, $\cdot$, $\cdot$
2. AD, BC, $\cdot$, $\cdot$
3. AD, BC, $\cdot$, $\cdot$
4. AD, BC, $\cdot$, $\cdot$

where the $\cdot$ symbols represent the remaining vertices. The algorithm is also quite simple: you need to choose two vertices that share exactly one vertex.

1. AD, BC, $B\mathbf{D}$, $C\mathbf{D}$
2. AD, BC, $\mathbf{B}D$, $A\mathbf{B}$
3. AD, BC, $\mathbf{A}B$, $\mathbf{A}C$
4. AD, BC, $A\mathbf{C}$, $\mathbf{C}D$



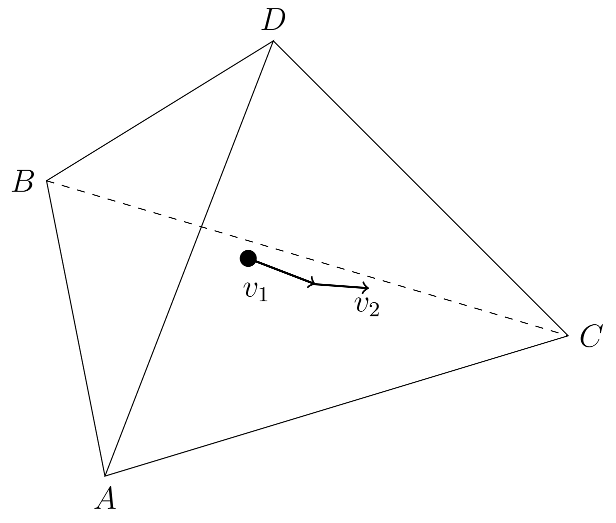
The good news is that the midpoints calculated initially do not need to be exact; they can be any points along the edges. This flexibility can lead to a noisy division.

1.3 Triangle Vertex Ordering

In computer graphics, the ordering of triangle vertices is crucial. When viewing a mesh from any position in the world, the triangles you see must be consistently ordered, either clockwise or counterclockwise. In this subsection, I provide a simple algorithm to order the triangles in your mesh. Suppose we have a list of triangles that make up our mesh. Using the glm library, the following code illustrates this:

```cpp
bool IsClockwise(const Triangle& triangle, const glm::vec3& polyCenter)
{
    glm::vec3 triangleCenter = (triangle.v1 + triangle.v2 + triangle.v3) / 3.0f;
    glm::vec3 outVector = triangleCenter - polyCenter;

    glm::vec3 v1 = triangle.v2 - triangle.v1;
    glm::vec3 v2 = triangle.v3 - triangle.v1;
    glm::vec3 normal = glm::cross(v1, v2);

    return glm::dot(normal, outVector) < 0;
}
```

Consider a convex polyhedron. We can easily calculate its center point by adding up all the vertices and then dividing by their count. For simplicity, I have chosen a tetrahedron:



First of all, we know that given three triangle vertices, they are either ordered clockwise or counterclockwise. To ensure the desired ordering, you can examine the current order and, if necessary, swap two vertices in the triangle. The key to determining the order is the cross product of the triangle's edge vectors. The cross product of two vectors always results in a vector that is perpendicular to the plane defined by the vectors, following the right-hand rule.
We can calculate the normal vector of a triangle by computing the cross product of two vectors along its edges. However, the order of the vectors matters, as it can change the

sign of the resulting normal vector. We need to determine whether the normal is pointing outwards or inwards relative to our polyhedron.
As shown in the image above, the vector is pointing outwards. There are different methods to check if the normal is pointing outwards, but I chose to check the dot product with the vector from the polyhedron center to the triangle center. By examining the sign of the dot product, we can determine the direction of the normal vector. If the dot product is negative, it means that the angle between the two vectors is greater than 90 degrees, indicating that the normal vector is pointing outwards.

2 Cube builder

In this section I will explain the methods, algorithms that I have used while creating the cube builder project. This involves a simplified 3D DDA algorithm, ray intersection with cubes, and other ideas.

2.1 Intersection of cubes

First, let us see a simple method for examining ray intersection with cubes. In this project I used finite length rays (technically segments would be a more precise name for this). We can define a ray with it's starting point $a \in \mathbb{R}^3$ and direction vector $\vec{n} \in \mathbb{R}^3$.

$$r(t) := \vec{n} \cdot t + a \quad (t \in \mathbb{R}_0^+)$$

Segments can be defined with two points a, $b \in \mathbb{R}^3$ as follows:

$$s(t) := (b - a) \cdot t + a \quad (t \in [0,\, 1])$$

where a is the starting point and b is the endpoint. Our goal now is to be able to examine if our segment intersects a cube or not. Let us define a cube in space with two points c_1, $c_2 \in \mathbb{R}^3$. Let us make a restriction when it comes to defining cubes with 2 points: the points $c_1 \in \mathbb{R}^3$ should hold the rule that

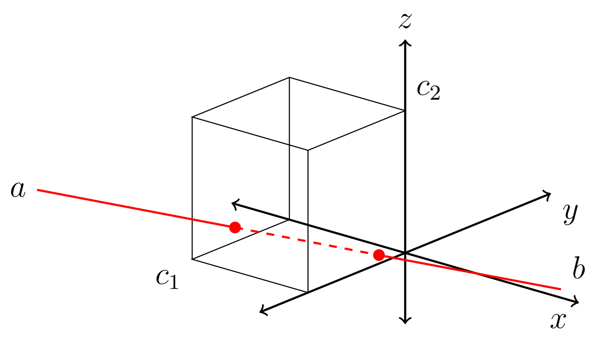
$$c_1 + \begin{pmatrix} t \\ t \\ t \end{pmatrix} = c_2 \quad (t > 0).$$

This will easen our task, with avoiding ambiguity. With this restriction we can say the following: our segment $s(t)$ $(t \in [0,\, 1])$ intersects our cube (that is represented as above) if and only if there exits a parameter $t \in [0,\, 1]$ such that

$$(c_1)_x \le s(t)_x \le (c_2)_x$$
$$(c_1)_y \le s(t)_y \le (c_2)_y$$
$$(c_1)_z \le s(t)_z \le (c_2)_z$$

Let us see an example:

$$c_1 := \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix},\ c_2 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ a := \begin{pmatrix} -1.5 \\ -2 \\ 0.65 \end{pmatrix},\ b := \begin{pmatrix} 0.8 \\ 0.65 \\ -0.25 \end{pmatrix}$$



$$s(t) = (b-a)\cdot t + a = \begin{pmatrix} 2.3 \\ 2.65 \\ -0.9 \end{pmatrix} \cdot t + \begin{pmatrix} -1.5 \\ -2 \\ 0.65 \end{pmatrix} \quad (t \in [0,\, 1])$$

If we have two points $u,\, v \in \mathbb{R}^3$ let $\overset{\star}{\leq}$ denote the comparison which is considered true if

$$x_x \leq y_x \text{ and } x_y \leq y_y \text{ and } x_z \leq y_z.$$

Now the segment we have defined earlier intersects the cube if and only if there exists $t \in [0,\, 1]$ such that

$$c_1 \overset{\star}{\leq} s(t) \overset{\star}{\leq} c_2$$
$$\Longleftrightarrow$$
$$\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \overset{\star}{\leq} \begin{pmatrix} 2.3 \\ 2.65 \\ -0.9 \end{pmatrix} \cdot t + \begin{pmatrix} -1.5 \\ -2 \\ 0.65 \end{pmatrix} \overset{\star}{\leq} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$\Longleftrightarrow$$
$$\begin{pmatrix} 0.5 \\ 1 \\ -0.65 \end{pmatrix} \overset{\star}{\leq} \begin{pmatrix} 2.3 \\ 2.65 \\ -0.9 \end{pmatrix} \cdot t \overset{\star}{\leq} \begin{pmatrix} 1.5 \\ 2 \\ 0.35 \end{pmatrix}.$$

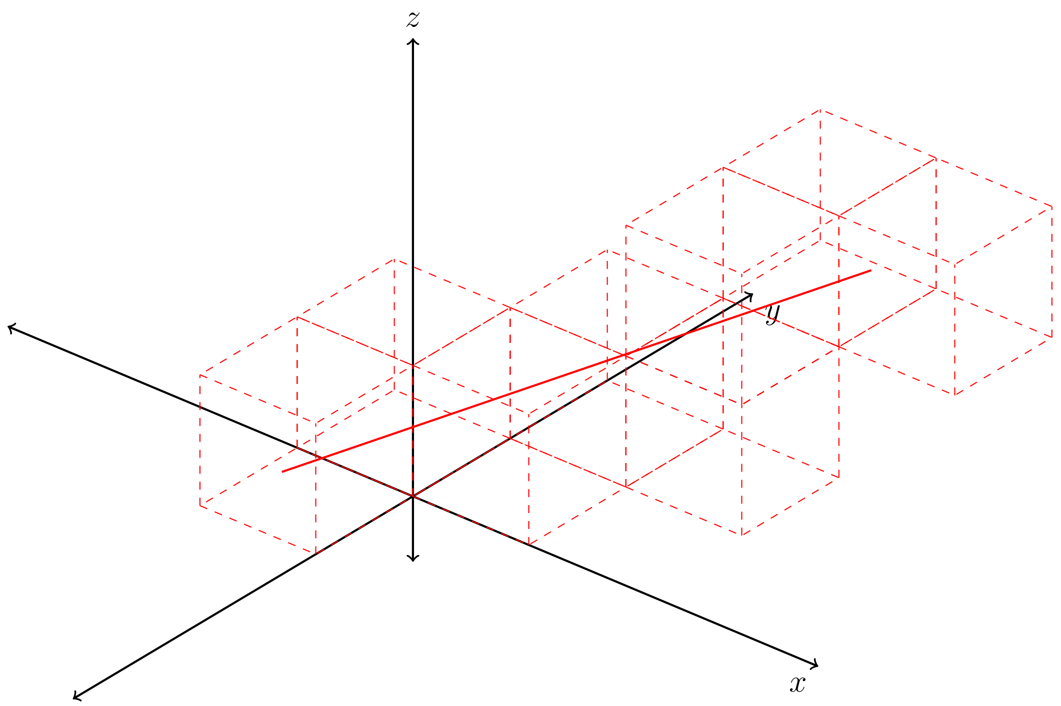
2.2 Ray-Voxel path

Let us say, we have a finite length ray. We would like to determine all the cubes in our world that our ray intersects. I have tried to search for algorithms written by others but I only found the DDA algorithm, which takes two arguments (two cubes in our space) and return a path of cubes. However I want to input two points in space that determine a finite length ray and then output the cubes that are intersected. You can image that we have two segments in space where each start and endpoint are in the same two cubes but the intersection path is different. It is easy to do this in 2D, but there are a lot more cases to consider when it comes to 3D calculations.

Let us start with the simpler solutions. First, let us have two points in space $p_1,\, p_2 \in \mathbb{R}^3$. These define the following segment:

$$s(t) := (p_2 - p_1)\cdot t + p_1 \quad (t \in [0,\, 1]).$$

We would like to find all the cubes that are intersected. The brute-force method would be to iterate through all the cubes in our space and check for intersection, with the algorithm in the previous section.



A slightly better approach would be to narrow the list of cubes which we examine. This could be done easily with calculating the smallest rectangular prism that contains all the cubes that our segment intersects. We first calculate the 2 corner cubes and iterate through the rectangular prism.