2017 P2Q1 (I)

(a) $\int_0^1 y_n y_m w\,dx = \delta_{nm}$

(b) $\int_0^1 w(x) f(x)^2 dx = \int_0^1 w(x)\left(\sum_{n=0}^{\infty} a_n y_n(x)\right)^2 dx$

$= \int_0^1 w(x) \sum_{n=0}^{\infty} a_n y_n(x) \sum_{m=0}^{\infty} a_m y_m(x)\,dx$

$= \int_0^1 w(x) \sum_{n=0}^{\infty} a_n a_n y_n y_n\,dx + \underbrace{\int_0^1 \sum_{n=0}^{\infty} \sum_{m=0}^{n-1} w a_n a_m y_n y_m\,dx}_{\text{0 USING ORTHONORMALITY}}$

$= \int_0^1 w(x) \sum_{n=0}^{\infty} a_n^2 y_n^2\,dx$

$= \sum_{n=0}^{\infty} a_n^2 \underbrace{\int_0^1 w(x) y_n^2(x)\,dx}_{\text{1, USING ORTHONORMALITY}} = \sum_{n=0}^{\infty} a_n^2$

(c)(i) $\mathcal{L} Y(x) = w(x)[\alpha Y(x) + f(x)]$

$f(x) = \sum_{n=0}^{\infty} a_n y_n(x)$

$\mathcal{L} Y = w[\alpha Y + \sum a_n y_n]$

$\mathcal{L} Y = \mathcal{L} \sum_{n=0}^{\infty} b_n y_n = \sum_{n=0}^{\infty} b_n \mathcal{L} y_n = \sum_{n=0}^{\infty} b_n \lambda_n w y_n$

COMBINE LAST TWO LINES, CANCEL w:

$\sum_{n=0}^{\infty} b_n \lambda_n y_n = \alpha Y + \sum_{n=0}^{\infty} a_n y_n$

$= \sum_{n=0}^{\infty} \alpha b_n y_n + a_n y_n = \sum_{n=0}^{\infty} (\alpha b_n + a_n) y_n$

$\Rightarrow \sum_{n=0}^{\infty} (\lambda_n - \alpha) b_n = \sum_{n=0}^{\infty} a_n$

$\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{a_n}{\lambda_n - \alpha}$

$\Rightarrow Y = \sum_{n=0}^{\infty} b_n y_n = \sum_{n=0}^{\infty} \frac{a_n}{\lambda_n - \alpha} y_n$

2017 P2 Q1 (II)

(c)(ii) $\lambda_0 = 1 \quad \lambda_1 = 2 \quad f = 2[y_0 + y_1] - \alpha y_1$

$$a_0 = \int_0^1 f y_0 w \, dx = 2 \qquad a_1 = \int_0^1 f y_1 w \, dx = 2 - \alpha$$

USING RESULT OF (c)(i):

$$Y = \frac{a_0}{1-\alpha} y_0 + \frac{a_1}{2-\alpha} y_1$$

- $\int_0^1 w(x) Y(x)^2 dx = 2$ ~~ORTHOGONALITY OF EIG~~
- ORTHOGONALITY OF EIGENFUNCTIONS

$$\Rightarrow \int_0^1 w\left(\frac{a_0}{1-\alpha}\right)^2 y_0^2 + w\left(\frac{a_1}{2-\alpha}\right)^2 y_1^2 = 2$$

(No $y_0 y_1$ TERMS BECAUSE THEY INTEGRATE TO 0)

$$\left(\frac{a_0}{1-\alpha}\right)^2 + \left(\frac{a_1}{2-\alpha}\right)^2 = 2$$

$$\left(\frac{\cancel{a_0}}{1-\alpha}\right)^2 + \left(\frac{2-\alpha}{2-\alpha}\right)^2 = 2$$

$$\left(\frac{2}{1-\alpha}\right)^2 + 1 = 2$$

$$\Rightarrow \alpha = 3$$

$$Y = \frac{2}{1-3} y_0 + \frac{2-3}{2-3} y_1$$

$$= -y_0 + y_1$$