Test 2 EAS595

Name:..
Person number:..

Nov 15, 2019, 3:00-5:20 pm

1. (5 points) What is the mathematical definition of Moment generating function for continuous random variables?

$$E[e^{sx}] = \int_{-\infty}^{\infty} e^{sx} f_x(x)\, dx$$

2. (10 points) Let X be a continuous random variable defined on $[0\ \infty)$ with a moment generating function $Mx = \frac{1}{1-t}$. What is the correlation of X and X^2?

$$Corr(X, X^2) = \frac{Cov(X, X^2)}{\sqrt{Var(X)\, Var(X^2)}}$$

$$Var(X) = E[X^2] - E[X]^2, \quad Var(X^2) = E[X^4] - E[X^2]^2, \quad Cov(X, X^2) = E[X^3] - E[X]E[X^2]$$

$$M_X = \frac{1}{1-t} \rightarrow \frac{\partial M_X}{\partial t} = \frac{1}{(1-t)^2} \rightarrow \frac{\partial^2 M}{\partial t^2} = \frac{2}{(1-t)^3} \rightarrow \frac{\partial^3 M}{\partial t^3} = \frac{6}{(1-t)^4} \rightarrow \frac{\partial^4 M}{\partial t^4} = \frac{24}{(1-t)^5}$$

$$E[X^n] = \left.\frac{\partial^n M}{\partial t^n}\right|_{@t=0} \rightarrow @t=0 \quad E[X] = 1, \quad E[X^2] = 2, \quad E[X^3] = 6, \quad E[X^4] = 24$$

$$\rightarrow \begin{cases} Cov(X, X^2) = 6 - 2 = 4 \\ Var(X) = 2 - 1 = 1 \\ Var(X^2) = 24 - 2^2 = 20 \end{cases} \Rightarrow Corr(X, X^2) = \frac{4}{\sqrt{20}}$$

3. (15 points) We have a bar with an initial length of L cm. We randomly choose the point x with an equal probability along the length of the bar to cut and reduce it's length to x cm. We again choose another point (Y) with an equal probability to cut the remaining of the bar.

- What is the E[X-Y]?
- What is the var(X-Y)?

$X \sim U(0,L) \rightarrow E[X] = \frac{L}{2}$, $Var[X] = \frac{L^2}{12}$

$Y \sim U(0,X)$

a)

$$\left.\begin{array}{l} E[X-Y] = E[X] - E[Y] \\ E[Y] = E\big[E[Y|X]\big] = E[X/2] = \frac{1}{2}E[X] \end{array}\right\} \rightarrow E[X-Y] = \frac{L}{2} - \frac{L}{4} = \frac{L}{4}$$

b)

$$Var(X-Y) = Var(X) + Var(Y) - 2\,Cov(X,Y)$$

$$Var(Y) = E\big[Var(Y|X)\big] + Var\big(E[Y|X]\big) = E\left[\frac{X^2}{12}\right] + Var\left[\frac{X}{2}\right]$$

$$E[X^2] = Var[X] + E[X]^2 = \frac{L^2}{12} + \frac{L^2}{4} = \frac{L^2}{3}$$

$$\rightarrow Var(Y) = \frac{1}{12}E[X^2] + \frac{1}{4}Var(X) = \frac{L^2}{36} + \frac{L^2}{48}$$

$$Cov(X,Y) = \underbrace{E[XY]}_{E\big[E[XY|X]\big] = E\left[\frac{X^2}{2}\right]} - E[X]\underbrace{E[Y]}_{\frac{1}{2}E[X]} = \frac{1}{2}\left(E[X^2] - E[X]^2\right) = \frac{1}{2}Var(X)$$

$$\rightarrow Var(X-Y) = \cancel{Var(X)} + Var(Y) - \cancel{2\left(\frac{1}{2}Var(X)\right)} = Var(Y) = \frac{L^2}{36} + \frac{L^2}{48}$$

4. (25 points) X is a uniform random variable over the interval of [-1 1] and Y is an exponential random variable with $\lambda = 1$. If X and Y are independent :

- What is the PDF of $Z = X + Y$. $\rightarrow z \in [-1\ \infty]$
- What is $E[Z|x = 0.5]$?

$$f_X = \begin{cases} \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & \text{other} \end{cases} \qquad f_Y(z-x) = \begin{cases} e^{-z} \cdot e^{x} & x \leq z \\ 0 & \text{other} \end{cases}$$

a) $$f_z = \int f_X\, f_Y(z-x)\, dx$$

$$f_z \text{ is non-zero if } \begin{cases} R_1 : -1 \leq x \leq 1 \\ \quad \text{and} \\ R_2 \quad x \leq z \end{cases}$$

$\rightarrow$ if $z \geq 1$ $R_1 \cap R_2 = -1 \leq x \leq 1$: $f_z = \int_{-1}^{1} \frac{e^{-z}}{2} e^{x} dx = \frac{e^{-z}}{2} e^{x} \Big]_{-1}^{1} = \frac{e^{-z}}{2}\left(e - \frac{1}{e}\right)$

$\rightarrow$ if $-1 \leq z \leq 1$ $R_1 \cap R_2$ $-1 \leq x \leq z$: $f_z = \int_{-1}^{z} \frac{e^{-z}}{2} e^{x} dx = \frac{e^{-z}}{2} e^{x} \Big]_{-1}^{z} = \frac{e^{-z}}{2}\left(e^{z} - \frac{1}{e}\right)$

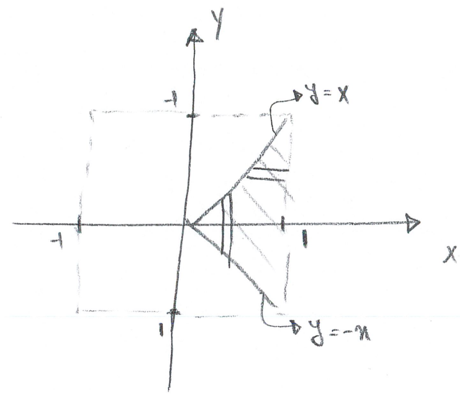
$$\rightarrow f_z = \begin{cases} \frac{e^{-z}}{2}\left(e - \frac{1}{e}\right) & z \geq 1 \\ \frac{e^{-z}}{2}\left(e^{z} - \frac{1}{e}\right) & -1 \leq z \leq 1 \end{cases}$$

b) $E[z \mid x = \frac{1}{2}] = E[X + Y \mid X = \frac{1}{2}] = E[\frac{1}{2} + Y] = \frac{1}{2} + \underbrace{E[Y]}_{\text{Table}} = 1.5$

5. (25 points) Consider random variables X,Y with the following joint PDF.

$$f_{X,Y} = \begin{cases} cxy^2 & |y| \le x \le 1, \quad -1 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

- Find $f_X(x)$, $f_Y(y)$



$$f_x = \int f_{xy}\,dy = \int_{-x}^{x} cxy^2\,dy = \left. \frac{cxy^3}{3} \right]_{-x}^{x} = \frac{2c}{3}x^4$$

$y > 0$ $$f_y = \int f_{xy}\,dx = \int_{y}^{1} cxy^2\,dx = \left. \frac{cx^2y^2}{2} \right]_{y}^{1} = \frac{c}{2}(y^2 - y^4)$$

$y < 0$ $$f_y = \int f_{xy}\,dx = \int_{-y}^{1} cxy^2\,dx = \left. \frac{cx^2y^2}{2} \right]_{-y}^{1} = \frac{c}{2}(y^2 - y^4)$$

Finding c $$\int_0^1 f_x\,dx = 1 \rightarrow \int_0^1 \frac{2c}{3}x^4\,dx = \left. \frac{2c}{3}\frac{x^5}{5} \right]_0^1 = 1 \rightarrow \frac{2c}{15} = 1 \rightarrow c = \frac{15}{2}$$

$$\rightarrow f_X(x) = \begin{cases} 5x^4 & 0 \le x \le 1 \\ 0 & \text{other} \end{cases} \qquad f_Y(y) = \begin{cases} \frac{15}{4}(y^2 - y^4) & -1 \le y \le 1 \\ 0 & \text{other} \end{cases}$$

6. (20 points) You are interested in measuring a physical quantity using a specific technique which is subjected to some measurement noise. Your professor will tell you that if you have multiple independent measurements (form an identical distribution) and then use their average, it will be more accurate comparing to a single measurement.

- Explain why an average of multiple measurements (iid samples) is more accurate than just using one measurement?
- Assume that all measurement are from an unknown distributions with a common but unknown mean (m) and a common standard deviation of 3 units. How many measurements do you need to make in order to be 95% confident that the estimated value of the mean (m) is within ±0.1 unit error?

a) The variance of the average is much smaller than every single observation that means the average is more trustable.

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \rightarrow Var(\bar{X}) = Var\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{n\,Var(X_i)}{n^2} = \frac{Var(X_i)}{n}$$

b) $$Pr(|\bar{X}-m| < 0.1) = 0.95 \Rightarrow Pr\Bigg(\underbrace{\frac{|\bar{X}-m|}{\frac{\sigma}{\sqrt{n}}}} < \underbrace{\frac{0.1}{\frac{\sigma}{\sqrt{n}}}}_{Z^*}\Bigg) = 0.95$$

This is standard Normal. let's call it Z

$$Pr(|Z| < Z^*) = Pr(-Z^* < Z < Z^*) = \Phi(Z^*) - \Phi(-Z^*)$$

$$\Phi(-Z^*) = 1 - \Phi(Z^*)$$

$$\Rightarrow 2\Phi(Z^*) - 1 = 0.95$$

$$\Phi(Z^*) = \frac{1.95}{2} = 0.975$$

from Table: $\Phi(Z^*) = 0.975 \rightarrow Z^* = 1.96$

$$Z^* = \frac{0.1}{3}\sqrt{n} = 1.96 \Rightarrow \sqrt{n} = 58.8 \Rightarrow n = 3457.4$$

You need 3458 samples