Lecture 17 Outline

Wednesday, October 23, 2019 1:24 PM

HW5 Discussion (Professional Code!)

Midterm Solutions

Review Session (Pi~~ck One~~)

~~F, Nov 8, 11:30am-12:50pm~~ Tu, Nov 12

~~F, Nov 15, 10am-11:20am~~ 3:30-4:50pm

Need to confirm

Normal Matrix Properties

Matrix Diagonalization

Normal Matrix Properties

Tuesday, October 22, 2019 3:04 PM

A Normal Matrix is one in which

$$\underline{A}^T \underline{A} = \underline{A}\,\underline{A}^T$$

This occurs for:

Symmetric $\underline{A}$ (i.e, $\underline{A} = \underline{A}^T$)

Skew-symmetric $\underline{A}$ (i.e, $\underline{A} = -\underline{A}^T$)

Orthogonal $\underline{A}$ (i.e, $\underline{A}^T \underline{A}$ is a diagonal matrix)

Next, consider real symmetric matrices (but discussion applies to other normal matrices)

Since $\underline{A}$ is real + symmetric, $\underline{A}$ is square

1. A real normal matrix, only has real eigenvalues

Proof: Let $\underline{A}$ be real + symmetric, and let λ be any (including possibly complex) eigenvalue, such that $\underline{A}\,\underline{x} = \lambda\,\underline{x}$

$$\underline{A}\,\underline{x} = \lambda\,\underline{x} \quad + \quad \underline{A}\,\bar{\underline{x}} = \bar{\lambda}\,\bar{\underline{x}}$$

Also, $(\underline{A}\,\bar{\underline{x}})^T = (\bar{\lambda}\,\bar{\underline{x}})^T \Rightarrow \bar{\underline{x}}^T \underline{A}^T = \bar{\underline{x}}^T \bar{\lambda}$

but $\underline{A}^T = \underline{A} \Rightarrow \bar{\underline{x}}^T \underline{A} = \bar{\underline{x}}^T \bar{\lambda}$

Next, take inner product of $\bar{\underline{x}}$ with $\underline{A}\,\underline{x} = \lambda\,\underline{x}$ and inner product of $\underline{x}$ with $\bar{\underline{x}}^T \underline{A} = \bar{\underline{x}}^T \bar{\lambda}$

Thus,

$$\underbrace{\bar{\underline{x}}^T (\underline{A}\,\underline{x})}_{\text{equal}} = \bar{\underline{x}}^T (\lambda\,\underline{x}) \quad + \quad \underbrace{(\bar{\underline{x}}^T \underline{A})\,\underline{x}}_{\text{equal}} = (\bar{\underline{x}}^T \bar{\lambda})\,\underline{x}$$

so that $\bar{\underline{x}}^T \lambda\,\underline{x} = \bar{\underline{x}}^T \bar{\lambda}\,\underline{x}$

$$\lambda\,\bar{\underline{x}}^T \underline{x} = \bar{\lambda}\,\bar{\underline{x}}^T \underline{x}$$

Since $\underline{\bar{x}}^T \underline{x} = |x_1|^2 + |x_2|^2 + \dots + |x_n^2| > 0$

$\rightarrow \lambda = \bar{\lambda}$ with $\lambda = a + ib$

$\bar{\lambda} = a - ib$

$\therefore b = 0$ and λ must be real

If $\underline{A}$ is real & symmetric, all eigenvalues λ must be real //

② All eigenvectors of a real symmetric matrix are orthogonal

Proof: Let $\underline{A}\underline{x} = \lambda_1 \underline{x}$ & $A\underline{y} = \lambda_2 \underline{y}$

for $\lambda_1 \neq \lambda_2$

Since $\underline{A}$ is real & symmetric, λ_1 & λ_2 are real. Then

$$(\lambda_1 \underline{x}) \cdot \underline{y} = (\lambda_1 \underline{x})^T \underline{y} = (\underline{A}\underline{x})^T \underline{y} = \underline{x}^T \underline{A}^T \underline{y}$$

$$= \underline{x}^T \underline{A}\, \underline{y} = \underline{x}^T (\lambda_2 \underline{y})$$

$$\Rightarrow \quad \underline{x}^T \lambda_1 \underline{y} = \underline{x}^T \lambda_2 \underline{y}$$

$$\lambda_1 (\underline{x}^T \underline{y}) = \lambda_2 (\underline{x}^T \underline{y})$$

Since $\lambda_1 \neq \lambda_2 \Rightarrow$ This is true iff $\underline{x}^T \underline{y} = 0 \rightarrow$ Eigenvectors are orthogonal //

③ One can also prove that the eigenvectors for a real, symmetric matrix $\underline{A}$ can be orthonormal

Note: Eigenvectors can be scaled by any non-zero constant, so that constant can be chosen to set the length of each eigenvector to unity

Matrix Diagonalization

Tuesday, October 22, 2019 5:19 PM

Matrix Diagonalization is the application of a matrix $\underline{P}$ such that

Lambda

$$\underline{P}^{-1}\underline{A}\,\underline{P} = \underline{\Lambda}$$, where $\underline{\Lambda}$ is a diagonal matrix

Consider the eigensystem of $\underline{A}$

$$\underline{A}\,\underline{x} = \lambda\,\underline{x}$$

Let $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n$ represent the eigenvectors of $\underline{A}$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

$$\underline{A}\,\underline{x}_1 = \lambda_1 \underline{x}_1$$

$$\underline{A}\,\underline{x}_2 = \lambda_2 \underline{x}_2$$

$$\vdots$$

$$\underline{A}\,\underline{x}_n = \lambda_n \underline{x}_n$$

Consider

$$\underline{A}\left[\underline{x}_1\ \underline{x}_2 \dots \underline{x}_n\right] = \underline{A}\,\underline{S} = \left[\lambda_1\underline{x}_1\ \lambda_2\underline{x}_2 \cdots \lambda_n\underline{x}_n\right]$$

Let

$$\underline{\Lambda} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$ diagonal!

Then

$$\left[\lambda_1\underline{x}_1\ \lambda_2\underline{x}_2 \cdots \lambda_n\underline{x}_n\right] = \underline{S}\,\underline{\Lambda}$$

$$\therefore\ \underline{A}\,\underline{S} = \underline{S}\,\underline{\Lambda}$$

a) Let $\underline{A}$ be non-defective $\Rightarrow$ All eigenvalues of $\underline{A}$ are complete

$\Rightarrow$ There are n independent eigenvectors

$\Rightarrow$ S^{-1} exists

$\Rightarrow \quad \underline{S}^{-1}$ exists

$$\Rightarrow \quad \underline{S}^{-1} \underline{A} \, \underline{S} = \underline{\Lambda}$$

or $\underline{A} = \underline{S} \, \underline{\Lambda} \, \underline{S}^{-1} \leftrightarrow$ eigendecomposition of $\underline{A}$

(or spectral decomposition)

b) If $\underline{A}$ is defective, then $\underline{S}^{-1}$ does not exist

$\Rightarrow$ If $\underline{A}$ is defective, one can not find an eigendecomposition

$\Rightarrow$ $\underline{A}$ is not diagonalizable

Let all eigenvalues of $\underline{A}$ be complete (thus $\underline{A}$ is complete) and consider $\underline{A}^2$

$$\underline{A}^2 = \underline{A} \, \underline{A} = (\underline{S} \, \underline{\Lambda} \, \underline{S}^{-1})(\underline{S} \, \underline{\Lambda} \, \underline{S}^{-1})$$

$$= \underline{S}\,\underline{\Lambda}^2\,\underline{S}^{-1}$$

with

$$\underline{\Lambda}^2 = \begin{bmatrix} \lambda_1^2 & & & 0 \\ & \lambda_2^2 & & \\ & & \ddots & \\ 0 & & & \lambda_n^2 \end{bmatrix}$$

More generally, $\underline{A}^n = \underline{S}\,\underline{\Lambda}^n\,\underline{S}^{-1}$

Example: Markov Chains

$\underline{P}_m = \underline{M}^m \underline{P}_0$ with $\underline{M}$ complete

$$= \underline{\vec{S}}_M \underline{\Lambda}_M^m \underline{\vec{S}}_M^{-1} \underline{P}_0$$

$$\underline{\Lambda}_M^m = \begin{bmatrix} \lambda_1^m & & & 0 \\ & \lambda_2^m & & \\ & & \ddots & \\ 0 & & & \lambda_n^m \end{bmatrix}$$

If all eigenvalues λ_i of $\underline{M}$ have $|\lambda_i| \leq 1$, then as $m \to \infty$, $\underline{M}$ will go to a steady state version $\underline{M}^{\infty}$ dominated by the largest eigenvalue (in an absolute value sense)

For the Markov transition matrix

$$\max |\lambda_i| = 1$$

Calling this λ_1, only the first column of $\underline{S}$ (i.e. the corresponding eigenvector $\underline{x}_1$) will determine the steady response

$$\underline{P}_{\infty} = \left(\underline{x}_1 \lambda_1 \underline{x}_1^T\right) \underline{P}_0$$

Example: Assume all eigenvalues of $\underline{A}$ have $|\lambda_i| < 1$

Then,

$$\lim_{m \to \infty} \underline{A}^m \underline{x} = \underline{0}$$