Lecture 16 Outline

Monday, October 21, 2019 1:40 PM

Eigensystems

Normal Matrix

Matrix Diagonalization

Eigensystems

Sunday, October 20, 2019 11:34 PM

$\underline{A}\,\underline{x} = \lambda \underline{x}$ Eigensystem of $\underline{A}$

λ : Eigenvalue

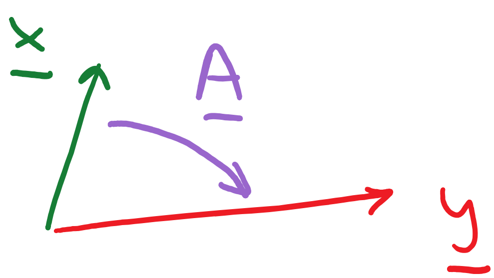
$\underline{x}$: Eigenvector

Frequently appears in applications

$\underline{A}\,\underline{x} - \lambda \underline{x} = \underline{0}$

$(\underline{A} - \lambda \underline{I})\vec{x} = \vec{0}$

What, in general, does a matrix do?

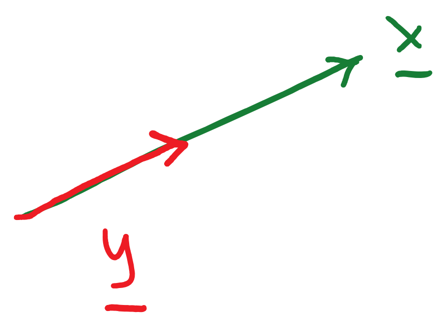


$\underline{A}\,\underline{x} = \underline{y}$

Matrix $\underline{A}$ transforms vector $\underline{x}$

into $\underline{y}$

Now, consider $\underline{A}\,\underline{x} = \underbrace{\lambda \underline{x}}_{\underline{y}}$



$\underline{x}$ is a special set of vectors of $\underline{A}$ such that applying $\underline{A}$ to $\underline{x}$ does nothing but scale $\underline{x}$

Consider

$$\underline{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\underline{A}\,\underline{x} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\underline{A}\,\underline{x} = \begin{bmatrix} \\ 3 \end{bmatrix} = \underset{\lambda}{\underset{\updownarrow}{3}} \underset{\underline{x}}{\underset{\updownarrow}{\begin{bmatrix} \\ 1 \end{bmatrix}}}$$

Eigenvalue $\lambda = 3$

Eigenvector $\underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Let $\underline{A} \in M_{n \times n}$ (Eigensystems such as these only valid for square matrices)

Given $\underline{A}$, find λ + $\underline{x}$

$$\underline{A}\,\underline{x} = \lambda \underline{x}$$

$$\underline{A}\,\underline{x} - \lambda \underline{x} = \underline{0}$$

$$(\underline{A} - \lambda \underline{I})\underline{x} = \underline{0}$$

Here $\underline{x}$ must be in the nullspace of $\underline{A} - \lambda \underline{I}$

We do not want the trivial solution. Thus,

$\Rightarrow \underline{A} - \lambda \underline{I}$ can not be of full rank

$\Rightarrow (\underline{A} - \lambda \underline{I})^{-1}$ does not exist

$\Rightarrow \det(\underline{A} - \lambda \underline{I}) = 0$

unknown eigenvalue

$\det(\underline{A} - \lambda \underline{I}) \Rightarrow$ Characteristic polynomial of matrix $\underline{A}$ (in terms of λ)

Eigensystem Procedure:

Given $\underline{A} \in M_{n \times n}$

① Solve for all λ's such that

$$\det(\underline{A} - \lambda \underline{I}) = 0$$

(Find roots of characteristic polynomial)

② For each λ_i, find the corresponding $\underline{x}_i$, such that

$$\underline{A}\,\underline{x}_i = \lambda_i \underline{x}_i$$

or

$$(\underline{A} - \lambda_i \underline{I})\underline{x}_i = \underline{0}$$

Example: Let $\underline{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$$\underline{A} - \lambda \underline{I} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$$

$$\det(\underline{A} - \lambda \underline{I}) = (1-\lambda)^2 - 4 = 0$$

$$(1-\lambda)^2 = 4, \quad 1-\lambda = \pm 2$$

$$\downarrow$$

$$\left.\begin{aligned} \lambda_1 &= -1 \\ \lambda_2 &= 3 \end{aligned}\right\} \text{Eigenvalues of } \underline{A}$$

For $\lambda_1 = -1$

Solve $(\underline{A} - \lambda_1 I)\underline{x}_1 = \underline{0}$

$$\begin{bmatrix} 1-(-1) & 2 \\ 2 & 1-(-1) \end{bmatrix} \underline{x}_1 = \underline{0}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Note: Rows are linearly dependent, $\det(\underline{A}-\lambda_1\underline{I})=0$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \underline{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Eigenvector associated with $\lambda_1 = -1$

Let $\lambda_2 = 3$

$$\begin{bmatrix} 1-\lambda_2 & 2 \\ 2 & 1-\lambda_2 \end{bmatrix}\underline{x}_2 = \underline{0}$$

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\therefore \underline{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Eigenvector associated with $\lambda_2 = 3$

Summary for $\underline{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Two eigenvalues & eigenvectors

$$\lambda_1 = -1, \quad \underline{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 3, \quad \underline{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Note: Eigenvectors can be multiplied by any non-zero constant & remain eigenvectors

Check:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\underline{A} \quad \underline{x}_1 \quad = \quad \lambda_1 \quad \underline{x}_1$$

$$\lambda_1 \quad \underline{x}_1$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = (3) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\underline{A} \quad \underline{x}_2 \quad = \quad \lambda_2 \underline{x}_2 \quad \checkmark$$

Summary:

λ is an eigenvalue of square matrix $\underline{A}$ with eigenvector $\underline{x}$ iff

$$\underline{A}\,\underline{x} = \lambda \underline{x}$$

To determine λ, we need

$$(\underline{A} - \lambda \underline{I})\underline{x} = \underline{0}$$

For non-trivial solutions

$$\det(\underline{A} - \lambda \underline{I}) = |\underline{A} - \lambda \underline{I}| = 0$$

characteristic polynomial of $\underline{A}$

Find roots λ of characteristic polynomial

Example 2: Let $\underline{A} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

$$|\underline{A} - \lambda \underline{I}| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 4 = 0$$

$$\lambda^2 - 4\lambda - 1 = 0 \Rightarrow \lambda = \frac{-b \pm (b^2 - 4ac)^{1/2}}{2a}$$

$$\lambda = \frac{4 \pm (16+4)^{1/2}}{2} = 2 \pm \sqrt{5}$$

To find eigenvector, use λ for each root

1) $\lambda = 2 + \sqrt{5}$

$$\underbrace{\begin{bmatrix} 1-(2+\sqrt{5}) & 2 \\ 2 & 3-(2+\sqrt{5}) \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\underbrace{\det(\underline{A} - \lambda \underline{I})} = 0 \rightarrow$ rows must be linearly dependent singular, rank < 2

$$(-1-\sqrt{5})x_1 + 2x_2 = 0 \rightarrow \underline{x} = \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

2) $\lambda = 2-\sqrt{5}$

Similar procedure, $\underline{x} = \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$

Summary:

$$\lambda_1 = 2+\sqrt{5}, \quad \underline{x}_1 = \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

\\\\

$$\lambda_2 = 2-\sqrt{5}, \quad \underline{x}_2 = \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

Properties of Eigensystems:

1. Eigenvalues of $\underline{A}^2$ are the square of the eigenvalues of $\underline{A}$, but eigenvectors are

exactly the same

Assume $\underline{A}\,\underline{x} = \lambda \underline{x}$ is known

Then,

$$\underline{A}^2 \underline{x} = \underline{A}(\underline{A}\,\underline{x}) = \underline{A}(\lambda \underline{x}) = \lambda \underline{A}\,\underline{x}$$

$$= \lambda(\lambda \underline{x}) = \lambda^2 \underline{x} \Rightarrow \underline{A}^2 \underline{x} = \lambda^2 \underline{x}$$

Higher powers:

$$\underline{A}^3 \underline{x} = \underline{A}(\underline{A}^2 \underline{x}) = \underline{A}(\lambda^2 \underline{x}) = \lambda^2 \underline{A}\,\underline{x} = \lambda^3 \underline{x}$$

$$\Rightarrow \underline{A}^3 \underline{x} = \lambda^3 \underline{x}$$

In general, eigenvalues of $\underline{A}^n$ are λ^n of

$$\underline{A}\,\underline{x} = \lambda \underline{x}$$

Recall Markov Chains:

$\underline{M}$ is transition matrix

Columns of $\underline{M}$ sum to one; all entries > 0

$\underline{P}_n$ probability of states after step n

$$\underline{P}_n = \underline{M}\,\underline{P}_{n-1}$$

$$\underline{P}_n = \underline{M}\left(\underline{M}\,\underline{P}_{n-2}\right)$$

$$\underline{P}_n = \underline{M}^n\,\underline{P}_0$$

may need to raise $\underline{M}$ to a large power; find eigensystem of $\underline{M}$

② Row reduction does not preserve eigenvalues

Row reduction involves the scaling + addition of the matrix rows

Example: $\underline{A} = \begin{bmatrix} 4 & -1 & 0 \\ 0 & 1 & 0 \\ \end{bmatrix}$

Example. $\underline{A} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -3 & 6 \end{bmatrix}$

$$\begin{vmatrix} 4-\lambda & -1 & 0 \\ 0 & 1-\lambda & 0 \\ 2 & 3 & 6-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda)(6-\lambda) = 0$$

$$\therefore \lambda = 4, 1, 6$$

Flip rows 1 & 3

$$\underline{B} = \begin{bmatrix} 2 & -3 & 6 \\ 0 & 1 & 0 \\ 4 & -1 & 0 \end{bmatrix}$$

$$\begin{vmatrix} 2-\lambda & -3 & 6 \\ 0 & 1-\lambda & 0 \\ 4 & -1 & -\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)(-\lambda) - 4(1-\lambda)6$$

$$= -(2-3\lambda+\lambda^2)\lambda - 24(1-\lambda)$$

$$= -\lambda^3 + 3\lambda^2 - 2\lambda + 24\lambda - 24$$

$$= -\left[\lambda^3 - 3\lambda^2 - 22\lambda + 24\right]$$

$$= -(\lambda-1)(\lambda+4)(\lambda-6)$$

$$\therefore \lambda = 1, \underline{\underline{-4}}, 6$$

$$\therefore \lambda \text{ of } \underline{B} \neq \lambda \text{ of } \underline{A}$$

③ The product of the eigenvalues of $\underline{A}$ equals $\det(\underline{A})$ and the sum of the eigenvalues equals $tr(\underline{A})$, where

$$tr(\underline{A}) = \text{trace of } \underline{A} = \text{sum of the diagonal}$$

Aside: ERO of swapping rows. What happens to $\det(\underline{A})$ after swap?

④ One can have imaginary (or complex) eigenvalues, even if $\underline{A}$ is real

Example:

$$\underline{A} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 2 \\ 4 & -1 & 0 \end{bmatrix} \Rightarrow \begin{vmatrix} 6-\lambda & 0 & 0 \\ 0 & 2-\lambda & 2 \\ 4 & -1 & -\lambda \end{vmatrix} = 0$$

$$= (6-\lambda)\left[(2-\lambda)(-\lambda)-(-1)(2)\right]=0$$

$$= (6-\lambda)\left(\lambda^2-2\lambda+2\right)=0$$

$$\lambda = 6 \qquad \lambda = \frac{2 \pm (4-8)^{1/2}}{2}$$

$$\lambda = 1 \pm i$$

$$\therefore \lambda = 6,\ 1+i,\ 1-i$$

Eigenvalues can be complex!

What are the corresponding eigenvectors?

Let $\lambda = 1+i$

$$(\underline{A} - \lambda \underline{I})\underline{x} = \underline{0}$$

$$\begin{bmatrix} 6-(1+i) & 0 & 0 \\ 0 & 2-(1+i) & 2 \\ 4 & -1 & -(i+1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

rref: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1+i \\ 0 & 0 & 0 \end{bmatrix}$ rank = 2

$\det(\underline{A} - \lambda \underline{I}) = 0$

$\therefore x_1 = 0$

Let $x_2 = 1$, $x_3 = -\frac{x_2}{1+i} \cdot \frac{1-i}{1-i} = -\frac{1-i}{2}$

$$\underline{x} = \begin{bmatrix} 0 \\ 1 \\ -\frac{1}{2}(1-i) \end{bmatrix} \quad \text{for } \lambda = 1+i$$

⑤ For a real matrix, all complex eigenvectors will come in complex conjugate pairs

If $\lambda = a+ib$ and $\underline{A}\,\underline{x} = \lambda \underline{x}$ gives the eigenvector $\underline{x}$, then

$$\left(\overline{\underline{A}\,\underline{x}}\right) = \left(\overline{\lambda x}\right) \Rightarrow \underline{\bar{A}}\,\underline{\bar{x}} = \bar{\lambda}\,\underline{\bar{x}}$$

for $\underline{A}$ real $\rightarrow$ $\underline{A}\,\underline{\bar{x}} = \bar{\lambda}\,\underline{\bar{x}}$

$\therefore \underline{\bar{\lambda}}$ has eigenvector $\underline{\bar{x}}$

$\bar{\lambda}$ has eigenvector $\bar{\underset{\sim}{x}}$

⑥ Repeated eigenvalues are possible

Example:

$$\underline{A} = \begin{bmatrix} 3 & -1 & 2 \\ 3 & -1 & 6 \\ -2 & 2 & -2 \end{bmatrix} \Rightarrow \lambda = -4, 2, 2$$

Then $\lambda = 2$ has a multiplicity of 2

(more precisely, this is the algebraic multiplicity)

Let $\lambda = 2$. Then

$$\underline{A} - \lambda \underline{I} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -3 & 6 \\ -2 & 2 & -4 \end{bmatrix} \overset{\text{rref}}{\sim} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ 2 free variables

$\Rightarrow \dim(N(\underline{A} - \lambda \underline{I})) = 2$

$\Rightarrow$ 2 eigenvectors

If the # of eigenvectors (or geometric multiplicity) equals the algebraic multiplicity, then that eigenvalue is said to be complete

If the # of eigenvectors is less than the (algebraic) multiplicity, then that eigenvalue is said to be defective

Example:

$$\underline{A} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \rightarrow \lambda = 3, 3$$

Algebraic multiplicity equals 2

$$\underline{A} - \lambda \underline{I} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$, only has

eigenvector $\underline{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ Geometric multiplicity equals 1

$\Rightarrow \lambda = 3$ is defective

If any eigenvalue of $\underline{A}$ is defective, then $\underline{A}$ is said to be defective

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$$\underline{A} - \lambda \underline{I} = \begin{bmatrix} 3-3 & 1 \\ 0 & 3-3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_2 = 0$$

Choose $x_1 = 1$

for $\lambda = 3$, $\underline{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$