6.2

$p = \frac{1}{4}$

a) $C_2^6 p^2 (1-p)^4 = \frac{6!}{4!\,2!} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^4$

b) Expected # of quizzes up till 3rd failure $= 12. = (3 \times 4)$

Dave will pass 9 $= (12 - 3)$

c) $P(B) = P(C) = \frac{1}{4}$ $\qquad P(A \cap B \cap C) = 7 \times \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^6 \times \frac{1}{4}^2$

$P(A) = C_1^7 \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^6$

d) Dave fail 2 before passing 2 $\to B$.

$P(B) = P(\text{Fail Fail}) + P(\text{Pass Fail Fail}) + P(\text{Fail Pass Fail Fail}) + \dots$

$= \left(\frac{1}{4}\right)^2 + \frac{3}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)\left(\frac{1}{4}^2\right) \dots$

$P(B) = \frac{\left(\frac{1}{4}\right)^2}{1 - \frac{1}{4} \cdot \frac{3}{4}} + \frac{\frac{3}{4}\left(\frac{1}{4}\right)^2}{1 - \frac{3}{4} \times \frac{1}{4}} = \frac{7}{52}$

6.3

$P_1 = P_{1|B} P_B = \frac{5}{6} \cdot \frac{2}{5} = \frac{1}{3}$

a) $P_1(1-P_1)^3 = \frac{1}{3}\left(\frac{2}{3}\right)^3$

b) Probability I1 was busy I2 was idle | 5 out of 10 were idle.

$P_B \cdot P_I = \frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$

c) $\frac{5}{P_{1|B}} = \frac{5}{2/5} = \frac{25}{2}$

d) Pascal RV of order 5. $P_{1|B} = \frac{2}{5}$, $T = B - 5$

$P_B(t) = C_{4}^{t-1} \left(\frac{2}{5}\right)^5 \left(1-\frac{2}{5}\right)^{t-5}$

$P_T(t) = \binom{t+4}{4} \left(\frac{2}{5}\right)^5 \left(1-\frac{2}{5}\right)^t$

$E[T] = E[B] - 5 = \frac{25}{2} - 5 = 7.5$

$Var(t) = \frac{5\left(1-\frac{2}{5}\right)}{\frac{16}{25}}$

6.16

a) no arrival in 2hr

$P(0.2) = e^{-0.6 \cdot 2} = 0.301$

b) $P(0.2)(1 - P(0;3)) = e^{-0.6 \cdot 2}(1 - e^{-0.6 \cdot 3}) = 0.251$

c) $\sum_{2}^{\infty} P(k.2) = 1 - P(0.2) - P(1.2) = 1 - e^{-0.6 \cdot 2} - (0.6 \cdot 2) e^{-0.6 \cdot 2}$

$= 0.337$

d) $\lambda = 0.6 \quad 2\lambda = 1.2$

E{number of fish caught after those 2 hrs} = $P(N=1) = P(0.2) = 0.301$.

$1.2 + 0.301 = 1.501$

e) Has been fishing for 4 hours.

$\rightarrow 4 + \frac{1}{0.6} = 5.667$

6.14

a) X = time till first bulb failure.

A = A bulb, B = B bulb.

$E[X] = E[X|A]P(A) + E[X|B]P(B) = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$.

b) D = no failures until time t.

$P(D) = P(D|A)P(A) + P(D|B)P(B) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t}$

c) $P(A|D) = \frac{P(A \cap D)}{P(D)} = \frac{1}{1+e^{-2t}}$

d) $E[X^2] = E[X^2|A]P(A) + E[X^2|B]P(B) = 1 + \frac{1}{9} = \frac{10}{9}$

$E[X] = \frac{2}{3}$

$Var[X] = \frac{10}{9} - \frac{4}{9} = \frac{2}{3}$

e)
$C^{11}_{3}\left(\frac{1}{2}\right)^{12}$

f) $\binom{12}{4}\left(\frac{1}{2}\right)^{12}$

g) $\left[\frac{1}{2}\left\{\frac{1}{1-s} + \frac{3}{3-s}\right\}\right]^{12}$

h) Y = first 2 type B

$f_Y(y) = 9ye^{-3y} \quad y \geq 0$

T = first type A

$f_T(t) = e^{-t}, \quad t \geq 0$

$P(T < Y) = \int_0^\infty 9ye^{-3y}(1-e^{-y})dy$

$= \frac{7}{16}$

i) V = Type B illumination time.

N = Type B out of 12

X_i = ith B bulb time

$E[N] = 6$, $Var(N) = 12 \times \frac{1}{4} = 3$

$E[X_i] = \frac{1}{3}$ & $Var(X_i) = \frac{1}{9}$

$E[V] = 2$

$Var(V) = \frac{1}{9} \cdot 6 + \frac{1}{9} \cdot 3 = 1$

6.14

(J.) $E[T|D] = t + 1 \cdot \frac{1}{1+e^{-2t}} + \frac{1}{3}\left(1 - \frac{1}{1+e^{-2t}}\right)$

$$= t + \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{1+e^{-2t}}$$

Extra

1. P(no arrivals in (2.4]) , $\lambda=0.4$.

a) $P(k,t) = P(0,2) = e^{-0.8}\frac{(0.4\times2)^0}{0!} = 0.4493$

b) P(1 arrival in [0,1), (3,5]) $\Rightarrow$ P(A) = [0,1), P(B) = (3,5]

$P(A) = P(k_1,T_1)\,P(k_2,T_2)$

$k_1 = 1$, $k_2 = 0$

$T_1 = 1$, $T_2 = 2$.

$P(A) = e^{-0.4}\frac{(0.4)}{1} \cdot e^{-0.8} = 0.1205$

$P(B) = e^{-0.4} \cdot e^{-0.8(0.8)} = 0.2410$

c) = P(1 arrival (0,1], 3 arrival (0,5]

$\rightarrow$ P(1 arrival (0,1], 2 arrival (1,5]

$P(k,T_1)\,P(k_2,T_2)$

$k_1 = 1$, $k_2 = 2$

$T_1 = 1$, $T_2 = 4$

$\rightarrow e^{-0.4}(0.4) - e^{-1.6}\frac{(1.6)^2}{2}$

$= 8\times0.064 \times e^{-2}$

$= 0.0693$

2 $N(t) =$ Poisson with $\lambda = \lambda_1 + \lambda_2 = 3$

$P(N(1) = 2, N(2) = 5)$

$= P(2 \text{ arrival in } (0,1), 3 \text{ arrival in } (1,2))$

$$= \left(\frac{e^{-3} \cdot 3^2}{2!}\right)\left(\frac{e^{3} \cdot 3^3}{3!}\right) = 0.05$$

$N(1) = 2 \rightarrow P(N(1) = 1, N(2) = 2)$

$$= \frac{P(N_1(1) = 1, N_2(1) = 1)}{P(N(1) = 2)}$$

$$= \frac{P(N_1(1) = 1) \cdot P(N_2(1) = 1)}{P(N(1) = 2)}$$

$$= \frac{e^{1} \cdot 2e^{2}}{\frac{e^{3} \cdot 3^2}{2!}} = \frac{4}{9}$$

$= 0.4444$