EAS595 HW2 Matthew Sahi

Q3 If T is the times the game is played.

a) To win the game T times, T-1 times must occur.

therefore

$$P(\text{Fischer Wins}) = \sum_{1}^{10} (0.3)^{T-1}(0.4) = 0.571425$$

b) The match has times played T with T-1 times occurred.

The probability that either sides win is 0.7

$$P(T) = \begin{cases} (0.3)^{T-1}(0.7) & T = 1 \sim 9 \\ (0.3)^{9} & T = 10 \\ 0 \end{cases}$$

Q18 $$P_X(x) = \begin{cases} 1/(b-a+1) \\ 0 \end{cases}$$

$$E[X] = \sum_{a}^{b} \frac{1}{b-a+1} 2^k = \frac{2^{b+1} - 2^a}{b-a+1}$$

$$\to E[X^2] = \sum_{a}^{b} \frac{1}{b-a+1} (2^k)^2 = \frac{4^{b+1} - 4^a}{3(b-a+1)}$$

$$\to var(X) = \frac{4^{b+1} - 4^a}{3(b-a+1)} - \left(\frac{2^{b+1} - 2^a}{b-a+1}\right)^2.$$

Q21

① $E[X] = \sum_{1}^{\infty} 2^{k} \cdot 2^{-k} = \infty$ therefore the expected value of a single game is infinite.

② Any value would suffice.

Q31 The PMF

$= P_Y(y) = \binom{4}{y}\left(\frac{1}{6}\right)^{y}\left(\frac{5}{6}\right)^{4-y}$, $y = 0, 1, \ldots 4$.

X is the 1's in the remaining 4-y rolls.

each can take 5 values of 1, 3, 4, 5, 6.

Therefore $P_{X|Y}(x|y) = \binom{4-y}{x}\left(\frac{1}{5}\right)^{x}\left(\frac{4}{5}\right)^{4-y-x}$

$= P_Y(y) P_{X|Y}(x|y)$

for all nonnegative x, y such that $0 \le x+y \le 4$

$= \binom{4}{y}\left(\frac{1}{6}\right)^{y}\left(\frac{5}{6}\right)^{4-y}\binom{4-y}{x}\left(\frac{1}{5}\right)^{x}\left(\frac{4}{5}\right)^{4-y-x}$

for all other values $P_{X,Y}(x,y) = 0$.

Q2.36

a) $P_{X,Y,Z}(x,y,z) = P(X=x, Y=y, Z=z)$

$= P(X=x)\, P(Y=y \mid X=x)\, (Z=z \mid X=x, Y=y)$

$= P_X(x)\, P_{Y|X}(y \mid x)\, P_{Z|X,Y}(z \mid x,y)$

b) $P(X=x, Y=y, Z=z) = P(X=x)\, P(Y=y \mid X=x)\, P(Z=z \mid Y=y, X=x).$

c) $P_{X_1}(x_1)\, P_{X_2|X_1}(x_2 \mid x_1) \ldots P_{X_n|X_1,X_2 \ldots X_{n-1}}(x_n \mid x_1, x_2 \ldots x_{n-1}).$

Additional Q1

$P(X=1) = P$

$P(X=0) = 1 - P(X=1) = 1 - P = q.$

PMF : $P(X=1) = P \quad P(X=0) = q$

$P_X(x) = p^x (1-p)^{1-x}, \; x = 0, 1$

$E(X) = \sum_x x \cdot p^x (1-p)^{1-x} = 1 \cdot p^1 (1-p)^0 + 0 \cdot p^0 (1-p)^1 = p$

$E(X^2) = \sum_x x^2 p^x (1-p)^{1-x} = 1^2 \cdot p \cdot (1-p)^0 + 0 = p$

$var(x) = p - p^2 = p(1-p)$

Q2

X = number of heads in 10 tosses.

a) $P(X=5) = \binom{10}{5}(0.8)^5(1-0.9)^{10-5} \approx 0.264$

b) 0.8

c) $P = \binom{9}{4}(0.8)^4(1-0.8)^{9-4} = 0.0165$

Q3 a) $P(X>10) = \int_{10}^{30} \frac{1}{30}dx = \left[\frac{1}{30}(x)\right]_{10}^{30} = \frac{2}{3}$

b) $P(10 \le x \le 20) = \int_{10}^{20} \frac{1}{30}dx = \frac{1}{30}\int_{10}^{20} dx = \frac{10}{30} = \frac{1}{3}$

Q4. Let T = the time the police catches the suspect

A_x = the distance between the police & the suspect

B_x = after 1 second the police & suspect is x units apart

For A_x we have $A_x = (A_x \cap B_x) \cup (A_x \cap B_{x+1}) \cup (A_x \cap B_{x-1})$
$x > 1$

using total expectation theorem if $x > 1$

$$E(T \mid A_x) = P(B_x \mid A_x) E[T \mid A_x \cap B_x] + P(B_{x-1} \mid A_x) E[T \mid A_x \cup B_{x-1}] + P(B_{x-1} \mid A_x) E[T \mid A_x \cup B_{x-2}].$$

$x = 1$

$$E(T \mid A_1) = P(B_1 \mid A_1) E[T \mid A_1 \cup B_1] + P(B_0 \mid A_1) E[T \mid A_1 \cup B_0]$$

→

so by applying $E[T|A_1] = \frac{1}{(1-P)/2}$.

for $X=2$ $E[T|A_2] = \frac{2}{(1-P)/2}$.

generalizing $x > 2$

$E[T|A_x] = P(1+E[T|A_x]) + ((1-P)/2)(1+E[T|A_{x-1}]) + P(1+E[T|A_{x-2}])$

Therefore $E[T|A_x]$ can be generated recursively for all distance X.

T can be obtained using PMF for X

$E[T] = \sum_x P_x(x) E[T|A_x]$.