Lecture 7 - Outline

Monday, September 16, 2019 8:17 PM

HW 1

Markov Chains

Vector Spaces + Subspaces

Markov Chains

Monday, September 16, 2019 9:46 AM

Study the evolution of states over time

Example: The banks A, B & C, which currently have a 40%, 10% + 50% of the investors, respectively

Let $P_0 = \begin{bmatrix} 0.40 \\ 0.10 \\ 0.50 \end{bmatrix}$

On a yearly basis, bank A retains 50% of their investors, while 25% go to B and the remaining 25% go to C

B retains 66.6%, while 16.7% go to A + C each

For C, the retention is 50%, with those going to A + B in equal numbers

Transition (Markov) matrix

$$\underset{\sim}{M} = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.5 & 0.167 & 0.25 \\ 0.25 & 0.666 & 0.25 \\ 0.25 & 0.167 & 0.50 \end{bmatrix} \end{array}$$

Columns must each sum to 1

What is new state after 1 year? n years?

$$\underline{P}_1 = \underline{M}\,\underline{P}_0 \quad \text{for} \quad \underline{P}_0^T = \begin{bmatrix} 0.4 & 0.1 & 0.5 \end{bmatrix}$$

$$\underline{P}_1 = \begin{bmatrix} 0.342 \\ 0.292 \\ 0.367 \end{bmatrix}$$

$$\underline{P}_2 = \underline{M}\,\underline{P}_1 = \underline{M}^2\,\underline{P}_0 = \begin{bmatrix} 0.312 \\ 0.372 \\ 0.318 \end{bmatrix}$$

Stochastic matrix

Square matrix with entries that are non-zero and each column sums to 1

Example: Markov matrix $\underline{M}$

Theorem

The product of a finite number of stochastic matrices is a stochastic matrix

Markov Chain

Distinct states $S_1, S_2, \ldots, S_n$

1) Each element resides in one of those states
2) Elements can move from one state to another
3) Probabilities of movement are fixed

In curent example

States S_1, S_2, S_3 are the banks A, B, C,

Elements are the investors

Theorem

After n steps $n \geq 1$, the probability is given by

$$\underline{P}_n = \underline{M}^n \underline{P}_0$$

Thus, given $\underline{M}$ and $\underline{P}_0$ all future steps are determined

What happens as $n \to \infty$?

$$\lim_{k \to \infty} \underline{P}_k = \lim_{k \to \infty} \underline{M}^k \underline{P}_0 = \underline{M}_\infty \underline{P}_0$$

In present bank example,

$$\begin{bmatrix} 0.286 & 0.286 & 0.286 \end{bmatrix}$$

$$\underline{M}_\infty = \begin{bmatrix} 0.286 & 0.286 & 0.286 \\ 0.429 & 0.429 & 0.429 \\ 0.286 & 0.286 & 0.286 \end{bmatrix}$$

Then

$$\underline{M}_\infty \underline{P}_0 = \begin{bmatrix} 0.286 \\ 0.429 \\ 0.286 \end{bmatrix}$$ Equilibrium probabilities

This also can be written as

$\underline{P}_\infty = \underline{M}\ \underline{P}_\infty$, which is called a fixed point, such that $f(x) = x$

If $\underline{M}$ is known, then just solve an eigenproblem

$$\underline{M}\ \underline{P} = \underline{P}$$

Vector Spaces & Subspaces

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Definition: A vector space is the collection of vectors with the same dimension that follows a set of rules

The vector space of vectors of real numbers is $\mathbb{R}^n$, with n as the dimension of the vectors

Examples:

- $\underline{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ is in $\mathbb{R}^3$
- Real scalars lie in $\mathbb{R}^1$ (or simply $\mathbb{R}$)
- π is in $\mathbb{R}$
- Complex vectors live in $\mathbb{C}^n$

$$\underline{u} = \begin{bmatrix} -i \\ 1+i \end{bmatrix} \text{ is in } \mathbb{C}^2$$

- Real matrices of dimension $m \times n$ live in the vector space $\mathbb{R}^{m \times n}$

  Typically denoted by M_{mn}

- Real functions live in some space F

Vector spaces defined by their collection and how operations take place

"Vectors" inside vector space remain inside

Rules of a Vector Space

Let $\underline{x}, \underline{y}, \underline{z}$ be in a particular vector space V with a + b as scalars in $\mathbb{R}$

All vectors spaces must obey:

① $\underline{x} + \underline{y} = \underline{y} + \underline{x}$ must be in V

② $\underline{x} + (\underline{y} + \underline{z}) = (\underline{x} + \underline{y}) + \underline{z}$ must be in V

③ unique zero vector exists, such that

$$\underline{0} + \underline{x} = \underline{x} + \underline{0} = \underline{x}$$

4) for every $\underline{x}$, there exists $-\underline{x}$, such that

$$\underline{x} + (-\underline{x}) = (-\underline{x}) + \underline{x} = \underline{0}$$

5) $a(\underline{x} + \underline{y}) = a\underline{x} + a\underline{y}$ must be in V

6) $(a+b)\underline{x} = a\underline{x} + b\underline{x}$ " "

7) $a(b\underline{x}) = b(a\underline{x})$ " "

8) $1\underline{x} = \underline{x}$

If all of these rules are followed, then the vector space is closed

Subspaces

A portion of a vector space is called a subset of that vector space

Denote this subset of a vector space V as W

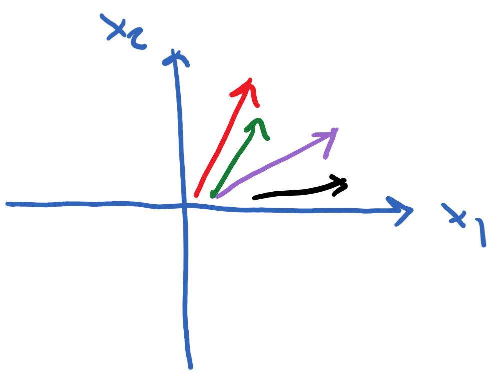
If W is closed under addition and multiplication, as defined above, then W is a subspace of V

Closed means that after addition and multiplication the result is in W

Examples:

$\boxed{1}$ All vectors in $\mathbb{R}^2$, such that

$$\underline{v} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{with } a \geq 0,\ b \geq 0$$



Is this subset W a vector space?

Check addition:

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$$

OK, still in W

Check multiplication:

$$k\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ka \\ kb \end{bmatrix}$$

Is $ka \geq 0$

Is $kb \geq 0$

for $k \in \mathbb{R}$

Result not in W for $k<0$

$\therefore$ W is a subset of U, but not a subspace

2. Let W be all vectors of the form

$$\left[a, b, \frac{a}{2} - 2b\right]$$

Is this a subspace of R^3?

Check addition:

$$\left[a, b, \frac{a}{2} - 2b\right] + \left[c, d, \frac{c}{2} - 2d\right]$$

$$= \left[a+c, b+d, \frac{a}{2} - 2b + \frac{c}{2} - 2d\right]$$

$$= \left[a+c, b+d, \frac{(a+c)}{2} - 2(b+d)\right]$$ ✓ ok

Check multiplication:

$$k\left[a, b, \frac{a}{2} - 2b\right]$$

$$= \left[ka, kb, \frac{ka}{2} - 2kb\right] \quad \text{for } k \in \mathbb{R}^1$$

✓ ok

$\therefore$ W is a subspace of $\mathbb{R}^3$

Span

Let S be a non-empty subset of vectors in vector space V. Then,

$$S \subseteq V$$

(is contained in)

All finite linear combinations of the vectors in S form the span of S, written span(S)

Examples:

$\boxed{1}$ Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Then, span(S) is all of $\mathbb{R}^2$

Any vector in $\mathbb{R}^2$ can be written as

$$a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}a\\b\end{bmatrix}$$

$\boxed{2}$ Let

$$S=\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$$

Then, span(S) is all vectors in $\mathbb{R}^4$ of the form $\begin{bmatrix}a\\0\\0\\b\end{bmatrix}$

Is this a subspace?

All vectors in this subset are

$$a\begin{bmatrix}1\\0\\0\\1\end{bmatrix}+b\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Check addition:

$$\left(a\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + b\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right) + \left(c\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + d\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right)$$

$$= (a+c)\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + (b+d)\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$ Yes, in span(S)

Check multiplication:

$$k\left(a\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + b\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right) = ka\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + kb\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$ Yes!

$\therefore$ span(S) is a subspace of $\mathbb{R}^4$

Ideas of subspace and span also applies to matrix and function spaces

Examples:

$\boxed{1}$ Let U_2 be set the of 2×2 upper triangular matrices and L_2 be the set of 2×2 lower triangular matrices

$$U_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

$$L_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Let $S = U_2 \underbrace{\cup}_{\text{union}} L_2$

Then, span(S) contains all 2×2 matrices, called $\underline{M}_{22}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$+ d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem

Let S be a non-empty subset of vector space V. Then,

① $S \subseteq \text{span}(S)$
(is contained in)

② $\text{span}(S)$ is a subspace of V

③ If W is a subspace of V with $S \subseteq W$, then $\text{span}(S) \subseteq W$

④ $\text{span}(S)$ is the smallest subspace of V containing S

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① Any vector in $S: \{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_n\}$ can be written as a linear combination of the subset S

$$\underline{v}_1 = 1\,\underline{v}_1 + 0\,\underline{v}_2 + \dots + 0\,\underline{v}_n$$

② $\text{span}(S)$ is a subspace of V

$\text{span}(S)$: $a_1\underline{v}_1 + a_2\underline{v}_2 + \dots + a_n\underline{v}_n$

$$(a_1 v_1 + a_2 v_2 + \dots + a_n \underline{v}_n) + (b_1 v_1 + b_2 v_2 + \dots + b_n \underline{v}_n)$$

$$= (a_1 + b_1)\underline{v}_1 + (a_2 + b_2)\underline{v}_2 + \dots + (a_n + b_n)\underline{v}_n$$

$$k(a_1\underline{v}_1 + a_2\underline{v}_2 + \dots + a_n\underline{v}_n)$$

$$= ka_1\underline{v}_1 + ka_2\underline{v}_2 + \dots + ka_n\underline{v}_n$$

$\rightarrow$ span(S) is a subspace

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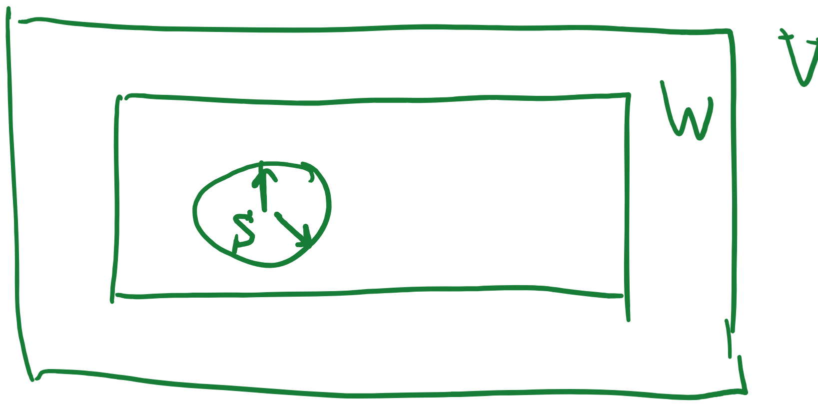

4 Summary of 1 $\rightarrow$ 3

Vector Independence

Let $\underline{a}_1$, $\underline{a}_2$ and $\underline{a}_3$ be vectors of same dimension

If the only combination of $\underline{a}_1, \underline{a}_2$ and $\underline{a}_3$ that results in the zero vector is

$$0\underline{a}_1 + 0\underline{a}_2 + 0\underline{a}_3 = \underline{0}$$

then $\underline{a}_1$, $\underline{a}_2$ and $\underline{a}_3$ are independent

On the other hand, if some non-trivial combination exits, then $\underline{a}_1$, $\underline{a}_2$ & $\underline{a}_3$ are dependent

Example: Is $\underline{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\underline{a}_2 = \begin{bmatrix} -1 \\ -4 \\ -5 \end{bmatrix}$, $\underline{a}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

independent?

$$2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 1\begin{bmatrix} -1 \\ -4 \\ -5 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ No!

$$\underline{a}_3 = 2\underline{a}_1 + \underline{a}_2$$

Linear Independence & Dependence

Let S be a subset of vector space V

S is linearly dependent, if some non-zero linear combination of S results in the zero vector

$S = \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$

Some $a_1\underline{v}_1 + a_2\underline{v}_2 + \dots + a_n\underline{v}_n = \underline{0}$

for some $a_1 \neq 0$, $a_2 \neq 0$, ... or $a_n \neq 0$

and the span (S) is linearly dependent.

Example:

$$S \in \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$$

However,

$$\overset{a_1}{(1)} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\underline{v}_1} + \overset{a_2}{(1)} \underbrace{\begin{bmatrix} -1 \\ -1 \end{bmatrix}}_{\underline{v}_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$ Linearly dependent

If not linearly dependent, then

linearly independent

Basis

B is a basis for vector space V iff

1. B spans all of V
2. B is linearly independent

Example:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is a basis for } \mathbb{R}^3$$

Example:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} \right\} \text{ also form a basis for } \mathbb{R}^3$$

(Basis is not unique)

Example:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis for M_{22}

Example:

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} = B$$

S is a subspace of $\mathbb{R}^4$ and is also a basis for that subspace

Dimension

The dimension of a vector space V is the minimum number of vectors needed in a basis B of V

If the number of vectors in B is finite, then $\dim(V) \rightarrow$ dimension of V, is finite

Otherwise V has infinite dimension

Example:

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$ is a basis

for $\mathbb{R}^3$, number of elements in the

set $\rightarrow \dim(\mathbb{R}^3) = |B| = 3$

set $\Rightarrow \dim(\mathbb{R}^3) = |B| = 3$

Example:

$$\dim(\mathbb{R}^n) = n$$

Example:

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$V = \text{span}(B) \Rightarrow \dim(V) = 3$$

Example:

Vector space P_3: All polynomials of order 3 & below

$P_3 : \{1, x, x^2, x^3\}$ basis for P_3

$$\Rightarrow a(1) + b(x) + c(x^2) + d(x^3)$$

$$\dim(P_3) = 4$$

More generally

$$\dim(P_n) = n+1$$

Example:

$$\dim(M_{22}) = 4$$

$$\dim(M_{mn}) = mn$$

Example: Infinite Dimensional Space

Taylor Series

$$B = \{ (x-a)^0, (x-a)^1, (x-a)^2, \ldots, (x-a)^n \ldots \}$$

$$F = \alpha (x-a)^0 + \beta (x-a)^1 + \gamma (x-a)^2 + \ldots$$

$$\dim(\text{Taylor Series}) = \infty$$