HW3 Matthew Sah

#1 $g(x) = \begin{cases} 1 & \text{if } x \le \frac{1}{3} \\ 2 & x > \frac{1}{3} \end{cases}$

$E[Y] = 1 \times \frac{1}{3} + 2 \times (1 - \frac{1}{3}) = \frac{5}{3}$ #

#3 Target Radius r.

a) Distance from Center X.

$F(x) = P(X \ge x) = \frac{\pi x^2}{\pi r^2} = \frac{x^2}{r^2}$ $\quad f(x) = \begin{cases} \frac{2x}{r^2} & 0 \le x \le r \\ 0 & \text{otherwise} \end{cases}$

$f(x) = F'(x) = \frac{x^2}{r^2} dx = \frac{2x}{r^2}$

$E[X] = x f(x) dx$

$= \int_0^r x \frac{2x}{r} dx = \int_0^r \frac{2x^2}{r^2} dx = \frac{2}{3} \frac{x^3}{r^2} \Big|_0^r \quad \frac{2}{3}\left(\frac{r^3}{r^2} - \frac{0}{0}\right)$

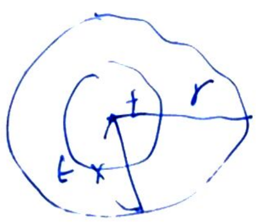
$E[x] = \frac{2}{3} r.$

$E[x^2] = \int x^2 f(x) dx = \int_0^r x^2 \frac{2x}{r^2} dx = \frac{1}{2} \frac{x^4}{r^2} \Big|_0^r = \frac{1}{2} r^2.$

$E[x^2] = \frac{1}{2} r^2$

$var\{x\} = E[x^2] - (E[x])^2 = \frac{r^2}{2} - \frac{4r^2}{9} = \frac{r^2}{18}$

b)



For $0 \le s < 1/t$ no score on target — Area of on target.

$F_S(s) = P(S \le s) = P(\text{no points on target}) = 1 - P(X \le t)$
$= 1 - \frac{t^2}{r^2}$

For $\frac{1}{t} < s$ score

$F_S(s) = P(S \le s) = P(X \le t) P(S \le s \mid X \le t) + P(X > t) P(S \le s \mid X > t)$

cont. #3.

$P(X \le t) = \frac{t^2}{r^2}$, $P(X > t) = 1 - \frac{t^2}{r^2}$

$\to P(S \le s \mid X \le t) = P(1/X \le s \mid X \le t) = \frac{P(\frac{1}{s} \le X \le t)}{P(X \le t)} = \frac{\pi t^2 - \pi (1/s)^2}{\pi t^2} = 1 - \frac{1}{s^2 t^2}$

$P(S \le s) = 1 - \frac{1}{s^2 r^2}$

the CDF of S

$F_S(s) = \begin{cases} 0 & \text{if } s < 0 \\ 1 - \frac{t^2}{r^2} & \text{if } 0 \le s \le 1/t \\ 1 - \frac{1}{s^2 r^2} & \text{if } 1/t \le s. \end{cases}$ S is not continuous.

#4.

a) $F_X(x) = P(X \le x) = P \times (Y \le x) + (1-P)(Z \le x)$
$= P F_Y(x) + (1-P) F_Z(x).$

$f(x) = F'(x) = p f_Y(x) + p f_Z(x)$

b) $f_X(x) = \begin{cases} p\lambda e^{\lambda x} & x < 0 \\ (1-P)\lambda e^{-\lambda x} & x \ge 0. \end{cases}$ $\to f_Y(y) \begin{cases} \lambda e^{\lambda y} & y < 0 \\ 0 & \text{otherwise} \end{cases}$

PDF

$\to f_Z(z) \begin{cases} 0 & \text{otherwise} \\ \lambda e^{-\lambda z} & y \ge 0 \end{cases}$

$\to$

#4 cont

CDF = $F_Y(y) = \begin{cases} e^{\lambda y} & y < 0 \\ 1 & y \geq 0 \end{cases}$ using (a))

$F_Z(z) = \begin{cases} 0 & z < 0 \\ 1 - e^{-\lambda z} & z \geq 0 \end{cases}$

$F_X(p) = \begin{cases} pe^{\lambda x} & x < 0 \\ 1 - (1-p)e^{-\lambda x} & x \geq 0. \end{cases}$

#5

a) X has mean 0. var = 1. X is normalized.

$P(X \leq 1.5) = \Phi(1.5) = 0.9332.$

$P(X \leq -1) = 1 - \Phi(1) = 0.1587$

b) the PDF of Y is the standard normal

c) using the standard normal of Y. as Z

$P(-1 \leq Y \leq 1) = P(-1 \leq Z \leq 0)$

$= \Phi(1) - \Phi(0)$

$= 0.3413.$

#6. $P(X \geq k\sigma) = P(Z \geq k) = 1 - \Phi(k).$

thus $P(X \geq k) = 0.1587$

$P(X \geq 2k) = 0.0228$

$P(X \geq 3k) = 0.0014.$

$P(|X| \leq k\sigma) = P(|Z| \leq k) = \Phi(k) - (1 - \Phi(k)) = 2\Phi(k) - 1$

$P(|X| \leq k) = 0.6826$

$P(|X| \leq 2k) = 0.9544$

$P(|X| \leq 3k) = 0.9972.$

#2. $f(x) = \begin{cases} \frac{2x}{55} & 3 \le x \le 8 \\ 0 & \text{otherwise.} \end{cases}$

a) $P(X \le 5) = \int_3^5 \frac{2x}{55} = \frac{2}{55} \times \frac{1}{2} \times x^2 \Big|_3^5 = \frac{16}{55}$

b) $E[X] = \int_3^8 x\, f(x)\, dx = \int_3^8 \frac{2x^2}{55}\, dx = \frac{2}{165} x^3 \Big|_3^8 = \frac{194}{33}$

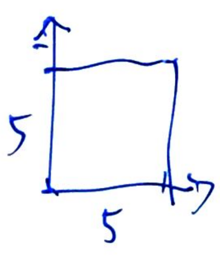
#7. $f_{x,y}(x,y) = \begin{cases} kxy^2 & 0<x<2,\ 0<y<1 \\ 0 & \text{otherwise.} \end{cases}$

$\rightarrow f_{x,y}(x,y) = 1 \longrightarrow \int_0^1 \int_0^2 kxy^2\, dx\, dy = 1$

$\int_0^1 \frac{1}{2} k x^2 y^2\, dy \Big|_0^2 = \int_0^1 k y^2 \left[\frac{y}{2} - \frac{0}{2}\right] dy. = \int_0^1 2ky^2\, dy.$

$= \left[2ky^3 \times \frac{1}{3}\right]\Big|_0^1 = \frac{2k}{3} = 1 \quad k = 1.5$

#8 $F(x,y) = \frac{1}{250}(20xy - x^2y - xy^2) \quad 0 \le x \le 5,\ 0 \le y \le 5$

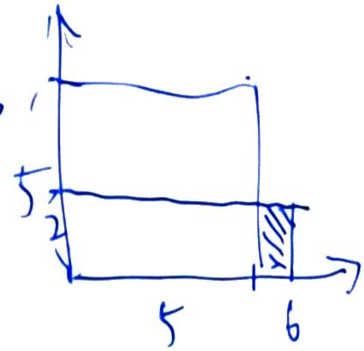


a) $P(1 \le y \le 3) \rightarrow F(5,3) - F(5,1)$

$= \frac{1}{250}\left(20 \times 5 \times 3 - 5^2 \times 3 - 5 \times 3^2 - (20 \times 5 \times 1 - 5^2 \times 1 - 5 \times 1^2)\right) = \frac{110}{250}$

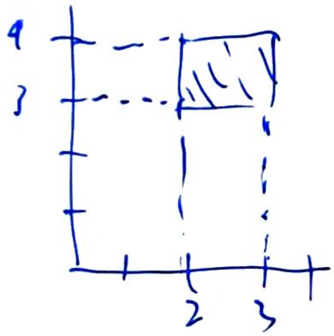
cont 8 b)

$F(6.2)$ →



$F(6.2) = 0$

c) $P[(2 \le X \le 3) \wedge (3 \le Y \le 4)]$



$= F(3.4) - F(3.3) - F(2.4) + F(2.3)$

using $F(x.y) = \frac{1}{250}(20xy - x^2y - y^2x)$

$$= \frac{1}{250}\big((240 - 36 - 48) - (180 - 27 - 27) - (160 - 16 - 32) + (120 - 12 - 18)\big)$$

$= 156 - 126 - 112 + 90 = \frac{8}{250}$

$= 0.032.$

#9

a) $f(x) = \begin{cases} \frac{x}{4} & 1 \le x \le 3 \\ 0 & \text{otherwise.} \end{cases}$

$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_1^3 \frac{x}{4} x\, dx = \frac{x^3}{12}\Big|_1^3 = \frac{26}{12}$

$f_{X|A}(x) = \begin{cases} \frac{f(x)}{P(A)} & x \in A \\ 0 & \text{otherwise} \end{cases}$

$P(A) = \int_2^3 \frac{x}{4} dx = \frac{x^2}{8}\Big|_2^3 = \frac{5}{8}.$

$E[X|A] = \int_2^3 x \cdot \frac{2x}{5} dx = \frac{2x^3}{15}\Big|_2^3 = \frac{38}{15}$

$= \begin{cases} \frac{2x}{5} & 2 \le x \le 3 \\ 0 & \text{otherwise.} \end{cases}$

b) $Y = X^2$

$E[Y] = E[X^2] = \int_1^3 x^2 \times f(x) \times dx = \int_1^3 \frac{x^3}{4} dx = 5$

$\text{var}(Y) = E[Y^2] - E[Y]^2 = \int_1^3 \frac{x^5}{4} dx - 5 = \frac{91}{3} - 5^2 = \frac{16}{3}.$

#10. stick length l

break length Y. $0 \le Y \le 1$

a) $f_{X,Y}(x,y) = f_Y(y) f_{X|Y}(x|y) = \begin{cases} \frac{1}{1} \cdot \frac{1}{y} & 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

b). $f_X(x) = \int f_{X,Y}(x,y) dy = \frac{1}{l} \ln(l/x) \quad 0 \le x \le 1$

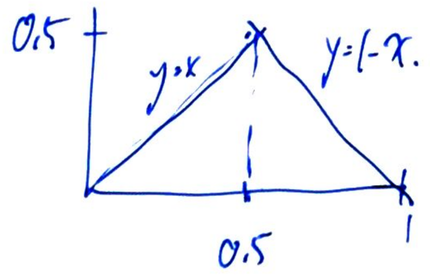
c) $E[X] = \int_0^1 x f(x) dx = \int_0^1 \frac{x}{l} \ln\left(\frac{l}{x}\right) dx = \frac{1}{4}$

d) $E[X] = E[Y] E\left[\frac{X}{Y}\right]$

$= \frac{l \times l}{2 \times 2} = \frac{l}{4}$

Final Question

$f_{X,Y}(x,y) = c$



then

$$\int_0^{\frac{1}{2}}\int_{y=0}^{y=x} c\,dy\,dx + \int_{\frac{1}{2}}^{1}\int_{y=0}^{y=1-x} c\,dy\,dx = 1.$$

$$\int_0^{\frac{1}{2}} cy\Big|_0^x dx + \int_{\frac{1}{2}}^{1} cy\Big|_0^{1-x} dx = \int_0^{\frac{1}{2}} cx\,dx + \int_{\frac{1}{2}}^{1} c(1-x)\,dx$$

$$= \frac{1}{2}cx^2\Big|_0^{\frac{1}{2}} + \left(cx - \frac{1}{2}cx^2\right)\Big|_{\frac{1}{2}}^{1} \quad = \frac{1}{8}c + c - \frac{1}{2}c - \frac{1}{2}c + \frac{1}{8}c = 1$$

$$c = 4.$$

$$f_{X|Y}(x) = \frac{f(x,y)}{f_Y(y)}.$$

$$f_Y(y) = \int_{x=y}^{x=\frac{1}{2}} 4\,dx + \int_{\frac{1}{2}}^{1} 4\,dx = 2 - 4y + 4(1-y) - 2$$

$$= -4y + 4 - 4y = -8y + 4$$

$$f_{X|Y}(x) = \frac{4}{-8y+4} = \frac{1}{-2y+1}$$