Lecture 5 - Outline

Tuesday, September 10, 2019 12:18 PM

Office Hours: Friday, Sept 13, 2:30pm-4:30pm

Properties of matrix operations

Operations on matrices

- Matrix transpose
- Matrix determinant
- Matrix powers
- Matrix inverse

Block matrices

Linear equation systems

Properties of Matrix Operations

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Commutative Property of Addition

$$\underline{A} + \underline{B} = \underline{B} + \underline{A} \quad , \quad a_{ij} + b_{ij} = b_{ij} + a_{ij}$$

Distributive Property of Scalar Multiplication

$$c(\underline{A} + \underline{B}) = c\underline{A} + c\underline{B} , \quad c(a_{ij} + b_{ij}) = ca_{ij} + cb_{ij}$$

Associative Property of Matrix Products

$$\underset{m \times n}{\underline{A}}(\underset{n \times p}{\underline{B}}\,\underset{p \times q}{\underline{C}}) = (\underline{A}\,\underline{B})\underline{C} \quad \text{with compatible } \underline{A}, \underline{B}, \underline{C}$$

Matrix Products not Commutative

In general, $\underline{A}\,\underline{B} \neq \underline{B}\,\underline{A}$

$\underline{A}\,\underline{B}$ may be compatible, but perhaps $\underline{B}\,\underline{A}$ is not!

Distributive Property of Matrix Products

$$\underline{C}(\underline{A} + \underline{B}) = \underline{C}\,\underline{A} + \underline{C}\,\underline{B}$$ Left

$$(\underline{A}+\underline{B})\underline{C} = \underline{A}\underline{C} + \underline{B}\underline{C}$$ Right

$$(\underline{A}+\underline{B})\underline{x} = \underline{A}\underline{x} + \underline{B}\underline{x}$$ Vector version

Operations on Matrices

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Matrix Transpose

Flip rows + columns; denote with ^{T}

Let $\underset{\sim}{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $\underset{\sim}{A} \in \mathbb{R}^{2 \times 3}$

$\underset{\sim}{A}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ $\underset{\sim}{A}^T \in \mathbb{R}^{3 \times 2}$

$(a_{ij})^T = a_{ji}$

If symmetric $(n \times n)$ $a_{ji} = a_{ij}$

skewsymmetric (anti) $a_{ji} = -a_{ij}$

Vector transpose

$\underline{x} = \begin{bmatrix} 1 \\ 6 \\ -5 \\ 4 \end{bmatrix}$ $\underline{x}^T = \begin{bmatrix} 1 & 6 & -5 & 4 \end{bmatrix}$

$(\underline{x}^T)^T = \underline{x}$

Dot products $\leftrightarrow$ Vector transpose

$$\underline{u} \cdot \underline{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

$$\underline{u}^T \underline{v} = \underbrace{\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}}_{1 \times n} \underbrace{\begin{bmatrix} v_1 \\ | \\ v_n \end{bmatrix}}_{n \times 1} = u_1 v_1 + \dots + u_n v_n$$

Inner product

$$\left.\begin{aligned} \underline{u} \cdot \underline{v} &= \underline{u}^T \underline{v} = \underline{v}^T \underline{u} \\ \underline{u} \cdot \underline{u} &= \underline{u}^T \underline{u} = \| \underline{u} \|^2 \end{aligned}\right\} \text{Scalar}$$

Outer product?

$$\underline{u} \otimes \underline{v} = \underbrace{\begin{bmatrix} u_1 \\ | \\ u_n \end{bmatrix}}_{n \times 1} \underbrace{\begin{bmatrix} v_1 & v_2 & - & v_n \end{bmatrix}}_{1 \times n}$$

$$= \begin{bmatrix} u_1 v_1 & u_1 v_2 & - \\ & & \\ u_n v_1 & u_n v_2 & - \end{bmatrix}$$

$$\begin{bmatrix} u_n v_1 & u_n v_2 & \text{—} \end{bmatrix}_{n \times n}$$

Matrix Determinant

Scalar info for a square matrix

$\det(\underline{A}) = |\underline{A}|$ with recursive definition

or $\det \underline{A}$

1×1 $\quad \underline{A} = [a_{11}] \quad \det(\underline{A}) = a_{11}$

2×2 $\quad \underline{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \det(\underline{A}) = a_{11}a_{22} - a_{21}a_{12}$

3×3 $\quad \underline{A} = \begin{bmatrix} a_{11} & \text{—} & a_{13} \\ | & & | \\ & \text{—} & a_{33} \end{bmatrix}$

$$\det(\underline{A}) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Laplace's expansion ($n \times n$)

$$\det(\underline{A}) = \sum_{j=1}^{n} a_{ij} (-1)^{i+j} M_{ij}$$

minor of $\underline{A}$ obtained by removing ith row & jth column

$$C_{ij} = (-1)^{i+j} M_{ij} \qquad \text{co-factor}$$

Pick any row i, result is same

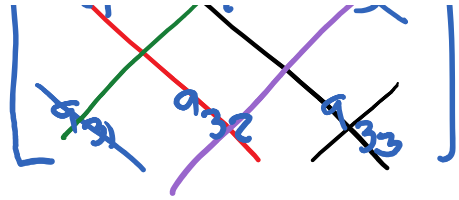
<u>or</u> $$\det(\underline{A}) = \sum_{i=1}^{n} a_{ij} C_{ij}$$

Pick any column j, result is same

Expand about row i (or column j)

Sarrus' method

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32}$$



$+ a_{13}a_{21}a_{32}$

$- a_{13}a_{22}a_{31} - a_{32}a_{23}a_{11}$

$- a_{21}a_{12}a_{33}$

Operation count

$\Theta(n!)$ ~ Laplace's expansion

NG for large n

Properties:

$$\det(\underline{A}\,\underline{B}) = \det(\underline{A})\det(\underline{B})$$

$$\det(\underline{I}_n) = 1$$

$$\det(\underbrace{\alpha}_{\text{scalar}}\underline{I}_n) = \alpha^n \quad \text{for } \alpha \in \mathbb{R}$$

$$\det(\alpha\underline{A}) = \det(\alpha\underline{I}_n\underline{A})$$
$$= \det(\alpha\underline{I}_n)\det(\underline{A})$$
$$= \alpha^n\det(\underline{A})$$

$$\det(\underline{A}^T) = \det(\underline{A})$$

det $(\underline{A})$ provides info on character of $\underline{A}$

Consider 2x2 case

$$\underline{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

What is det$(\underline{A}) = 0$?

$$\det(\underline{A}) = ad - bc = 0 \quad \therefore \; c = \frac{ad}{b}, \; b = \frac{ad}{c}$$

$$\underline{A} = \begin{bmatrix} a & b \\ \frac{ad}{b} & d \end{bmatrix} = \begin{bmatrix} a & b \\ a\underbrace{\left(\frac{d}{b}\right)}_{\alpha} & b\underbrace{\left(\frac{d}{b}\right)}_{\alpha} \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ \alpha a & \alpha b \end{bmatrix}$$

$$\underline{A} = \begin{bmatrix} a & \frac{ad}{c} \\ c & d \end{bmatrix} = \begin{bmatrix} a & a\left(\frac{d}{c}\right) \\ c & c\left(\frac{d}{c}\right) \end{bmatrix}$$

$$= \begin{bmatrix} a & \beta a \\ c & \beta c \end{bmatrix}$$

$$= \begin{bmatrix} c & \beta c \end{bmatrix}$$

Say $\underline{A}$ is singular

Rows are not linearly independent

Columns are not linearly independent

Matrix Powers

$$\underline{A}^p = \underbrace{\underline{A}\,\underline{A} \cdots \underline{A}}_{p \text{ times}}$$

$$\underline{A}^p \underline{A}^q = \underline{A}^{p+q} \qquad (\underline{A}^p)^q = \underline{A}^{pq}$$

For scalar algebra $c^0 = 1$

Then, for linear algebra $\underline{A}^0 = \underline{I}_n$

for $\underline{A} \in \mathbb{R}^{n \times n}$

$$\underline{A}\,\underline{I} = \underline{A} \quad , \underline{I}\,\underline{A} = \underline{A} , \underline{I}\,\underline{x} = \underline{x}$$

$$\underline{A}^1 \underline{A}^0 = \underline{A}^{1+0} = \underline{A}^1 = \underline{A}$$

How about $\ln(\underline{A})$? $e^{\underline{A}}$? $\sin(\underline{A})$?

Taylor series (Mcclaurin series)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{\underline{A}} = \sum_{n=0}^{\infty} \frac{\underline{A}^n}{n!}$$

Matrix Inverse

Let $\underline{A}$ be square $n \times n$

Multiplicative inverse

$$\underline{A}^{-1} \underline{A} = \underline{I}_n \qquad \left(\frac{1}{c}\right)c = c\left(\frac{1}{c}\right) = 1$$

$$\underline{A}\,\underline{A}^{-1} = \underline{I}_n$$

$$\underline{A}^1 \underline{A}^{-1} = \underline{A}^{(1-1)} = \underline{A}^0 = \underline{I}_n$$

Find solution to sets of eqns

$$\underline{A}\,\underline{x} = \underline{b}$$

$$\underbrace{\underline{A}^{-1}\,\underline{A}}\,\underline{x} = \underline{A}^{-1}\,\underline{b}$$

$$\underline{I}_n \underline{x} = \underline{x} = \underline{A}^{-1}\underline{b}$$

but very inefficient!

Determinant of inverse

$$\det(\underline{A}^{-1}) = \frac{1}{\det(\underline{A})}$$

Can use $\det(\underline{A})$ as a check on existance of $\underline{A}^{-1}$

If $\det(\underline{A}) \neq 0$, then $\underline{A}^{-1}$ exists

If $\det(\underline{A}) = 0$, then $\underline{A}^{-1}$ does not exist

& we say $\underline{A}$ is singular

Check 2x2 example,

Check 2x2 example,

$$\underline{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\underline{A}^{-1} = \frac{1}{\det(\underline{A})}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\underline{A}\,\underline{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad-bc}\begin{bmatrix} ad-bc & -ab+ba \\ cd-dc & -cb+ad \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underline{I}_2$$

Block Matrices

Usually, components are numbers

but not necessary

Components could be matrices themselves

$$\underline{A} = \begin{bmatrix} \underline{A}_{11} & \underline{A}_{12} & \underline{A}_{13} \\ \underline{A}_{21} & \underline{A}_{22} & \underline{A}_{23} \end{bmatrix}$$

Submatrices must be compatible

$$\begin{array}{ccc} A_{11} \rightarrow & A_{12}, & A_{13} \\ m \times n & m \times p & m \times q \\ \downarrow & \downarrow & \downarrow \\ A_{21} & A_{22} & A_{23} \\ r \times n & r \times p & r \times q \end{array}$$

$$\underline{A} = \begin{bmatrix} \underline{I}_2 & \underline{I}_2 & \underline{I}_2 \\ \underline{I}_2 & \underline{I}_2 & \underline{I}_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Can simplify calculations & notation

Usual rules apply

Linear Equation Systems

Wednesday, September 11, 2019 10:33 AM

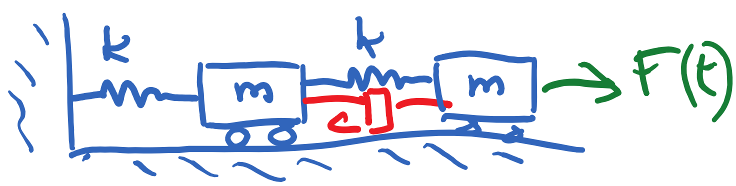
Many application across social + physical sciences and engineering

$\underline{A}\,\underline{x} = \underline{b}$ Systems of linear equations

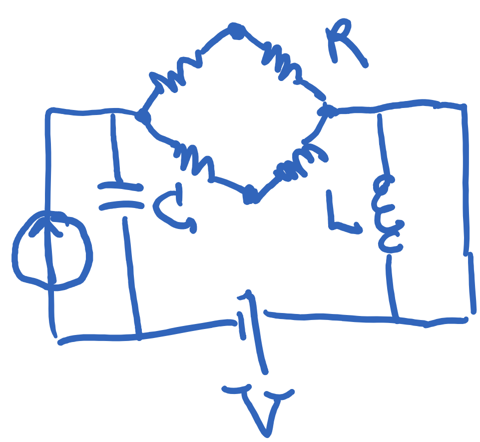
Linear transformation

Examples:

Structural systems



Electrical circuits



Networks

Social ; Technological (internet, power grid)
Biological (ecological, neural)

Adjacency matrix