EAS595 HW4 Matthew Sah.

4.2

let $Y = e^X$

$$PDF = P(Y \le y) = P(e^X \le y) = \begin{cases} P(X \le \ln y) & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{d}{dx} F_X(\ln y) & y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

when X is uniform at $[0, 1]$.

$$\rightarrow f_Y(y) = \begin{cases} \frac{1}{y} & 0 < y \le e \\ 0 & \text{otherwise.} \end{cases}$$

4.5

let $Z = |X - Y|$

$$F_Z(z) = P(|X-Y| \le z) = 1 - (1-z)^2$$

$$f_Z(z) = \begin{cases} 2(1-z) & 0 \le z \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

4.7 let $Z = \max(X, Y)$.

for $t \in [0, 1]$

$$P(Z \le t) = P(X \le t) P(Y \le t) = t^2.$$

$$f_Z(z) = \begin{cases} 0 & z < 0. \\ 2z & 0 \le z \le 1 \\ 0 & z \le 1 \end{cases}$$

$$\rightarrow E[Z] = \int_{-\infty}^{\infty} z f(z) dz$$

$$= \int_0^1 2z^2 dz = \frac{2}{3}.$$

distance $= 1 - \frac{2}{3} = \frac{1}{3}$.

4.17.

since covariance is unchanged after added constant.

X, Y has 0 mean.

$Cov(X-Y, X+Y) = E[(X-Y)(X+Y)] = Var(X) - Var(Y) = 0.$

X & Y have same variance.

4.24 let X = time professor devoted to task.

Y = Length of time from 9am of his arrival.

Since duration of task : $\frac{1}{(5-y)}$

a)

$E[X \mid Y=y] = \frac{1}{\lambda(y)} = 5-y = E[X \mid Y].$

$E[X] = E[E[X \mid Y]] = 5 - E[Y] = \underline{3}$ #

b) let Z = 9am until professor finishes his task. $(X+Y)$.

$E[Z] = \underline{5}$ #

c) W = length of time from 9:00am ~ PHD student arrive.

R = under condition of meeting the professor, time spent with professor

T = time spent between prof & student.

F = student finds professor.

$E[T \mid F] = E[R] = \frac{1}{2}$

$E[T] = E[T|F]P[F] = \frac{1}{3}P[F]$

$P(W > X+Y)$

$= \int_0^4 P(W > X+Y \mid Y = y) f_Y(y) dy$

$= \frac{12}{32} + \frac{1}{32}\int_0^4 (5-y) e^{-\frac{8-y}{5-y}} dy$

to find $P[F]$

$P[F] = P[Y \le W \le X+Y]$

$P(F) = 1 - P(W < Y) + P(W > X+Y)$

$= 1 - \frac{1}{4} + \left(\frac{12}{32} + \frac{1}{32} \times 1.7584\right)$

$= 0.32$

$E[T] = \frac{1}{2}P[F] = 0.16$

$\Rightarrow 9.6$ mins

$E[X+Y] = 5$

$E[X+Y+R] = 5 + E[R] = \frac{11}{2}$

$E[Z] = 0.68 \times 5 + 0.16 \times 0.11 = \underline{5.16}$

4.29

① $M_S = E[e^{SX}] = \frac{1}{2}e^{S} + \frac{1}{4}e^{2S} + \frac{1}{4}e^{3S}$

② $E[X] = \frac{1}{2} + \frac{2}{4} + \frac{3}{4} = \frac{7}{4}$

$E[X^2] = \frac{15}{4}$

$E[X^3] = \frac{37}{4}$

Extra 1.

$Z = X - Y^2 \rightarrow X = Z + Y^2.$

$F_Z(z) = P(X \le Y^2 + z)$

$-1 \le z \le 1 \;<\; \begin{matrix} -1 \le z \le 0 \\ 0 \le z \le 1 \end{matrix}$

$-1 \le z \le 0$

$$F_Z(z) = \int_{\sqrt{-z}}^{y=1} \int_{x=0}^{y^2+z} 1 \, dy\, dx$$

$$= \int_{y=\sqrt{-z}}^{y=1} y^2 + z \, dy$$

$$= \frac{1}{3} y^3 + zy \Big|_{\sqrt{-z}}^{1}$$

$$= \frac{1}{3}\left[1 - (-z)^{\frac{3}{2}}\right] + z\left[1 - (-z)^{\frac{1}{2}}\right]$$

$$f_Z(z) = 1 - \frac{3}{2}(-z)^{\frac{1}{2}} - \frac{z}{2}(-z)^{-\frac{1}{2}}$$

$0 \le z \le 1$

$$F_Z(z) = 1 - \int_0^{y=\sqrt{1-z}} \int_{x=y^2+z}^{x=1} 1 \, dx\, dy$$

$$= 1 - \int_0^{\sqrt{1-z}} (1 - y^2 - z)\, dy$$

$$= 1 - (1-z)^{\frac{1}{2}}(1-z) + \frac{1}{3}(1-z)^{\frac{3}{2}}$$

$$f_Z(z) = \frac{1}{2}\left[(1-z)^{-\frac{1}{2}}(1-z) + (1-z)^{\frac{1}{2}}\right]$$

$$f_Z(z) = \begin{cases} 1 - \frac{3}{2}(-z)^{\frac{1}{2}} - \frac{z}{2}(-z)^{-\frac{1}{2}} & -1 \le z \le 0 \\ \frac{1}{2}\left[(1-z)^{-\frac{1}{2}}(1-z) + (1-z)^{\frac{1}{2}}\right] & 0 \le z \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Extra 2

$y = z - x$

$-2 \le z \le 2$

$-2 \le z \le 0$

$f_Z(z) = \frac{1}{4}\int_{-1}^{z+1} (x+1)(1-z+x)\,dx$

$= -\frac{1}{24}(z-4)(z+2)^2$

$0 \le z \le 2$

$f_Z(z) = \frac{1}{4}\int_{z-1}^{1} (x+1)(1-z+x)\,dx$

$= \frac{1}{24}(z-2)^2(z+4)$

PDF $f_Z(z) = \begin{cases} -\frac{1}{24}(z-4)(z+2)^2 & -2 \le z \le 0 \\ \frac{1}{24}(z-2)^2(z+4) & 0 \le z \le 2 \\ 0 & \text{otherwise} \end{cases}$

Exam 3.

$M_x(t) = E[e^{ex}]$

a)
$$= \int_{-1}^{1} e^{tx} \cdot \left(\frac{1+x}{2}\right) dx$$

$$= \frac{1}{2}\left[\int_{-1}^{1} e^{tx}\, dx + \int_{-1}^{1} x e^{tx}\, dx\right]$$

$$= \frac{1}{2}\left[\left(\frac{e^{tx}}{t}\right)\Big|_{-1}^{1} + \left(\frac{xe^{tx}}{t} - \frac{1}{t^2}e^{tx}\right)\Big|_{-1}^{1}\right]$$

$$= \frac{1}{2}\left[\frac{e^{t}}{t}(2-t) - \frac{e^{-t}}{t}\left(-\frac{1}{t}\right)\right]$$

$$= \frac{1}{2}\left[\frac{e^{t}}{t}(2-t) + \frac{e^{-t}}{t^2}\right] \qquad M_x(t) = \frac{1}{2t}\left[e^{t}(2-t) + \left(\frac{e^{-t}}{t}\right)\right]$$

b)
$$E[x] = \frac{d}{dt} M_x(t)$$

$$= \frac{d}{dt}\left\{\frac{1}{2t}\left(e^{t}(2-t) + \left(\frac{e^{-t}}{t}\right)\right)\right]\Bigg|_{t=0}$$

$$= t^{-1}e^{t} + e^{t}(-1)t^{-2} - \frac{e^{t}}{2} + \frac{1}{2}\left[-e^{-t}(t^{-2}) + e^{-t}(-2)t^{-3}\right]\Bigg|_{t=0}$$

$$= \text{N} - \frac{1}{2} + \text{N}$$

cannot find mean or variance.