Lecture 4 - Outline

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Vectors

Dot product & inner angle

Schwartz + triangle inequalities

Matrices

Definitions

Operations

Vectors (continued)

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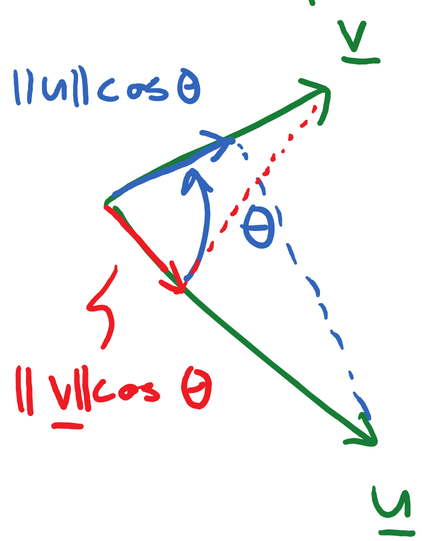
Linear Algebra is essentially a manipulation of vectors & groups of vectors

Dot Product (revisited)

Gives the inner angle between two vectors in the plane defined by those vectors



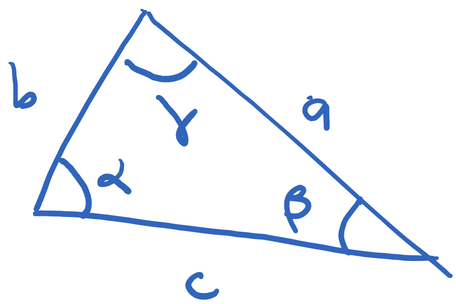
Show that

$$\underline{u}\cdot\underline{v} = \|u\|\|v\|\cos\theta$$

Cosine is even

$$\cos(-\theta) = \cos\theta$$

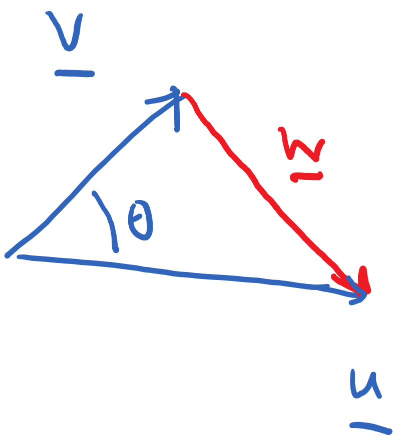
Law of cosines



$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

$$a^2 = \_\_$$

$$b^2 = \_\_$$



$$\underline{u} = \underline{v} + \underline{w}$$

$$\underline{w} = \underline{u} - \underline{v}$$

$$\|\underline{w}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2 - 2\|\underline{u}\|\|\underline{v}\|\cos\theta$$

$$\|\underline{u} - \underline{v}\|^2 = (\underline{u} - \underline{v})\cdot(\underline{u} - \underline{v})$$

$$= \underline{u}\cdot\underline{u} - \underline{v}\cdot\underline{u} - \underline{u}\cdot\underline{v} + \underline{v}\cdot\underline{v}$$

$$= \|\underline{u}\|^2 - 2\underline{u}\cdot\underline{v} + \|\underline{v}\|^2$$

$$\cancel{\|\underline{u}\|^2} \cancel{-2}\,\underline{u}\cdot\underline{v} + \cancel{\|\underline{v}\|^2} = \cancel{\|\underline{u}\|^2} + \cancel{\|\underline{v}\|^2} \cancel{-2}\|\underline{u}\|\|\underline{v}\|\cos\theta$$

$$\underline{u}\cdot\underline{v} = \|\underline{u}\|\|\underline{v}\|\cos\theta$$

Show relation between $\underline{u}\cdot\underline{v} = 0$ and

whether $\underline{u}$ & $\underline{v}$ are perpendicular $\perp$
(orthogonal)

If $\underline{u}$ & $\underline{v}$ are $\perp$, then

$$\underline{u} \cdot \underline{v} = \|\underline{u}\| \|\underline{v}\| \cos\theta \text{ for } \theta = \pm\frac{\pi}{2}$$

$$\cos\left(\pm\frac{\pi}{2}\right) \stackrel{?}{=} 0$$

and $\underline{u} \cdot \underline{v} = 0$

If $\underline{u} \cdot \underline{v} = 0$, then are $\underline{u}$ & $\underline{v}$ $\perp$?

$$\underline{u} \cdot \underline{v} = \|\underline{u}\| \|\underline{v}\| \cos\theta = 0$$

In general $\|\underline{u}\| \neq 0$, $\|\underline{v}\| \neq 0$ (Assume neither $\underline{u}$ or $\underline{v}$ are zero vector)

Thus $\cos\theta = 0 \Rightarrow \theta = \pm\pi/2$

$\therefore$ $\underline{u} \cdot \underline{v} = 0$ iff $\underline{u}$ & $\underline{v}$ are $\perp$ (or orthogonal)

iff = If and only if

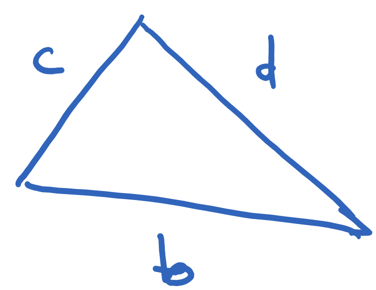
Consider $\underline{u} \cdot \underline{v} = \|\underline{u}\| \|\underline{v}\| \cos\theta$

$$|\underline{u}\cdot\underline{v}| = \big|\, \|\underline{u}\|\,\|\underline{v}\|\cos\theta \,\big|$$

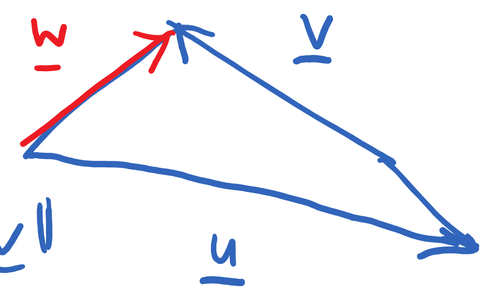
$$= \|\underline{u}\|\,\|\underline{v}\|\,|\cos\theta|$$

Schwartz inequality $\quad |\underline{u}\cdot\underline{v}| \leq \|\underline{u}\|\,\|\underline{v}\|$

Recall $\quad |c+d| \leq |c| + |d|$



Triangle Inequality of Vector



$$\|\underline{u}+\underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$$

Consider whether

$$\|\underline{u}+\underline{v}\|^2 \overset{?}{\leq} \left(\|\underline{u}\| + \|\underline{v}\|\right)^2 \qquad (1)$$

LHS $\|\underline{u}+\underline{v}\|^2 = (\underline{u}+\underline{v})\cdot(\underline{u}+\underline{v})$

$$= \|\underline{u}\|^2 + 2\underline{u}\cdot\underline{v} + \|\underline{v}\|^2$$

RHS $(\|\underline{u}\|+\|\underline{v}\|)^2 = \|\underline{u}\|^2 + 2\|\underline{u}\|\|\underline{v}\| + \|\underline{v}\|^2$

Combine LHS + RHS into (1)

$$\cancel{\|\underline{u}\|^2} + \cancel{2}\underline{u}\cdot\underline{v} + \cancel{\|\underline{v}\|^2} \overset{?}{\leq} \cancel{\|\underline{u}\|^2} + \cancel{2}\|\underline{u}\|\|\underline{v}\| + \cancel{\|\underline{v}\|^2}$$

$$\underline{u}\cdot\underline{v} \overset{?}{\leq} \|\underline{u}\|\|\underline{v}\|$$

but

$$\underline{u}\cdot\underline{v} \leq |\underline{u}\cdot\underline{v}|$$

+ from Schwartz inequality

$$|\underline{u}\cdot\underline{v}| \leq \|\underline{u}\|\|\underline{v}\|$$

$\therefore$ $\underline{u}\cdot\underline{v} \leq \|\underline{u}\|\|\underline{v}\|$ and

$$\|\underline{u}+\underline{v}\|^2 \leq (\|\underline{u}\| + \|\underline{v}\|)^2$$

$$\|\underline{u}+\underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$$ Triangle inequality

inequality

length of
head-to-tail
summation

sum of the
length of
individual
vectors

Matrices

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Definition

Ordered collection of elements or components (usually numbers, either real or complex) arranged into rows and columns

For m rows and n columns, matrix is of dimension $m \times n$

Note:

$$\left.\begin{array}{l}\text{Column vector } \underline{v} \text{ is of dimension } m \times 1 \\ \text{Row vector } \underline{u} \text{ is of dimension } 1 \times n\end{array}\right\} \text{Vectors as matrices}$$

Notation

$\underline{A}$, $\mathbf{A}$, a_{ij}

bold

indicial notation

capital letters for matrices

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

3x2

With real components

$\underset{\sim}{A} \in \mathbb{R}^{3\times 2}$

Special Matrices (assumed square nxn)

Identity matrix

$$\underset{=}{I}_n = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

Ones on main diagonal; zero elsewhere

Diagonal matrix

$$\underset{=}{D} = \begin{bmatrix} \alpha & & & 0 \\ & \beta & & \\ & & \gamma & \\ 0 & & & \ddots & \\ & & & & \delta \end{bmatrix}$$

Only non-zero components are on main diagonal

Symmetric matrix $s_{ij} = s_{ji}$

$$\underset{=}{S} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & \cdots & s_{1n} \\ s_{12} & s_{22} & & & \\ s_{13} & & \ddots & & \\ & & & \ddots & \\ s_{1n} & & & & s_{nn} \end{bmatrix}$$

Symmetric about main diagonal

Skew-symmetric matrix $z_{ij} = -z_{ji}$

$$\underline{z} = \begin{bmatrix} z_{11} & z_{12} & z_{13} & - & z_{1n} \\ -z_{12} & & & & \\ -z_{13} & & & & \\ & & & & \\ -z_{1n} & & & & \end{bmatrix}$$

$z_{11} = -z_{11}$

$\therefore z_{11} = 0$

$z_{ii} = 0$

no sum

Matrix Operations

Matrix Addition

$$\underbrace{\underline{A} + \underline{B}}_{\substack{\text{Must be of} \\ \text{same dimension} \\ m \times n}} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \\ a_{31}+b_{31} & a_{32}+b_{32} \end{bmatrix} = \underline{C}$$

Here $m=3$, $n=2$

$a_{ij} + b_{ij} = c_{ij}$

Compatible wrt matrix addition

Commutative? Yes $\underline{A} + \underline{B} = \underline{B} + \underline{A}$

Matrix Subtraction

$\underline{A} - \underline{B}$ ~ component-by-component subtraction

$a_{ij} - b_{ij}$

Matrix-Scalar Products

$$c\underline{A} = \begin{bmatrix} ca_{11} & - & ca_{1n} \\ | & \ddots & | \\ ca_{m1} & - & ca_{mn} \end{bmatrix} = \underline{A}c$$

Commutative

ca_{ij}

Matrix-Vector Products

$$\underline{A}\,\underline{x} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

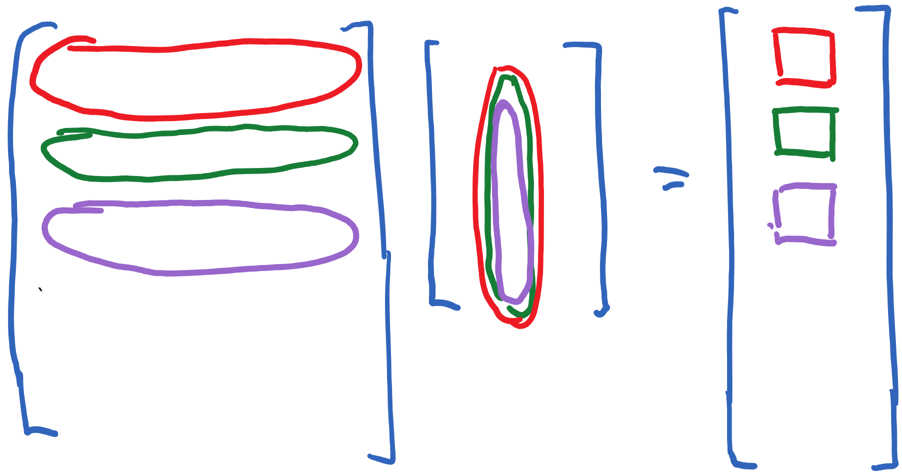
3×2 $\quad$ 2×1

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \\ a_{31}x_1 + a_{32}x_2 \end{bmatrix}$$

$\underbrace{3 \times 1}$

$$A_{ij} x_j = y_i \qquad \underline{A}\,\underline{x} = \underline{y}$$

sum over j



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In this view, matrix-vector products are a bunch of dot products of the rows of $\underline{A}$ with the vector $\underline{x}$

$\underline{A}$ + $\underline{x}$ must be compatible

A Different View:

$$\underline{A}\,\underline{x} = \begin{bmatrix} a_{11} x_1 + a_{12} x_2 \\ a_{21} x_1 + a_{22} x_2 \end{bmatrix}$$

$$= x_1 \underline{a}_1 + x_2 \underline{a}_2$$

sum of vector-scalar products

Let columns of $\underline{A}$ be represents as vectors

$$\underline{A} = \begin{bmatrix} \underline{u} & \underline{v} & \underline{w} \end{bmatrix} \text{ with } \underline{x} = \begin{bmatrix} b \\ c \\ d \end{bmatrix}$$

$$\underline{A}\,\underline{x} = b\underline{u} + c\underline{v} + d\underline{w}$$

Linear combination of vectors

Matrix-Matrix Products

$$\underline{A}\,\underline{B} = ?$$

Let $\underline{B} = \begin{bmatrix} \underline{b}_1 & \underline{b}_2 & \cdots & \underline{b}_q \end{bmatrix}$

each vector is of dimension p

$p \times q$

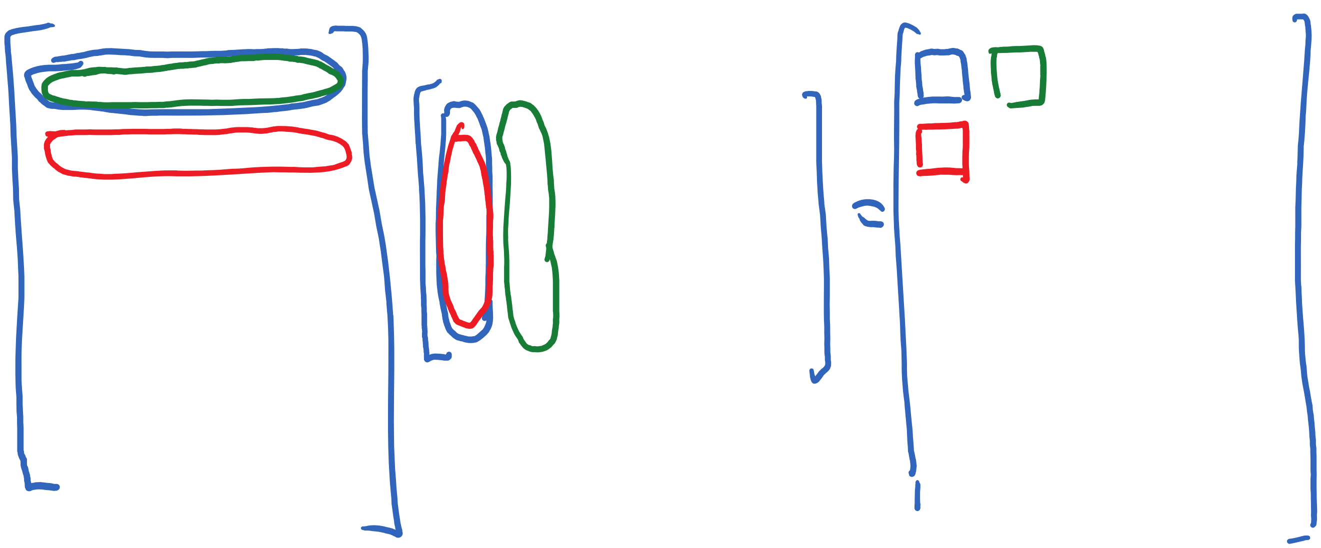
$$\underset{m\times p}{\underline{A}}\ \underset{p\times q}{\underline{B}} = \underline{A}\,[\underline{b}_1\ \underline{b}_2\ \cdots\ \underline{b}_q]$$

$$= \underbrace{\Big[\underbrace{\underline{A}\underline{b}_1}_{\substack{\underline{c}_1 \\ m\times 1}}\ \ \underbrace{\underline{A}\underline{b}_2}_{\substack{\underline{c}_2 \\ m\times 1}}\ \cdots\ \underbrace{\underline{A}\underline{b}_q}_{\substack{\underline{c}_q \\ m\times 1}}\Big]}_{\substack{\underline{C} \\ m\times q}}$$

$$\underset{m\times p}{\underline{A}}\ \underset{p\times q}{\underline{B}} = \underset{m\times q}{\underline{C}}$$

$$a_{ij}\, b_{jk} = c_{ik}$$

sum $j = 1, \dots, p$



$\underset{\sim}{A}$ + $\underset{\sim}{B}$ must be compatible

$$\underset{\sim}{A} \in \mathbb{R}^{m \times p}$$

$$\underset{\sim}{B} \in \mathbb{R}^{p \times q}$$

$$\underset{\sim}{C} \in \mathbb{R}^{m \times q}$$