Lecture 3 - Outline

Tuesday, September 3, 2019 9:46 PM

Matlab: anonymous functions

Floating point: special cases

Vectors

Floating Point Representations

Wednesday, September 4, 2019 9:34 AM

Lecture 2 Representation

sign	signed exponent	mantissa
$\pm$	$\underbrace{\pm\, \varepsilon_1 \varepsilon_2 \text{ — } \varepsilon_m}_{\varepsilon}$	$\underbrace{d_1 d_2 \text{ — } d_p}_{M}$

e.g. 0.1101011

$$\therefore\ r = \pm M B^{\varepsilon}$$

base of number system

IEEE 754 Representation

sign	biased exponent	mantissa
$\pm$	$\underbrace{\varepsilon_1 \varepsilon_2 \text{ — } \varepsilon_m}_{\varepsilon}$	$\underbrace{d_1 d_2 \text{ — } d_p}_{M}$

$$r = \pm M B^{\varepsilon - \varepsilon_0}$$

$\varepsilon - \varepsilon_0$: actual exponent

$-\varepsilon_0$: offset

Type	Sign	Actual Exponent	Exp (biased)	Exponent field	Fraction field	Value
Zero	0	−126	0	0000 0000	000 0000 0000 0000 0000 0000	0.0
Negative zero	1	−126	0	0000 0000	000 0000 0000 0000 0000 0000	−0.0
One	0	0	127	0111 1111	000 0000 0000 0000 0000 0000	1.0
Minus One	1	0	127	0111 1111	000 0000 0000 0000 0000 0000	−1.0
Smallest denormalized number	*	−126	0	0000 0000	000 0000 0000 0000 0000 0001	$\pm 2^{-23} \times 2^{-126} = \pm 2^{-149} \approx \pm 1.4 \times 10^{-45}$
"Middle" denormalized number	*	−126	0	0000 0000	100 0000 0000 0000 0000 0000	$\pm 2^{-1} \times 2^{-126} = \pm 2^{-127} \approx \pm 5.88 \times 10^{-39}$
Largest denormalized number	*	−126	0	0000 0000	111 1111 1111 1111 1111 1111	$\pm(1-2^{-23}) \times 2^{-126} \approx \pm 1.18 \times 10^{-38}$
Smallest normalized number	*	−126	1	0000 0001	000 0000 0000 0000 0000 0000	$\pm 2^{-126} \approx \pm 1.18 \times 10^{-38}$
Largest normalized number	*	127	254	1111 1110	111 1111 1111 1111 1111 1111	$\pm(2-2^{-23}) \times 2^{127} \approx \pm 3.4 \times 10^{38}$
Positive infinity	0	128	255	1111 1111	000 0000 0000 0000 0000 0000	+∞
Negative infinity	1	128	255	1111 1111	000 0000 0000 0000 0000 0000	−∞
Not a number	*	128	255	1111 1111	non zero	NaN
* Sign bit can be either 0 or 1 .						

Comparing floating-point numbers

Every possible bit combination is either a NaN or a number with a unique value in the affinely extended real number system with its associated order, except for the two bit combinations negative zero and positive zero, which sometimes require special attention (see below). The binary representation has the special property that, excluding NaNs, any two numbers can be compared as sign and magnitude integers (endianness issues apply). When comparing as 2's-complement integers: If the sign bits differ, the negative number precedes the positive number, so 2's complement gives the correct result (except that negative zero and positive zero should be considered equal). If both values are positive, the 2's complement comparison again gives the correct result.

Vectors

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Definition

Ordered collection of element or components (usually number, either real or complex)

Also called a tuple

Dimension of vector denotes number of components

Vector with n components is an n-tuple

Notation

$\vec{v}$, $\underset{\sim}{v}$, $\underline{v}$, v_i

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Components v_1 + v_2

v_i indicial notation

Algebraic Operations

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Addition (+) $c+d$

Subtraction (−) $c-d$

Multiplication (×) $c\times d$ (or cd)

Division (÷) $c\div d$ (or $\frac{c}{d}$)

Properties of algebraic operations

	Commutativity	Associativity
Addition	$c+d=d+c$	$(b+c)+d=b+(c+d)$
Subtraction	×	
Multiplication	$cd=dc$	$(bc)d=b(cd)$
Division	×	

$\underline{v}$ above is column vector

$\underline{u} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$ row vector

Vector Operations

Addition, subtraction

Multiplication by scalars

Linear combinations

Products (inner + outer)

Let $\underline{u}$ + $\underline{v}$ arbitrary vectors of same dimension (assume column vectors)

c + d real scalars ($c, d \in \mathbb{R}$)

Vector addition

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

$$\begin{bmatrix} u_4 \end{bmatrix} \qquad \begin{bmatrix} v_4 \end{bmatrix}$$

$$\underline{u} + \underline{v} = \begin{bmatrix} u_1 \\ | \\ u_4 \end{bmatrix} + \begin{bmatrix} v_1 \\ | \\ v_4 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ | \\ u_4 + v_4 \end{bmatrix}$$

$$\underline{w} = \underline{u} + \underline{v} = \begin{bmatrix} w_1 \\ | \\ w_4 \end{bmatrix}$$

Commutative?

$$\underline{u} + \underline{v} = \underline{v} + \underline{u}$$

$$\underset{4 \times 1}{\begin{bmatrix} u_1 + v_1 \\ | \\ u_4 + v_4 \end{bmatrix}} = \begin{bmatrix} v_1 + u_1 \\ | \\ v_4 + u_4 \end{bmatrix}$$

$$w_i = u_i + v_i = v_i + u_i$$

$$\underline{x} = \underset{1 \times 4}{\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}} \qquad \underline{u} + \underline{x} \to \text{undefined}$$

Vector Subtraction

$$\underline{u} - \underline{v} = \underline{z} = \begin{bmatrix} u_1 - v_1 \\ | \\ u_n - v_n \end{bmatrix}$$

$$u_i - v_i = z_i$$

Commutative? No!

Additive inverse

$$\underline{u} + \underline{y} = \underline{0}$$

Then $+\underline{y}$ is additive inverse

$$\underline{y} = -\underline{u}$$

Special vectors

$$\underline{0} = \begin{bmatrix} 0 \\ 0 \\ | \\ 0 \end{bmatrix}$$ Zero Vector

$$\underline{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad \text{One vector}$$

Multiplication by Scalars

$$\underline{y} = c\,\underline{u} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_4 \end{bmatrix} = \underline{u}\,c$$

Commutative? Yes

$$\underline{y} = -\underline{u} = c\underline{u} \quad \text{with } c = -1$$

Associative

$$(\underline{u} + \underline{v}) + \underline{w} \overset{?}{=} \underline{u} + (\underline{v} + \underline{w}) \quad \text{Yes}$$

$$(u_i + v_i) + w_i \overset{?}{=} u_i + (v_i + w_i) \quad \text{Yes}$$

Linear Combinations of Vectors (very important)

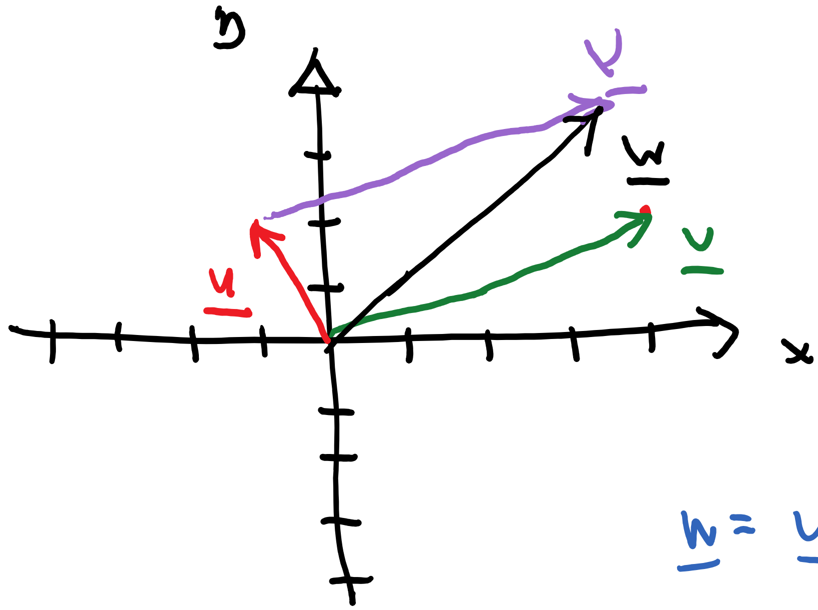
$$c\underline{u} + d\underline{v} = \underline{w}$$

$$cu_i + dv_i = w_i$$

Graphical Representations

Let $n = 2$ (2D)

$\underline{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \underline{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$



$\underline{w} = \underline{u} + \underline{v}$

Head-to-tail

$\underline{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Let $c = d = 0$

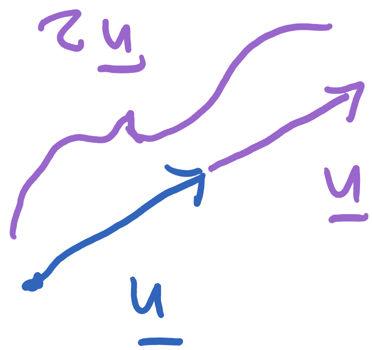
zero vector

$$0\underline{u} + 0\underline{v} = \underline{0}$$

General c

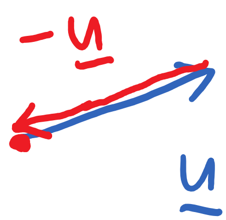
$c\underline{u} + 0\underline{v} = c\underline{u}$

$2\underline{u} = \underline{u} + \underline{u}$



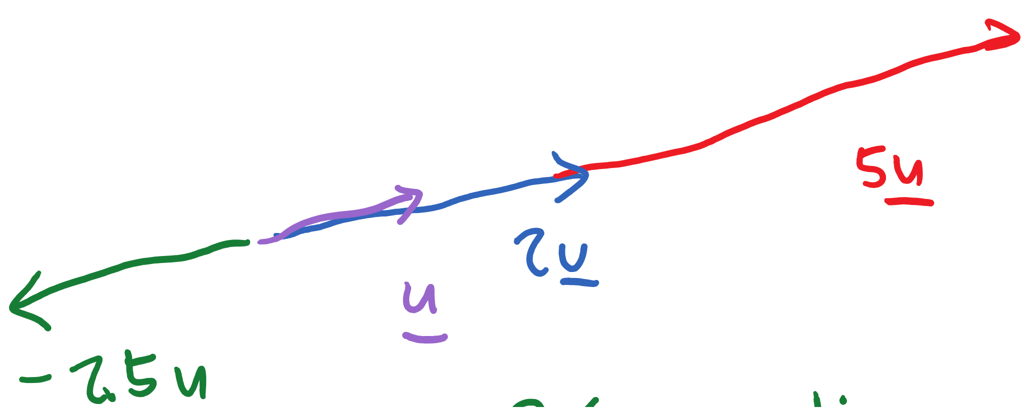
Head-to-tail

$\underline{u} - \underline{u} = \underline{0}$



What are all combinations $c\underline{u}$?

with $c \in [-\infty, +\infty]$

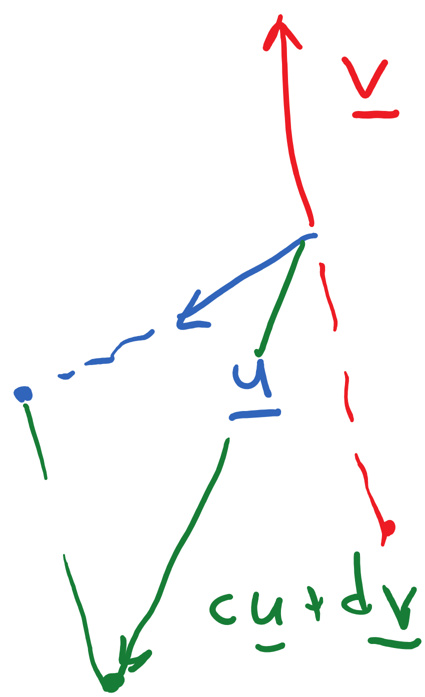


Defines a line

Linear combination of two vectors $\underline{u}$ + $\underline{v}$

$$\underline{w} = c\underline{u} + d\underline{v}$$

$c, d \in [-\infty, +\infty]$



Defines plane that contains both $\underline{u}$ + $\underline{v}$

$\underline{u}$ + $\underline{v}$ should not be colinear (same direction)

Similarly, linear combination of 3 vectors of dimension 3

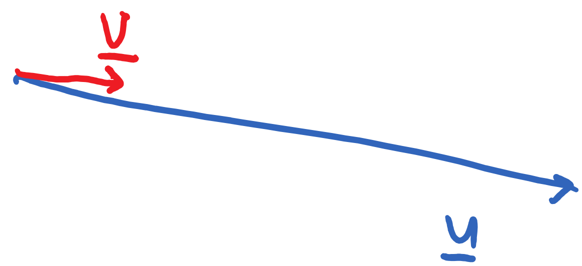
$\underline{u}, \underline{v}, \underline{w}$

$$\underline{z} = c\underline{u} + d\underline{v} + e\underline{w} \qquad c, d, e \in \mathbb{R}$$

$[-\infty, +\infty]$

Described 3D space (or volume)

Any stipulations? $\underline{u}, \underline{v} + \underline{w}$ cannot be co-planar



Multiplication of Vectors

$$\underline{u} = \begin{bmatrix} u_1 \\ | \\ u_4 \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} v_1 \\ | \\ v_4 \end{bmatrix}$$

How do we multiply vectors?

$\underline{u} \cdot \underline{v}$ Dot product (Inner product)

$$\underline{u} \cdot \underline{v} = \begin{array}{l} u_1 v_1 \\ + u_2 v_2 \\ + \quad | \\ + u_4 v_4 \end{array}$$

$$\underline{u} \cdot \underline{v} = \sum_{i=1}^{n} u_i v_i$$

^{T} $\begin{bmatrix} v_1 \end{bmatrix}$

$$\underline{u}^T \underline{v} = \begin{array}{c} \quad i=1 \\ \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \end{array} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

$$\underline{u} \cdot \underline{v} = u_i v_i$$

sum over i is implied for repeated indices

Result is a <u>scalar</u>

Example:

$$\underline{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\underline{u} \cdot \underline{v} = (-1)(4) + (2)(-1) = -6$$

Commutative!

Let's dot $\underline{u}$ with itself

$$\underline{u} \cdot \underline{u} = u_i u_i$$
$$= u_1^2 + u_2^2 + \ldots + u_n^2 = \|\underline{u}\|^2$$

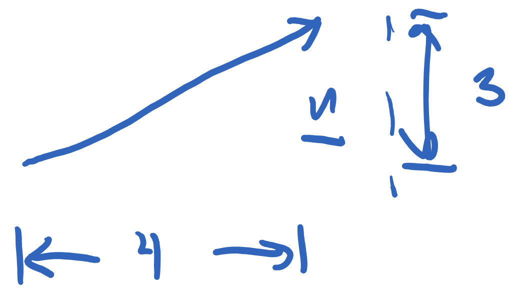
$= \text{length of } \underline{u} \text{ squared}$

$l = (u_1^2 + u_2^2)^{1/2}$ in 2D

$\|\underline{u}\|$: magnitude or length of $\underline{u}$

called the 2-norm

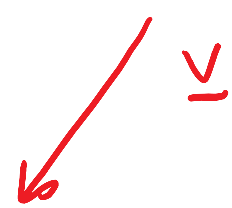
$\underline{u} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$



$\underline{u} \cdot \underline{u} = 4^2 + 3^2 = 25$

$\|\underline{u}\| = 5$ is length

$\underline{v} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$



$\|\underline{v}\| = (4 + 9)^{1/2} = \sqrt{13}$

If $\underline{u} \cdot \underline{u} = \|\underline{u}\| = 1$, then $\underline{u}$ is a unit vector

Let

$$\underline{u} = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{4}\underline{1}$$

$$\underline{u} \cdot \underline{u} = 4\left(\frac{1}{4} \cdot \frac{1}{4}\right) = \frac{1}{4} = \|\underline{u}\|^2$$

$$\|\underline{u}\| = \frac{1}{2}$$

$$\underline{v} = \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \end{bmatrix} = \frac{1}{2}\underline{1}$$

$\|\underline{v}\| = 1$ $\quad$ $\underline{v}$ is unit vector

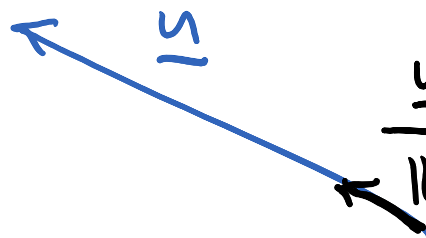
Convert any vector into a unit vector

$$\left(\frac{1}{\|\underline{u}\|}\right)\underline{u} = \frac{\underline{u}}{\|\underline{u}\|} = \underline{v}$$

$$\underline{v} \cdot \underline{v} = \frac{\underline{u}}{\|\underline{u}\|} \cdot \frac{\underline{u}}{\|\underline{u}\|} = \frac{1}{\|\underline{u}\|^2}\underline{u} \cdot \underline{u}$$

$$\|\underline{u}\| \qquad \|\underline{u}\| \qquad \|\underline{u}\|^2 = \underline{u}$$

$$= \frac{\|\underline{u}\|^2}{\|\underline{u}\|^2} = 1$$



$$\frac{\underline{u}}{\|\underline{u}\|}$$

Any exception

Cannot be zero vector

If $\underline{u} = \underline{0}$

$$\frac{\underline{0}}{\|\underline{0}\|}$$

direction not defined

divide by zero

Unit vector examples:

Cartesian directions

$$\underline{e}_x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \underline{e}_y = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{2D}$$

$$\underline{e}_x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \underline{e}_y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \underline{e}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{3D}$$

What dot product of these vectors?

$$\underline{e}_x \cdot \underline{e}_x = 1$$

$$\underline{e}_x \cdot \underline{e}_y = 0$$