Lecture 10 Outline

Thursday, September 26, 2019 3:29 PM

Linear Trasformations

- Rank - Nullity
- Orthogonality

Numerical Solutions

- Matrix + Vector norms
- Condition number
- LU decomposition
- Gaussian elimination

Linear Transformations

Thursday, September 26, 2019 9:29 PM

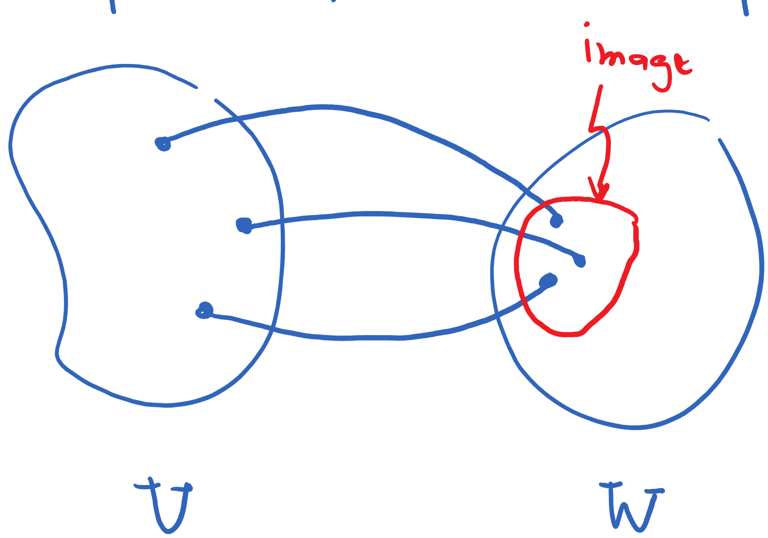
Let V + W be vector spaces with a linear transformation $L: V \to W$

Kernel of L: The $\ker(L)$ is the subspace of V, such that $\ker(L) = \{ \underline{v} \in V : L\underline{v} = 0_W \}$

The size of the $\ker(L)$ is called the nullity of L: $\text{nullity}(L)$

Rank of L: The rank of a linear operator, $\text{rank}(L)$, is the dimension of its image.

Recall that the image of a vector space is the portion of W that it maps into



Rank-Nullity or Dimension Theorem

Let V + W be vector spaces with a linear transformation $L: V \to W$

Then

$$\operatorname{rank}(L) + \operatorname{nullity}(L) = |L|$$

dimension (or size) of V

Apply to matrices: $\underline{A} \in M_{mn}$

$\underline{A}\,\underline{x} = \underline{b}$ $\quad \underset{\sim}{x} \in V \in \mathbb{R}^n$

$\quad \underline{b} \in W \in \mathbb{R}^m$

$\underline{b}$ is the image of $\underline{x}$ under the linear transformation of $\underline{A}$

Linear combinations of $C(\underline{A})$ give all vectors in the image

$$\Rightarrow \text{rank}(\underline{A}) = |\mathcal{C}(\underline{A})|$$

↳ dimension (or size) of the column space of $\underline{A}$

Note: Each vector in $\mathcal{C}(\underline{A})$ must be an independent vector (i.e., $\mathcal{C}(\underline{A})$ contains the minimum # of vectors to span the columns of $\underline{A}$)

Example:

$$\underline{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Do not say

$$\mathcal{C}(\underline{A}) \neq \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

but rather

$$\mathcal{C}(\underline{A}) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\Rightarrow \text{rank}(\underline{A}) = 3$$

All linear combinations of $N(\underline{A})$ give vectors $\underline{b} = \underline{0} = \underline{0}_w$

$$\underline{A}\,\underline{x} = \underline{0}$$

$$\Rightarrow \text{nullity}(\underline{A}) = |N(\underline{A})|$$

dimension (or size) of the null space of $\underline{A}$

Since for $\underline{A}\,\underline{x} = \underline{b}$, $\underline{x} \in \mathbb{R}^n$

Then $\text{rank}(\underline{A}) + \text{nullity}(\underline{A}) = n$

$= \#$ of columns

Theorem: Let $\underline{A} \in M_{mn}$, $C(\underline{A})$ be the column space, $N(\underline{A})$ be the nullspace, $C(\underline{A}^T)$ be the row space and $N(\underline{A}^T)$ be the left nullspace.

1) $\text{rank}(\underline{A}) = |C(\underline{A})| = |C(\underline{A}^T)|$

2) $|N(\underline{A})| = n - \text{rank}(\underline{A})$

3) $|N(\underline{A}^T)| = m - \text{rank}(\underline{A})$

Example: Let

$$\underline{A} = \begin{bmatrix} 8 & 2 & 1 & 23 \\ 4 & 2 & 9 & 19 \\ 10 & 1 & 6 & 35 \end{bmatrix}$$

$\underset{3\times 4}{\underline{A}}\,\underset{4\times p}{\underline{x}} = \underset{3\times p}{\underline{b}}$

$\underline{x} \in \mathbb{R}^4$

$\underline{b} \in \mathbb{R}^3$

rref ↳ reduced row eshelon form

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\underbrace{\hphantom{1 \quad 0 \quad 0}}_{\text{Independent columns}}$ free variable

$$\text{rref}(\underline{A}^T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\underbrace{\hphantom{1 \quad 0 \quad 0}}_{\text{Independent columns}}$ no free variables

$\Rightarrow \text{rank}(\underline{A}) = |C(\underline{A})| = 3 = |C(\underline{A}^T)|$

$\Rightarrow |N(\underline{A})| = n - \text{rank}(\underline{A}) = 4 - 3 = 1$

$$\Rightarrow |N(\underline{A}^T)| = m - \text{rank}(\underline{A}) = 3-3 = 0$$

Find all subspaces (Matlab check: rref)

$$C(\underline{A}) = \left\{ \begin{bmatrix} 8 \\ 4 \\ 10 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ 6 \end{bmatrix} \right\}$$

$$N(\underline{A}): \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $x_4 = 1$ (free variable)

Then
$$\begin{aligned} x_1 + 3 &= 0 \\ x_2 - 1 &= 0 \\ x_3 + 1 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= -3 \\ x_2 &= 1 \\ x_3 &= -1 \end{aligned}$$

$$\therefore N(\underline{A}) = \left\{ \begin{bmatrix} -3 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Also, $N(\underline{A})$ contains the zero vector $\underline{0}_4$

Check

$[-3]$

Check

$$\begin{bmatrix} 8 & 2 & 1 & 23 \\ 4 & 2 & 9 & 19 \\ 10 & 1 & 6 & 35 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -24 + 2 - 1 + 23 \\ -12 + 2 - 9 + 19 \\ -30 + 1 - 6 + 35 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{C}(\underline{A}^T) = \left\{ \begin{bmatrix} 8 \\ 2 \\ 1 \\ 23 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 9 \\ 19 \end{bmatrix}, \begin{bmatrix} 10 \\ 1 \\ 6 \\ 35 \end{bmatrix} \right\}$$

$$N(\underline{A}^T) = \{ \quad \}$$ Contains $\underline{0}_3$ vector

Let $\underline{A} \in M_{mn}$

(1) The matrix $\underline{A}$ has full column rank if rank$(\underline{A}) = n$. If $\underline{A}$ has full column rank, then the following holds

a) All columns of $\underline{A}$ are independent

b) Only vector in $N(\underline{A})$ is $\underline{0}$

c) If $\underline{A}^{-1}$ exists, then the solution to $\underline{A}\,\underline{x} = \underline{b}$ is unique (i.e., only an $\underline{x}$ such that $\underline{A}\,\underline{x} = \underline{b}$)

(2) The matrix $\underline{A}$ has full row rank if $\operatorname{rank}(\underline{A}) = m$. Furthermore,

a) All rows of $\underline{A}$ are independent

b) $C(\underline{A})$ spans all of $\mathbb{R}^m$ $(\underline{b} \in \mathbb{R}^m)$

c) $\underline{A}\,\underline{x} = \underline{b}$ has at least one solution for any $\underline{b}$

$C(\underline{A})$ spans all of $\mathbb{R}^m$

$\Rightarrow$ Any vector in $\mathbb{R}^m$ can be written as a linear combination of the columns of $\underline{A}$

$\Rightarrow$ Any $\underline{b}$ must be in $\mathbb{R}^m$ (for $\underline{A}\,\underline{x} = \underline{b}$)

$\Rightarrow \underline{x}$ is that linear combination of the columns of $\underline{A}$ that gives $\underline{b}$

$$\underline{A}\underline{x} = [\underline{a}_1 \ \underline{a}_2 \ \cdots \ \underline{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ | \\ x_n \end{bmatrix}$$

$$= x_1 \underline{a}_1 + x_2 \underline{a}_2 + \cdots + x_n \underline{a}_n$$

(3) Now let $\underline{A} \in M_{nn}$ (square matrix)

The matrix $\underline{A}$ has full rank if

rank$(\underline{A}) = n$ (i.e., both full row + column rank)

If $\underline{A}$ is full rank, then

a) $\underline{A}\underline{x} = \underline{b}$ has a solution for any $\underline{b}$

b) $C(\underline{A})$ span all of $\mathbb{R}^n$

c) $N(\underline{A})$ is only the $\underline{0}$

d) $\underline{A}\underline{x} = \underline{b}$ only has one solution for any $\underline{b}$

In other words, if $\underline{A}$ is full rank, then $\underline{A}^{-1}$ exists

Only one solution: $\underline{A}\,\underline{x} = \underline{b} \Rightarrow \underline{x} = \underline{A}^{-1}\,\underline{b}$

Now, all of the following are equivalent statements:

1. $\underline{A}$ is invertible
2. The columns of $\underline{A}$ are independent
3. The rows of $\underline{A}$ are independent
4. $\det(\underline{A}) \neq 0$
5. $\underline{A}\,\underline{x} = \underline{0}$ only has $\underline{x} = \underline{0}$ as a solution
6. $\underline{A}$ has n pivots for $\underline{A} \in M_{nn}$
7. $\underline{A}$ is full rank (i.e. $\text{rank}(\underline{A}) = n$)
8. $\text{rref}(\underline{A}) = \underline{I}$
9. $C(\underline{A})$ spans all of $\mathbb{R}^n$

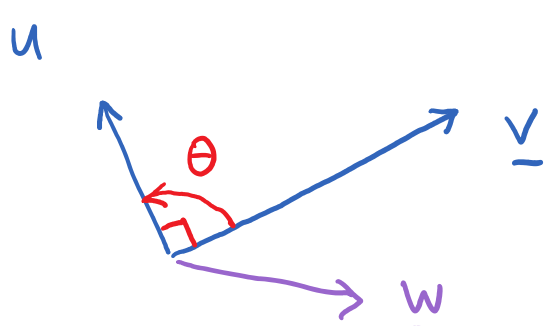
(10) $C(\underline{A}^T)$ spans all of $\mathbb{R}^n$

If any one of these is true, then all are true for square matrix $\underline{A}$

Orthogonality

Recall: Two vectors are orthogonal (perpendicular) to each other iff

$$\underline{u} \cdot \underline{v} = \underline{u}^T \underline{v} = 0$$



Here, $\underline{u}$ and $\underline{v}$ are orthogonal

$$\underline{u} \cdot \underline{v} = \|\underline{u}\| \|\underline{v}\| \cos\theta = 0$$

while $\underline{u}$ and $\underline{w}$ are not orthogonal

Two subspaces are orthogonal to each other if any vector in one subspace is orthogonal to all vectors in the other subspace.

If $\underline{u}$ is in subspace S and $\underline{v}$ is in subspace T, then if for any $\underline{u} \in S$ & $\underline{v} \in T$, we have $\underline{u} \cdot \underline{v} = 0$, then S and T are orthogonal.

For a matrix $\underline{A} \in M_{mn}$

(1) The row space $C(\underline{A}^T)$ is an orthogonal subspace in $\mathbb{R}^n$ of the nullspace $N(\underline{A})$

(a) To show this, consider

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

Let $\underline{x} \in N(\underline{A}) \Rightarrow \underline{A}\,\underline{x} = \underline{0}$

$$\underline{A}\,\underline{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} \underline{\text{row}1} \cdot \underline{x} \\ \underline{\text{row}2} \cdot \underline{x} \\ \vdots \\ \underline{\text{row}m} \cdot \underline{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The row space is the linear combination of the rows of $\underline{A}$

Since any $\underline{x}$ in $N(\underline{A})$ gives $\underline{A}\,\underline{x} = \underline{0}$ and since $\underline{A}\,\underline{x}$ is simply a dot product between rows of $\underline{A}$ and $\underline{x}$

$\Rightarrow C(\underline{A}^T)$ + $N(\underline{A})$ are othogonal to each other

(b) Another way to show this:

Let $\underline{y}$ be any vector compatible with $\underset{n \times m}{\underline{A}^T}$. Then

$$\underset{n \times m}{\underline{A}^T} \underset{m \times 1}{\underline{y}}$$ is a linear combination of the rows of $\underline{A}$

Let $\underline{x}$ be in $N(\underline{A})$. Then

$$\underline{x} \cdot (\underline{A}^T \underline{y}) = \underline{x}^T \underline{A}^T \underline{y} = (\underline{A}\,\underline{x})^T \underline{y}$$

$$= \underline{0}^T \underline{y} = 0$$

(2) The column space $C(\underline{A})$ is an orthogonal subspace in $\mathbb{R}^m$ of the left nullspace $N(\underline{A}^T)$

Let $\underset{m \times n}{\underline{A}}\,\underset{n \times 1}{\underline{y}}$ represent any vector in $C(\underline{A})$

and let $\underline{x} \in N(\underline{A}^T)$: $\underline{x}^T \underline{A} = \underline{0} = \underset{n \times m}{\underline{A}^T}\,\underset{m \times 1}{\underline{x}}$

Then,

$$\underline{x} \cdot (\underline{A}\,\underline{y}) = \underline{x}^T \underline{A}\,\underline{y} = (\underline{x}^T \underline{A})\,\underline{y} = \underline{0}^T \underline{y} = 0$$

$m \times 1$ $\underbrace{\overbrace{m \times n \;\; n \times 1}}_{m \times 1}$

$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$

Now, one step further

(1) $N(\underline{A})$ is the orthogonal complement of $C(\underline{A}^T)$ in $\mathbb{R}^n$

(2) $N(\underline{A}^T)$ is the orthogonal complement of $C(\underline{A})$ in $\mathbb{R}^m$

The orthogonal complement to a subspace contains every possible vector that is perpendicular (orthogonal) to that subspace.

Example: continued from above

$$\underline{A} = \begin{bmatrix} 8 & 2 & 1 & 23 \\ 4 & 2 & 9 & 19 \\ 10 & 1 & 6 & 35 \end{bmatrix}$$

Recall

$\left(\begin{bmatrix} 8 \\ \end{bmatrix} \begin{bmatrix} 4 \\ \end{bmatrix} \begin{bmatrix} 10 \\ \end{bmatrix} \right.$

Recall

$$\underset{\text{row space}}{C(\underline{A}^T)} = \left\{ \begin{bmatrix} 8 \\ 2 \\ 1 \\ 23 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 9 \\ 19 \end{bmatrix}, \begin{bmatrix} 10 \\ 1 \\ 6 \\ 35 \end{bmatrix} \right\}$$

$$N(\underline{A}) = \left\{ \begin{bmatrix} -3 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} -3 \\ 1 \\ -1 \\ 1 \end{bmatrix}^T \begin{bmatrix} 8 \\ 2 \\ 1 \\ 23 \end{bmatrix} = -24 + 2 - 1 + 23 = 0 \checkmark$$

$$\begin{bmatrix} -3 \\ 1 \\ -1 \\ 1 \end{bmatrix}^T \begin{bmatrix} 4 \\ 2 \\ 9 \\ 19 \end{bmatrix} = -12 + 2 - 9 + 19 = 0 \checkmark$$

$$\begin{bmatrix} -3 \\ 1 \\ -1 \\ 1 \end{bmatrix}^T \begin{bmatrix} 10 \\ 1 \\ 6 \\ 35 \end{bmatrix} = -30 + 1 - 6 + 35 = 0 \checkmark$$

$\therefore N(\underline{A})$ is the orthogonal complement to $C(\underline{A}^T)$

On the other hand,

$$C(\underline{A}) = \left\{ \begin{bmatrix} 8 \\ 4 \\ 10 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ 6 \end{bmatrix} \right\}$$

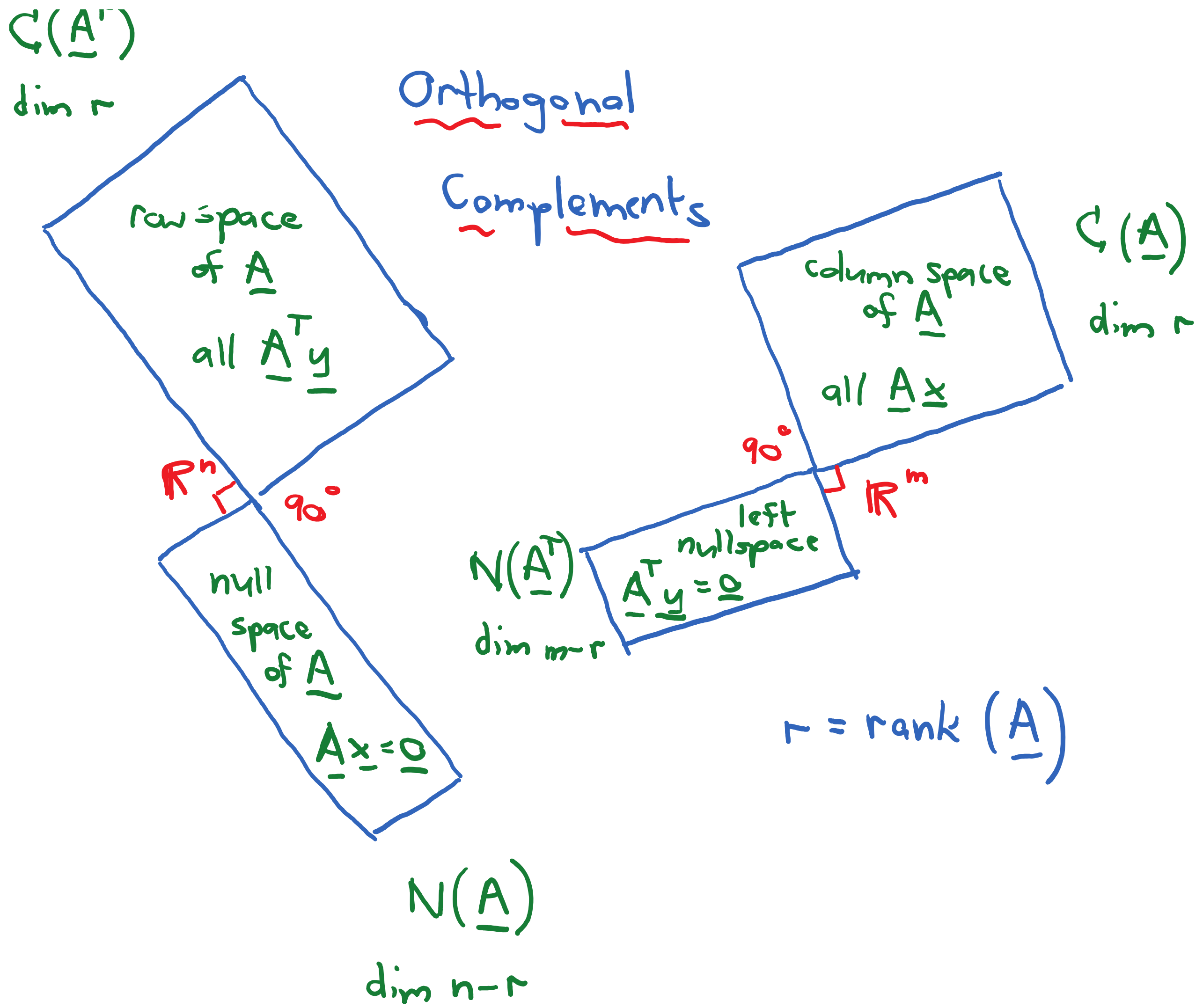
which spans all of $\mathbb{R}^3$. Consequently, there is <u>no</u> other vector orthogonal to $C(\underline{A})$ in $\mathbb{R}^3$ and the left nullspace $N(\underline{A}^T)$ consists of only the zero vector, that is,

$$N(\underline{A}^T) = \{ \ \} \quad \text{or} \quad N(\underline{A}^T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Pictorial View of the Four Subspaces of $\underline{A}$ (Strang, 2019)

$C(\underline{A}^T)$

Orthogonal



Orthogonal
Complements
$C(\underline{A}^T)$
dim r
row space of $\underline{A}$
all $\underline{A}^T\underline{y}$
$\mathbb{R}^n$
90°
null space of $\underline{A}$
$\underline{A}\underline{x} = \underline{0}$
$N(\underline{A})$
dim n−r
$C(\underline{A})$
dim r
column space of $\underline{A}$
all $\underline{A}\underline{x}$
90°
$\mathbb{R}^m$
left nullspace
$\underline{A}^T\underline{y} = \underline{0}$
$N(\underline{A}^T)$
dim m−r
r = rank $(\underline{A})$