Lecture 8 - Outline

Saturday, September 21, 2019 12:21 PM

HW1 - Function Call

Functions

Introduction

Composition

Inverse

Linear Transformations

Functions

Saturday, September 21, 2019 12:32 PM

Introduction

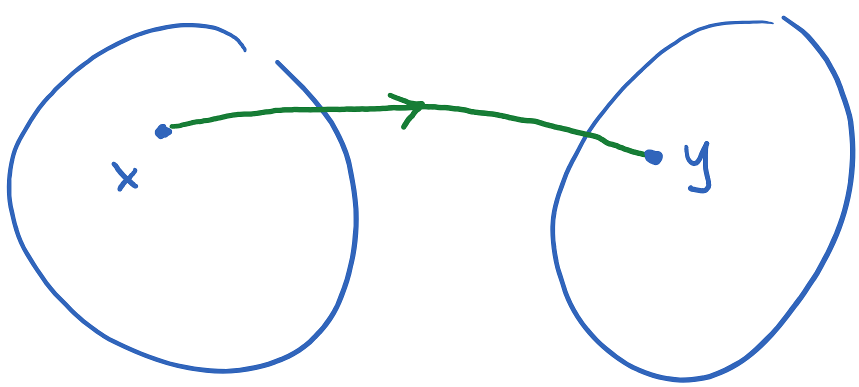
A function is the assignment of an element in the domain X into the codomain Y

Write as $f: X \to Y$ or $f(x) = y$

element in X element in Y

Also called a mapping

Each element in X is assigned to a single element in Y



domain X codomain Y

X Y

Example: Let $X = \mathbb{Z}$, $X = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$f(x) = x^2$ assigns every element in X to one and only one element in Y, given by x^2

Multiple elements in X may be assigned to the same element in Y

$$(-2)^2 = 4, \quad (2)^2 = 4$$

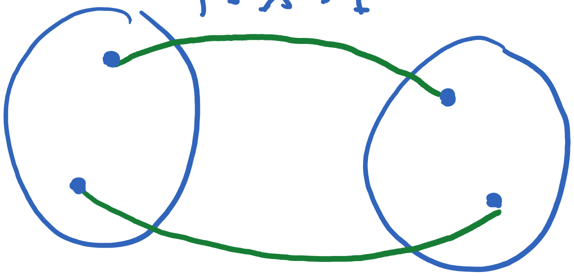
Example: $f(x) = \pm\sqrt{x}$

$f(4) = \pm 2$; A single element of X is assigned to multiple elements of Y

$\therefore$ $f(x)$ is not a function

Pictoral Representation

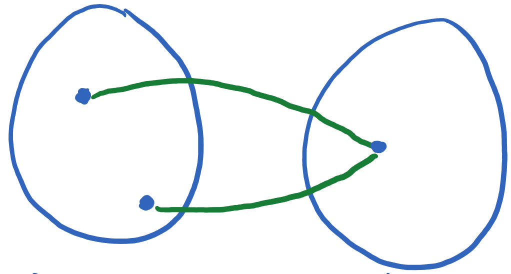
$f: X \to Y$



domain X codomain Y

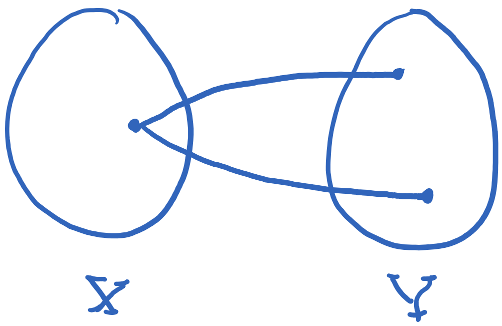
Function? Yes

$f: X \to Y$

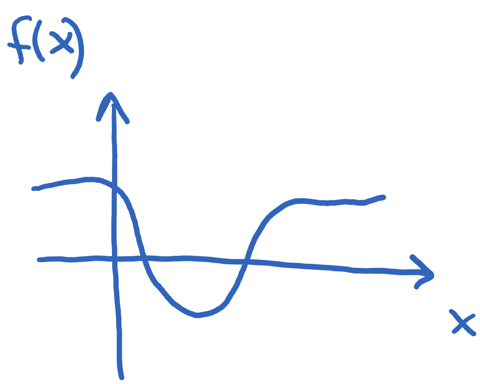


domain X codomain Y

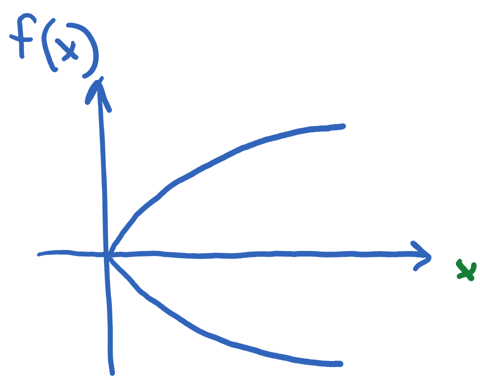
Function? Yes



Function? No!



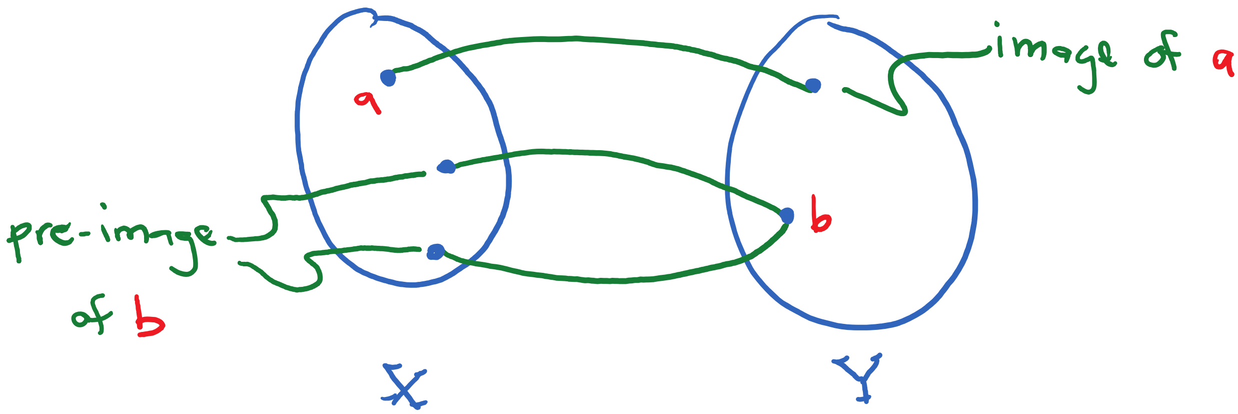
Valid function



Not a function

The image of a domain element is the unique codomain element

The pre-image of a codomain element are the domain elements that map to it



Example: $f(x) = x^2$

$X = \mathbb{Z}$

$Y = \mathbb{Z}$

4 is the image of 2

+2, −2 are the pre-images of 4

Not all elements in Y have a pre-image

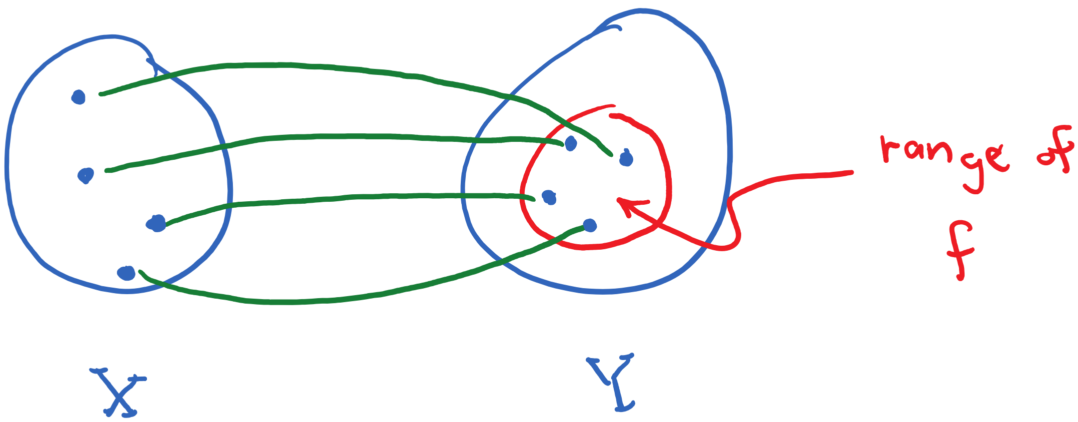
e.g. 5 in Y has no pre-image

If S is a subset of X, then the image of that subset is $f(S)$

The pre-image of a subset of Y, call it T, is given by $f^{-1}(T)$

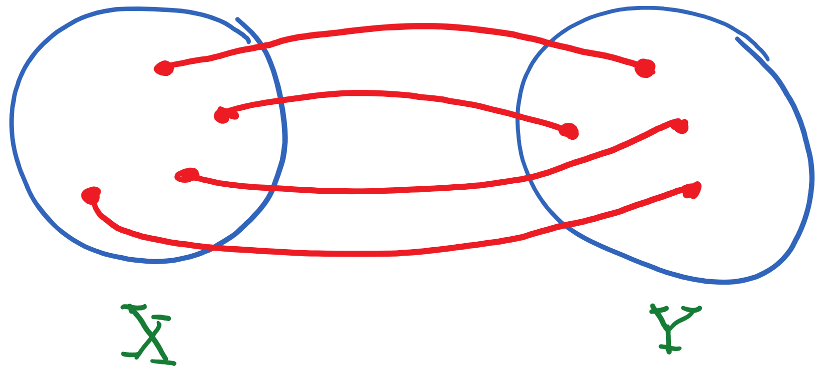
given by $f^{-1}(T)$

The image of the entire domain is called the **range** of the function



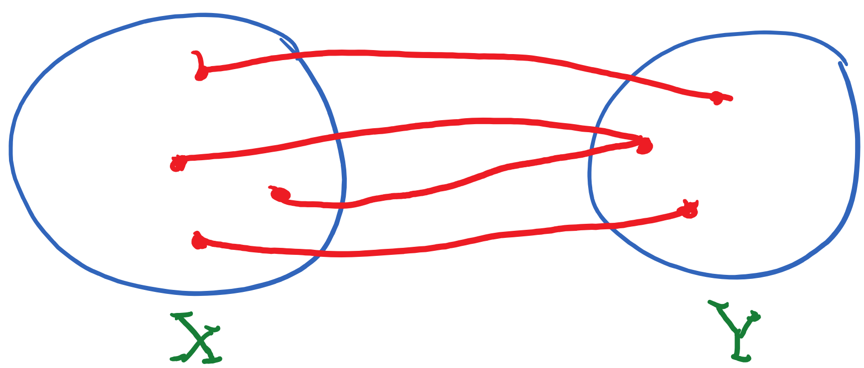
$f(X)$ can <u>never</u> produce an element outside of its range in Y

Functions are called **one-to-one** iff every point in X goes to distinct elements in Y



To show that a function is one-to-one (1-to-1), one must show that if $f(x_1) = f(x_2)$ for any $x_1, x_2 \in X$, then $x_1 = x_2$

Functions are onto iff every element of Y is an image of some element of X $\rightarrow$ range(f) = codomain(f)



Example:

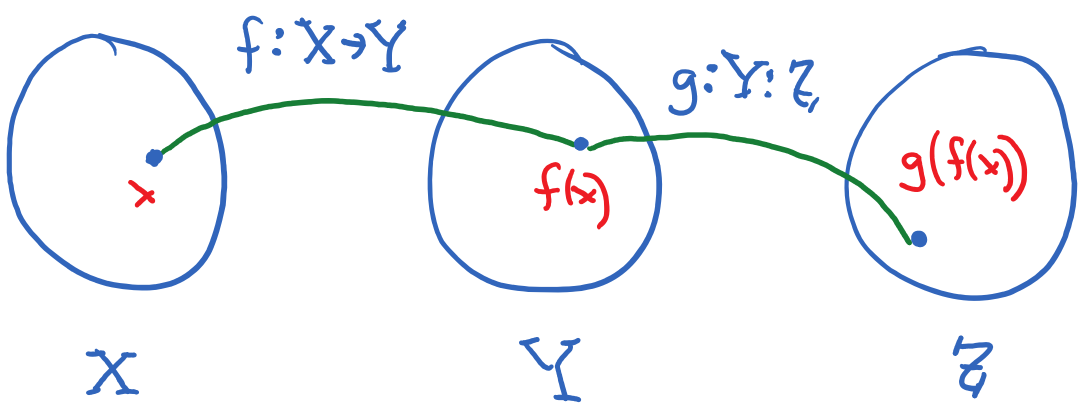
$x \in \mathbb{R}$, $f(x) = 2x$ is onto since every value $y = f(x)$ in the codomain has a value $x = \frac{1}{2}y$ in the domain

Composition

Let $f: X \to Y$, $g: Y \to Z$

Then composition, $g \circ f: X \to Z$

given $g(f(x))$



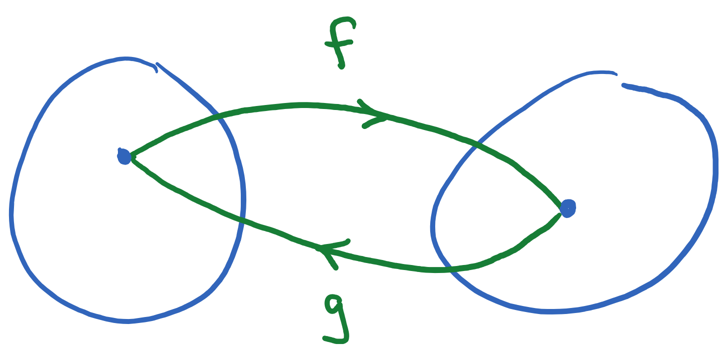
Theorem: If $f: X \to Y$ and $g: Y \to Z$ are both one-to-one, then $g \circ f$ is also one-to-one

Theorem: If $f: X \to Y$ and $g: Y \to Z$ are both onto, then $g \circ f$ is alsa onto

Inverse

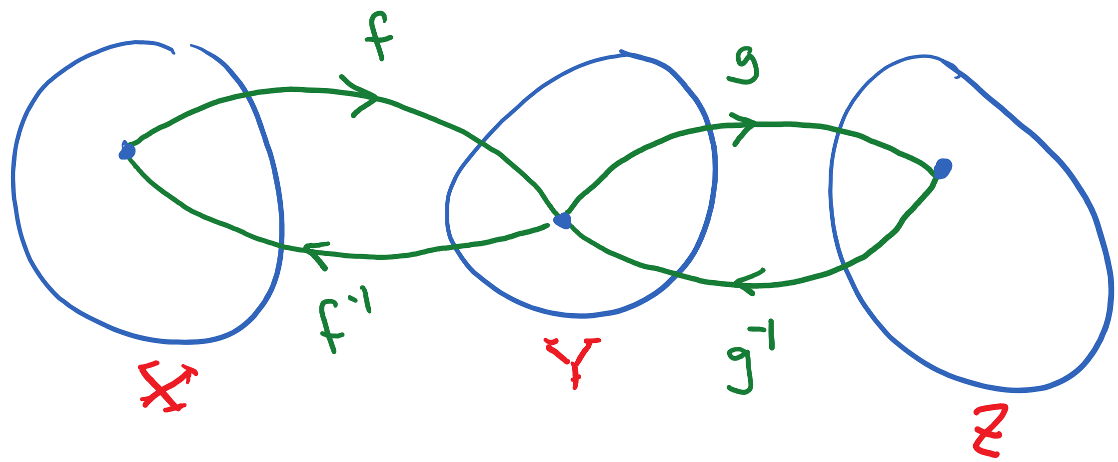
$f: X \to Y$, $g: Y \to X$ are inverses of each other, if $(g \circ f)x = x$ and $(f \circ g)y = y$ for any $x \in X$ and $y \in Y$

Theorem: $f: X \to Y$ has an inverse $g: Y \to X$ iff f is both one-to-one and onto.



Theorem: If g is the inverse of f, then g is the only inverse of f.

Theorem: If f and g both have inverses, the inverse of the function composition $g \circ f$ is $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$



Linear Transformations

Linear transformations are functions on vector spaces that follow two rules:

Let V and W be vector spaces such that $f: V \to W$. Then, f is a linear transformation iff

(1) $f(\underline{v}_1 + \underline{v}_2) = f(\underline{v}_1) + f(\underline{v}_2)$

(2) $f(a\underline{v}_1) = a f(\underline{v}_1)$

for any $\underline{v}_1, \underline{v}_2 \in V$ and $a \in \mathbb{R}$

Example: $f: M_{mn} \to M_{nm}$ (transpose operator)

$f(\underline{A}) = \underline{A}^T$

Is this a linear transformation?

(1) $f(\underline{A}+\underline{B}) = f(\underline{A}) + f(\underline{B})$

$$(\underline{A}+\underline{B})^T = \underline{A}^T + \underline{B}^T \quad \checkmark$$

(2) $f(c\underline{A}) = c\, f(\underline{A})$

$$(c\underline{A})^T = \underline{A}^T c = c\,\underline{A}^T \quad \checkmark$$

Yes, the transpose operator is linear

Theorem: Let V and W be vector spaces and let $L: V \to W$ be a linear transformation. Let $\underline{0}_V$ be the zero vector in V and $\underline{0}_W$ be the zero vector in W. Then

(1) $L(\underline{0}_V) = \underline{0}_W$

(2) $L(-\underline{v}) = -L(\underline{v})$ for all $\underline{v} \in V$

(3) $L(a_1\underline{v}_1 + a_2\underline{v}_2 + \dots + a_n\underline{v}_n) =$

$a_1 L(\underline{v}_1) + a_2 L(\underline{v}_2) + \dots + a_n L(\underline{v}_n)$

for all $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n \in V$ and

$a_1, a_2, \dots, a_n \in \mathbb{R}$

Proof

(1) $L(\underline{0}_V) = L(0\,\underline{0}_V) \overset{\text{from definition}}{=} 0\,L(\underline{0}_V) = \underline{0}_W$

(2) $L(-\underline{v}) = L(-1\,\underline{v}) = -1\,L(\underline{v}) = -L(\underline{v})$

(3) $L(a_1\underline{v}_1 + a_2\underline{v}_2) = L(a_1\underline{v}_1) + L(a_2\underline{v}_2)$

$= a_1 L(v_1) + a_2 L(\underline{v}_2)$ + similarly for higher n

Theorem: Let $L_1 : V_1 \to V_2$ + $L_2 : V_2 \to V_3$ be two linear transformations

Then $L_2 \circ L_1 : V_1 \to V_3$

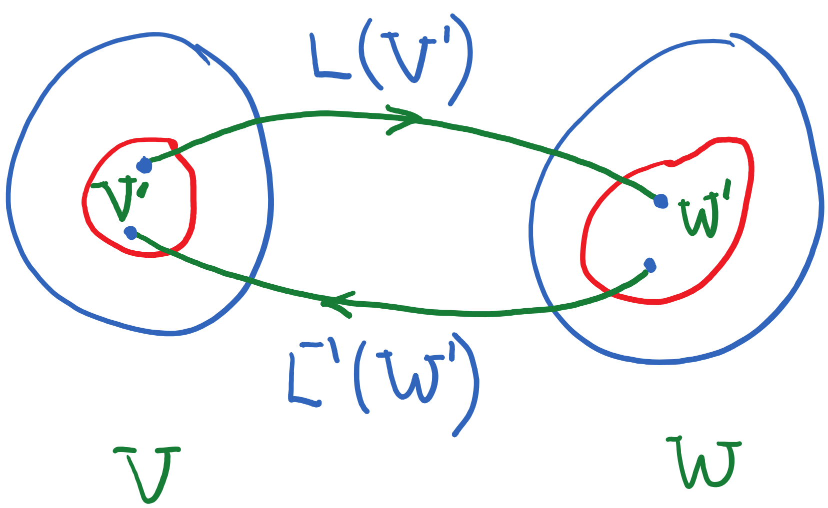
$\Rightarrow (L_2 \circ L_1)(V) = L_2(L_1(V))$

is also a linear transformation

Theorem: Let $L: V \to W$ be a linear transformation

(1) If V' is a subspace of V, then $L(V')$ is a subspace of W

(2) If W' is a subspace of W, then $L^{-1}(W')$ is a subspace of V'



Example: Let $L: M_{22} \to \mathbb{R}^3$

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = [a, 0, c]$$

Is this a linear transformation?

(1) $L\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$

$= L\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right) = [a_1+a_2, 0, c_1+c_2]$

$= [a_1, 0, c_1] + [a_2, 0, c_2]$

$= L\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$ ✓

(2) $L\left(\alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = L\left(\begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}\right)$

$= [\alpha a, 0, \alpha c]$

$= \alpha [a, 0, c] = \alpha L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ ✓

The range of L given by

$\{ [a, 0, c]$ for $a, c \in \mathbb{R} \}$ forms

a subspace of $\mathbb{R}^3$

A linear transformation is determined by its actions on the basis of a vector space

Basis A minimum set of unique and independent vectors that span the vector space

Example: Let

$$\underline{b}_1 = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 1 \end{bmatrix}, \underline{b}_2 = \begin{bmatrix} -2 \\ 5 \\ 0 \\ 1 \end{bmatrix}, \underline{b}_3 = \begin{bmatrix} -3 \\ 5 \\ 1 \\ 1 \end{bmatrix}, \underline{b}_4 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$B = \{ \underline{b}_1, \underline{b}_2, \underline{b}_3, \underline{b}_4 \}$ form a basis for $\mathbb{R}^4$

Let $L : \mathbb{R}^4 \to \mathbb{R}^3$, such that

$$L(\underline{b}_1) = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, L(\underline{b}_2) = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, L(\underline{b}_3) = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}, L(\underline{b}_4) = \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix}$$

What is $L\left(\begin{bmatrix} -4 \\ 14 \\ 1 \\ 5 \end{bmatrix}\right)$?

Recall that if $\mathcal{B}$ is a basis for $\mathbb{R}^4$, then all vectors in $\mathbb{R}^4$ can be written as

$$k_1 \underline{b}_1 + k_2 \underline{b}_2 + k_3 \underline{b}_3 + k_4 \underline{b}_4 = \underline{v} \in \mathbb{R}^4$$

Then

$$L(\underline{v}) = L\left(k_1 \underline{b}_1 + k_2 \underline{b}_2 + k_3 \underline{b}_3 + k_4 \underline{b}_4\right)$$

$$= k_1 L(\underline{b}_1) + k_2 L(\underline{b}_2) + k_3 L(\underline{b}_3) + k_4 L(\underline{b}_4)$$

$$= k_1 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + k_3 \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} + k_4 \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix}$$

Here, we need

$$k_1 \begin{bmatrix} 0 \\ 4 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -2 \\ 5 \\ 0 \\ 1 \end{bmatrix} + k_3 \begin{bmatrix} -3 \\ 5 \\ 1 \\ 1 \end{bmatrix} + k_4 \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 14 \\ 1 \\ 5 \end{bmatrix}$$

$\therefore \; k_1 = 2, \; k_2 = -1, \; k_3 = 1, \; k_4 = 3$

Then,

$$L\left(\begin{bmatrix} -4 \\ 14 \\ \vdots \end{bmatrix}\right) = 2\begin{bmatrix} 3 \\ \vdots \end{bmatrix} - \begin{bmatrix} 2 \\ \vdots \end{bmatrix} + \begin{bmatrix} -4 \\ \vdots \end{bmatrix} + 3\begin{bmatrix} 6 \\ \vdots \end{bmatrix}$$

$$L\left(\begin{bmatrix}-4\\14\\1\\5\end{bmatrix}\right) = 2\begin{bmatrix}3\\1\\2\end{bmatrix} - \begin{bmatrix}2\\-1\\1\end{bmatrix} + \begin{bmatrix}-4\\3\\0\end{bmatrix} + 3\begin{bmatrix}6\\1\\-1\end{bmatrix}$$

$$= \begin{bmatrix}18\\9\\0\end{bmatrix}$$

Theorem: Let $\mathcal{B} = \{\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_n\}$ be a basis for vector space V. Let $\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_n$ be any n vectors in vector space W, then there is a unique linear transformation $L: V \to W$, such that $L(\underline{b}_1) = \underline{w}_1$, $L(\underline{b}_2) = w_2, \ldots, L(\underline{b}_n) = w_n$

This transformation can be written as a matrix-vector product!

Consider $\mathbb{R}^4$ example

$$\underline{v} = k_1 \underline{b}_1 + k_2 \underline{b}_2 + k_3 \underline{b}_3 + k_4 b_4$$

$$L(\underline{v}) = k_1 L(\underline{b}_1) + k_2 L(\underline{b}_2) + k_3 L(\underline{b}_3) + k_4 L(\underline{b}_4)$$

Let $\underline{A} = \left[L(\underline{b}_1),\ L(\underline{b}_2),\ L(\underline{b}_3),\ L(\underline{b}_4) \right]$

Then,

$$L(\underline{v}) = \underline{A}\,\underline{k} \quad \text{with} \quad k = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix}$$

Matrix of linear transformation

In above problem,

$$\underline{A} = \begin{bmatrix} 3 & 2 & -4 & 6 \\ 1 & -1 & 3 & 1 \\ 2 & 1 & 0 & -1 \end{bmatrix}$$

Theorem: Let B be a basis for V and C be a basis for W. For the linear transformation $L: V \to W$, there exists a matrix $\underline{A}_{BC}$, such that

$$\underline{A}_{BC} \underbrace{[\underline{v}]_B}_{\text{in basis } B} = \underbrace{[L(\underline{v})]_C}_{\text{in basis } C}$$

Examples: Geometric operators in $\mathbb{R}^3$

$$\underline{A} = \left[L(\underline{e}_1), L(\underline{e}_2), L(\underline{e}_3) \right]$$

- Reflection in x-y plane

$$L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ a_2 \\ -a_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

- Scaling

$$L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} ca_1 \\ ca_2 \\ ca_3 \end{bmatrix} = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

- Rotation about z

$$L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 \cos\theta - a_2 \sin\theta \\ a_1 \sin\theta + a_2 \cos\theta \\ \end{bmatrix}$$

about z

$$\left(\begin{bmatrix} q_2 \\ q_3 \end{bmatrix}\right) = \begin{bmatrix} q_1 \sin\theta + q_2 \cos\theta \\ q_3 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$