Lecture 9 Outline

Monday, September 23, 2019 3:41 PM

Linear Transformation

Example (revisited)

Four Subspaces of a Matrix

Column Space

Null space

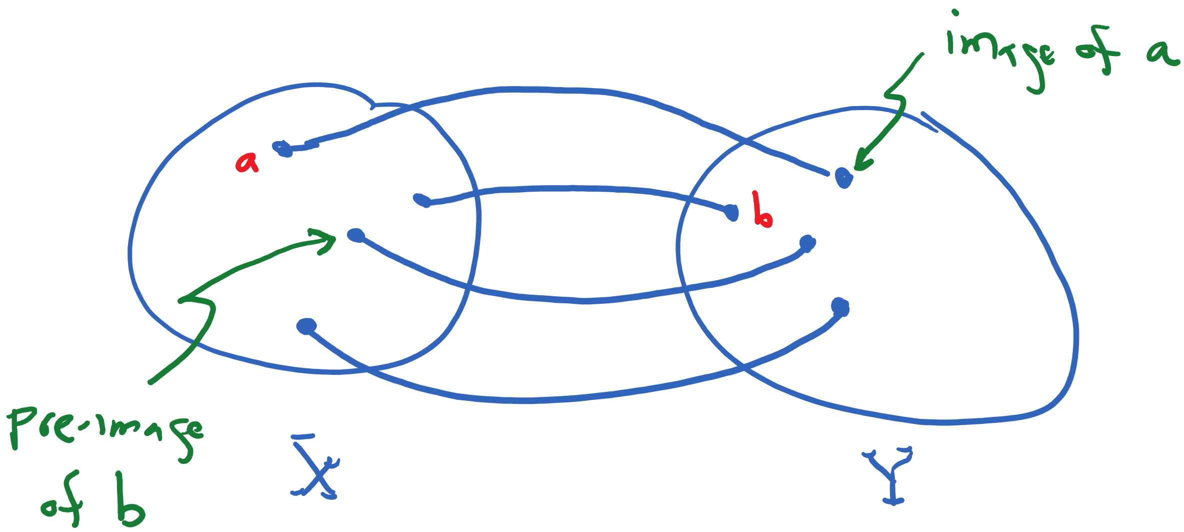
Row Space

Left Nullspace

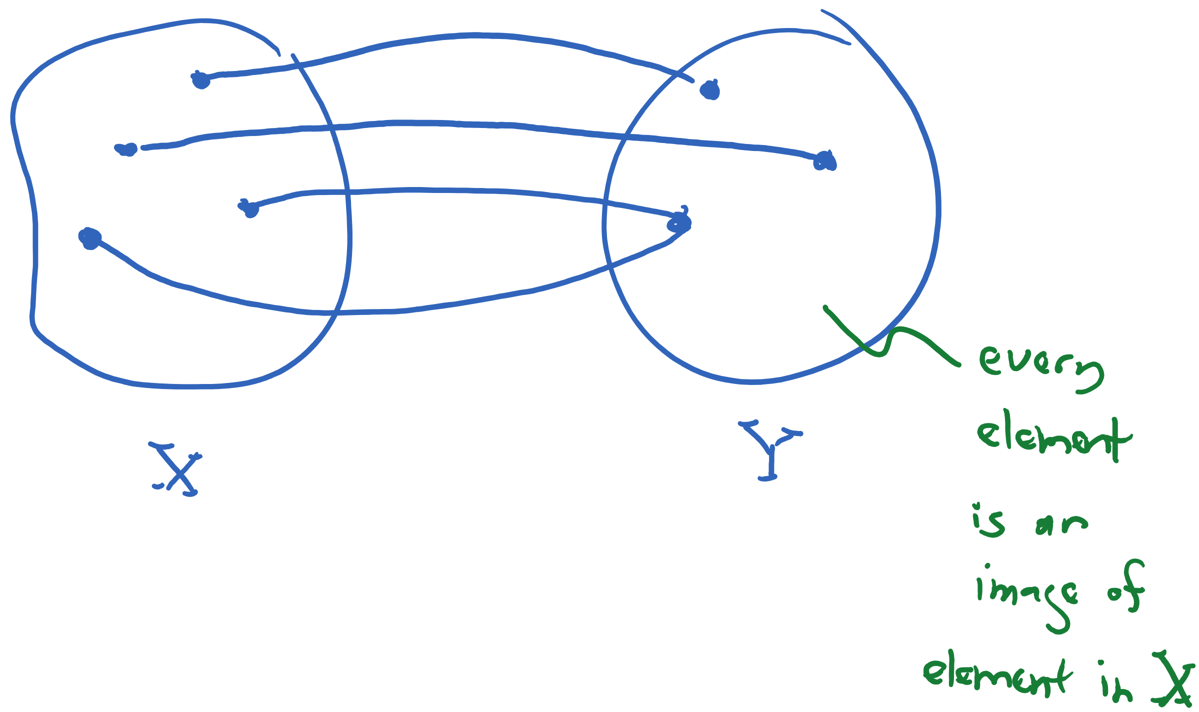
Definitions

Wednesday, September 25, 2019 10:36 AM

Functions are one-to-one iff every element in X goes to a distinct element in Y



Functions are onto iff every element of Y is an image of some element of X and thus range(f) = codomain(f)


X
Y
every element is an image of element in X

Linear Transformation Example

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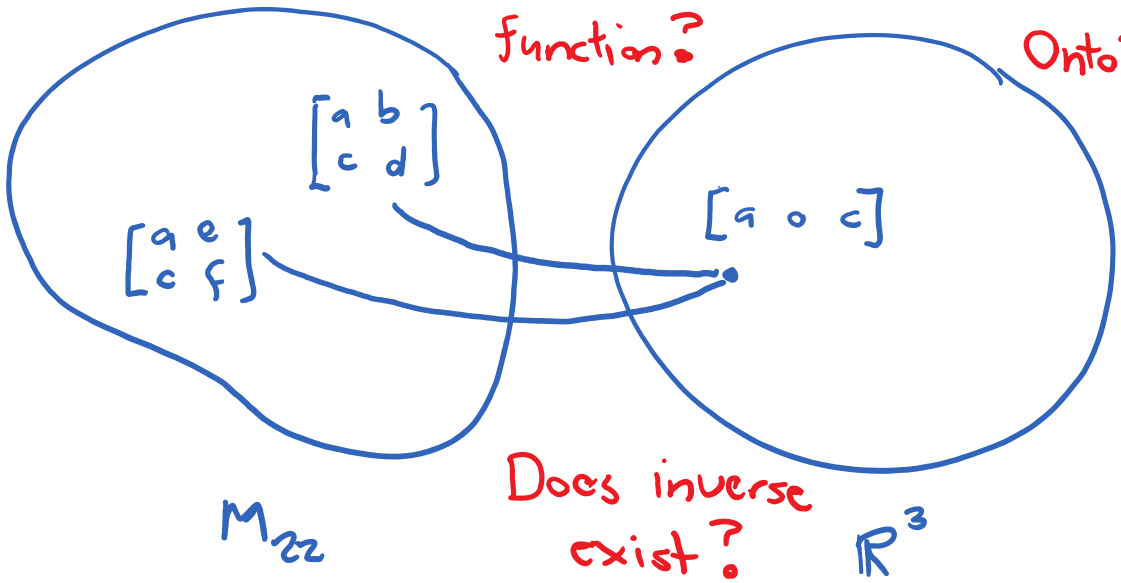
Example: Let $L : M_{22} \to \mathbb{R}^3$ (Linear Transformation)

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = [a, 0, c]$$ (from last lecture)

Is this a function?

One-to-one?

Onto?



Range of L?

Does inverse exist?

Let $\underline{M} = K_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + K_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + K_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + K_4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Use B_{11}, B_{12}, B_{21} + B_{22} as basis

$$\underline{M} = K_1 \underline{B}_{11} + K_2 \underline{B}_{12} + K_3 \underline{B}_{21} + K_4 \underline{B}_{22}$$

$$K_1 \underline{B}_{11} + K_2 \underline{B}_{12} + K_3 \underline{B}_{21} + K_4 \underline{B}_{22} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\therefore \quad K_1 + K_4 = a, \; K_2 = b, \; K_3 = c, \; K_4 = d$

$\quad + K_1 = a - d$

$L(\underline{B}_{11}) = [1 \quad 0 \quad 0], \quad L(\underline{B}_{12}) = [0 \quad 0 \quad 0]$

$L(\underline{B}_{21}) = [0 \quad 0 \quad 1], \quad L(\underline{B}_{22}) = [1 \quad 0 \quad 0]$

$$\underline{A} = \left[L(\underline{B}_{11})^T, L(\underline{B}_{12})^T, L(\underline{B}_{21})^T, L(\underline{B}_{22})^T \right]$$

$$\underline{A}\,\underline{K} = L(\underline{M})$$

$$(\underline{A}\,\underline{K})^T = \left(\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a-d \\ b \\ c \\ d \end{bmatrix} \right)^T$$

$$= [a \quad 0 \quad c]$$

Four Subspaces of a Matrix

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Review Example:

$$\underline{A}\,\underline{x} = \underline{b}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 2 \\ 1 & 1 & 2 & 4 \end{array}\right]$$ Augmented matrix

$\downarrow$ Row reduce

pivots

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\downarrow$

$$x_1 + x_3 = 0$$

$$x_2 + x_3 = 0$$

$$0 x_3 = 1 \leftarrow ???$$

This exists because:

a) $\det(\underline{A}) = 0 \rightarrow \underline{A}^{-1}$ does not exist

b) $\underline{A}$ has a non-trivial nullspace, where the nullspace of $\underline{A}$ is all vectors $\underline{v}$, such that $\underline{A}\,\underline{v} = \underline{0}$

Here, one finds

$$\underline{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\underline{A}\,\underline{v} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1-1 \\ 2+1-3 \\ 1+1-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and the columns of $\underline{A}$ are not independent!

In addition to nullspace, it is useful to identify the column space of $\underline{A}$.

Altogether, there are four subspaces of a matrix:

(1) Column space

(2) Nullspace

(3) Row space

(4) Left nullspace

Column Space

Recall that a matrix-vector product is simply a linear combination of the matrix columns:

$$\underbrace{\underline{A}}_{m\times n \text{ matrix}} \overbrace{\underline{x}}^{n\times 1 \text{ vector}} = \left[\underbrace{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n}_{m\times 1 \text{ vectors}} \right] \begin{bmatrix} x_1 \\ x_2 \\ | \\ x_n \end{bmatrix} = \underbrace{\underline{b}}_{m\times 1 \text{ vector}}$$

$$\underline{b} = x_1 \underline{a}_1 + x_2 \underline{a}_2 + \dots + x_n \underline{a}_n$$

All possible vectors $\underline{b}$ exist in the column space $C(\underline{A})$ of the matrix

The column space is a subspace of $\mathbb{R}^m$

The column space is very important when solving $\underline{A}\,\underline{x} = \underline{b}$

Theorem: The system $\underline{A}\,\underline{x} = \underline{b}$ has at least one solution iff $\underline{b}$ is in the column space of $\underline{A}$

Note: At least one solution!

Case 1: $\underline{A}^{-1}$ exists

$\underline{A}\,\underline{x} = \underline{b}$ with $\underline{b}$ in $C(\underline{A})$

then $\underline{x} = \underline{A}^{-1}\underline{b} \Rightarrow$ a solution

$\Rightarrow$ any $\underline{b}$ is a valid right-hand side (rhs)

With $\underline{A} \in M_{nn}$ (square matrix)

$\Rightarrow$ If $\underline{A}^{-1}$ exists, then $\underline{A}$ has n independent columns, which means it spans all of $\mathbb{R}^n$

Example:

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\underline{A}^{-1}$ exists; all 3 columns are linearly independent; columns span all of $\mathbb{R}^3$

Case 2: $\underline{A}^{-1}$ does not exist

then $\underline{b}$ must be in the column space $C(\underline{A})$

to have a solution

Example: $\underline{A}\,\underline{x} = \underline{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Only two independent columns; $\underline{A}^{-1}$ does not exist

Can one write

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} ?$$

Try $x = 1,\ y = 1,\ z = a$

$$(1)\begin{bmatrix} \ \\ \ \\ \ \end{bmatrix} + (1)\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + a \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

ok ✓

$\bar{b}$ is in the column space $C(\underline{A})$ and has at least one solution, actually an infinity of solutions

Example: $\underline{A}\,\underline{x} = \underline{c}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

changed only this element

Can one write

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} ?$$

Is $\underline{c}$ in $C(\underline{A})$? No!

No x, y, z to satisfy above equation

$\therefore$ No solution

Nullspace

Another important subspace is the nullspace, given by $N(\underline{A})$

The nullspace is all vectors $\underline{v}$, such that

$$\underline{A}\,\underline{v} = \underline{0}$$

Is this a vector space?

Let $\underline{v}$ & $\underline{w}$ be in $N(\underline{A})$

$(\underline{A}\,\underline{v} = \underline{0}$ and $\underline{A}\,\underline{w} = \underline{0})$

① $\underline{A}(\underline{v} + \underline{w}) = \underline{A}\,\underline{v} + \underline{A}\,\underline{w} = \underline{0} + \underline{0} = \underline{0}$ ✓

② $\underline{A}(c\underline{v}) = c(\underline{A}\,\underline{v}) = c\underline{0} = \underline{0}$ ✓

Yes, the nullspace is a vector space

The nullspace is always non-empty;

$\underline{0}$ is always in the nullspace

$$\underline{A}\,\underline{0} = \underline{0}$$

If $\underline{A}^{-1}$ exists, then

$$\underline{A}\,\underline{v} = \underline{0} \Rightarrow \underline{v} = \underline{A}^{-1}\underline{0} = \underline{0}$$

$\Rightarrow$ If $\underline{A}^{-1}$ exists, then the only vector in $N(\underline{A})$ is $\underline{0}$

If the nullspace has any vector in addition to $\underline{0}$, then $\underline{A}^{-1}$ does not exist.

Example: To obtain $N(\underline{A})$

$$\underline{A} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix} \qquad \text{Solve } \underline{A}\,\underline{x} = \underline{0}$$

Two possibilities:

(1)

$$\underline{A} \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow N(\underline{A}) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Only trivial $N(\underline{A}) \Rightarrow \underline{A}^{-1}$ exists

(2)

$$\underline{A} \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

or something similar

$$\underline{A} \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

or something similar

Example:

$$\underline{A} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\Big\downarrow \text{rref}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & \boxed{0} & 0 \end{array}\right]$$

free column/variable

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $x_3 = c$ for $c \in \mathbb{R}$

or simply $x_3 = 1$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \end{bmatrix} \quad \begin{bmatrix} 0 \end{bmatrix}$$

$$x_1 - 1 = 0 \Rightarrow x_1 = 1$$

$$x_2 - 2 = 0 \Rightarrow x_2 = 2$$

$\therefore\ \underline{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is in nullspace of $\underline{A}$

such that $\underline{A}\,\underline{x} = \underline{0}$

But so is

$$2\underline{x} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, -3\underline{x} = \begin{bmatrix} -3 \\ -6 \\ -3 \end{bmatrix}, \ldots$$

Nullspace is not limited to square matrices.

Example:

$$\underline{A} = \begin{bmatrix} 1 & 3 & 2 & 3 \\ 2 & 6 & 8 & 10 \\ 3 & 9 & 10 & 13 \end{bmatrix}$$

Find $N(\underset{\sim}{A})$: All vectors which span $N(\underset{\sim}{A})$

$$\begin{bmatrix} 1 & 3 & 2 & 3 \\ 2 & 6 & 8 & 10 \\ 3 & 9 & 10 & 13 \end{bmatrix}$$

$$\Big\downarrow \text{rref}$$

$$\begin{array}{cccc} & & & \\ \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\uparrow$ free column (variable) $\qquad \uparrow$ free column (variable)

$x_1, x_3 \Rightarrow$ "fixed"

$x_2, x_4 \Rightarrow$ "free"

To determine the nullspace, set one free variable to 1, the others to zero & find fixed variables (x_1 & x_3 here)

Set $x_2 = 1$, $x_4 = 0$

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 1 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} x_1 + 3 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{array}{l} x_1 = -3 \\ x_3 = 0 \end{array}$$

$$\rightarrow \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ is in } N(\underline{A})$$

Set $x_2 = 0$, $x_4 = 1$

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} x_1 + 1 \\ x_3 + 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{array}{l} x_1 = -1 \\ x_3 = -1 \end{array}$$

$$\rightarrow \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \text{ is in } N(\underline{A})$$

$$\begin{bmatrix} 1 \end{bmatrix}$$

$$\therefore N(\underline{A}) = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Recall, with span, any linear combination of these vectors is in $N(\underline{A})$

Matrix Subspaces: Recap

Let $\underline{A} \in M_{\underset{\#\text{rows}}{m}\,\underset{\#\text{columns}}{n}}$

1. Column Space $C(\underline{A})$ is the subspace of $\mathbb{R}^m$ that is spanned by the columns of $\underline{A}$. (Also called the Range Space)

$$\underline{A}\,\underline{x} = \underline{b}$$

$\underline{b}$ is a linear combination of the

$\underline{s}$ $\in \mathbb{R}^n$ $\quad$ $\underline{h}$ $\in \mathbb{R}^m$

$\underline{b}$ is a linear combination of the columns of $\underline{A}$

② Nullspace $N(\underline{A})$ is the subspace of $\mathbb{R}^n$ that is spanned by all vectors, which are solutions of $\underline{A}\,\underline{x} = \underline{0}$. (Also called the Kernel Space)

All matrices have a nullspace, because $\underline{0}$ is always in $N(\underline{A})$

$$\underline{A}\,\underline{0} = \underline{0}$$

③ Row Space $C(\underline{A}^T)$ is the subspace of $\mathbb{R}^n$ that is spanned by the rows of $\underline{A}$.

④ Left Nullspace $N(\underline{A}^T)$ is the subspace of

$\mathbb{R}^m$ that is spanned by all vectors, which are solutions of $\underline{A}^T \underline{x} = \underline{0}$

Note that

$$\underline{A}^T \underline{x} = \underline{0} \Rightarrow (\underline{A}^T \underline{x})^T = \underline{0}^T$$

$$\underline{x}^T \underline{A} = \underline{0}^T$$

Summary

Let $\underline{A} \in M_{mn}$

Matrix Subspace	Notation	Subspace of
Column Space	$C(\underline{A})$	$\mathbb{R}^m$
Nullspace	$N(\underline{A})$	$\mathbb{R}^n$
Row Space	$C(\underline{A}^T)$	$\mathbb{R}^n$
Left Nullspace	$N(\underline{A}^T)$	$\mathbb{R}^m$