

# Rendering Equation in Water Column

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Last Update

2018-01-19.

## 1 Model

### 1.1 Rendering Equation

The interaction of incident, reflected, and emitted light at any given point is described by the general Rendering Equation,

$$L_o = L_e + L_r \quad (1)$$

Where,  $L_o$  is the light leaving the particular point,  $L_e$  is the light emitted from the point (if it is a source), and  $L_r$  is the light reflected from that point.

Since we will be working with non-light generating disks, we set,  $L_e = 0$ . The light reflected off of the surface of a disk, can be calculated using the Bidirectional Reflectance Distribution Function (BRDF)[3]. For a given point on a surface, the general BRDF is,

$$\rho_d(\phi_0, \theta_0, \phi_f, \theta_f) = \frac{L_r(\phi_f, \theta_f)}{L_i(\phi_0, \theta_0)} \quad (2)$$

Where  $\phi_0, \theta_0$  are the incident azimuthal and polar angle of the light source, and  $\phi_f, \theta_f$  are the azimuthal and polar angle with respect to the viewer and reflected point.

From the general rendering equation, the reflected radiance is given by

$$L_r = \int_{\phi_i=0}^{2\pi} \int_{\theta_i=0}^{\frac{\pi}{2}} f_r L_s T(d) \cos(\theta_i) \sin(\theta_i) d\theta_i d\phi_i \quad (3)$$

Where, the  $f_r$  is Lambertian BRDF.

The transmission function  $T(d)$  can be related to the depth of the reflecting surface.

### 1.2 Beer-Lambert Law

The Beer-Lambert provides us with a linear relationship between incident light, absorbed light, and concentration of absorbed light.

The wavelength dependent relation is given as,

$$\epsilon dc = \log_{10} \frac{I_0}{I_f} \quad (4)$$

where,  $\epsilon$  is the molar absorptivity in  $L \cdot cm^{-1} \cdot mol^{-1}$ ,  $d$  is the depth in  $cm$ ,  $C$  is the concentration in  $mol \cdot L^{-1}$ .  $I_0$  is the initial light intensity, and  $I_f$  is the final intensity after absorption.

Given some incident light of radiance,  $L_0$ , on the water surface, we have the decomposed radiance as

$$L_0 = L_{sa} + L_s$$

where,  $L_{sa}$  is the absorbed light radiation.  $L_s$  is the light incident on the point under water and being reflected by the disk.

According to the Beer-Lambert Law we have the relation,

$$\text{Log}_{10} \left( \frac{L_0}{L_s} \right) = \frac{\epsilon d C}{\cos(\theta)} \quad (5)$$

and,

$$T(d) = 10^{\frac{-\epsilon d C}{\cos(\theta)}} \quad (6)$$

### 1.3 Reflectance

Since we want to specifically take into account the absorbance due to chlorophyll a, we limit the consideration to considering the radiance of light at  $\lambda = 662nm$

Even though the incident light is polychromatic, but since we are considering the specific wavelength and the incident light is so spread out, we can reasonably treat it as monochromatic. (?)

The reflectance in the particular frequency then becomes,

$$L_{r\lambda} = \int_{\phi_i=0}^{2\pi} \int_{\theta_i=0}^{\theta_c} f_r L_{s\lambda} 10^{\frac{-\epsilon d C}{\cos(\theta)}} \cos(\theta_i) \sin(\theta_i) d\theta_i d\phi_i \quad (7)$$

Here we sum the incident angle upto some cutoff angle  $\theta_c$ , as the higher incident lights have negligible effect on the reflected light.

We can further make an assumption that the secchi disk is perfectly matte with rotationally invariant reflectance. The Lambertian BRDF is thus,

$$f_r = \frac{\rho_D}{\pi} \quad (8)$$

Where,  $\rho_D$  is the albedo of the secchi disk with  $0 \leq \rho_D \leq 1$  Combining (4) and (5) we get the reflectance off of the surface of secchi disk as,

$$L_{r\lambda} = \int_{\phi_i=0}^{2\pi} \int_{\theta_i=0}^{\theta_c} \frac{\rho_D}{\pi} L_{s\lambda} 10^{\frac{-\epsilon d C}{\cos(\theta)}} \cos(\theta_i) \sin(\theta_i) d\theta_i d\phi_i \quad (9)$$

Chlorophyll a has an extinction coefficient of  $\epsilon_{663} = 0.088 \frac{cm^{-1}}{g}$  at 663nm. Since the light collected by the camera will have had to travel up the water column we must add another absorbance factor determined by the equation 2.

The light incident on the camera is therefore,

$$L_{i663} = \int_{\phi_i=0}^{2\pi} \int_{\theta_i=0}^{\theta_c} \frac{\rho_D}{\pi} L_{s663} 10^{\frac{-2\epsilon d C}{\cos(\theta)}} \cos(\theta_i) \sin(\theta_i) d\theta_i d\phi_i \quad (10)$$

The cutoff angle should ideally vary with the azimuthal position of the sun, time of day and the latitude of position.

## 2 Analytic Solution

Equation (10) can be solved analytically for a light source which is at angle  $\theta_0$  to  $\theta_c$ . The concentration of the chlorophyll,  $c$ , can be retrieved by,

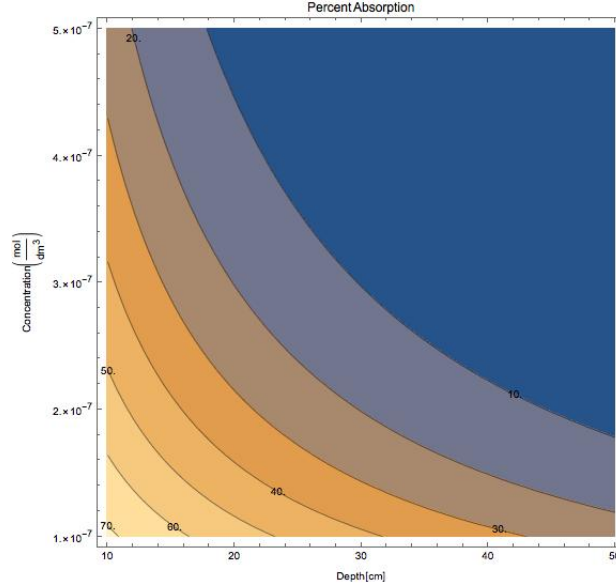
$$C = \text{Log}_{10} \left[ \int_{\theta_0}^{\theta_c} \int_0^{2\pi} \frac{\rho_D}{\pi} \frac{L_{s663}}{L_{i663}} 10^{\frac{-2\epsilon d}{\cos(\theta)}} \cos(\theta_i) \sin(\theta_i) d\theta_i d\phi_i \right] \quad (11)$$

$$= \text{Log}_{10} \left[ 2\rho_D \frac{L_{s663}}{L_{i663}} 10^{-2\epsilon d} \int_{\theta_0}^{\theta_c} 10^{\frac{1}{\cos(\theta)}} \cos(\theta_i) \sin(\theta_i) d\theta_i \right] \quad (12)$$

$$= \text{Log}_{10} \left[ 2\rho_D \frac{L_{s663}}{L_{i663}} 10^{-2\epsilon d} \left( \frac{-\ln(10)\cos(\theta_i) + 1}{\ln^2(10)10^{\cos(\theta_i)}} \right) \Big|_{\theta_0}^{\theta_c} \right] \quad (13)$$

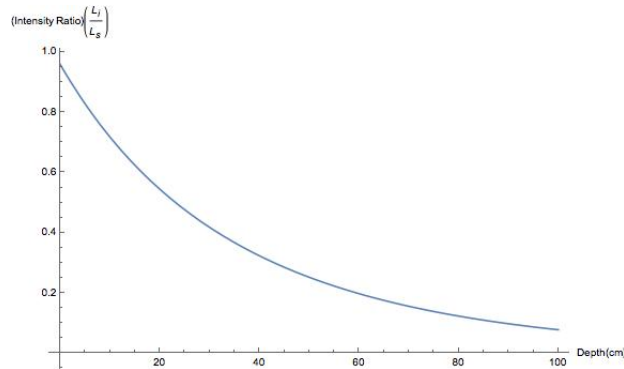
## 3 Numerical Simulations

Carrying out a simplistic modelling of the rendering equation with the secchi disk albedo,  $\rho_d = 1$ , we can carry out the model the intensity percent ratio of reflected light absorbed by the camera and incident light on the submerged secchi disk,  $\frac{L_{r663}}{L_{i663}}$  with,

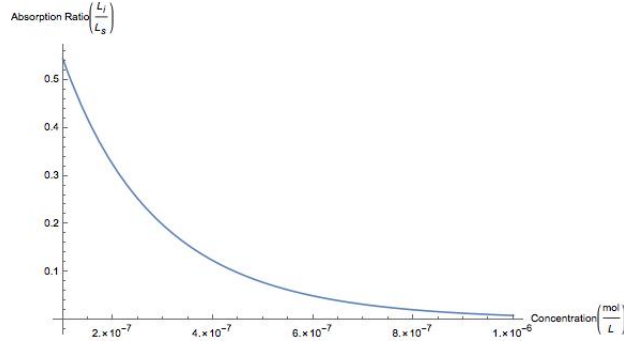


The concentration ranges from  $1 \cdot 10^{-7} \text{ mol} \cdot L^{-1}$  to  $5 \cdot 10^{-7} \text{ mol} \cdot L^{-1}$ . The depth being modeled range from  $10\text{cm}$  to  $50\text{cm}$

Holding the concentration constant at  $1 \cdot 10^{-7} \text{ mol} \cdot L^{-1}$ , we can model the relationship between  $\frac{L_{r663}}{L_{i663}}$  and depth.



Following the same procedure as before, keeping the depth constant at  $20\text{cm}$ , we have the relationship between the concentration and intensity ratio ( $\frac{L_{r663}}{L_{i663}}$ ).



The simulations were carried out with numerical approximations in mathematica and the cutoff angle used was  $\frac{\pi}{2.1}$ . ( $\frac{\pi}{2}$  creates a discontinuity in the integrand)

## 4 Notes/Definitions

**Radiance:** Radiant power per unit foreshortened area per unit solid angle. "throuput for light given point in a given direction" [3]

**Irradiance:** Incident power per unit area. (Light incident on surface or point)

**Bidirectinal Reflectance Distribution Function (BRDF):** The ratio of irradiance and radiance on a surface.

## 5 References

- [1]Harold H. Strain, Mary R. Thomas, Joseph J. Katz, Spectral absorption properties of ordinary and fully deuteriated chlorophylls a and b, In Biochimica et Biophysica Acta, Volume 75, 1963, Pages 306-311, ISSN 0006-3002, [https://doi.org/10.1016/0006-3002\(63\)90617-6](https://doi.org/10.1016/0006-3002(63)90617-6).
- [2]<https://www.cs.princeton.edu/smr/cs348c-97/surveypaper.html>
- [3] <https://www2.cs.arizona.edu/classes/cs433/fall08/lectures/BRDFandGlobalIllum.pdf>
- [4] Beer-Lambert Law (<http://life.nthu.edu.tw/labcjw/BioPhyChem/Spectroscopy/beerslaw.htm>)