# Rendering Equation in Water Column

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## 1 Model

### 1.1 Fresnels Reflection and Transmittance

If we consider the intensity of the light detected by the sensor, $L_{sensor}$, we can effectively decompose the sources into two components: the reflection off of the air-water interface (air-to-air) $L_{reflected}$, and a proportion of the reflectance off of the disk underwater, $K \cdot L_{transimitted}$. Where $K$ is determined by Rendering equation as the concentration of particulates as a parameter.

Given the ambient atmospheric light intenstiy, $L_{atm}$, by energy considerations we can say

$$L_{atm} = L_{reflected} + L_{transmitted}$$

#### 1.1.1 Reflectance

Since the incident light is unpolarized, the reflectance is given by the average of S-polarized and P-polarized light. As an approximation for a given incident angle $\theta$, we can use Schlick's approximation to obtain reflectance R.

$$R(\theta) = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 + \left(1 - \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2\right)(1 - cos(\theta))^5$$

where, $n_1$ is the index of refraction of the incident media (air in our case) and $n_2$ is the index of refracton of the reflected media (water = 1.33).

Since the reflectance is wavelength independent, we have,

$$L_{reflected} = R \cdot L_{atm}$$

#### 1.1.2 Transmittance

We can relate the transmittance $T$ of the unpolarized light with the reflectance $R$ as

$$T = 1 - R$$

We have the transmitted light into the water as,

$$L_{transmitted} = T \cdot L_{atm}$$

The attenuation $L_{transmitted}$ due to chlorophyll absoprtion and reflection on the disk is discussed in the following sections.

In practice considering the reflection may be trivial as at $\theta = 0$ the maximum reflectance is $\approx 3\%$ and has a significant effect at higher incident angles only ($> 60°$).

## 1.2 Rendering Equation

The interaction of incident, reflected, and emmited light at any given point on a given surface is described by the general Rendering Equation,

$$L_o = L_e + L_r \tag{1}$$

Where, $L_o$ is the light leaving the particular point, $L_e$ is the light emitted from the point (if it is a source), and $L_r$ is the light reflected from that point.

Since we will be working with non-light generating disks, we set, $L_e = 0$. The light reflected off of the surface of a disk, can be calculated using the Bidirectional Reflectance Distribution Function (BRDF)[3]. For a given point on a surface, the general BRDF is,

$$\rho_d(\phi_0, \theta_0, \phi_f, \theta_f) = \frac{L_r \phi_f, \theta_f}{L_I \phi_0, \theta_0} \tag{2}$$

Where $\phi_0, \theta_0$ are the incident azimuthal and polar angle of the light source, and $\phi_f, \theta_0$ are the azimuthal and polar angle with respect to the viewer and and relfected point.

From the general rendering equation, the reflected radiance is given by

$$L_r = \int_{\phi_i=0}^{2\pi} \int_{\theta_i=0}^{\frac{\pi}{2}} f_r L_s T(d) cos(\theta_i) sin(\theta_i) d\theta_i d\phi_i \tag{3}$$

Where, the $f_r$ is Lambertian BRDF.

The transmission function $T(d)$ can be related to the depth of the reflecting surface.

## 1.3 Beer-Lambert Law

The Beer-Lambert provides us with a linear relationship betweem incident light, absorbed light, and concentration of absorbed light.

The wavelength dependent relation is given as,

$$\epsilon dc = Log_{10} \frac{I_0}{I_f} \tag{4}$$

where, $\epsilon$ is the molar absorptivity in $L \cdot cm^{-1} \cdot mol^{-1}$, $d$ is the depth in $cm$, $C$ is the concentration in $mol \cdot L^{-1}$. $I_0$ is the initial light intensity, and $I_f$ is the final intensity after absorption.

Given some incident light of radiance, $L_0$, on the water surface,

we have the decomposed radiance as

$$L_0 = L_{sa} + L_s$$

where, $L_{sa}$ is the absorbed light radiation. $L_s$ is the light incident on the point under water and being reflected by the disk.

According to the Beer-Lambert Law we have the relation,

$$Log_{10} \left( \frac{L_0}{L_s} \right) = \frac{\epsilon dC}{cos(\theta)} \tag{5}$$

and,

$$T(d) = 10^{\frac{-\epsilon dC}{cos(\theta)}} \tag{6}$$

## 1.4 Reflectance

Since we want to specifically take into account the absorbance due to chlorophyll a, we limit the consideration to considering the radiance of light at $\lambda = 662nm$

Even though the incident light is poloychromatic, but since we are considering the specific wavelength and the incident light is so spread out, we can reasonably treat it as monochromatic. (?)

The reflectance in the particular frequency then becomes,

$$L_{r\lambda} = \int_{\phi_i=0}^{2\pi} \int_{\theta_i=0}^{\theta_c} f_r L_{s\lambda} 10^{\frac{-\epsilon dC}{cos(\theta)}} cos(\theta_i) sin(\theta_i) d\theta_i d\phi_i \tag{7}$$

Here we sum the incident angle upto some cutoff angle $\theta_c$, as the higher incident lights have negligible effect on the reflected light.

The cutoff angle should ideally vary with the azimuthal position of the sun, time of day and the lattitude of position.

We can further make an assumption that the secchi disk is perfectly matte with rotationally invariant reflectance. The Lambertian BRDF is thus,

$$f_r = \frac{\rho_D}{\pi} \tag{8}$$

Where, $\rho_D$ is the albedo of the secchi disk with $0 \leq \rho_D \leq 1$ Combining (4) and (5) we get the reflectance off of the surface of secchi disk as,

$$L_{r\lambda} = \int_{\phi_i=0}^{2\pi} \int_{\theta_i=0}^{\theta_c} \frac{\rho_D}{\pi} L_{s\lambda} 10^{\frac{-\epsilon dC}{cos(\theta)}} cos(\theta_i) sin(\theta_i) d\theta_i d\phi_i \tag{9}$$

Chlorophyll a has an extinction coefficient of $\epsilon_{663} = 0.088 \frac{cm^{-1}}{g}$ at 663nm. Since the light collected by the camera will have had to travel up the water column we must add another absorbance factor determined by the equation 2.

The light incident on the camera is therefore,

$$L_{i663} = \int_{\phi_i=0}^{2\pi} \int_{\theta_i=0}^{\theta_c} \frac{\rho_D}{\pi} L_{s663} 10^{\frac{-2\epsilon dC}{cos(\theta)}} cos(\theta_i) sin(\theta_i) d\theta_i d\phi_i \tag{10}$$

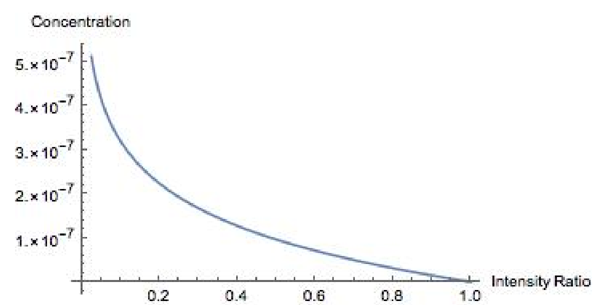
## 1.5 Analytical Solution

### 1.5.1 At High Cutoff Angles

At high cutoff angles $\theta_c > 60°$, we can reasonably account for reflectance off of the water surface to be sufficiently low ($\leq 3\%$). The correction to the incident is sufficiently low and can be discounted.

Equation (10) can be solved analytically to the first order approximation of the taylor expansion of $10^{\frac{-\epsilon dC}{cos(\theta_i)}}$, for a light source which is at angle $\theta_0$ to $\theta_c$. The concentration of the chlorophyll, c, can be retrieved by,

$$C = \frac{ln\left(\frac{L_{i663}}{2\rho_d L_{s663}\left(\frac{\cos^2(\theta_i)}{2}\Big|_{\theta_0}^{\theta_c}\right)}\right)}{\epsilon d ln(10^{-2})} \tag{11}$$




Notice that as $d \to 0$ the equation reduces to a function of incident angle and properties of the material and will correspond to the concentration $C \to 0$. The coupling of $\epsilon$ and $d$ also shows that in a particulate free medium ($\epsilon = 0$), the reflectance is affected by the depth $d$ and is analogous to the case where $d = 0$ for any depth.

In this case, $\theta_c$ is less than the $\frac{\pi}{3}$ so we can assume the difference transmitted intensity is the same as the incident atmospheric intensity. The addition of higher incident angles would make equation unsolvable analytically (But I believe we can approximate using perturbation theory).

### 1.6 Second Order approximation

(I'm still grappling with the math on this one) The Appendix has the math carried out but the last approximation is troubling me.


### 1.7 Possible Concerns

- The issues with non-constant air-water interface (waves) is that both the incident light intenstiy and reflected light intensity can very dramatically over the the surface of the secchi disk and result in patches of high reflectance and distorted disk image. Imposing a periodic boundary condition, by defining the water surface as a plane wave could be a better approximation. There would be a significant increase in calculation complexity in that case.
- Another siginificant ommitance is back scattering from the reflected light, being reflected back onto the disk. (For an approximation, I believe the correction due to this would be marginal, but I have to carry out a sample calculation to have a concrete value).

## 2 Sensor Baseline

### 2.1 Compensating for Exposure

#### 2.1.1 Aperture and Intensity

The aperture of, measured in f-stops, is the opening of the camera sensor. the f-stop measures the amount of sensors surface open to collect photons. Larger apertures create a smaller depth of field (should be considered when deciding on the baseline).

The intensity of the incident light is inversely proportional to the square of the f-number.

$$I \propto \frac{1}{f^2}$$

#### 2.1.2 Shutter Speed and Intensity

We can assume that the shutter speed (how long the sensor collects photons), is inversely proportional to the measured intensity.

$$I \propto \frac{1}{T}$$

#### 2.1.3 Pixel values

If the pixel value is an estimate of the power per solid angle per area of the sensor, we can combine the shutter speed and aperture relations as

$$I \propto \frac{1}{T \cdot f^2}$$

# 3 Wavelength Intensity

The raw data obtained from the camera provides us with the three values in the R,G,B channels per pixel (see mathemtica for example). Given a specific wavelength, $\lambda$, we want to obtain the intensity at that wavelenth.

## 3.1 Spectral Sensitivity Function

Given constant ISO and aperture, at each wavelenth, $\lambda$, the spectral sensitivity in RGB channels is calculated by

$$c(\lambda) = \frac{d(\lambda)}{r(\lambda)t(\lambda)}$$

where, $r(\lambda)$ is the radiance. $t(\lambda)$ is the exposure time. $d(\lambda)$ is the data.

Therefore rdiance at a wavelength would be,

$$r(\lambda) = \frac{d(\lambda)}{c(\lambda)t'(\lambda, f)}$$

with $t'(\lambda)$ is a function of the wavelength and aperture as well.

## 3.2 RGB Image Analysis

For a Lambartian diffuser, the most general theoretical digital value for each pixel, $g_i$,

$$g_i = \alpha_i \left[ \int_\lambda I_\lambda O_\lambda L_r F_{i\lambda} C_\lambda d\lambda \right]^{\gamma_i} + \beta_i \tag{12}$$

where i = Red, Green, Blue. $O_\lambda$ is the spectral reflectance of the object (in our case the secchi disk), $C_\lambda$ is the spectral sensitivity of the image sensor at the particular wavelength. $F_\lambda$ is the spectral transmittance of the color filters. $\gamma_i$ is the gamma correction done by each channel and $\beta_i$ is the error factor due to dark current

If no processing is done by the camera electronics during the photo we can set $\gamma_i = \alpha_i = F_{i\lambda} = 1$ we get the equation of the form,

$$g_i = \int_\lambda I_\lambda O_\lambda L_r C_\lambda d\lambda + O(\text{error}) \tag{13}$$

Without post-processig the pixel value is given by

$$g_i$$

The incident light on the seonsor for any wevalength is given by, $I_\lambda O_\lambda L$.

The spectral response of the seons $C_\lambda$ is non-linear not readily available. If we consider corresponding picels of images of unsubmerged and submerged Secchi diskds, $g$ and $g^{\Gamma, wecanobtaintherelativeinensities.}$

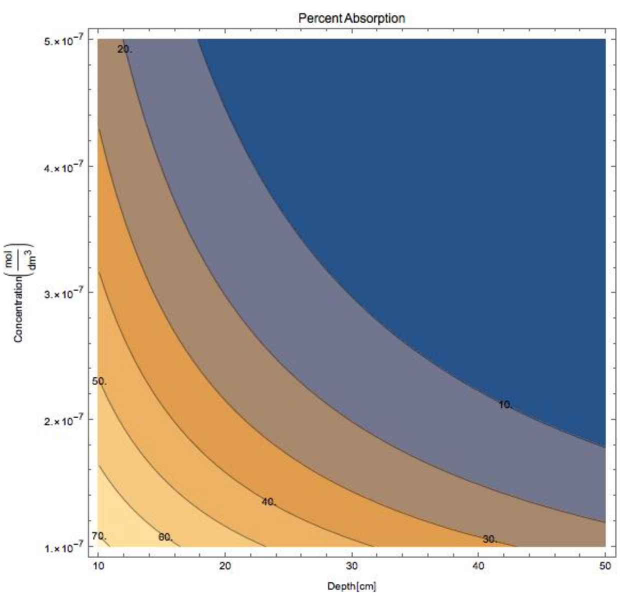
# 4 Experiment

## 4.1 Set Up

Using a lamp as a point source, we can measure the efficacy of the model. The set up for the experiment is with a secchi disk positioned submerged at depth $d$, in a solution of water and particulates - possible food coloring or something similar- with known absorption coefficients $\epsilon$. We can use a lamp as the point source of light. Placing the lamp at height $H$ above the secchi disk, at an angle $\theta_a$.

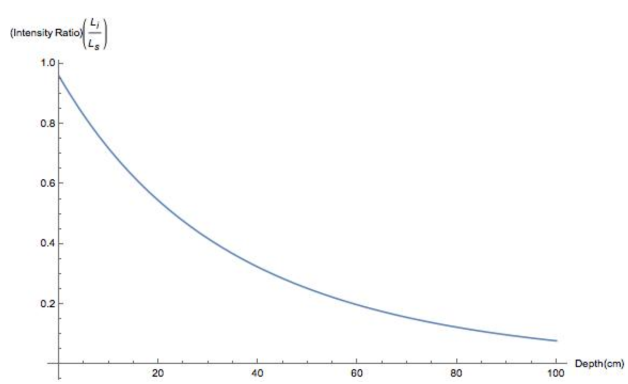
# 5 Numerical Simulations

Carrying out a simplistic modelling of the rendering equation with the secchi disk albedo, $\rho_d = 1$, we can carry out the model the intensity percent ratio of reflected light absorbed by the camera and incident light on the submerged secchi disk, $\frac{L_{r663}}{L_{i663}}$ with,

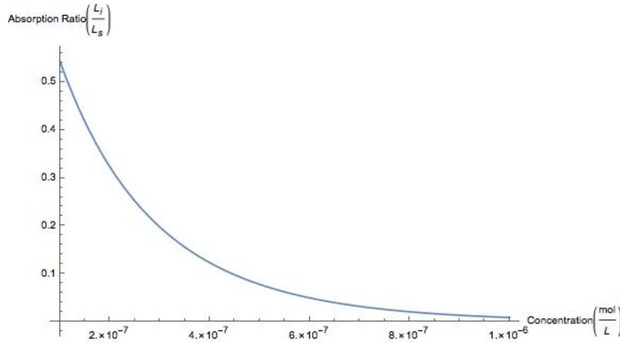



The concentration ranges from $1 \cdot 10^{-7} mol \cdot L^{-1}$ to $5 \cdot 10^{-7} mol \cdot L^{-1}$. The depth being modeled range from $10cm$ to $50cm$

Holding the concentration consant at $1 \cdot 10^{-7} mol \cdot L^{-1}$, we can model the relationship between $\frac{L_{r663}}{L_{i663}}$ and depth.




Following the same procedure as before, keeping the depth constant at $20cm$, we have the relationship between the concentration and intensity ratio ($\frac{L_{r663}}{L_{i663}}$).




The simulations were carried out with numerical approximations in mathemtica and the cutoff angle angle used was $\frac{\pi}{2.1}$. ($\frac{\pi}{2}$ creates a discontinuity in the integrand)

# 6 Appendix

## 6.1 Concentration Function calculation

The taylor expansion of $10^{\frac{-2\epsilon dC}{cos(\theta)}}$ is given by

$$10^{\frac{-2\epsilon dC}{cos(\theta)}} = 10^{-2\epsilon dC} - 10^{-2\epsilon dC}\epsilon dCx^2ln(10)$$

We can use the taylor expantion to solve for rendering equation analytically up to the second order,

$$\begin{aligned}
L_{i663} &= \int_{\phi_i=0}^{2\pi}\int_{\theta_i=0}^{\theta_c}\frac{\rho_D}{\pi}L_{s663}10^{\frac{-2\epsilon dC}{cos(\theta)}}cos(\theta_i)sin(\theta_i)d\theta_i d\phi_i \\
&= \int_{\phi_i=0}^{2\pi}\int_{\theta_i=0}^{\theta_c}\frac{\rho_D}{\pi}L_{s663}\left(10^{-2\epsilon dC} - 10^{-2\epsilon dC}\epsilon dC\theta_i^2 ln(10)\right)cos(\theta_i)sin(\theta_i)d\theta_i d\phi_i \\
&= \int_{\theta_i=0}^{\theta_c}2\rho_D L_{s663}\left(10^{-2\epsilon dC} - 10^{-2\epsilon dC}\epsilon dC\theta_i^2 ln(10)\right)cos(\theta_i)sin(\theta_i)d\theta_i \\
&= 2\rho_D L_{s663}10^{-2\epsilon dC}\left(\int_{\theta_i=0}^{\theta_c}cos(\theta_i)sin(\theta_i)d\theta_i - \epsilon dCln(10)\int_{\theta_i=0}^{\theta_c}\theta_i^2 cos(\theta_i)sin(\theta_i)d\theta_i\right) \\
&= 2\rho_D L_{s663}10^{-2\epsilon dC}\left(\left.\frac{-\cos^2(\theta_i)}{2}\right|_{\theta_0}^{\theta_c} - (\epsilon dCln(10))\left.\frac{2\theta_i\sin(2\theta_i) + \left(1-2\theta_i^2\right)\cos(2\theta_i)}{8}\right|_{\theta_0}^{\theta_c}\right)
\end{aligned}$$

Defining the angular constats as k and q, we can approximate the Concentration function to the second order,

$$L_{i663} = 2\rho_D L_{s663}10^{-2\epsilon dC}(k - Cq)$$

$$\begin{aligned}
\frac{L_{i663}}{2\rho_D L_{s663}} &= 10^{2\epsilon dC}(k - Cq) \\
ln\left(\frac{L_{i663}}{2\rho_D L_{s663}}\right) &= \epsilon dC \cdot ln(10^{-2}) + ln(k - Cq) \\
ln\left(\frac{L_{i663}}{2\rho_D L_{s663}}\right) + k - 1 &= C \\
\frac{ln\left(\frac{L_{i663}10^{2\epsilon d}}{2\rho_D L_{s663}}\right) + k - 1}{q - ln(10)} &= C
\end{aligned}$$

# 7 Notes/Definitions

**Radiance:** Radiant power per unit foreshortened area per unit solid angle. "throuput for light given point in a given direction" [3]
**Irradiance:** Incident power per unit area. (Light incident on surface or point)
**Bidirectional Reflectance Distribution Function (BRDF):** The ratio of irradiance and radiance on a surface. **Dark Current:** Residual current on camera sensor without any incident light. (related to degradation of sensor possibly?)

# 8 References

[1] Harold H. Strain, Mary R. Thomas, Joseph J. Katz, Spectral absorption properties of ordinary and fully deuteriated chlorophylls a and b, In Biochimica et Biophysica Acta, Volume 75, 1963, Pages 306-311, ISSN 0006-3002, https://doi.org/10.1016/0006-3002(63)90617-6.
[2] https://www.cs.princeton.edu/ smr/cs348c-97/surveypaper.html
[3] https://www2.cs.arizona.edu/classes/cs433/fall08/lectures/BRDFandGlobalIllum.pdf
[4] Beer-Lambert Law (`http://life.nthu.edu.tw/~labcjw/BioPhyChem/Spectroscopy/beerslaw.htm`)
[5] Incomplete Gamma Function (`http://people.math.sfu.ca/~cbm/aands/page_260.htm`)