Rendering Equation in Water Column

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1 Rendering Equation

From the general radiance equation,the reflected radiance is given by

$$L_r = \int_{\phi_i=0}^{2\pi} \int_{\theta_i=0}^{\frac{\pi}{2}} f_r L_s T(d) cos(\theta_i) sin(\theta_i) d\theta_i d\phi_i \tag{1}$$

Where, the f_r is Lambertian BRDF.

The transmission function $T(d)$ can be related to the depth of the reflecting surface.

Given an incident light of radiance, L_0, we have the decomposed radiance as

$$L_{sa} + L_s = L_0$$

where, L_{sa} is the absorbed light radiation.

According to the Beer-Lambert Law we have the relation,

$$Log_{10}\left(\frac{L_0}{L_s}\right) = \frac{-\epsilon dC}{cos(\theta)} \tag{2}$$

and,

$$T(d) = 10^{\frac{-\epsilon dC}{cos(\theta)}} \tag{3}$$

where, ϵ is the molar absorptivity in $L \cdot cm^{-1} \cdot mol^{-1}$, d is the depth in cm, C is the concentration in $mol \cdot L^{-1}$.

Since we want to specifically take into account the absorbance due to chlorophyll a, we limit the consideration to considering the radiance of light at $\lambda = 662nm$

Even though the incident light is poloychromatic, but since we are considering the specific wavelength and the incident light is so spread out, we can reasonably treat it as monochromatic. (?)

The reflectance in the particular frequency then becomes,

$$L_{r\lambda} = \int_{\phi_i=0}^{2\pi} \int_{\theta_i=0}^{\theta_c} f_r L_{s\lambda} 10^{\frac{-\epsilon dC}{cos(\theta)}} cos(\theta_i) sin(\theta_i) d\theta_i d\phi_i \tag{4}$$

Here we sum the incident angle upto some cutoff angle θ_c, as the higher incident lights have negligible effect on the reflected light.

The cutoff angle should ideally vary with the azimuthal position of the sun, time of day and the lattitude of position.

We can further make an assumption that the secchi disk is perfectly matte with rotationally invariant reflectance. The Lambertian BRDF is thus,

$$f_r = \frac{\rho_D}{\pi} \tag{5}$$

Where, ρ_D is the albedo of the secchi disk with $0 \leq \rho_D \leq 1$ Combining (4) and (5) we get the reflectance off of the surface of secchi disk as,

$$L_{r\lambda} = \int_{\phi_i=0}^{2\pi} \int_{\theta_i=0}^{\theta_c} \frac{\rho_D}{\pi} L_{s\lambda} 10^{\frac{-\epsilon dC}{cos(\theta)}} cos(\theta_i) sin(\theta_i) d\theta_i d\phi_i \tag{6}$$

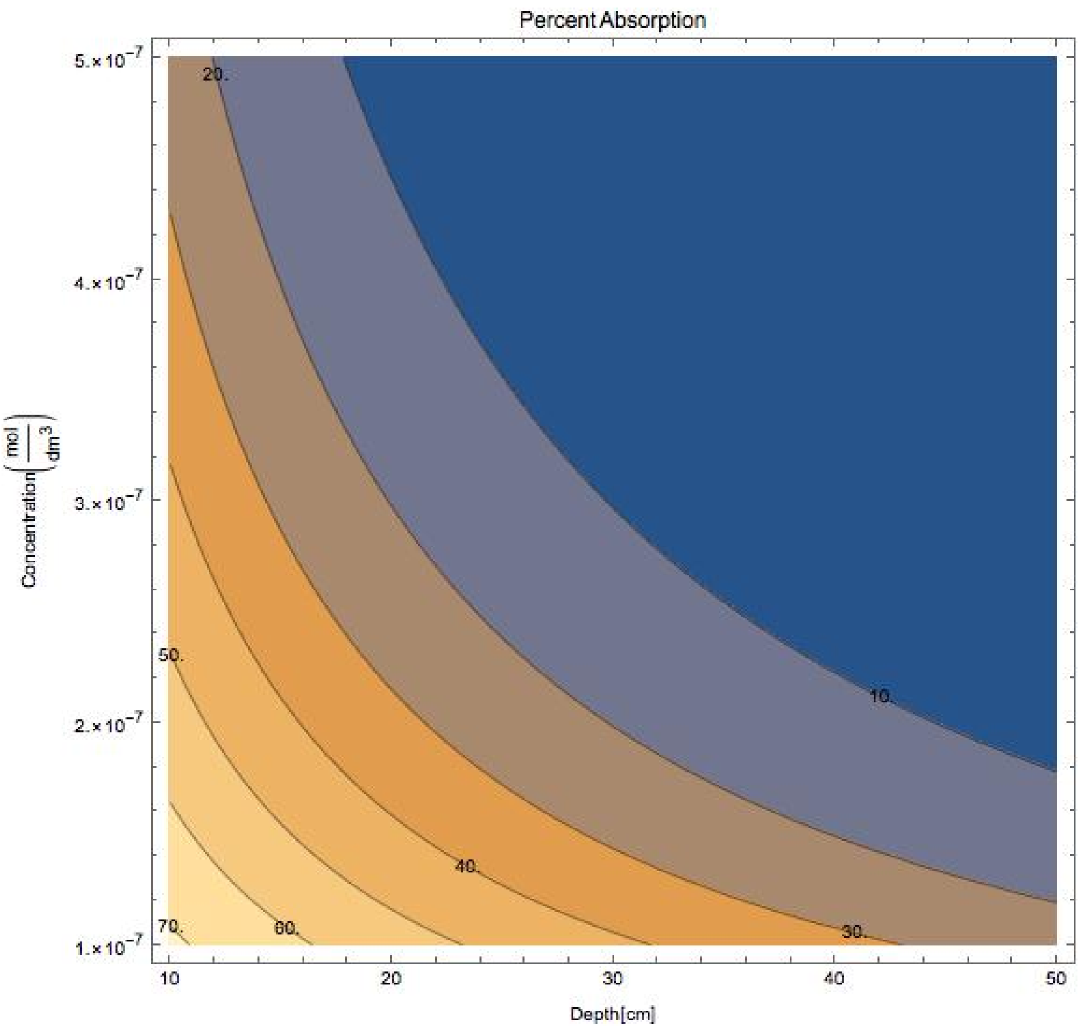
Chlorophyll a has an extinction coefficient of $\epsilon_{663} = 0.088 \frac{cm^{-1}}{g}$ at 663nm. Since the light collected by the camera will have had to travel up the water column we must add another absorbance factor determined by the equation 2.

The light incident on the camera is therefore,

$$L_{i663} = \int_{\phi_i=0}^{2\pi} \int_{\theta_i=0}^{\theta_c} \frac{\rho_D}{\pi} L_{s663} 10^{\frac{-2\epsilon dC}{cos(\theta)}} cos(\theta_i) sin(\theta_i) d\theta_i d\phi_i \tag{7}$$

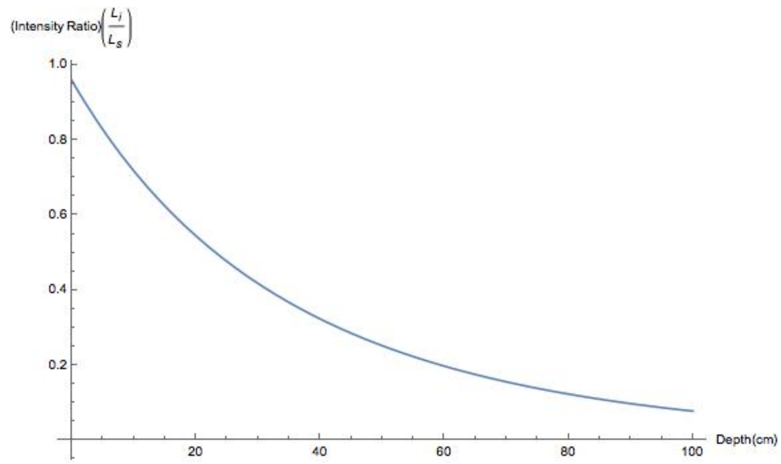
2 Model

Carrying out a simplistic modelling of the rendering equation with the secchi disk albedo, $\rho_d = 1$, we can carry out the model the intensity percent ratio of reflected light absorbed by the camera and incident light on the submerged secchi disk, $\frac{L_{r663}}{L_{i663}}$ with,

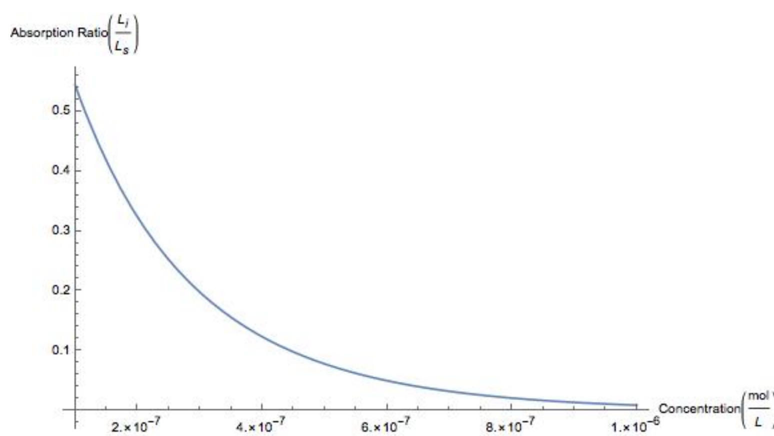


The concentration ranges from $1 \cdot 10^{-7} mol \cdot L^{-1}$ to $5 \cdot 10^{-7} mol \cdot L^{-1}$. The depth being modeled range from $10cm$ to $50cm$

Holding the concentration consant at $1 \cdot 10^{-7} mol \cdot L^{-1}$, we can model the relationship between $\frac{L_{r663}}{L_{i663}}$ and depth.



Following the same procedure as before, keeping the depth constant at $20cm$, we have the relationship between the concentration and intensity ratio ($\frac{L_{r663}}{L_{i663}}$).



The simulations were carried out with numerical approximations in mathemtica and the cutoff angle angle used was $\frac{\pi}{2.1}$. ($\frac{\pi}{2}$ creates a discontinuity in the integrand)

3 References

Harold H. Strain, Mary R. Thomas, Joseph J. Katz, Spectral absorption properties of ordinary and fully deuteriated chlorophylls a and b, In Biochimica et Biophysica Acta, Volume 75, 1963, Pages 306-311, ISSN 0006-3002, https://doi.org/10.1016/0006-3002(63)90617-6.