Rendering Equation in Water Column

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1 Model

1.1 Fresnels Reflection and Transmittance

If we consider the intensity of the light detected by the sensor, L_{sensor}, we can effectively decompose the sources into two components: the reflection off of the air-water interface (air-to-air) $L_{reflected}$, and a proportion of the reflectance off of the disk underwater, $K \cdot L_{transimitted}$. Where K is determined by Rendering equation as the concentration of particulates as a parameter.

Given the ambient atmospheric light intenstiy, L_{atm}, by energy considerations we can say

$$L_{atm} = L_{reflected} + L_{transmitted}$$

1.1.1 Reflectance

Since the incident light is unpolarized, the reflectance is given by the average of S-polarized and P-polarized light. As an approximation for a given incident angle θ, we can use Schlick's approximation to obtain reflectance R.

$$R(\theta) = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 + \left(1 - \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2\right)(1 - cos(\theta))^5$$

where, n_1 is the index of refraction of the incident media (air in our case) and n_2 is the index of refracton of the reflected media (water = 1.33).

Since the reflectance is wavelength independent, we have,

$$L_{reflected} = R \cdot L_{atm}$$

1.1.2 Transmittance

We can relate the transmittance T of the unpolarized light with the reflectance R as

$$T = 1 - R$$

We have the transmitted light into the water as,

$$L_{transmitted} = T \cdot L_{atm}$$

The attenuation $L_{transmitted}$ due to chlorophyll absoprtion and reflection on the disk is discussed in the following sections.

In practice considering the reflection may be trivial as at $\theta = 0$ the maximum reflectance is $\approx 3\%$ and has a significant effect at higher incident angles only ($> 60°$).

1.2 Rendering Equation

The interaction of incident, reflected, and emmited light at any given point on a given surface is described by the general Rendering Equation,

$$L_o = L_e + L_r \tag{1}$$

Where, L_o is the light leaving the particular point, L_e is the light emitted from the point (if it is a source), and L_r is the light reflected from that point.

Since we will be working with non-light generating disks, we set, $L_e = 0$. The light reflected off of the surface of a disk, can be calculated using the Bidirectional Reflectance Distribution Function (BRDF)[3]. For a given point on a surface, the general BRDF is,

$$\rho_d(\phi_0, \theta_0, \phi_f, \theta_f) = \frac{L_r \phi_f, \theta_f}{L_I \phi_0, \theta_0} \tag{2}$$

Where ϕ_0, θ_0 are the incident azimuthal and polar angle of the light source, and ϕ_f, θ_0 are the azimuthal and polar angle with respect to the viewer and and relfected point.

From the general rendering equation, the reflected radiance is given by

$$L_r = \int_{\phi_i=0}^{2\pi} \int_{\theta_i=0}^{\frac{\pi}{2}} f_r L_s T(d) cos(\theta_i) sin(\theta_i) d\theta_i d\phi_i \tag{3}$$

Where, the f_r is Lambertian BRDF.

The transmission function $T(d)$ can be related to the depth of the reflecting surface.

1.3 Beer-Lambert Law

The Beer-Lambert provides us with a linear relationship betweem incident light, absorbed light, and concentration of absorbed light.

The wavelength dependent relation is given as,

$$\epsilon dc = Log_{10} \frac{I_0}{I_f} \tag{4}$$

where, ϵ is the molar absorptivity in $L \cdot cm^{-1} \cdot mol^{-1}$, d is the depth in cm, C is the concentration in $mol \cdot L^{-1}$. I_0 is the initial light intensity, and I_f is the final intensity after absorption.

Given some incident light of radiance, L_0, on the water surface,

we have the decomposed radiance as

$$L_0 = L_{sa} + L_s$$

where, L_{sa} is the absorbed light radiation. L_s is the light incident on the point under water and being reflected by the disk.

According to the Beer-Lambert Law we have the relation,

$$Log_{10}\left(\frac{L_0}{L_s}\right) = \frac{\epsilon dC}{cos(\theta)} \tag{5}$$

and,

$$T(d) = 10^{\frac{-\epsilon dC}{cos(\theta)}} \tag{6}$$

1.4 Reflectance

Since we want to specifically take into account the absorbance due to chlorophyll a, we limit the consideration to considering the radiance of light at $\lambda = 662nm$

Even though the incident light is poloychromatic, but since we are considering the specific wavelength and the incident light is so spread out, we can reasonably treat it as monochromatic. (?)

The reflectance in the particular frequency then becomes,

$$L_{r\lambda} = \int_{\phi_i=0}^{2\pi} \int_{\theta_i=0}^{\theta_c} f_r L_{s\lambda} 10^{\frac{-\epsilon dC}{cos(\theta)}} cos(\theta_i) sin(\theta_i) d\theta_i d\phi_i \tag{7}$$

Here we sum the incident angle upto some cutoff angle θ_c, as the higher incident lights have negligible effect on the reflected light.

The cutoff angle should ideally vary with the azimuthal position of the sun, time of day and the lattitude of position.

We can further make an assumption that the secchi disk is perfectly matte with rotationally invariant reflectance. The Lambertian BRDF is thus,

$$f_r = \frac{\rho_D}{\pi} \tag{8}$$

Where, ρ_D is the albedo of the secchi disk with $0 \leq \rho_D \leq 1$ Combining (4) and (5) we get the reflectance off of the surface of secchi disk as,

$$L_{r\lambda} = \int_{\phi_i=0}^{2\pi} \int_{\theta_i=0}^{\theta_c} \frac{\rho_D}{\pi} L_{s\lambda} 10^{\frac{-\epsilon dC}{cos(\theta)}} cos(\theta_i) sin(\theta_i) d\theta_i d\phi_i \tag{9}$$

Chlorophyll a has an extinction coefficient of $\epsilon_{663} = 0.088 \frac{cm^{-1}}{g}$ at 663nm. Since the light collected by the camera will have had to travel up the water column we must add another absorbance factor determined by the equation 2.

The light incident on the camera is therefore,

$$L_{i663} = \int_{\phi_i=0}^{2\pi} \int_{\theta_i=0}^{\theta_c} \frac{\rho_D}{\pi} L_{s663} 10^{\frac{-2\epsilon dC}{cos(\theta)}} cos(\theta_i) sin(\theta_i) d\theta_i d\phi_i \tag{10}$$

1.5 Analytic Solution

Equation (10) can be solved analytically for a light source which is at angle θ_0 to θ_c. The concentration of the chlorophyll, c, can be retrieved by,

$$\begin{aligned}
C &= Log_{10} \left[\int_{\theta_0}^{\theta_c} \int_0^{2\pi} \frac{\rho_D}{\pi} \frac{L_{s663}}{L_{i663}} 10^{\frac{-2\epsilon d}{cos(\theta)}} cos(\theta_i) sin(\theta_i) d\theta_i d\phi_i \right] && (11) \\
&= Log_{10} \left[2\rho_D \frac{L_{s663}}{L_{i663}} 10^{-2\epsilon d} \int_{\theta_0}^{\theta_c} 10^{\frac{1}{cos(\theta)}} cos(\theta_i) sin(\theta_i) d\theta_i \right] && (12) \\
&= Log_{10} \left[2\rho_D \frac{L_{s663}}{L_{i663}} 10^{-2\epsilon d} \left(\frac{-ln(10) cos(\theta_i) + 1}{ln^2(10) 10^{cos(\theta_i)}} \Bigg|_{\theta_0}^{\theta_c} \right) \right] && (13)
\end{aligned}$$

In this case, θ_c is less than the $\frac{\pi}{3}$ so we can assume the difference transmitted intensity is the same as the incident atmospheric intensity. The addition of higher incident angles would make equation unsolvable analytically (But I believe we can approximate using perturbation theory).

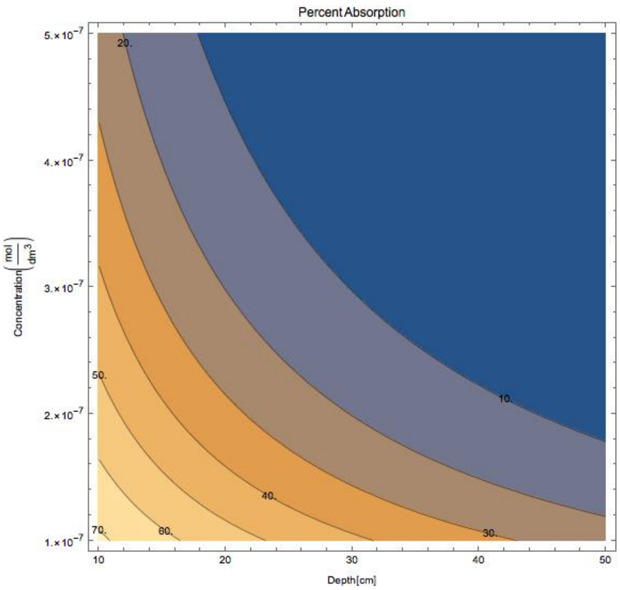
2 Experiment

2.1 Set Up

Using a lamp as a point source, we can measure the efficacy of the model. The set up for the experiment is with a secchi disk positioned submerged at depth d, in a solution of water and particulates - possible food coloring or something similar- with known absorption coefficients ϵ. We can use a lamp as the point source of light. Placing the lamp at height H above the secchi disk, at an angle θ_a.

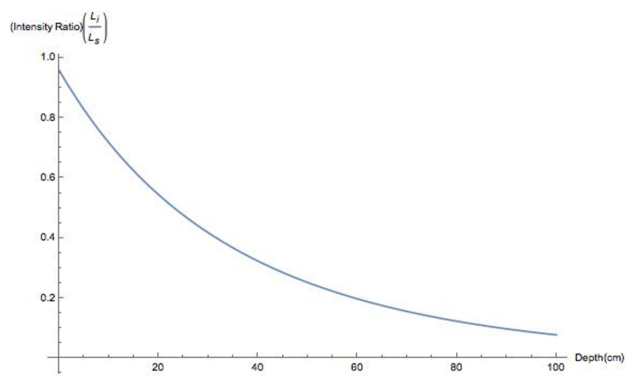
3 Numerical Simulations

Carrying out a simplistic modelling of the rendering equation with the secchi disk albedo, $\rho_d = 1$, we can carry out the model the intensity percent ratio of reflected light absorbed by the camera and incident light on the submerged secchi disk, $\frac{L_{r663}}{L_{i663}}$ with,

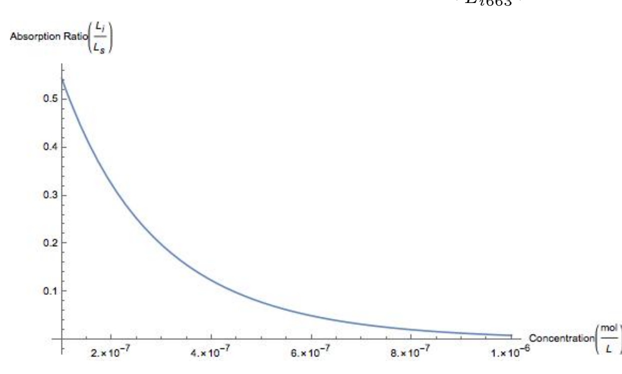


The concentration ranges from $1 \cdot 10^{-7} mol \cdot L^{-1}$ to $5 \cdot 10^{-7} mol \cdot L^{-1}$. The depth being modeled range from $10cm$ to $50cm$

Holding the concentration consant at $1 \cdot 10^{-7} mol \cdot L^{-1}$, we can model the relationship between $\frac{L_{r663}}{L_{i663}}$ and depth.



Following the same procedure as before, keeping the depth constant at $20cm$, we have the relationship between the concentration and intensity ratio ($\frac{L_{r663}}{L_{i663}}$).



The simulations were carried out with numerical approximations in mathemtica and the cutoff angle angle used was $\frac{\pi}{2.1}$. ($\frac{\pi}{2}$ creates a discontinuity in the integrand)

4 Notes/Definitions

Radiance: Radiant power per unit foreshortened area per unit solid angle. "throuput for light given point in a given direction" [3]
Irradiance: Incident power per unit area. (Light incident on surface or point)
Bidirectional Reflectance Distribution Function (BRDF): The ratio of irradiance and radiance on a surface.

5 References

[1] Harold H. Strain, Mary R. Thomas, Joseph J. Katz, Spectral absorption properties of ordinary and fully deuteriated chlorophylls a and b, In Biochimica et Biophysica Acta, Volume 75, 1963, Pages 306-311, ISSN 0006-3002, https://doi.org/10.1016/0006-3002(63)90617-6.
[2] https://www.cs.princeton.edu/ smr/cs348c-97/surveypaper.html
[3] https://www2.cs.arizona.edu/classes/cs433/fall08/lectures/BRDFandGlobalIllum.pdf
[4] Beer-Lambert Law (http://life.nthu.edu.tw/ labcjw/BioPhyChem/Spectroscopy/beerslaw.htm)