Mean Field Approach to BCS Theory

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1 BCS Hamiltonian

$$H_I = \frac{-V}{\Omega} \sum_{\vec{k}\vec{p}} c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} c_{-\vec{p}\uparrow} c_{-\vec{p}\downarrow}$$

$$H_K = \sum_{\vec{k}\sigma} \xi(\vec{k}) c^\dagger_{\vec{k}\sigma} c_{\vec{k}\sigma}$$

$$\xi(\vec{k}) = \frac{\hbar^2 k^2}{2m} - \mu$$

Order Parameter:

$$\Delta = \frac{V}{\Omega} \sum_{\vec{k}} \langle c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} \rangle$$

2 Mean Field Decoupling

Mean Field Decouple on dagger and non-dagger terms to get:

$$c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} = \langle c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} \rangle + \left(c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} - \langle c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} \rangle \right)$$

$$c_{-\vec{p}\uparrow} c_{-\vec{p}\downarrow} = \langle c_{-\vec{p}\uparrow} c_{-\vec{p}\downarrow} \rangle + \left(c_{-\vec{p}\uparrow} c_{-\vec{p}\downarrow} - \langle c_{-\vec{p}\uparrow} c_{-\vec{p}\downarrow} \rangle \right)$$

Define for simplicity of notation, for $\vec{k} \neq \vec{p}$:

$$a^\dagger_{\vec{k}} = c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow}$$

$$a_{\vec{p}} = c_{-\vec{p}\uparrow} c_{-\vec{p}\downarrow}$$

Rewrite the intercation Hamiltonian as,

$$H_I = \frac{-V}{\Omega} \sum_{\vec{k},\vec{p}} \left[\langle a^\dagger_{\vec{k}} \rangle + \left(a^\dagger_{\vec{k}} - \langle a^\dagger_{\vec{k}} \rangle \right) \right] \left[\langle a_{\vec{p}} \rangle + \left(a_{\vec{p}} - \langle a_{\vec{p}} \rangle \right) \right]$$

Multiplying out and ignoring the double fluctuation term for mean field approximation,

$$H_I = \frac{-V}{\Omega} \left(\sum_{\vec{k}\vec{p}} \langle a^\dagger_{\vec{k}} \rangle \langle a_{\vec{p}} \rangle + \langle a_{\vec{p}} \rangle a^\dagger_{\vec{k}} - \langle a_{\vec{p}} \rangle \langle a^\dagger_{\vec{k}} \rangle + \langle a^\dagger_{\vec{k}} \rangle a_{\vec{p}} - \langle a^\dagger_{\vec{k}} \rangle a_{\vec{p}} \right)$$

Using the order parameter notation, we get,

$$H_I = \sum_{k} -\Delta^\dagger c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow} - \Delta c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\uparrow} - \frac{\Omega}{V} \Delta^\dagger \Delta$$

Adding the site hopping Hamiltonian,

$$H_k = \sum_k \xi(\vec{k}) c^\dagger_{\vec{k}\uparrow} c_{\vec{(k)}\uparrow} + \xi(\vec{k}) c^\dagger_{\vec{k}\downarrow} c_{\vec{k}\downarrow}$$

Thus for the full Hamiltonian,

$$H = H_k + H_I = \sum_k -\Delta^\dagger c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow} - \Delta c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\uparrow} - \frac{\Omega}{V}\Delta^\dagger \Delta + \xi(\vec{k}) c^\dagger_{\vec{k}\uparrow} c_{\vec{(k)}\uparrow} + \xi(\vec{k}) c^\dagger_{\vec{k}\downarrow} c_{\vec{k}\downarrow}$$

Using the Nambu Spinor defined as,

$$\Psi = \begin{bmatrix} c_{\vec{k}\uparrow} \\ c^\dagger_{-\vec{k}\downarrow} \end{bmatrix} \Psi^\dagger = \begin{bmatrix} c^\dagger_{\vec{k}\uparrow} & c_{-\vec{k}\downarrow} \end{bmatrix} \tag{1}$$

The full Hamiltonian can be written as

$$H + \xi(\vec{k}) = \sum_{\vec{k}} \Psi^\dagger_{\vec{k}} \begin{bmatrix} \xi(\vec{k}) & -\Delta \\ -\Delta^\dagger & -\xi(\vec{k}) \end{bmatrix} \Psi_{\vec{k}} - \frac{\Omega}{V}\Delta^\dagger \Delta$$

3 Brogoliubuv Transformation

4 Diagonalization