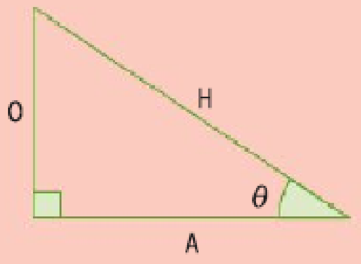
Trigonometry – theory

→ For any right-angled triangle with an angle θ:

$$\text{sine } \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\text{O}}{\text{H}}$$

$$\text{cosine } \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{A}}{\text{H}}$$

$$\text{tangent } \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\text{O}}{\text{A}}$$



Since:

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{b} \longrightarrow$$

It means that:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Ex.

For this triangle, find the length of side *a*.

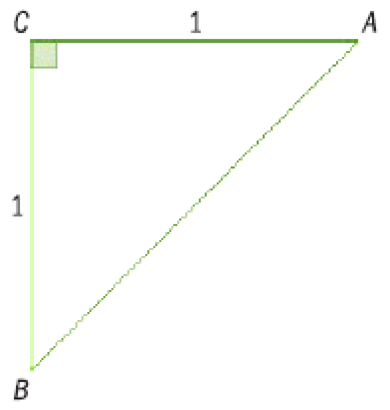
Ans: 4.05

Inverse functions:

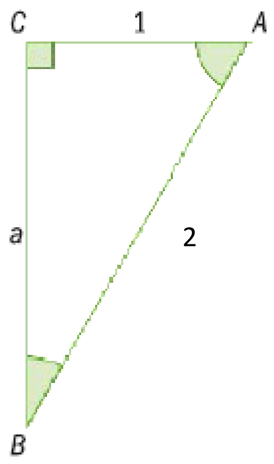
What if I know $\cos x = 0.75$. I do I find x? Theoretically, I shall compute $x = \frac{0.75}{cos}$, however this makes no mathematical sense. Correctly, it is simply written $x = \sin^{-1} 0.75$ or $x = \arcsin 0.75$. The latter might be more useful as it avoids confusion with other exponents. The same applies to sin, which becomes arcsin, or tan, which becomes arctan.

The notation arccos, arcsin are simply the inverse

Special right-angled triangles

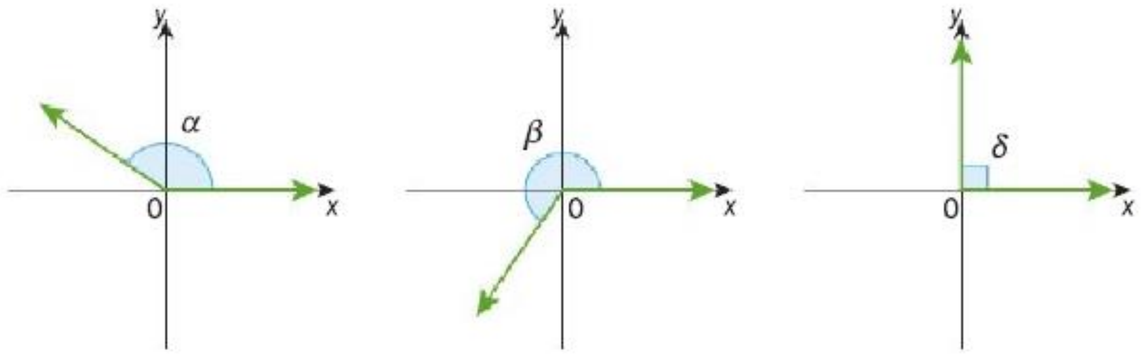


Calculate sin, cos and tan of these triangles.



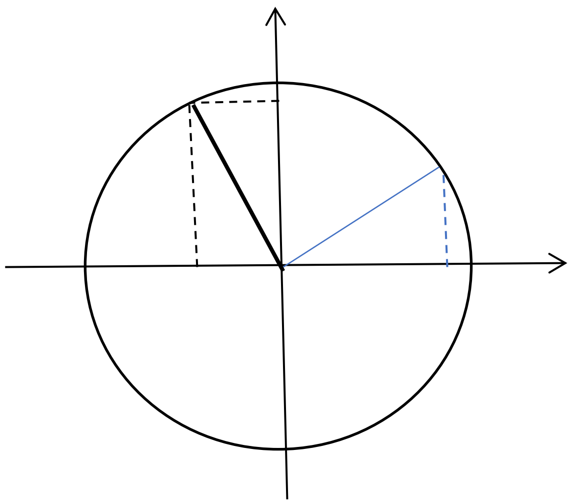
What if the angle is 0? And 90 degrees?

Using coordinate axes in trigonometry



This helps visualising a lot faster. For example, find the cos, sin e tan of 120 degrees clockwise.

Clockwise vs anticlockwise (counterclockwise). Radius of the circle is 1.



From the graph, using Pythagorean theorem, we obtain that: $\sin^2\theta + \cos^2\theta = 1$

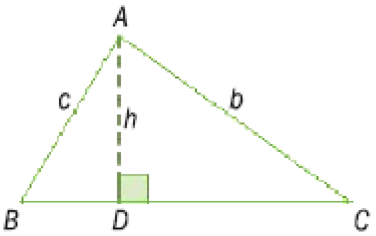
The sine rule

The sine rule

For any $\triangle ABC$, where a is the length of the side opposite $\hat{A}$, b is the length of the side opposite $\hat{B}$, and c is the length of the side opposite $\hat{C}$,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \text{ or } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

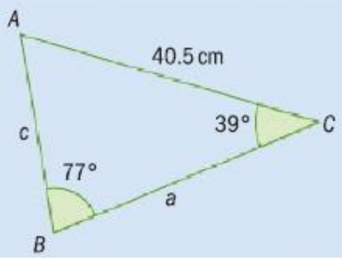
Why?



This is particularly useful when dealing with non right-angled triangles.

EX.

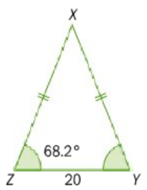
Find the missing angles and sides in this triangle, rounding your answers to 2 decimal places.



A ship is sailing due north. The captain sees a lighthouse 10 km away on a bearing of 032°. Later, the captain observes that the bearing of the lighthouse is 132°. How far did the ship travel between these two observations?

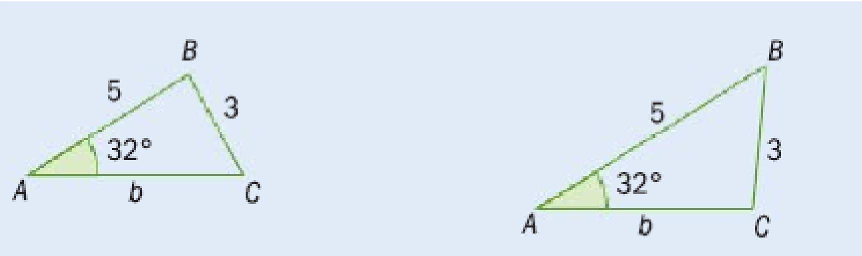
ANS: 13.25 km

An isosceles triangle has base 20 cm, and base angles of 68.2°, as shown. Use the sine rule to find the length of sides XY and XZ.



Adam and Kevin are standing 35 metres apart, on opposite sides of a flagpole. From Adam's position, the angle of elevation of the top of the flagpole is 36°. From Kevin's position, the angle of elevation is 50°. How high is the flagpole?

Ambiguous triangles



This occurs because of supplementary angles, which have the same value of sin x !!

Ex.

A ship is sailing due west when the captain sees a lighthouse at a distance of 20 km on a bearing of 230°.

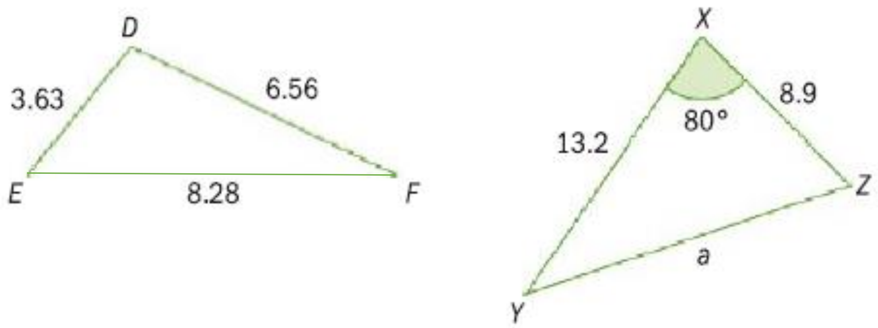
a Draw a diagram to model this situation.

b How far must the ship sail before the lighthouse is 16 km away?

c How far must the ship sail beyond this point before the lighthouse is again at a distance of 16 km from the ship?

d What is the bearing of the lighthouse from the ship the second time the two are 16 km apart?

Cosine rule



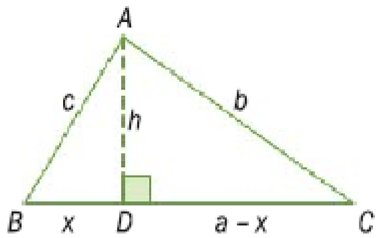
These are unsolvable using the rules we have seen so far.

The cosine rule

For $\triangle ABC$, where a is the length of the side opposite $\hat{A}$, b is the length of the side opposite $\hat{B}$, and c is the length of the side opposite $\hat{C}$:

$$a^2 = b^2 + c^2 - 2bc\cos A \text{ or}$$
$$b^2 = a^2 + c^2 - 2ac\cos B \text{ or}$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Demonstration:



In triangle ACD, Pythagoras' theorem gives

$$b^2 = h^2 + (a - x)^2 = h^2 + a^2 - 2ax + x^2$$

In triangle ABD,

$$h^2 + x^2 = c^2$$

so $h^2 = c^2 - x^2$

Substitute for h^2 in the first equation to get

$$b^2 = c^2 - x^2 + a^2 - 2ax + x^2$$
$$= c^2 + a^2 - 2ax$$

In triangle ABD, $\cos B = \dfrac{x}{c}$, so $x = c\cos B$

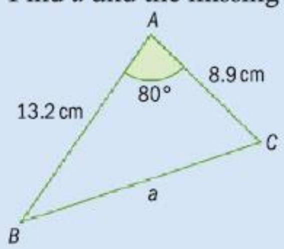
By substituting for x, you get

$$b^2 = a^2 + c^2 - 2ac\cos B$$

This equation is one form of the **cosine rule**.

Ex.

Find a and the missing angles in this triangle.



Find angles A, B and C.

The diagonals of a parallelogram form an acute angle of 62°.
The lengths of the diagonals are 6 cm and 9 cm.
Find the lengths of the sides of the parallelogram.

Ship A leaves port and sails due east for 28 km. Ship B leaves from the same port and sails 49 km. The ships are then 36 km apart. On what bearing was ship B sailing?

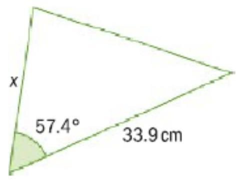
Area of a triangle

The area of any triangle ABC is given by the formula:

area $= \frac{1}{2}bc\sin A$ or area $= \frac{1}{2}ac\sin B$ or area $= \frac{1}{2}ab\sin C$

Ex.

The triangle shown has an area of $324\ \text{cm}^2$.
Find the value of x.



The triangle shown has an area of $30\ \text{cm}^2$.
Find the value of x.