F-2d-05-4

OPTIMAL CONTROL SOLUTIONS FOR THE BALL-AND-BEAM PROBLEM

Mark W. Palmer
Marquette University
Department of Electrical and Computer Engineering
P.O. Box 1881
Milwaukee, WI 53202-1881
Email: palmer@spiff.eece.mu.edu, USA

Dr. Xin Feng
Marquette University
Department of Electrical and Computer Engineering, USA

Abstract: The ball-and-beam problem is a commonly recurring nonlinear control application used by authors to demonstrate the effectiveness of various control methods. We present theory and methods for finding optimal fixed final time solutions to the ball-and-beam problem. Optimal control theory is applied to the ball-and-beam state equations and cost function to obtain a system of eight first-order differential equations. This system is solved numerically using the FORTRAN 77 routine COLNEW to obtain the optimal solution. The optimal solution provides an objective benchmark to which other controls can be compared. *Copyright © 1999 IFAC*

Keywords: optimal control, nonlinear control, boundary-value problem, numerical solutions

1. INTRODUCTION

The ball-and-beam problem is a commonly recurring control application in the engineering literature. It was recommended as early as 1984 as a nontrivial control application by Astrom and Wittenmark, and has been used since then as a test for various control techniques. (For examples, see Li, *et. al.* (1995), Wang (1998), Sastry and Tomlin).

Considering that the ball-and-beam application has emerged as a benchmark test for evaluating control effectiveness, the authors suggest that it can be helpful to know the "ideal," or optimal control for a given set of evaluation standards. Once the optimal control is found, it can serve as an objective baseline against which other proposed control solutions are rated.

In this paper, optimal control theory is applied to the ball-and-beam system, obtaining solutions for the fixed final time optimal control case. Section 2 shows how to derive the boundary-value problem given the dynamics of the ball-and-beam system and optimization criteria. Section 3 describes how the resulting boundary-value problem can be solved numerically with public domain FORTRAN 77 code, with results shown in Section 4.

2. THEORY

The ball-and-beam experiment is depicted in Figure 1. The beam rotates in a vertical plane in response to input torque applied at the fulcrum. The ball is free to roll along the beam, but it is required that the ball roll without slipping or losing contact with the beam. State is described by the vector $\mathbf{x} = \begin{bmatrix} r & \dot{r} & \theta & \dot{\theta} \end{bmatrix}^T$, with

 ISBN: 0 08 043248 4

Figure 1: The ball and beam system.

the positive senses of r and θ shown in Figure 1. Positive torque is the counterclockwise direction, which increases θ. Typically, the objective is to manipulate the input in such a way that the ball reaches a stable state at the origin ($\mathbf{x}=\begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}^T$).

The dynamics of the ball-and-beam system can be written as

$$\dot{\mathbf{x}} = \begin{bmatrix} x_2 \\ \alpha(x_1 x_4^2 - \beta\sin(x_3)) \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u, \qquad (1)$$

where u is the input and β is the acceleration due to gravity. The factor α describes the mass and moment of inertia of the ball (Hauser, *et. al.*, 1992):

$$\alpha = \frac{M}{\frac{I}{R^2} + M}. \qquad (2)$$

In (2), I is the moment of inertia of the ball, M is the mass of the ball, and R is the ball's radius.

In the general continuous-time optimization problem, a performance index, or "cost" J is associated with the system

$$J(t_0) = \phi(x(T),T) + \int_{t_0}^{T} L(x,u,t)\,dt\,, \qquad (3)$$

where L is a function of the trajectory and ϕ is the penalty cost applied to deviation from the targeted final state. For our example, no deviation is allowed at the final time from the stationary point $\mathbf{x}=0$, so $\phi(x(T),T)=0$. We choose L to be

$$L(x,u,t) = x_1^2(t) + x_2^2(t) + x_3^2(t) + x_4^2(t) + u^2(t)\,, \qquad (4)$$

so that total state deviation from zero and control magnitude, integrated over time, is to be minimized. We want to reach the final state at T=6 seconds from the starting point $\mathbf{x}(0)=\begin{bmatrix}2 & 0 & 0 & 0\end{bmatrix}^T$. The goal in this optimization problem is to find the input $u(t)$ over the time $0 \le t \le 6$ that drives the ball-and-beam system from the starting point to $\mathbf{x}(T)=\begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}^T$, while minimizing the cost function (3).

To obtain the optimal controller, start by writing the Hamiltonian equation:

$$H(x,u,t) = L(x,u,t) + \lambda^T f(x,u,t) \qquad (5)$$

$$\begin{aligned} H = {} & x_1^2(t) + x_2^2(t) + x_3^2(t) + x_4^2(t) + u^2(t) + \\ & \lambda_1 x_2 + \lambda_2 \alpha x_1 x_4^2 - \lambda_2 \alpha\beta \sin(x_3) + \\ & \lambda_3 x_4 + \lambda_4 u \end{aligned} \qquad (6)$$

This introduces the costates, written as λ in the notation used by Lewis and Syrmos (1995). The costate equations are found using

$$\dot{\lambda} = -\frac{\partial H}{\partial x}, \qquad (7)$$

which gives

$$\begin{aligned} \dot{\lambda}_1 &= -\alpha x_4^2 \lambda_2 - 2x_1 \\ \dot{\lambda}_2 &= -\lambda_1 - 2x_2 \\ \dot{\lambda}_3 &= \alpha\beta\cos(x_3)\lambda_2 - 2x_3 \\ \dot{\lambda}_4 &= -2\alpha x_1 x_4 \lambda_2 - \lambda_3 - 2x_4 \end{aligned}. \qquad (8)$$

The system of equations in (8) when combined with (1) gives eight equations in nine variables. The stationarity condition

$$\frac{\partial H}{\partial u} = 0 \qquad (9)$$
$$2u + \lambda_4 = 0,$$

allows us to reexpress u in terms of the state and costate

$$u = -\frac{1}{2}\lambda_4\,, \qquad (10)$$

so the final system has eight first-order differential equations in eight variables.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \alpha x_1 x_4^2 - \alpha\beta\sin(x_3) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -\frac{1}{2}\lambda_4 \\ \dot{\lambda}_1 &= -\alpha x_4^2 \lambda_2 - 2x_1 \\ \dot{\lambda}_2 &= -\lambda_1 - 2x_2 \\ \dot{\lambda}_3 &= \alpha\beta\cos(x_3)\lambda_2 - 2x_3 \\ \dot{\lambda}_4 &= -2\alpha x_1 x_4 \lambda_2 - \lambda_3 - 2x_4 \end{aligned}. \qquad (11)$$

 ISBN: 0 08 043248 4

Since the initial and final states are each four conditions, a total of eight boundary conditions have been specified, and the boundary-value problem is complete.

3. METHODS

Due to the instability in the system of differential equations shown in (11), initial-value based methods such as shooting methods can be unreliable, computationally intensive, and may fail to give results. Instead of these methods, the use of the FORTRAN 77 boundary-value solver COLNEW, an updated version of COLSYS written by U. Ascher and G. Bader, is suggested (Ascher, *et. al.*, 1981). This software is available online from *http://gams.nist.gov*.

In the notation of COLNEW, x replaces t as the independent variable, and u will replace x as the state variable. The differential equations should be written and ordered according to

$$u_n^{(m_n)}(x) = F_n(x;\mathbf{z}(\mathbf{u})) \quad n = 1,\ldots,d \quad a < x < b . \tag{12}$$

The subscript n stands for the "differential equation number n." There are a total of d differential equations in the problem. The equations are ordered in such a way that the order m_n of the equations is nondecreasing. In the case of the optimal ball-and-beam problem, there are d=8 equations all of order m_n=1. The variables a and b are limits of the range of the independent variable, 0 and 6.

Since all the equations are first order, no state derivatives appear on the right-hand side of the equations. This is not true of all problems that COLNEW encounters, so one further transformation of variables is made which orders all state variables and their derivatives into a new vector, $\mathbf{z}$. The u elements have a fixed ordering in the $\mathbf{z}$ vector:

$$\mathbf{z}(\mathbf{u}) = (u_1, u_1', \ldots, u_1^{(m_1-1)}, u_2, \ldots, u_d, \ldots, u_d^{(m_d-1)}) . \tag{13}$$

But again, since all of the equations in this problem are first order, this just becomes

$$\mathbf{z}(\mathbf{u}) = (u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8) . \tag{14}$$

The final format of the system of differential equations is

$$\begin{aligned} F_1 &= z_2 \\ F_2 &= \alpha z_1 z_4^2 - \alpha\beta \sin(z_3) \\ F_3 &= z_4 \\ F_4 &= -\frac{1}{2} z_8 \\ F_5 &= -\alpha z_4^2 z_6 - 2z_1 \\ F_6 &= -z_5 - 2z_2 \\ F_7 &= \alpha\beta \cos(z_3) z_6 - 2z_3 \\ F_8 &= -2\alpha z_1 z_4 z_6 - z_7 - 2z_4 \end{aligned} \quad . \tag{15}$$

For the boundary conditions, the following format applies:

$$g_j(\varsigma_j;\mathbf{z}(\mathbf{u})) = 0 \qquad j = 1, \ldots, m^* \tag{16}$$

where m^* is

$$m^* = \sum_{n=1}^{d} m_n , \tag{17}$$

which for this problem equals eight. Boundary conditions must be ordered so that the locations of the conditions are nondecreasing from a to b:

$$a \le \varsigma_1 \le \varsigma_2 \le \cdots \le \varsigma_{m^*} \le b . \tag{18}$$

The conditions are written as functions of $\mathbf{z}$ set equal to zero. The ζ values and g expressions which completely describe the eight boundary conditions are

$$\varsigma = \begin{bmatrix} 0 & 0 & 0 & 0 & 6 & 6 & 6 & 6 \end{bmatrix}^T \tag{19}$$

$$\mathbf{g} = \begin{bmatrix} z_1 - 2 & z_2 & z_3 & z_4 & z_1 & z_2 & z_3 & z_4 \end{bmatrix}^T .$$

Lastly, COLNEW requires that the Jacobians of the system of equations (15) and the boundary conditions (19) be available.

The FORTRAN front-end code for setting up this problem for COLNEW is available from the authors via email. The code returns state and costate for the optimal solution trajectory by calling the routine APPSLN (included in the COLNEW routines) after the COLNEW routine has completed. To obtain an entire solution trajectory, simply call APPSLN from a loop that varies X from 0 to 6.

4. RESULTS

For parameters α=0.7143 and β=9.81, the initial state-costate vector returned by COLNEW for this problem is z(0) = [2 0 0 0 8.525772371443 6.960732795714 -22.53455679375 -4.002366083995]^T. Plotting ball position, velocity, beam angle and control input u from

 ISBN: 0 08 043248 4

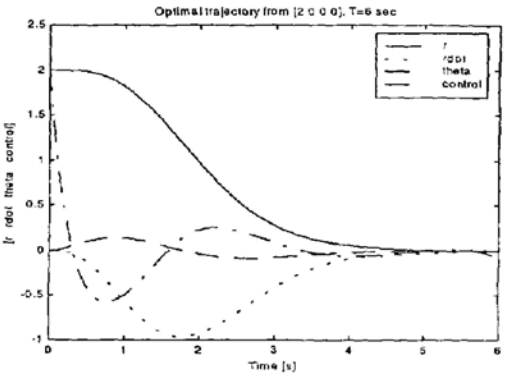

Figure 2: Selected state trajectories and control for the optimal ball-and-beam problem.

time 0 to 6 seconds gives the results shown in Figure 2. Recall that the control u is computed from the costate using (10).

It is possible to use one initial state-costate vector from APPSLN as the initial condition in an ordinary differential equation solver to get the trajectories shown in Figure 2. However, care must be taken to keep the errors small. Accumulated error will depend on the precision of the initial state-costate vector, the error tolerance used in the ODE solver, and the amount of time over which a solution is computed. In this example where T is only 6 seconds, the error is not noticeable so long as the initial state-costate and solver computations are accurate to about nine digits. However, for problems where T is greater than 12 seconds, error growth will be noticeable even using full double-precision accuracy. In all cases, it is important to look at the final computed state to verify that the solver has reached the expected target.

5. CONCLUSION

This paper presented the computation of optimal control and solution trajectories for the fixed final time case of the ball-and-beam problem. Optimal control theory was applied to the ball-and-beam equations to formulate a system of eight first-order differential equations with eight boundary values. The FORTRAN boundary-value solver COLNEW solved the system. Optimal control solutions to benchmark control problems such as the ball-and-beam provide objective performance ideals that various control strategies can be rated against.

REFERENCES

Ascher, U., J. Christiansen, R. D. Russell. Collocation Software for Boundary-Value ODEs. *ACM Transactions on Mathematical Software*, **7**, no. 2, 209-222.

Astrom, K. J. and B. Wittenmark. *Computer Controlled Systems, Theory and Design.* Prentice Hall, Englewood Cliffs, NJ, 1984.

Hauser, J., S. Sastry, P. Kokotovic. Nonlinear control via approximate input-output linearization: The ball and beam example. *IEEE Transactions on Automatic Control*, **37**, no. 3, March 1992, 392-398.

Lewis, F. L. and V. L. Syrmos. *Optimal Control (2nd Edition)*, J. Wiley, New York, 1995, 134.

Li, H. H., N. Godfrey, and Y. Ji. A Beam-and-Ball Controller Prototype. *IEEE Micro*, **15**, no. 6, 64.

Sastry, S. S. and C. J. Tomlin. "Switching through Singularities," Department of Electrical Engineering and Computer Sciences, University of California, Berkeley.

Wang, L. X. Stable and Optimal Fuzzy Control of Linear Systems. *IEEE Transactions on Fuzzy Systems*, **6**, no. 1, 137-143.

ISBN: 0 08 043248 4