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Optimal Control of a Ball and Beam System through LQR and LQG

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***Abstract*—This paper presents the design of an optimal control strategy for a 2 degree of freedom standard laboratory system - Ball and Beam. The system is an open loop and non-linear system, which is inherently unstable. A linear quadratic regulator (LQR) is designed and implemented with an objective to control the position of ball on the beam by varying the angular position of beam. As all states of system are not measurable, therefore, an observer based state feedback is applied. The modelling and simulation of the LQR strategy for controlling the system is carried out on MATLAB SIMULINK platform. The LQR strategy is very effective in stabilizing the system. A comparative analysis is also presented for LQR and Linear Quadratic Gaussian (LQG) control under the effect of sensor noises in simulation.**

***Keywords*— Ball and beam system, LQR, Observer, LQG**

I. Introduction

The ball and beam system is a standard laboratory equipment that can be used to verify and validate different control strategies. This system finds its use very common as this being a simple, well defined and standard structure [1]. The position of ball on beam is systematically controlled by varying the input angle of beam, which is the most important action in this system [1-2]. It is an open loop unstable system. This system finds it use in horizontally stabilizing the aerial vehicle during the time of landing and in balancing the goods carried by a robot.

Due to the system being highly complex and non-linear, some model free control and optimization design approaches such as Fuzzy control, Neural network, Phase lead compensation, particle swarm optimization etc. have been used to set and stabilize the ball position on beam and the beam angular position [3]. In order to stabilize the system, there is the requirement of phase advance in the control system, but the presence of noise is more at high frequencies, so, trade-off is required [4]. Various control concepts have been used in various studies of the ball and beam system. Because of the ball and beam system being inherently unstable, the position of ball can be adjusted in absence of limit for an unchangeable input angular position of beam [5]. This feature makes the ball and beam system a suitable equipment for verifying the various control design strategies. The systems has two degree of freedom (DOFs).

This work implements the state feedback translation of the linear quadratic regulator (LQR) control algorithm. The controller formulations are developed using the linearized prototype of ball and beam system. The optimal control design algorithm of LQG, which is a composition of a linear quadratic estimator (Kalman filter) and a LQR, is used for optimal control of ball and beam system. The simple and established LQR control algorithms are widely used for controlling the linear system dynamics and the same is made use of in this paper. The system performance has been comparatively studied with and without noise and presented in this paper. The comparative analysis of the results demonstrate the effectiveness of the control approaches.

The remaining sections of this paper are structured as follows. Second section discusses the mathematical modeling of ball and beam system, followed by third section, which describes the optimal control using LQR, elaborated briefly. Fourth section highlights the state estimation (Kalman filter) and LQG (Full state observer), whereas, in fifth section, the SIMULINK model and simulation results are presented. Lastly, in sixth section, the conclusion of this paper is presented, while at the end, a concise list of reference is given.

II. Ball and Beam System

A. System Description

Ball and beam system comprises, as its main elements, the base, a ball, a beam, gear, support block, and motor etc., which are the mechanical parts of this system, as shown in Fig.1. One end of beam is rigid by a shaft, whereas, the other end rotates up and down on which a ball is free to move [1]. Here, the inclination angle of beam is controlled by making the pulley run through controlling the position of DC servomotor.

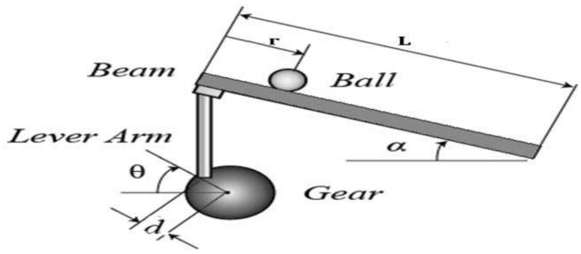

Fig. 1: Schematic of a ball beam system

B. Mathematical modelling

In modeling the system mathematically, it is assumed that the beam length as L, beam angular position is θ, which is limited, d is the distance between contact point and the center. Accordingly, the mathematical expression between inclination angle (α) and angular position of beam (θ) can be described as [1,3]

$$\alpha = \frac{d}{L}\theta \tag{1}$$

To stop the ball at a particular location on the beam is accomplished by controller through adjusting the angle of gear. The movement of ball on the beam can be expressed by a kinematic expression as under:

$$\left(\frac{J_b}{R_b^2} + \mathrm{M}\right)\ddot{r} + \mathrm{Mg}\sin\alpha = \mathrm{Mr}(\alpha)^2 \tag{2}$$

Where, g is the gravity constant, M is the mass of ball, J_b is the rotational inertia of ball, r is the ball position on beam and R_b is the radius of ball. Here, the friction is considered to be negligible when ball moves on the beam [1]. Since the inclination angle (α) is very close to the zero, therefore, a linearized model at α = 0 degree is expressed as:

$$\ddot{r} = -\frac{Mg}{\frac{J_b}{R_b^2}+M}\alpha = -\frac{Mgd}{L(\frac{J_b}{R_b^2}+\mathrm{M})}\theta \tag{3}$$

Now, taking the Laplace transform of equation (3), we get

$$\frac{R(s)}{\theta(s)} = -\frac{Mgd}{L(\frac{J_b}{R_b^2}+\mathrm{M})}\cdot\frac{1}{s^2} \tag{4}$$

The transfer function of system is expressed by the expression:

$$\frac{R(s)}{\theta(s)} = -\frac{0.7}{s^2} \tag{5}$$

The desired ball position is at the centre i.e. at 20 cm and error allowance is presumed to be $\pm 1\,cm$. For the proposed method of control, the ball and beam system is considered as an ideal investigation model.

The non-linear version of ball and beam system can be linearized about the equilibrium point (r = 0, $\dot{r} = 0$, $\theta = 0, \dot{\theta} = 0$) leading to the following equations representing the linear version of ball and beam system

$$\dot{x_1} = x_2 \tag{6}$$

$$\dot{x_2} = -\alpha x_3 \tag{7}$$

$$\dot{x_3} = x_4 \tag{8}$$

$$\dot{x_4} = -\beta x_3 + \gamma x_4 \tag{9}$$

where, x_1= r, x_2= $\dot{r}$, $x_3 = \theta$, $x_4 = \dot{\theta}$

In above equations, the parameters used are defined as below and their nominal values are given in Table I [1].

$$\alpha = \frac{Mg}{\frac{J_b}{R_b^2}+M},\ \beta = \frac{Mg}{J+J_b},\ \gamma = \frac{1}{J+J_b}$$

TABLE I. PARAMETERS OF BALL BEAM SYSTEM

Parameters	Value
M	0.05 k_g
R_b	0.01m
G	9.81 m/s^2
L	40 cm
D	4 cm
J	2.0 ×10^{-2} kgm^2
J_b	2.0 ×10^{-6} kgm^2

Putting the values of the parameters from Table I in (6), (7), (8) and (9), the state space model can be developed as given in (10) and (11).

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -7.0071 & 0\\ 0 & 0 & 0 & 1\\ -24.5225 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ 49.9950\end{bmatrix}u \tag{10}$$

$$y = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} \tag{11}$$

The controller is then designed utilizing the linear model of the ball and beam system. The next section describes the LQG controller.

III. LINEAR QUADRATIC REGULATOR

The synthesize of a control process if leads to extremization of performance then it is referred to as optimal control, which provides the best possible behavior. The main aim of the optimal control is to satisfy, to the best possible extent, the physical constraints and at the same time to extremize (Maximize or Minimize) a suitable performance index or cost function [6]. Optimal control is obtained using LQR, which is one of the best strategies. To exercise control, LQR takes the dynamic system states and control input. Since, LQR is simple, optimal and robust, therefore, is used most commonly [7]. For the LQR design, system matrices A and B must be controllable.

Linearization is done about the equilibrium point with the initial conditions as x[0]= $[0,0,0.1,0]^T$. Consequently, the linear state space equations are given as

$$\dot{x} = \mathrm{A}x + \mathrm{Bu} \tag{12}$$

where, x $= [\mathrm{r}, \dot{r}, \theta, \dot{\theta}]^T$

The state feedback control $u = -Kx$ tends to

$$\dot{x} = (A - BK)x \tag{13}$$

Where, K is obtained from minimization of performance index

$$\mathrm{J} = \int (x^T Q x + u^T R u)\, dt \tag{14}$$

Where, Q is positive semi definite symmetric constant matrix and R is positive definite symmetric constant matrix.

K is then obtained as

$$K = R^{-1}B^{T}P \tag{15}$$

Where, P is the positive definite symmetric constant matrix, which is evaluated from the solution of Algebraical Ricatti Equation i.e.

$$A^{T}P + PA - PBR^{-1}B^{T}P + Q = 0 \tag{16}$$

In the process of using LQR approach in the design of optimal control of ball and beam system, all the dynamic system states i.e. ball position (r), ball velocity ($\dot{r}$), beam angular position(θ), and beam angular velocity ($\dot{\theta}$) are considered free for calculation. These are then used in the LQR. The LQR is designed making use of the linear state space model of dynamic system. LQR control involves changing the poles location of the system to optimal location on which depends the time response, overshoot, steady state. It is therefore, obvious that LQR strategy is superior to the pole placement methodologies as LQR method gives more accurate results.

Because, the ball position in the ball and beam system is the only measurable state and the rest of the states are not measurable, therefore, an observer is to be designed for this system for which system matrices A and C must be observable [8]. Then the control vector is given by $U = -K\hat{x}$, where $\hat{x}$ is the estimated state vector.

IV. STATE ESTIMATION

This section is primarily about estimating all-state variables of ball and beam system, which is denoted by $\hat{x}$ and the control vector, consequently is given by $U = -K\hat{x}$.

A. Full - order state Observer

The estimation of state variables is accomplished a state observer or merely an observer, as depicted in Fig. 2. The state equation of system is given by (10) where, A and B are n×n and n×1 real constant matrices, respectively.

The measurement equation of system is given by $y = Cx$ where, C is $1\times n$ real constant matrix.

To estimate the all-state variables, we construct a model of the system as

$$\dot{\hat{x}} = A\hat{x} + Bu \tag{17}$$

Where, $\hat{x}$ is the estimate of actual state. The difference between measured and estimated output is applied back for expediting the estimation process and with this error signal, the model is continuously corrected.

The state equations for this is given as

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \tag{18}$$

where L is n× 1 real constant matrix.

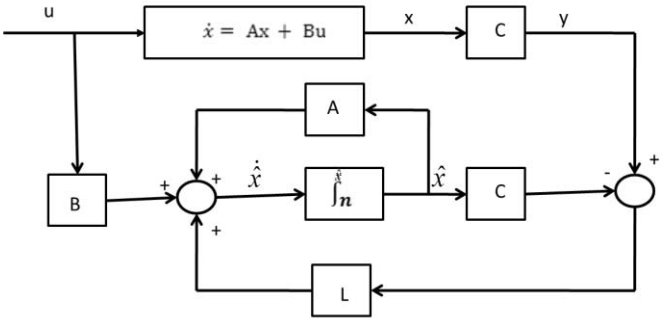

Fig. 2: Block diagram of Luenberger observer

State vector is given as $\tilde{x} = x - \hat{x}$, on differentiating both sides, we get

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} \tag{19}$$

From above equations, we get

$$\dot{\tilde{x}} = (A - LC)\tilde{x} \tag{20}$$

The characteristic equation of error is given by

$$| sI - (A - LC) | = 0 \tag{21}$$

Here, L must be chosen so as to ensure that (A-LC) has stable roots. The convergence of $\hat{x}$(t) must be independent of $\hat{x}$(0). Here, equation (21) is without input control vector because A, B and C are taken as identical in both the plant and observer. Therefore, convergence of estimation error $\tilde{x}$ tends to zero regardless of input control vector. L must be selected in a similar manner as K is selected in control design method [9].

We assume observer error root at desired location, which is specified as $s = \Lambda_1, \Lambda_2 \ldots\ldots\ldots\ldots\ldots \Lambda_n$ leading to desired observer characteristic equation as

$$(s - \Lambda_1)(s - \Lambda_2)\ldots\ldots\ldots(s - \Lambda_n) = 0 \tag{22}$$

To evaluate L, the coefficients in (21) & (22) need to be compared and this is possible when plant is completely observable i.e.(A, C) must be observable.

B. Kalman filtering

Kalman filtering is extensively used to estimate instantaneous position of moving objects and in target tracking, navigation, digital image processing, pattern recognition etc. Kalman filter is a least-variant estimation algorithm, which derive optimal state estimation. A linear addition of x(t), u(t) and w(t) represents the state equation, whereas, a linear addition of x(t) and v(t) represents the observation equation. Therefore, with the help of state equation and observation equation, a dynamical model can be constructed as [10].

State equation is defined as:

$$x(t) = Ax(t-1) + Bu(t-1) + w(t) \tag{23}$$

Observation equation is defined as:

$$\mathrm{y(t) = Cx(t) + v(t)} \tag{24}$$

where, x(t) is state vector, u(t) is control vector, y(t) is observation vector, A is state transition matrix, C is observation matrix, w(t) is system noise vector and v(t) is observation noise matrix. Here, w(t) and v(t) are supposed to be positive, symmetric & uncorrelated, with mean zero white noise vectors. Mean and covariance of w(t) and v(t) are defined as follows:

$E\{w(t)\} = 0, E\{w(t)w^T(t)\} = Q$

$E\{v(t)\} = 0, E\{v(t)v^T(t)\} = R$

$E\{w(t)v^T(t)\} = E\{v(t)w^T(t)\} = 0$

From the Kalman filtering theory, also known as Bayesian filtering theory, a prediction equations (Time update) and a correction equations (Measurement update) are obtained.

Prediction equations (Time update) are defined as follows:

$$\hat{x}^-(t) = A\hat{x}(t-1) + Bu(t-1) \qquad (25)$$

$$P^-(t) = AP(t-1)A^T + Q \qquad (26)$$

Correction equations (Measurement update) are defined as follows:

$$K_t = P_t^- C^T(CP_t^- C^T + R)^{-1} \qquad (27)$$

$$\hat{x}_{t-1} = \hat{x}^-{}_t + K_t(y_t - C\hat{x}^-{}_t) \qquad (28)$$

$$P_t = (1 - K_t C)P_t^- \qquad (29)$$

Where, K_t is Kalman gain matrix, $\hat{x}_t$ is maximum filter measure, P_t is filter deflection matrix, $\hat{x}^-(t)$ is the prior estimate and $P^-(t)$ is the prior error covariance.

Since kalman filter with the LQR is known as the LQG control, so, in this work we also design a LQG control.

For the system

$$\dot{x} = Ax + Bu + w \qquad (30)$$

$$y = Cx + v \qquad (31)$$

LQG regulator minimizes the cost function of the following type

$$J = E\{\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau [x^T, u^T]\, Q_{xu} \begin{bmatrix} x \\ u \end{bmatrix} dt\} \qquad (32)$$

Where, the process noise *w* and measurement noise *v* are Gaussian white noises with covariance Q_{wv} = E([w;v] * [w^T,v^T]) and Q_{xu} = E([x,u]*[x^T uT]) .

V. Simulation and Results

This section presents the modelling and simulation of the LQR and LQG controllers on the considered system. The ball position on the beam and the angular position of the beam are utilized as the state feedback to the controller to control the ball and beam system. Since ball position and angular position of beam are the only two measurable states, so to obtain the remaining states an observer is designed. For this first we checked the observability of system via Kalman test. After performing the observability test we found that the system is observable. Then we checked the controllability of the system which was satisfied. The observer poles are chosen as -20,-25,-30,-35 and the observer gain (L) is computed as

$$L = \begin{bmatrix} 56.032 & 766.969 & -9.68 & -152.34 \\ -5.105 & -147.89 & 53.967 & 707.99 \end{bmatrix}^T$$

For this system, first we computed the LQR gain matrix by using Q and R matrices. After some trial, we got suitable Q and R matrices, as following, for the system.

$$Q = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 100 \end{bmatrix}, R = [1]$$

Now, on solving algebraic riccatti equation, the optimal feedback gain was obtained as

$$K = [-3.6906 \;\; -11.7158 \;\; 40.7948 \;\; 10.043]$$

Putting these values in Simulink model, given by Fig-3, with initial states; x[0] =[0 0 0.1 0]^T , the simulation is carried out. The response of LQR without noise is given in Fig-4. The response of LQR shows that the system is stable with settling time of 7.28 sec for ball position and 2.25 sec for angular position of beam and steady state error is zero for both states.

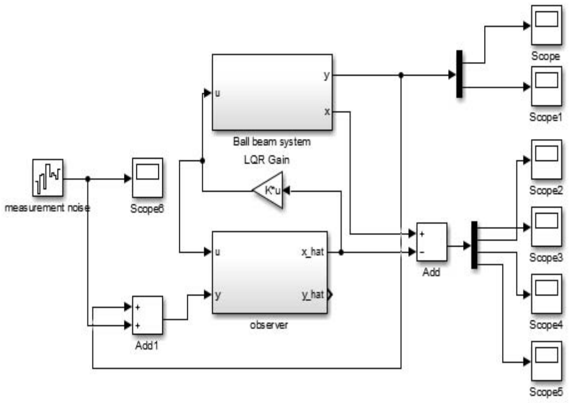

Fig. 3: Simulink implementation of LQR with observer

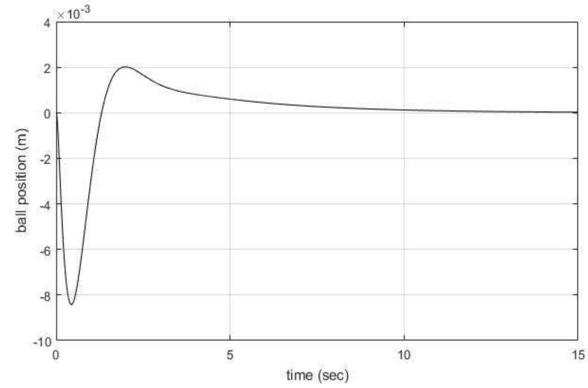

Fig. 4(a): Ball Position

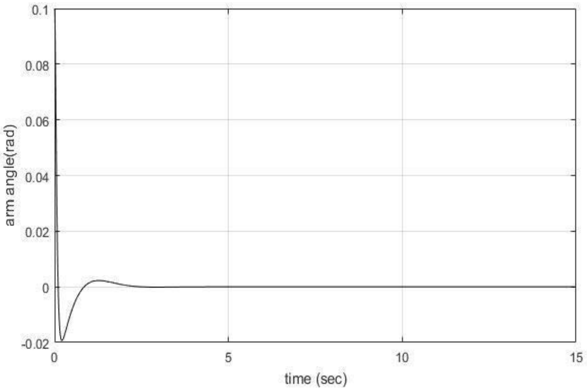

Fig. 4(b): Beam Angle

Simulation runs are again carried out after introducing a measurement noise, as in Fig. 5, in the sensor and the response is again verified, which is given by Fig-6.

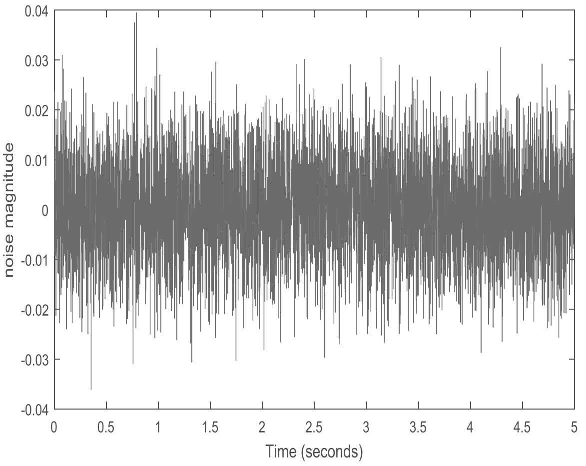

Fig. 5: Measurement noise

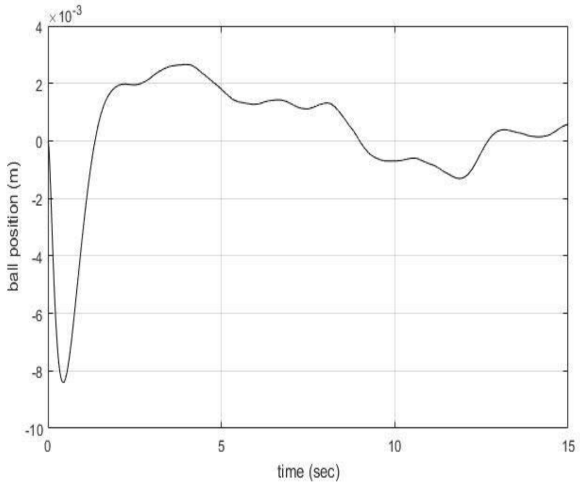

Fig. 6(a): Ball Position

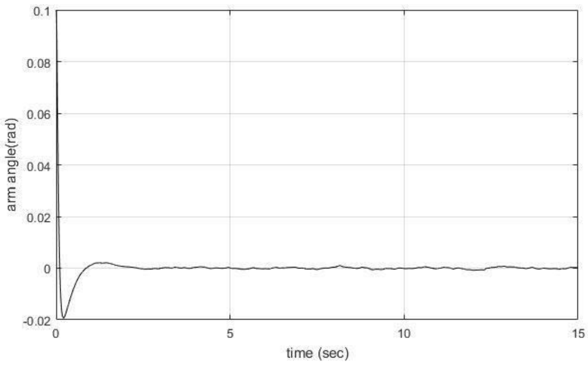

Fig. 6(b): Beam Angle

The response of LQR with the noise shows that the system is unstable with ball vibrating around mean position.

Extending the work further, a LQR controller with Kalman filter for estimation of state was designed, which is also known as LQG control. The matrices chosen for LQG are

$$Q_{XU} = diag(0.1,0.1,0.1,0.1,1)$$

$$Q_{WV} = diag(10,100,10,100,1,1)$$

Solving for the LQG, the solution is a state space model of LQG as given by

$$\dot{\hat{x}} = \begin{bmatrix} -5.383 & 1 & 2.283 & 0 \\ -12.09 & 0 & 1.678 & 0 \\ 2.283 & 0 & -7.566 & 1 \\ 50.06 & 32.22 & -120.9 & -20.96 \end{bmatrix} \hat{x} + \begin{bmatrix} 5.383 & -2.283 \\ 12.09 & -8.685 \\ -2.283 & 7.566 \\ -20.88 & 26.23 \end{bmatrix} y$$

$$y = \begin{bmatrix} 1.074 & 0.6445 & -1.894 & -0.4192 \end{bmatrix} \hat{x}$$

Implementing LQG strategy, as per depiction in Fig. 7, on the system under investigation, the response is obtained under the impact of measurement noise, which is given in Fig.8.

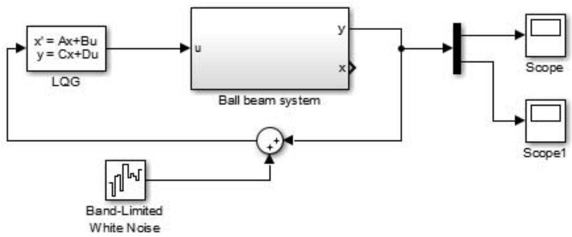

Fig. 7: Simulink implementation of LQG

In case of both control strategies, it can be seen that the stability has been achieved with and without the sensor noise, but, LQG proves more effective in handling the noise as against LQR with observer. For quantitative analysis, the settling time and steady stare error have been tabulated in Table II.

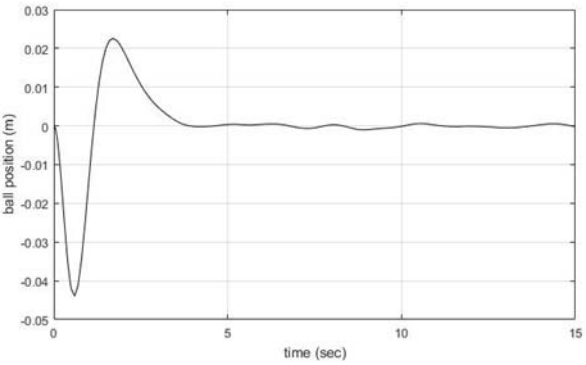

Fig. 8(a): Ball Position

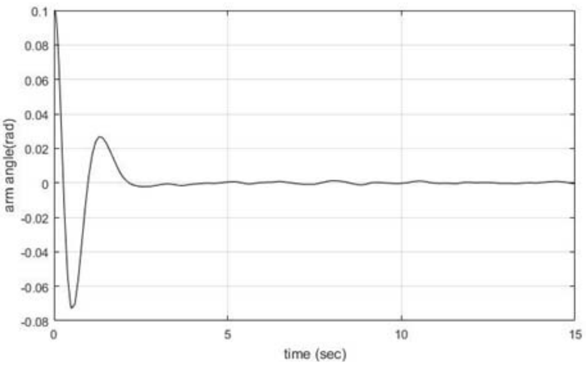

Fig. 8(b): Beam Angle

TABLE II. QUANTITATIVE RESPONSE CHARACTERISTICS

Controller	State	Settling time(sec)	Steady state error
LQR with observer	Ball position	7.28	0
	angular position of beam	2.24	0
LQR with observer with noise	Ball position	-	0.00005
	angular position of beam	2.25	0.00001
LQG	Ball position	4.25	0.00002
	angular position of beam	2.20	0

VI. CONCLUSION

The LQR controller has been designed and implemented for a ball and beam standard nonlinear and unstable system to stabilize it in the presence and absence of sensor noise. For comparative analysis, LQG control was also designed and implemented on to the same system under same conditions. It is observed that the LQR, designed via arbitrary choice of weighting matrices, has been successful to stabilize the system. The state estimation was also performed, as required for LQG control. When the system is corrupted with the sensor noise, the LQR with observer is not able to achieve stability and the ball is seen moving back and forth. While on the other hand, the LQG control shows better response, as it takes care of the measurement noise and stabilizes the system successfully.

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