Lesson 3. Second-order Linear Ordinary Differential Equations

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Table of Contents

A **second-order linear ordinary differential equation** for the function y is

$$y'' + a_1(x)y' + a_0(x)y = b(x); \tag{1}$$

where a_1, a_0, b are given functions on the interval $I \subset \mathbb{R}$.
The eq. (1) above:

- is **homogeneous** if the source $b(x) = 0$ for all $x \in \mathbb{R}$;
- has **constant coefficients** if a_1 and a_0 are constants;
- has **variable coefficients** if either a_1 or a_0 is not constant.

An **initial-value problem** for the second-order differential equation (1) consists of finding a solution of the differential equation that also satisfies initial conditions of the form

$$y(x_0) = y_0, \quad y'(x_0) = y_1 \tag{2}$$

where y_0 and y_1 are given constants.

Note that the form of the initial conditions (2) involves the specification of both $y(x)$ and its derivative $y'(x)$ at an initial point x_0.

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Homogeneous Constant Coefficients:

$$y'' + 4y' + 5y = 0$$

Inhomogeneous Constant Coefficients:

$$y'' - 3y' + 2y = e^x$$

Homogeneous Variable Coefficients:

$$x^2y'' + xy' - y = 0$$

Inhomogeneous Variable Coefficients:

$$x^2y'' + 2xy' + y = \sin(x)$$

The Initial-Value Problem

$$y'' + y = 0, \quad y(0) = 2, y'(0) = 3$$

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Classic Examples: Mechanical Vibrations

Spring-mass system

With mass m, position $y(t)$, spring constant k, viscous damping γ, and external force $F(t)$

- Unforced, undamped oscillator:

$$my'' + ky = 0$$

- Unforced, damped oscillator:

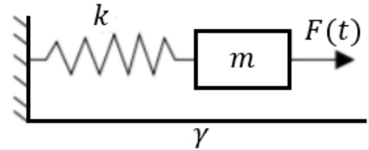
$$my'' + \gamma y' + ky = 0$$

- Forced, undamped oscillator:

$$my'' + ky = F(t)$$

- Forced, damped oscillator:

$$my'' + \gamma y' + ky = F(t)$$



Classic Examples: Mechanical Vibrations

Pendulum Motion

With mass m, drag c, length L, $\gamma = \frac{c}{mL}$, $\omega^2 = \frac{g}{L}$, angle $\theta(t)$

- Nonlinear Pendulum: $L\ddot{\theta} + g\sin\theta = 0$
- Damped Nonlinear Pendulum:

$$L\ddot{\theta} + b\dot{\theta} + g\sin\theta = 0$$

- Linear Pendulum: $L\ddot{\theta} + g\theta = 0$
- Damped Linear Pendulum:

$$L\ddot{\theta} + b\dot{\theta} + g\theta = 0$$

- Forced Damped Nonlinear Pendulum:

$$L\ddot{\theta} + b\dot{\theta} + g\sin\theta = F\cos\omega t$$

- Forced Damped Linear Pendulum:

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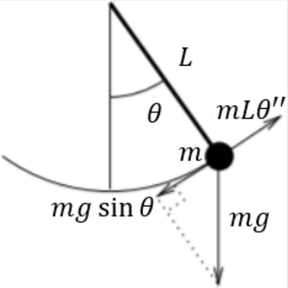

Solutions to the IVP

Theorem (Existence and Uniqueness)
If the functions a_1, a_0, b are continuous on a closed interval $I \subset \mathbb{R}$, the constant $x_0 \in I$, and $y_0, y_1 \in \mathbb{R}$ are arbitrary constants, then there is a unique solution y, defined on I, of the initial value problem

$$y'' + a_1(x)y' + a_0(x)y = b(x), \quad y(x_0) = y_0, \quad y'(x_0) = y_1.$$

Example : Find the domain of the solution to the initial value problem

$$(t-1)y'' - 3ty' + \frac{4(t-1)}{(t-3)}y = t(t-1), \quad y(2) = 1, \quad y'(2) = 0.$$

Solution:
We first write the equation above in the form given in the Theorem above,

$$y'' - \frac{3t}{(t-1)}y' + \frac{4}{(t-3)}y = t$$

The equation coefficients are defined on the domain

$$(-\infty, 1) \cup (1, 3) \cup (3, \infty)$$

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Solution:

So the solution may not be defined at $t = 1$ or $t = 3$. That is, the solution is defined in

$$(-\infty, 1) \text{ or } (1, 3) \text{ or } (3, \infty)$$

Since the initial condition is at $t_0 = 2 \in (1, 3)$, then the domain of the solution is

$$D = (1, 3)$$

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Properties of Homogeneous Equations

The Superposition Principle

If $y = y_1(x)$ and $y = y_2(x)$ are two solutions of the differential equation then any linear combination

$$y = C_1 y_1(x) + C_2 y_2(x)$$

of $y_1(x)$ and $y_2(x)$, where C_1 and C_2 are constants, is also a solution of the differential equation.

This result is not true for nonhomogeneous equations.

Linearly dependent and Linearly independent functions

The functions $y_1(x)$ and $y_2(x)$ are linearly independent on the interval $I = (\alpha, \beta)$, if their quotient in this segment is not identically equal to a constant:

$$\frac{y_1(x)}{y_2(x)} \neq \text{const}$$

Otherwise, these functions are linearly dependent.

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Properties of Homogeneous Equations

General Solution

If $y_1(x)$ and $y_2(x)$ are linearly independent solutions of the differential equation

$$y'' + a_1(x)y' + a_0(x)y = 0 \text{ on an interval } I = (\alpha, \beta)$$

where a_1, a_0 are continuous functions on I, then there are unique constants C_1, C_2 such that every solution y of the differential equation can be written as a linear combination

$$y(t) = C_1 y_1(x) + C_2 y_2(x)$$

Fundamental solutions

Two solutions $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions to the differential equation if every solution of

$$y'' + a_1(x)y' + a_0(x)y = 0$$

can be expressed as a linear combination of $y_1(x)$ and $y_2(x)$.

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Homogeneous Constant Coefficient Equations

Consider the second-order homogeneous linear differential equation with constant coefficients:

$$y'' + a_1 y' + a_0 y = 0$$

where a_1 and a_0 are constants.

General Solution Form

Assume a solution of the form:

$$y(t) = e^{\lambda x}$$

where λ is a constant to be determined.
Substitute $y(t) = e^{\lambda x}$ into the differential equation:

$$a(\lambda^2 e^{\lambda x}) + b(\lambda e^{\lambda x}) + c(e^{\lambda x}) = 0$$

Factor out $e^{\lambda x}$:

$$e^{\lambda x}(a\lambda^2 + b\lambda + c) = 0$$

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Since $e^{\lambda x} \neq 0$, we get the auxiliary equation(or characteristic equation):

$$a\lambda^2 + b\lambda + c = 0$$

The roots of the characteristic equation $a\lambda^2 + b\lambda + c = 0$ are given by:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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Homogeneous Constant Coefficient Equations

Let λ_1 and λ_2 be the roots of the characteristic equation:

$$a\lambda^2 + b\lambda + c = 0$$

Then the general solution of the homogeneous differential equation

$$y'' + a_1 y' + a_0 y = 0$$

satisfies:

Roots	General Solution
$\lambda_1 \neq \lambda_2$ (real)	$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$
$\lambda_1 = \lambda_2$	$y(x) = C_1 e^{\lambda_1 x} + C_2 t e^{\lambda_1 x}$
$\lambda_{1,2} = \alpha \pm i\beta$ (complex)	$y(x) = C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x)$

Example 1: $y'' + y' - 6y = 0$

Solve the equation

$$y'' + y' - 6y = 0$$

Solution

The auxiliary equation is

$$\lambda^2 + \lambda - 6 = 0$$

whose roots are 2 and -3.
Therefore, the general solution of the given differential equation is

$$y = C_1 e^{2x} + C_2 e^{-3x}$$

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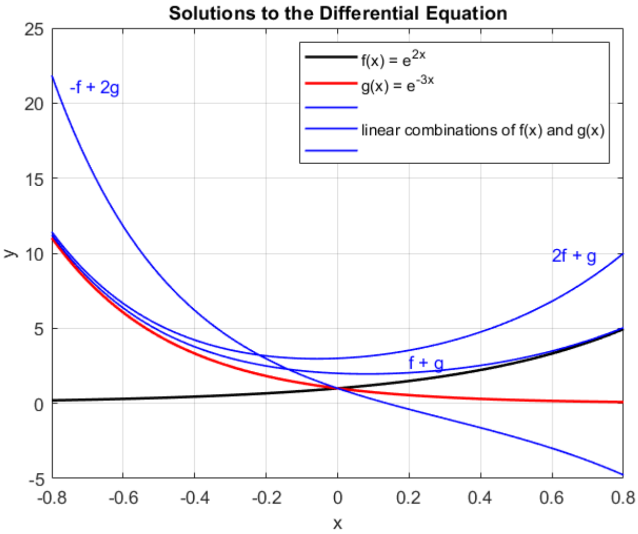

Example 2: $4y'' + 12y' + 9y = 0$

Solve the equation

$$4y'' + 12y' + 9y = 0$$

Solution

The auxiliary equation is

$$4\lambda^2 + 12\lambda + 9 = 0$$

can be factored as

$$(2\lambda + 3)^2 = 0$$

The only root is $-\frac{3}{2}$. Therefore, the general solution of the given differential equation is

$$y = C_1 e^{-\frac{3}{2}x} + C_2 x e^{-\frac{3}{2}x}$$

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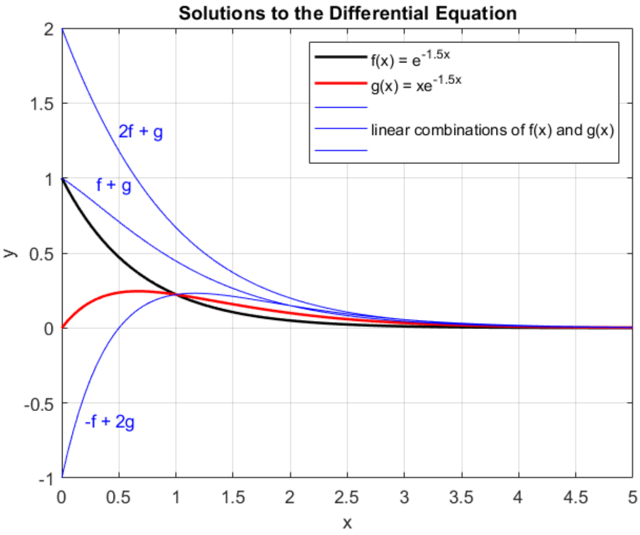

Example 3: $y'' - 6y' + 13y = 0$

Solve the equation

$$y'' - 6y' + 13y = 0$$

Solution

The auxiliary equation is

$$\lambda^2 - 6\lambda + 13 = 0$$

By the quadratic formula, the roots are

$$\lambda = \frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i$$

Therefore, the general solution of the given differential equation is

$$y(t) = C_1 e^{3x} \cos(2x) + C_2 e^{3x} \sin(2x)$$

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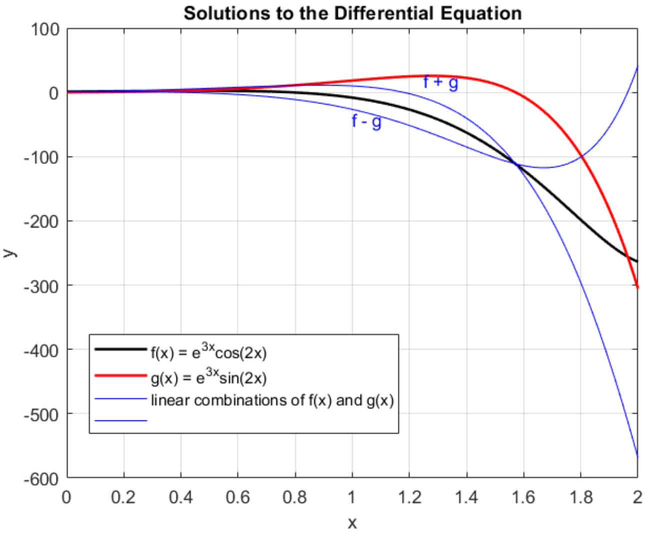

Table of Contents

The Wronskian

Given two functions $y_1(x)$ and $y_2(x)$, their Wronskian is defined as:

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x)$$

Linear Independence

Two functions $y_1(x)$ and $y_2(x)$ are linearly independent on an interval I if and only if their Wronskian is not identically zero on I.

Consider the functions $y_1(x) = e^x$ and $y_2(x) = e^{-x}$. Their Wronskian is:

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Since the Wronskian is non-zero, $y_1(x)$ and $y_2(x)$ are linearly independent.

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Method of variation of parameters

Consider DE

$$y'' + a_1 y' + a_0 y = f(x), \tag{1}$$

The related homogeneous equation

$$y'' + a_1 y' + a_0 y = 0 \tag{2}$$

is called the **complementary equation**

Theorem (The general solution of the nonhomogeneous equation)

The general solution of the nonhomogeneous differential equation (1) is represented as the sum of some particular solution and the general solution of the complementary equation:

$$y(x) = y_p(x) + y_c(x), \tag{3}$$

where $y_p(x)$ is a particular solution of (1) and $y_c(x)$ is the general solution of the complementary equation (2).

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Thus, if we know the general solution of the homogeneous equation (2), the basic difficulty lies in finding some particular solution. We shall give a general method for finding the particular solution of nonhomogeneous equation (1).

The Method of Variation of Arbitrary Constants (Parameters)

We write the general solution of the complementary equation (2):

$$y_c(x) = C_1 y_1(x) + C_2 y_2(x) \tag{4}$$

where C_1, C_2 are arbitrary constants.
We shall seek for a particular solution to (1) in the form (4), considering the constants C_1, C_2 as some undetermined functions $C_1(x), C_2(x)$.

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The Method of Variation of Arbitrary Constants (Parameters)

The function (4) will be a solution of the nonhomogeneous equation (1) provided the functions $C_1(x), C_2(x)$ satisfy the system:

$$\begin{cases} C_1'(x)y_1(x) + C_2'(x)y_2(x) = 0 \\ C_1'(x)y_1'(x) + C_2'(x)y_2'(x) = f(x) \end{cases}$$

The main determinant of this system is the Wronskian of the functions $y_1(x)$ and $y_2(x)$, which is not equal to zero due to the linear independence of the solutions $y_1(x)$ and $y_2(x)$. Therefore, this system of equations always has a unique solution.
The final formulas for $C_1'(x)$ and $C_2'(x)$ have the form

$$C_1'(x) = -\frac{y_2(x)f(x)}{W(y_1, y_2)},$$

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$$C_2(x) = \int \frac{y_1(x)f(x)}{W(y_1,y_2)}\,dx + \tilde{C}_2, \tag{6}$$

where $\tilde{C}_1, \tilde{C}_2$ are constants of integration.

Knowing the expressions for C_1 and C_2, we find an integral that is dependent on the two arbitrary constants $\tilde{C}_1, \tilde{C}_2$; that is, we found the general solution of nonhomogeneous equation.

(If we put $\tilde{C}_1 = \tilde{C}_2 = 0$, we get the particular solution of the equation (1)).

Then the general solution of the original nonhomogeneous equation will be expressed by the formula (3):

$$y(x) = y_2(x)\int \frac{y_1(x)f(x)}{W(y_1,y_2)}\,dx - y_1(x)\int \frac{y_2(x)f(x)}{W(y_1,y_2)}\,dx + C_1y_1(x) + C_2y_2(x). \tag{7}$$

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Solve the equation:

$$y'' + y = \tan(x), \quad 0 < x < \frac{\pi}{2}$$

using method of variation of parameters

Step 1: Find the general solution of the complementary equation

The complementary equation is:

$$y'' + y = 0$$

The characteristic equation is:

$$\lambda^2 + 1 = 0 \implies \lambda = \pm i$$

The general solution of the complementary equation is:

$$y_c(x) = C_1 \sin(x) + C_2 \cos(x)$$

Thus, we identify: $y_1(x) = \sin(x)$, $y_2(x) = \cos(x)$

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Step 3: Using the method of variation of parameters

The particular solution using variation of parameters is given by:

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Method of variation of parameters: Example 1

Step 3: Using the method of variation of parameters

Compute the first integral:

$$-\cos(x)\int\sin(x)\tan(x)\,dx = -\cos(x)\int\sin^2(x)\sec(x)\,dx =$$

$$= -\cos(x)\int(1-\cos^2(x))\sec(x)\,dx =$$

$$= -\cos(x)\int(\sec(x) - cos(x))\,dx =$$

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$$= -\cos(x)\ln(\tan(x)+\sec(x)) + \cos(x)\sin(x)$$

Method of variation of parameters: Example 1

Step 3: Using the method of variation of parameters

Compute the second integral:

$$\sin(x)\int\cos(x)\tan(x)\,dx=\sin(x)\int\sin(x)\,dx=-\sin(x)\cos(x)$$

Substitute both integrals back into the expression for $y_p(x)$:

$$y_p(x)=-\cos(x)\ln(\tan(x)+\sec(x))+\cos(x)\sin(x)-\sin(x)\cos(x)$$

$$y_p(x)=-\cos(x)\ln(\tan(x)+\sec(x))$$

Method of variation of parameters: Example 1

Step 3: Using the method of variation of parameters

Compute the second integral:

$$\sin(x)\int\cos(x)\tan(x)\,dx = \sin(x)\int\sin(x)\,dx = -\sin(x)\cos(x)$$

Substitute both integrals back into the expression for $y_p(x)$:

$$y_p(x) = -\cos(x)\ln(\tan(x)+\sec(x)) + \cos(x)\sin(x) - \sin(x)\cos(x)$$

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Method of variation of parameters: Example 1

Step 3: Using the method of variation of parameters

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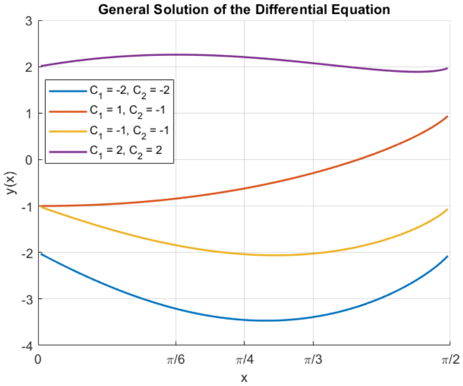
Method of variation of parameters: Example 1

Step 4: Final Solution

The general solution to the given differential equation is:

$$y(x) = -\cos(x)\ln(\tan(x) + \sec(x)) + C_1 \sin(x) + C_2 \cos(x)$$



Method of variation of parameters: Example 2

Solve the equation:

$$y'' + 3y' + 2y = \sin(e^x)$$

using method of variation of parameters

Step 1: Find the general solution of the complementary equation

The complementary equation is:

$$y'' + 3y' + 2y = 0$$

The characteristic equation is:

$$\lambda^2 + 3\lambda + 2 = 0 \implies \lambda_{1,2} = -1, -2$$

The general solution of the complementary equation is:

$$y_c(x) = C_1 e^{-x} + C_2 e^{-2x}$$

Thus, we identify: $y_1(x) = e^{-x}$, $y_2(x) = e^{-2x}$

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The Wronskian $W(y_1, y_2)$ is computed as follows:

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x} = -e^{-3x}$$

Step 3: Using the method of variation of parameters

The particular solution using variation of parameters is given by:

$$y_p(x) = y_2(x) \int \frac{y_1(x) f(x)}{W(y_1, y_2)} \, dx - y_1(x) \int \frac{y_2(x) f(x)}{W(y_1, y_2)} \, dx$$

Substituting $y_1(x) = e^{-x}$, $y_2(x) = e^{-2x}$, $W(y_1, y_2) = -e^{-3x}$, and $f(x) = \sin(e^x)$:

$$y_p(x) = e^{-2x} \int \frac{e^{-x} \sin(e^x)}{-e^{-3x}} \, dx - e^{-x} \int \frac{e^{-2x} \sin(e^x)}{-e^{-3x}} \, dx$$

$$y_p(x) = -e^{-2x} \int e^{2x} \sin(e^x) \, dx + e^{-x} \int e^x \sin(e^x) \, dx$$

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Method of variation of parameters: Example 2

Step 3: Using the method of variation of parameters

Compute the first integral:

$$-e^{-2x}\int e^{2x}\sin(e^x)\,dx =$$

Let $u = e^x$, so $du = e^x dx$.

$$= -e^{-2x}\int u^2\sin(u)\frac{du}{u} = -e^{-2x}\int u\sin(u)\,du$$

Integrate by parts, let $v = u$ and $dw = \sin(u)\,du$:

$$-e^{-2x}\int u\sin(u)\,du = -e^{-2x}(-u\cos(u) + \int\cos(u)\,du) =$$

$$= -e^{-2x}(-u\cos(u) + \sin(u)) =$$

Undo substitution $u = e^x$:

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Method of variation of parameters: Example 2

Step 3: Using the method of variation of parameters

Compute the second integral:

$$e^{-x}\int e^{x}\sin(e^{x})\,dx =$$

Let $u = e^x$, so $du = e^x dx$.

$$= e^{-x}\int u\sin(u)\frac{du}{u} = e^{-x}\int \sin(u)\,du = e^{-x}(-\cos(u)) =$$

Undo substitution $u = e^x$:

$$= e^{-x}(-\cos(e^{x})) = -e^{-x}\cos(e^{x})$$

Substitute both integrals back into the expression for $y_p(x)$:

$$y_p(x) = -e^{-2x}\sin(e^{x}) + e^{-2x}e^{x}\cos(e^{x}) - e^{-x}\cos(e^{x}) = -e^{-2x}\sin(e^{x})$$

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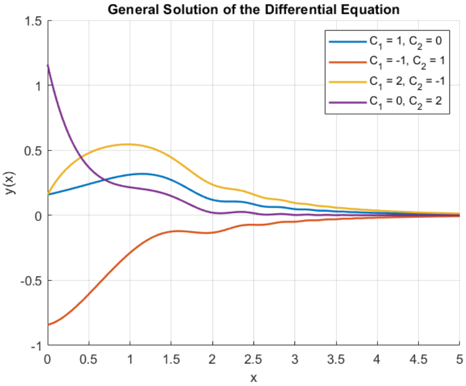
Method of variation of parameters: Example 2

Step 4: Final Solution

The general solution to the given differential equation is:

$$y(x) = C_1 e^{-x} + C_2 e^{-2x} - e^{-2x}\sin(e^{x})$$



Method of variation of parameters: Example 3

Solve the equation:

$$ty'' - (t+1)y' + y = t^2$$

using method of variation of parameters given that

$$y_1(t) = e^t \text{ and } y_2(t) = t + 1$$

form a fundamental set of solutions.

Step 0: Standard form

First, divide by a t:

$$y'' - \left(1 + \frac{1}{t}\right) y' + \frac{1}{t} y = t$$

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Method of variation of parameters: Example 3

Step 1: Find the general solution of the complementary equation

The complementary equation is:

$$y'' - \left(1 + \frac{1}{t}\right) y' + \frac{1}{t} y = 0$$

Given that $y_1(t) = e^t$ and $y_2(t) = t + 1$ form a fundamental set of solutions, the general solution to the homogeneous equation is:

$$y_c(t) = C_1 y_1(t) + C_2 y_2(t) = C_1 e^t + C_2(t+1)$$

Step 2: Compute the Wronskian

The Wronskian $W(y_1, y_2)$ is:

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Method of variation of parameters: Example 3

Step 3: Using the method of variation of parameters

The particular solution using variation of parameters is given by:

$$y_p(t) = y_2(t) \int \frac{y_1(t) f(t)}{W(y_1, y_2)} \, dt - y_1(t) \int \frac{y_2(t) f(t)}{W(y_1, y_2)} \, dt$$

Substituting $y_1(t) = e^t$, $y_2(t) = t + 1$, $W(y_1, y_2) = -te^t$, and $f(t) = t$:

$$y_p(t) = (t+1) \int \frac{e^t \cdot t}{-te^t} \, dt - e^t \int \frac{(t+1)t}{-te^t} \, dt$$

$$y_p(t) = -(t+1) \int dt + e^t \int (t+1)e^{-t} \, dt$$

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Method of variation of parameters: Example 3

Step 3: Using the method of variation of parameters

Compute the first integral:

$$-(t+1)\int dt = -(t+1)t$$

Second integral:

$$e^t \int (t+1)e^{-t}\,dt =$$

Using integration by parts:
Let $u = t+1$ and $dv = e^{-t}dt$:

$$du = dt \quad \text{and} \quad v = -e^{-t}$$

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$$e^t \int (t+1)e^{-t}\, dt = -e^t(t+1)e^{-t} - e^t \int -e^{-t}\, dt =$$

$$= -(t+1) + e^t \int e^{-t}\, dt = -t - 1 - e^t e^{-t} = -t - 2$$

So the particular solution becomes:

$$y_p(t) = -(t+1)t - t - 2 = -t^2 - 2t - 2$$

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Using the integration by parts formula $\int u\,dv = uv - \int v\,du$:

$$e^t \int (t+1)e^{-t}\,dt = -e^t(t+1)e^{-t} - e^t \int -e^{-t}\,dt =$$

$$= -(t+1) + e^t \int e^{-t}\,dt = -t - 1 - e^t e^{-t} = -t - 2$$

So the particular solution becomes:

$$y_p(t) = -(t+1)t - t - 2 = -t^2 - 2t - 2$$

Method of variation of parameters: Example 3

Step 3: Using the method of variation of parameters

Compute the first integral:

$$-(t+1)\int dt = -(t+1)t$$

Second integral:

$$e^t \int (t+1)e^{-t}\,dt =$$

Using the integration by parts formula $\int u\,dv = uv - \int v\,du$:

$$e^t \int (t+1)e^{-t}\,dt = -e^t(t+1)e^{-t} - e^t \int -e^{-t}\,dt =$$

$$= -(t+1) + e^t \int e^{-t}\,dt = -t - 1 - e^t e^{-t} = -t - 2$$

So the particular solution becomes:

$$y_p(t) = -(t+1)t - t - 2 = -t^2 - 2t - 2$$

Method of variation of parameters: Example 3

Step 4: Final Solution

The general solution to the given differential equation is:

$$y(x) = C_1 e^t + C_2(t+1) - t^2 - 2t - 2$$