Lesson 3. Second-order Linear Ordinary Differential Equations

University of Budapest

Faculty of Informatics, Department of Numerical Analysis

2024

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Regular and Singular points

Consider an inhomogeneous second-order linear ODE with variable coefficients

$$y'' + a_1(x)y' + a_0(x)y = f(x) \tag{1}$$

A point $x = x_0$ is called a **regular point** of the DE (1) if the functions $a_1(x)$, $a_0(x)$ and $f(x)$ are analytic at $x = x_0$ (Remember this means each function has a Taylor series expansion in powers of $x - x_0$ valid in a neighborhood of x_0).

A point $x = x_0$ is called a **singular point** of DE (1) if one or more of the functions $a_1(x)$, $a_0(x)$ and $f(x)$ is not analytic at $x = x_0$.

Singular points can be further classified into:

- **Regular Singular Point**: If $x = x_0$ is singular point and if the multiplication $a_1(x)(x - x_0)$ and $a_0(x)(x - x_0)^2$ result in functions, each of which is analytic at $x = x_0$ then the point $x = x_0$ is called **regular singular point**;
- **Irregular Singular Point**: If $x = x_0$ is a singular point and not a regular singular point and the above conditions for a regular singular point are not met then the point $x = x_0$ is called **irregular singular point**

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Regular and Singular points

Another way of defining singular points is following:

Let

$$y'' + a_1(x)y' + a_0(x)y = 0$$

be a linear second order differential equation. Then x_0 is called a **regular singular point** if

$$\lim_{x \to x_0} a_1(x)$$

and

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are both ***finite***.
A singular point that is not a regular singular point is called an **irregular singular point**.

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Regular and Singular points

Example 1: Find all singular points of the given equation and determine whether each one is regular or irregular

$$(x-1)y'' + \frac{1}{x}y' - 2y = 0$$

Solution

Dividing the give differential equation by $(x-1)$ we obtain

$$y'' + \frac{1}{x(x-1)}y' - \frac{2}{(x-1)}y = 0$$

We identify the coefficients:

$$a_1(x) = \frac{1}{x(x-1)},$$
$$a_0(x) = -\frac{2}{(x-1)}.$$

The singular points are $x = 0$ and $x = 1$.

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Regular and Singular points

Solution

- Consider the first point $x = 0$:

$$a_1(x)(x - x_0) = \frac{1}{x(x-1)}(x - 0) = \frac{1}{x-1}$$

The Taylor series expantion is of the form:

$$\frac{1}{x-1} = -(1 + x + x^2 + x^3 + ...) \quad |x| < 1$$

This geometric series converges for |x|<1, showing that $\frac{1}{(x-1)}$ is **analytic** around $x = 0$.

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Solution

- Consider the first point $x = 0$:

$$a_0(x)(x - x_0)^2 = -\frac{2}{(x-1)}(x-0)^2 = -\frac{2x^2}{x-1}$$

The Taylor series expantion is of the form:

$$-\frac{2x^2}{x-1} = 2(x^2 + x^3 + x^4 + ...) \quad |x| < 1$$

The function $\frac{-2x^2}{x-1}$ can be expressed as a power series:

$$\frac{-2x^2}{x-1} = \sum_{n=0}^{\infty} 2x^{n+2}$$

which converges for $|x| < 1$. Therefore, $\frac{-2x^2}{x-1}$ is **analytic** around $x = 0$.

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Solution

- Consider the second point $x = 1$:

$$a_1(x)(x - x_0) = \frac{1}{x(x-1)}(x-1) = \frac{1}{x}$$

The function $\frac{1}{x}$ can be expressed as a power series:

$$\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

which converges for $|x - 1| < 1$. Therefore, $\frac{1}{x}$ is **analytic** around $x = 1$.

Regular and Singular points

Solution

- Consider the second point $x = 1$:

$$a_0(x)(x - x_0)^2 = -\frac{2}{(x-1)}(x-1)^2 = -2(x-1)$$

The function $-2(x-1)$ is a linear polynomial:

$$-2(x-1) = -2x + 2$$

Polynomial functions are **analytic** everywhere because they can be represented exactly by their Taylor series expansions.

Thus, $x = 1$ is regular singular point.

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Example 2: Find all singular points of the given equation and determine whether each one is regular or irregular

$$(x-1)^2 y'' + \frac{1}{x^2} y' + 2y = 0$$

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Dividing the give differential equation by $(x-1)^2$ we obtain

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We compute the following limit:

$$\lim_{x\to 0} a_1(x)(x - x_0) = \lim_{x\to 0} \frac{1}{x(x-1)^2} = \infty$$

Since the first limit is not finite, $x = 0$ **is irregular singular point**.

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Classic Examples

The Euler-Cauchy Equation:

$$ax^2y'' + bxy' + cy = 0$$

- **Regular singular point at** $x = 0$.

The Legendre's Equation:

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$$

- **Regular singular points at** $x = \pm 1$.

The Chebyshev's Equation:

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Classic Examples

The Bessel's Equation:

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

- **Regular singular points at** $x = 0$ **and** $x = \infty$.

The Hermite Equation:

$$y'' - 2xy' + \lambda y = 0, \quad -\infty < x < \infty$$

- **Regular singular point at** $x = \infty$.

The Airy Equation:

$$y'' - xy = 0$$

- **Irregular singular point at** $x = \infty$.

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Euler Equations

Consider the Euler equation:

$$ax^2y'' + bxy' + cy = 0, \tag{1}$$

where a, b, and c are real constants and $a \neq 0$.

Euler equation has solutions defined on $(0, \infty)$ and $(-\infty, 0)$, since it can be rewritten as

$$ay'' + \frac{b}{x}y' + \frac{c}{x^2}y = 0. \tag{2}$$

Consider the interval $(0, \infty)$

We will look for the solution in the form

$$x^r, \quad x > 0$$

Substituting $y = x^r$ into Equation (1) produces

$$ax^2(x^r)'' + bx(x^r)' + cx^r = ax^2r(r-1)x^{r-2} + bxrx^{r-1} + cx^r = (ar(r-1) + br + c)x^r. \tag{3}$$

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Euler Equations

The polynomial

$$p(r) = ar(r-1) + br + c$$

is called the **indicial polynomial** of Equation (1), and $p(r) = 0$ is its **indicial equation**.

Real, Distinct Roots

If the indicial equation $p(r) = 0$ has real roots r_1 and r_2, with $r_1 \neq r_2$, then the solutions of the differential equation are:

$$y_1(x) = x^{r_1} \quad \text{and} \quad y_2(x) = x^{r_2}$$

The Wronskian of these solutions is given by:

$$W[x^{r_1}, x^{r_2}] = (r_2 - r_1)x^{r_1+r_2-1} \neq 0$$

Then the general solution of the differential equation is:

$$y = C_1 x^{r_1} + C_2 x^{r_2}, \quad x > 0$$

Note: if r is not a rational number, then x^r is defined by $x^r = e^{r \ln x}$

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$$W[x^{r_1}, x^{r_2}] = (r_2 - r_1)x^{r_1+r_2-1} \neq 0$$

Then the general solution of the differential equation is:

$$y = C_1 x^{r_1} + C_2 x^{r_2}, \quad x > 0$$

Note: if r is not a rational number, then x^r is defined by $x^r = e^{r \ln x}$

Euler Equations

The polynomial

$$p(r) = ar(r-1) + br + c$$

is called the **indicial polynomial** of Equation (1), and $p(r) = 0$ is its **indicial equation**.

Real, Distinct Roots

If the indicial equation $p(r) = 0$ has real roots r_1 and r_2, with $r_1 \neq r_2$, then the solutions of the differential equation are:

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The Wronskian of these solutions is given by:

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Note: if r is not a rational number, then x^r is defined by $x^r = e^{r \ln x}$

Euler Equations

Let r_1 and r_2 be the roots of the indicial equation:

$$ar(r-1)+br+c=0$$

Then the general solution of the Euler equation

$$ax^2y''+bxy'+cy=0$$

on $(0,\infty)$ satisfies:

Roots	General Solution
$r_1 \neq r_2$ (real)	$y(x)=C_1x^{r_1}+C_2x^{r_2}$
$r_1=r_2$	$y(x)=x^r(C_1+C_2\ln(x))$
$r_{1,2}=\alpha\pm i\beta$ (complex)	$y(x)=x^\alpha(C_1\cos(\beta\ln(x))+C_2\sin(\beta\ln(x)))$

Example 1: $6x^2y'' + 5xy' - y = 0$

Find the general solution of

$$6x^2y'' + 5xy' - y = 0$$

on $(0, \infty)$.

Solution

The indicial polynomial is

$$p(r) = 6r(r-1) + 5r - 1 = (2r-1)(3r+1).$$

whose roots are $\frac{1}{2}$ and $-\frac{1}{3}$.
Therefore, the general solution of the given equation on $(0, \infty)$ is

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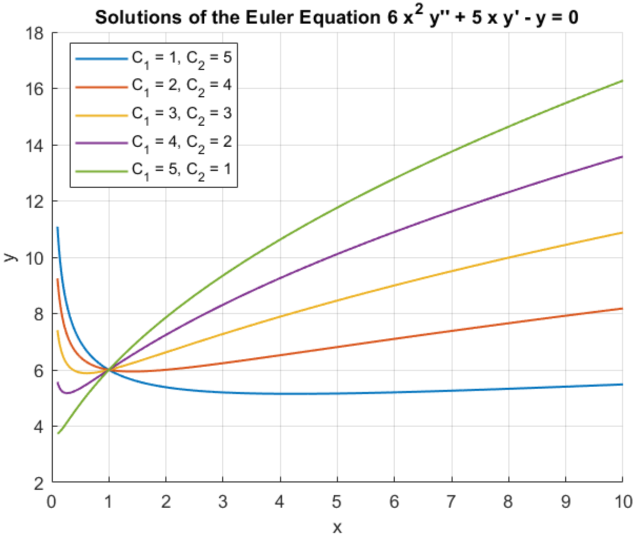

Example 2: $x^2y'' - 5xy' + 9y = 0$

Find the general solution of

$$x^2y'' - 5xy' + 9y = 0$$

on $(0, \infty)$.

Solution

The indicial polynomial is

$$p(r) = r(r-1) - 5r + 9 = (r-3)^2,$$

whose root is 3.
Therefore, the general solution of the given equation on $(0, \infty)$ is

$$y = x^3(C_1 + C_2 \ln x).$$

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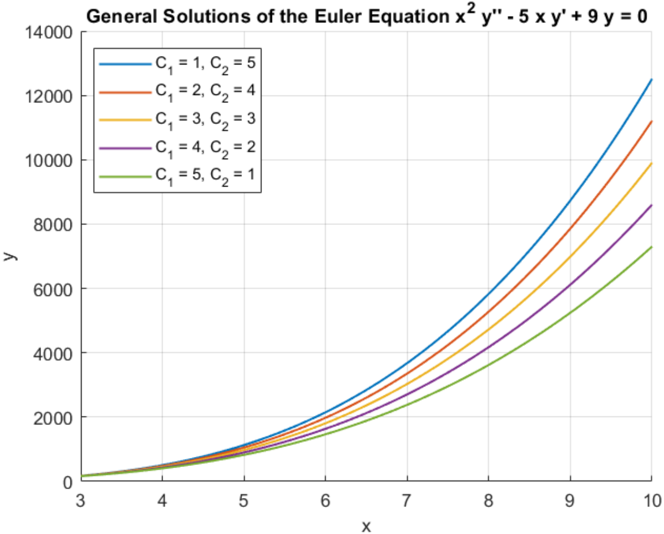

Example 3: $x^2y'' + 3xy' + 2y = 0$

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on $(0, \infty)$.

Solution

The indicial polynomial is

$$p(r) = r(r-1) + 3r + 2 = (r+1)^2 + 1.$$

The roots of the indicial equation are $r = -1 \pm i$
Therefore, the general solution of the given equation on $(0, \infty)$ is

$$y = x^{-1}\left[c_1 \cos(\ln x) + c_2 \sin(\ln x)\right].$$

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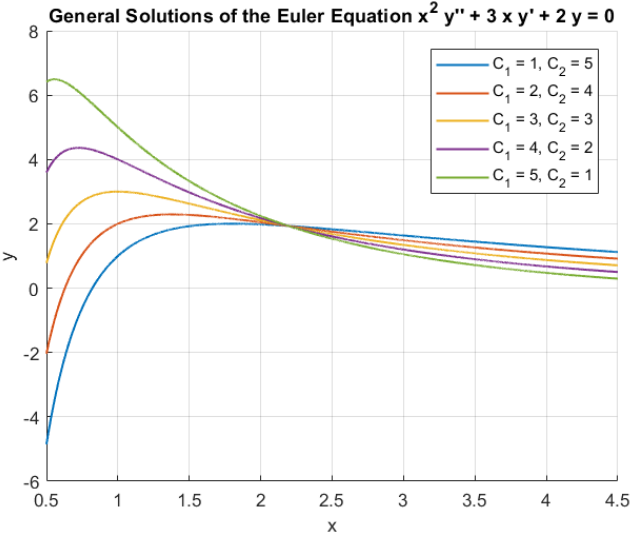

Euler equations

Consider the Euler equation of the form:

$$a(x-x_0)^2 y'' + b(x-x_0)y' + cy = 0$$

We look for solutions of the form

$$y = (x-x_0)^r$$

The general solution depends on the nature of the roots of the indicial equation.

Roots	General Solution
$r_1 \neq r_2$ (real)	$y = C_1\lvert x-x_0\rvert^{r_1} + C_2\lvert x-x_0\rvert^{r_2}$
$r_1 = r_2$	$y = (C_1 + C_2 \ln\lvert x-x_0\rvert)\lvert x-x_0\rvert^{r_1}$
$r_{1,2} = \alpha \pm i\beta$ (complex)	$y = \lvert x-x_0\rvert^{\alpha}(C_1\cos(\beta\ln\lvert x-x_0\rvert) + C_2\sin(\beta\ln\lvert x-x_0\rvert))$

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Example: $3(x+6)^2y'' + 25(x+6)y' - 16y = 0$

Find the solution to the following differential equation on any interval not containing $x_0 = -6$

$$3(x+6)^2y'' + 25(x+6)y' - 16y = 0$$

Solution:

The indicial polynomial is

$$3r(r-1) + 25r - 16 = 0 \Rightarrow 3r^2 + 22r - 16 = 0 \Rightarrow (3r-2)(r+8) = 0$$

Thus, the roots are:

$$r_1 = \frac{2}{3}, \quad r_2 = -8$$

The general solution is then:

$$y(x) = c_1|x+6|^{\frac{2}{3}} + c_2|x+6|^{-8}$$

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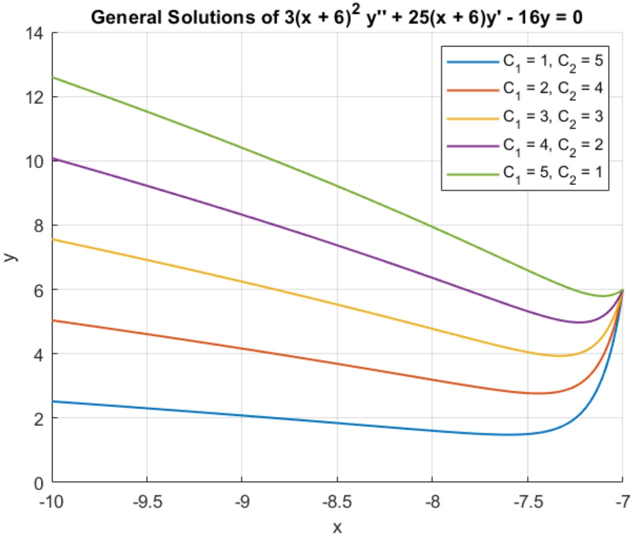

Table of Contents

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Solve the following initial-value problem:

$$y'' + 6y' + 13y = 0, \quad y(0) = 0, \quad y'(0) = 2$$

Solution:

The general solution :

$$y(x) = e^{-3x}(c_1 \cos 2x + c_2 \sin 2x).$$

Differentiate the general solution:

$$y'(x) = e^{-3x}(-2c_1 \sin 2x + 2c_2 \cos 2x) - 3e^{-3x}(c_1 \cos 2x + c_2 \sin 2x).$$

Applying the initial conditions, we obtain

$$\begin{cases} y(0) = c_1 = 0, \\ y'(0) = 2c_2 - 3c_1 = 2. \end{cases}$$

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Solution:

Therefore, $c_1 = 0$ and $c_2 = 1$, and the solution to the initial value problem is

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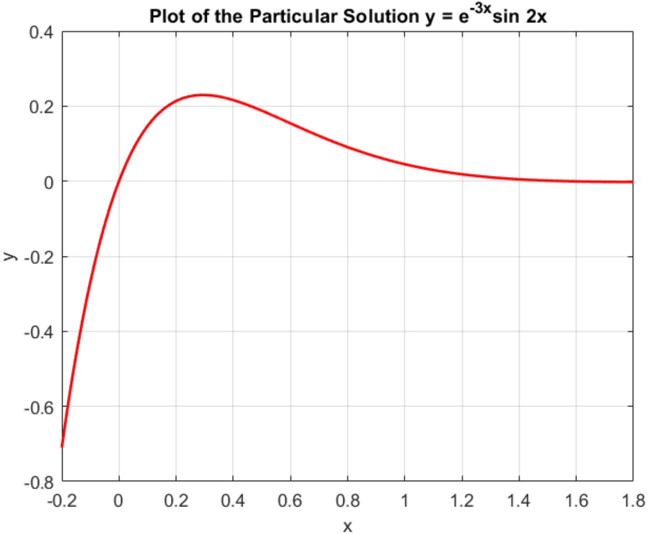

Example 2: $2x^2y'' + 3xy' - 15y = 0, \quad y(1) = 0, \quad y'(1) = 1$

Solve the following initial-value problem:

$$2x^2y'' + 3xy' - 15y = 0, \quad y(1) = 0, \quad y'(1) = 1$$

Solution:

We first need to find the roots to the indicial equation:

$$2r(r-1) + 3r - 15 = 0 \Rightarrow 2r^2 + r - 15 = 0 \Rightarrow (2r-5)(r+3) = 0$$

Thus, the roots are:

$$r_1 = \frac{5}{2}, \quad r_2 = -3$$

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Differentiate the general solution:

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Solving the system of equations, we get:

$$c_1 = \frac{2}{11}, \quad c_2 = -\frac{2}{11}$$

The particular solution to the given IVP:

$$y(x) = \frac{2}{11}x^{\frac{5}{2}} - \frac{2}{11}x^{-3}$$

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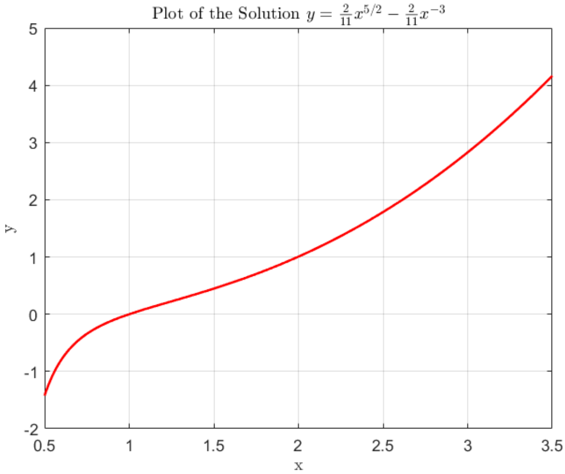

Taylor Series Solutions of Initial Value Problems

Consider the initial value problem:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \quad y(x_0) = y_0, y'(x_0) = y_0'$$

Assume that the solution $y(x)$ can be written as a Taylor series expanded about x_0:

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \cdots \tag{1}$$

The initial conditions will allow us to solve for the first and second coefficients of (1). Use the method of successive differentiation (as we did in the previous lesson) to find the rest of the coefficients in the series that represents the solution.

To derive the coefficients, differentiate the given differential equation successively and evaluate at $x = x_0$. This will generate a system of equations that can be solved for the higher-order derivatives $y_0^{(n)}$.

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Taylor Series Solutions of Initial Value Problems

Consider the initial value problem:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \quad y(x_0) = y_0, y'(x_0) = y'_0$$

Assume that the solution $y(x)$ can be written as a Taylor series expanded about x_0:

$$y(x) = y_0 + (x - x_0)y'_0 + \frac{(x - x_0)^2}{2!}y''_0 + \frac{(x - x_0)^3}{3!}y'''_0 + \cdots \tag{1}$$

The initial conditions will allow us to solve for the first and second coefficients of (1). Use the method of successive differentiation (as we did in the previous lesson) to find the rest of the coefficients in the series that represents the solution.

To derive the coefficients, differentiate the given differential equation successively and evaluate at $x = x_0$. This will generate a system of equations that can be solved for the higher-order derivatives $y_0^{(n)}$.

Taylor Series Solution for $y'' - 2xy' + x^2y = 0, y(0) = 1, y'(0) = -1$

Solve the IVP using Taylor series of the 4th degree:

$$y'' - 2xy' + x^2y = 0, \quad y(0) = 1, y'(0) = -1.$$

Solution:

The Taylor series expansion of $y(x)$ around $x = 0$ is:

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{(4)}(0) + \cdots$$

Given:

$$y(0) = 1, \quad y'(0) = -1$$

we have:

$$y(x) = 1 - x + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{(4)}(0) + \cdots$$

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Solution:

From the ODE:

$$y'' = 2xy' - x^2 y$$

Substitute $x = 0$:

$$y''(0) = 2(0)(-1) - (0)^2(1) = 0$$

Thus:

$$y(x) = 1 - x + 0 + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!} y^{(4)}(0) + \cdots$$

Taylor Series Solution for $y'' - 2xy' + x^2y = 0, y(0) = 1, y'(0) = -1$

Solution:

Next, find $y'''(0)$:
Differentiate $y'' = 2xy' - x^2y$:

$$y''' = 2y' + 2xy'' - 2xy - x^2y'$$

Substitute $x = 0$:

$$y'''(0) = 2(-1) + 2(0)(0) - 2(0)(1) - (0)^2(-1) = -2$$

Thus:

$$y(x) = 1 - x - \frac{2x^3}{3!} + \frac{x^4}{4!}y^{(4)}(0) + \cdots = 1 - x - \frac{x^3}{3} + \frac{x^4}{4!}y^{(4)}(0) + \cdots$$

Taylor Series Solution for $y'' - 2xy' + x^2 y = 0, y(0) = 1, y'(0) = -1$

Solution:

Finally, find $y^{(4)}(0)$: Differentiate $y''' = 2y' + 2xy'' - 2xy - x^2 y'$:

$$y^{(4)} = 4y'' + 2xy''' - 2y - 4xy' - x^2 y''$$

Substitute $x = 0$:

$$y^{(4)}(0) = 4(0) + 2(0)(-2) - 2(1) - 0 - 0 = -2$$

Thus:

$$y(x) = 1 - x - \frac{x^3}{3} - \frac{2x^4}{4!} + \cdots = 1 - x - \frac{x^3}{3} - \frac{x^4}{12} + \cdots$$

The degree 4 Taylor polynomial solution is:

$$y(x) = 1 - x - \frac{x^3}{3} - \frac{x^4}{12}$$

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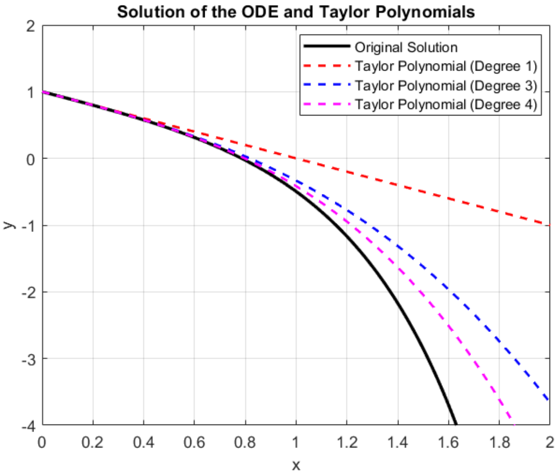

Taylor Series Solution for $y'' + xy = 0, y(0) = 0, y'(0) = 1$

Solve the IVP using Taylor series of the 5th degree:

$$y'' + xy = 0, \quad y(0) = 0, y'(0) = 1$$

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Given:

$$y(0) = 0, y'(0) = 1$$

we have:

$$y(x) = 0 + x \cdot 1 + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{(4)}(0) + \frac{x^5}{5!}y^{(5)}(0) + \cdots =$$

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Taylor Series Solution for $y'' + xy = 0, y(0) = 0, y'(0) = 1$

Solution:

From the ODE:

$$y'' = -xy$$

Substitute $x = 0$:

$$y''(0) = -0 \cdot y(0) = 0$$

Thus:

$$y(x) = x + 0 + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{(4)}(0) + \frac{x^5}{5!}y^{(5)}(0) + \cdots =$$

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Taylor Series Solution for $y'' + xy = 0, y(0) = 0, y'(0) = 1$

Solution:

Next, find $y'''(0)$:
Differentiate $y'' = -xy$ to find y''':

$$y''' = -y - xy'$$

Substitute $x = 0$:

$$y'''(0) = -y(0) - 0 \cdot y'(0) = 0$$

Thus:

$$y(x) = x + 0 + 0 + \frac{x^4}{4!}y^{(4)}(0) + \frac{x^5}{5!}y^{(5)}(0) + \cdots =$$

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Solution:

Next, find $y^{(4)}(0)$:
Differentiate $y''' = -y - xy'$ to find $y^{(4)}$:

$$y^{(4)} = -y' - (y' + xy'') = -2y' - xy''$$

Substitute $x = 0$:

$$y^{(4)}(0) = -2y'(0) - 0 \cdot y''(0) = -2 \cdot 1 - 0 = -2$$

Thus:

$$y(x) = x - \frac{2x^4}{24} + \frac{x^5}{5!}y^{(5)}(0) + \cdots =$$

$$= x - \frac{x^4}{12} + \frac{x^5}{5!}y^{(5)}(0) + \cdots$$

Taylor Series Solution for $y'' + xy = 0, y(0) = 0, y'(0) = 1$

Solution:

Finally, find $y^{(5)}(0)$: Differentiate $y^{(4)} = -2y' - xy''$ to find $y^{(5)}$:

$$y^{(5)} = -2y'' - (y'' + xy''') = -3y'' - xy'''$$

Substitute $x = 0$:

$$y^{(5)}(0) = -3y''(0) - 0 \cdot y'''(0) = -3 \cdot 0 - 0 = 0$$

Thus:

$$y(x) = x - \frac{x^4}{12} + 0 + \cdots = x - \frac{x^4}{12}$$

The degree 5 Taylor polynomial solution for the given ODE is:

$$y(x) = x - \frac{x^4}{12}$$

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Example 2: $y'' + xy = 0, y(0) = 0, y'(0) = 1$

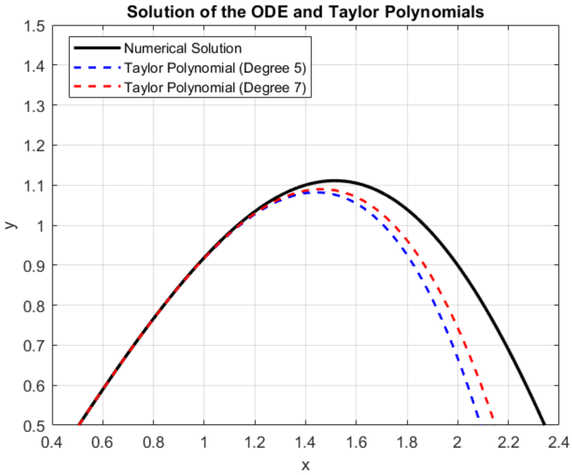

Table of Contents

Boundary value problem

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$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x), \quad x \in [a, b], \tag{1}$$

subject to the **boundary conditions**

$$\begin{aligned} y(a) = \alpha, y(b) = \beta \\ y'(a) = \alpha, y'(b) = \beta \\ y'(a) = \alpha, y(b) = \beta \\ y(a) = \alpha, y'(b) = \beta \end{aligned} \tag{2}$$

where $a_2(x)$, $a_1(x)$, $a_0(x)$, and $f(x)$ are given functions, and α and β are specified constants.
We call a and b **boundary points**.

The next three examples show that the question of existence and uniqueness for solutions of boundary value problems is more complicated than for initial value problems.

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BVP: Example 1

Example 1: Consider the boundary value problem:

$$y'' + y = 1, \quad y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 0.$$

Solution:

The general solution of $y'' + y = 1$ is

$$y = 1 + c_1 \sin x + c_2 \cos x,$$

so $y(0) = 0$ if and only if $c_2 = -1$ and $y\left(\frac{\pi}{2}\right) = 0$ if and only if $c_1 = -1$.
Therefore,

$$y = 1 - \sin x - \cos x$$

is the **unique solution** of the boundary value problem.

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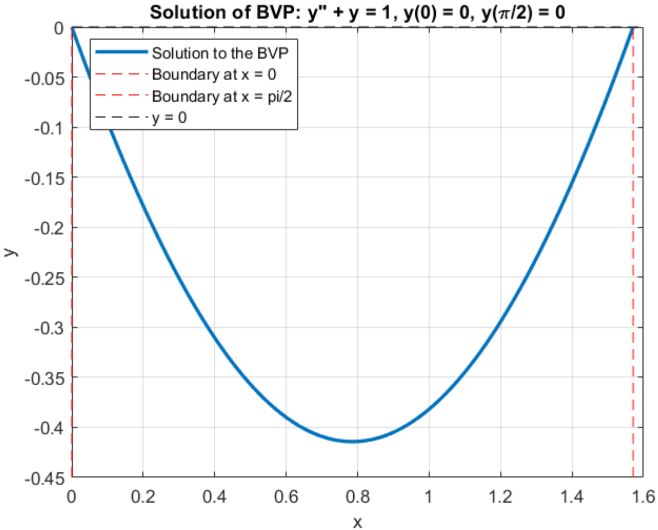
Solution of BVP: y" + y = 1, y(0) = 0, y(π/2) = 0
Solution to the BVP
Boundary at x = 0
Boundary at x = pi/2
y = 0
y
x
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6

BVP: Example 2

Example 2: Consider the boundary value problem:

$$y'' + y = 1, \quad y(0) = 0, \quad y(\pi) = 0.$$

Solution:

The general solution of $y'' + y = 1$ is

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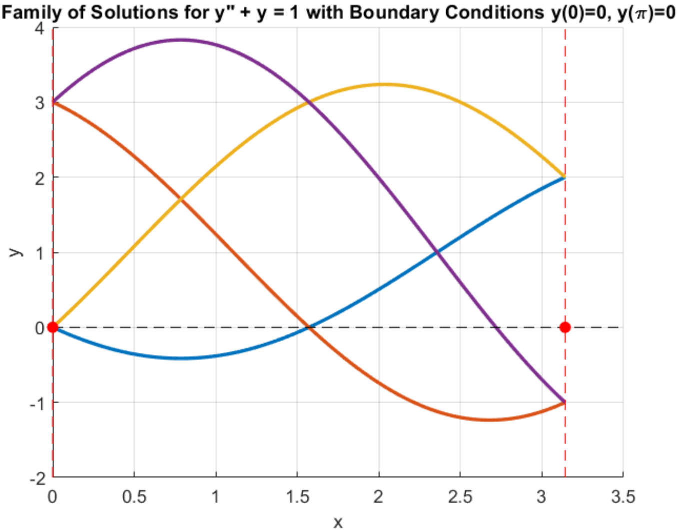
Family of Solutions for y" + y = 1 with Boundary Conditions y(0)=0, y(π)=0
y
x
4
3
2
1
0
-1
-2
0
0.5
1
1.5
2
2.5
3
3.5

BVP: Example 3

Example 3: Consider the boundary value problem:

$$y'' + y = \sin 2x, \quad y(0) = 0, \quad y(\pi) = 0.$$

Solution:

The general solution of $y'' + y = \sin 2x$ is

$$y = -\frac{\sin 2x}{3} + c_1 \sin x + c_2 \cos x.$$

The boundary conditions $y(0) = 0$ and $y(\pi) = 0$ both require that $c_2 = 0$, but they don't restrict c_1.

Therefore the boundary value problem has **infinitely many solutions**:

$$y = -\frac{\sin 2x}{3} + c_1 \sin x,$$

where c_1 is arbitrary.

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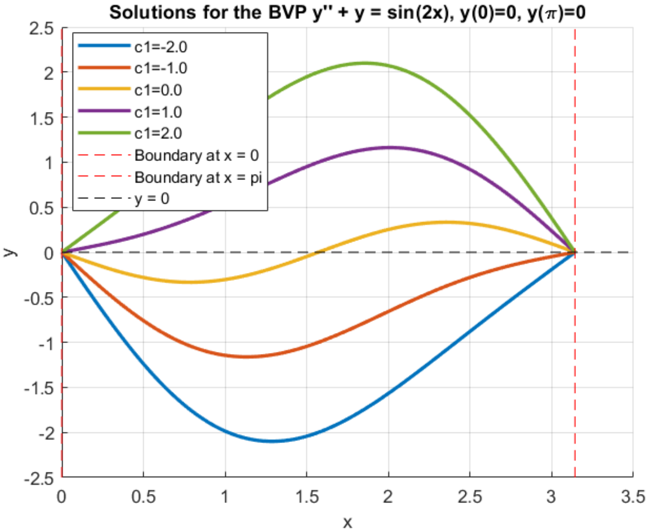
where c_1 is arbitrary.

Example 3


Solutions for the BVP y'' + y = sin(2x), y(0)=0, y(π)=0
c1=-2.0
c1=-1.0
c1=0.0
c1=1.0
c1=2.0
Boundary at x = 0
Boundary at x = pi
y = 0
y
x
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
0.5
1
1.5
2
2.5
3
3.5