Lesson 1. First-order Ordinary Differential Equations I

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2025

Contents

1 Fundamentals of Differential Equations

1.1 Overview of Differential Equations

Definition 1.1. Equations in which the unknown function or the vector function appears under the sign of the derivative or the differential are called **differential equations**.

Differential equations are used to model a wide range of real-world phenomena, particularly those involving **rates of change**. For instance:

- velocity is the rate of change of displacement: $v(t) = x'(t)$;
- acceleration is the rate of change of velocity: $a(t) = v'(t) = x''(t)$.

Example 1.1 (Newton's Second Law of Motion)**.**

$$m\frac{d^2x}{dt^2}(t) = f\left(t, x(t), \frac{dx}{dt}(t)\right) \tag{1}$$

This equation governs the motion of the object, and solving it will give the object's position $x(t)$ as a function of time under the influence of the specified force.

Example 1.2 (Radioactive Decay)**.** The amount u of a radioactive material changes in time as follows,

$$\frac{du}{dt}(t) = -ku(t), \quad k > 0, \tag{2}$$

where k is a positive constant representing the radioactive properties of the material.

Example 1.3 (The Heat Equation)**.** Describes how heat diffuses through a given region over time:

$$\frac{\partial T}{\partial t} = k\nabla^2 T, \tag{3}$$

where:

- $T = T(t, \mathbf{x})$ is the temperature at time t and position $\mathbf{x} = (x, y, z)$,
- k is a positive constant representing the thermal conductivity of the material,
- ∇^2 is the Laplacian operator.

Example 1.4 (The Wave Equation)**.** Describes how a wave propagates through a medium over time. In this case, $u = u(t, \mathbf{x})$ represents the wave disturbance at time t and position $\mathbf{x} = (x, y, z)$, and c is the speed of the wave:

$$\frac{\partial^2 u(t, \mathbf{x})}{\partial t^2} = c^2\nabla^2 u(t, \mathbf{x}) \tag{4}$$

The wave equation can be used to model different types of wave propagation: Vibrating String (1D Wave Equation); Sound Waves (3D Wave Equation); etc.

Example 1.5 (Schrödinger equation)**.** The Schrödinger equation in Quantum Mechanics, in one space dimension, stationary, is

$$-\frac{\hbar^2}{2m}\psi'' + V(x)\psi = E\psi, \tag{5}$$

where $\psi(x)$ is the probability density of finding a particle of mass m at the position x having energy E under a potential $V(x)$, and $\hbar$ is Planck's constant divided by 2π.

Definition 1.2. If there are several equations working together, we have a so-called **system of differential equations**.

Example 1.6 (Maxwell's equations for electromagnetics)**.** Maxwell's equations are comprised of four partial differential equations, each representing a different aspect of electromagnetism. These equations are:

$$\operatorname{div}\mathbf{E} = \frac{\rho}{\varepsilon_0} \quad (1) \qquad\qquad \operatorname{div}\mathbf{B} = 0 \quad (2)$$
$$\operatorname{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (3) \qquad \operatorname{curl}\mathbf{B} = \mu_0\mathbf{J} + \mu_0\varepsilon_0\frac{\partial \mathbf{E}}{\partial t} \quad (4)$$

Gauss's law for electricity (1) describes how electric charges produce electric fields. **Gauss's law for magnetism** (2) states that there are no magnetic monopoles in nature. **Faraday's law** (3) describes how a changing magnetic field creates an electric field. **Ampere's law with Maxwell's addition** (4) combines the effects of electric currents (Ampere's law) and changing electric fields (Maxwell's addition).

1.2 Classification of Differential Equations

Definition 1.3. If an equation involves the derivative of one variable with respect to another, then the former is called a **dependent variable** and the latter an **independent variable**.

Definition 1.4. If in a differential equation the dependent variables are functions of one independent variable, then the differential equation is called **ordinary (ODE)**. But if the unknown function appearing in the differential equation is a function of two or more independent variables, the differential equation is called a **partial differential equation (PDE)**.

- The equations in examples (1), (2) and (5) are ODE – the unknown function depends on a single independent variable, t or x.
- The equations in examples (3) and (4) are PDE – the unknown function depends on two or more independent variables, t, x, y, and z, and their partial derivatives appear in the equations.

Definition 1.5. The **order** of a differential equation (or system) is the order of the highest derivative in the equation (system).

- Newton's equation in example (1) is second order;
- The time decay equation in example (2) is first order;
- Schrödinger equation in example (5) is second order;
- The heat equation in example (3) is first order in time and second order in space variables;
- The wave equation in example (4) is second order in time and space variables.

Definition 1.6. The **degree** of a differential equation (system) is the highest power of the highest-order derivative in the equation (system).

Example 1.7 (Second degree first-order ODE)**.**

$$(y')^2 = x - y^3$$

We also distinguish how the **dependent variables** appear in the equation (or system).

Definition 1.7. An equation is **linear** if the dependent variable (or variables) and their derivatives appear linearly, that is only as first powers, they are not multiplied together, and no other functions of the dependent variables appear. In other words, the equation is a sum of terms, where each term is some function of the independent variables or some function of the independent variables multiplied by a dependent variable or its derivative. Otherwise, the equation is called **nonlinear**.

An ordinary differential equation is linear if it can be put into the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x). \tag{1.1}$$

The functions $a_0(x), a_1(x), \ldots, a_n(x)$ are called **the coefficients**.

For a function $u(x, y)$, a second order linear PDE is of the form

$$\begin{aligned} a_1(x, y)\, u_{xx} + a_2(x, y)\, u_{xy} + a_3(x, y)\, u_{yx} + a_4(x, y)\, u_{yy} \\ + \; a_5(x, y)\, u_x + a_6(x, y)\, u_y + a_7(x, y)\, u = f(x, y), \end{aligned} \tag{1.2}$$

where a_i and f are functions of the independent variables x and y only.

Definition 1.8. When the coefficients of a linear DE do not depend on x, the DE is said to have **constant coefficients**.

Definition 1.9. A linear equation (1.1)–(1.2) may further be called **homogeneous** if all terms depend on the dependent variable. That is, if no term is a function of the independent variables alone. Otherwise, the equation is called **nonhomogeneous** or **inhomogeneous**.

Example 1.8 (Linear and nonlinear equations)**.**

- Mass-Spring-Damper System (linear, nonhomogeneous, constant coefficients):

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

- Simple Pendulum (nonlinear):

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0$$

- Transport equation (linear, homogeneous, constant coefficients):

$$u_x + u_y = 0$$

Definition 1.10. An equation (or system) is called **autonomous** if the equation does not depend on the independent variable.

Example 1.9 (Autonomous equations and systems)**.**

- Newton's Law of Cooling:
$$\frac{dT}{dt} = -k(T - T_{out})$$
- Logistic Growth Model:
$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$
- Predator-Prey Model (Lotka-Volterra Equations):
$$\frac{dx}{dt} = ax - bxy, \qquad \frac{dy}{dt} = -cy + dxy,$$
where x and y represent the prey and predator populations, respectively.

1.3 Fundamentals of Differential Equations

Definition 1.11. A **solution** of a differential equation is a function which, when substituted into the differential equation, reduces it to an identity.

Definition 1.12. Initial Condition(s) are a condition, or set of conditions, on the solution that will allow us to determine which solution that we are after. Initial conditions are of the form
$$y(t_0) = y_0 \text{ and/or } y^{(k)}(t_0) = y_k.$$
So, in other words, initial conditions are values of the solution and/or its derivative(s) at specific points. *The number of initial conditions that are required for a given differential equation will depend upon the order of the differential equation.*

Definition 1.13. The general solution to a differential equation is the most general form that the solution can take and doesn't take any initial conditions into account.

Definition 1.14. The particular solution to a differential equation is the specific solution that not only satisfies the differential equation, but also satisfies the given initial condition(s).

Definition 1.15. An **Initial Value Problem** (or **IVP**) is a differential equation along with an appropriate number of initial conditions.

Definition 1.16. A **singular solution** is the solution of a differential equation that cannot be obtained from the general solution gotten by the usual method of solving the differential equation.

Definition 1.17. An **equilibrium solution** is a solution to a DE whose derivative is zero everywhere. On a graph an equilibrium solution looks like a horizontal line.

Definition 1.18. An equilibrium solution is said to be **Asymptotically Stable** if on both sides of this equilibrium solution, there exist other solutions which approach this equilibrium solution.

Definition 1.19. An equilibrium solution is said to be **Semi-Stable** if on one side of this equilibrium solution there exist other solutions which approach this equilibrium solution, and on the other side of the equilibrium solution other solutions diverge from this equilibrium solution.

Definition 1.20. An equilibrium solution is said to be **Unstable** if on both sides of this equilibrium solution other solutions diverge from this equilibrium solution.

1.4 Notation for Derivatives

Leibniz's Notation:

$$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \ldots, \quad \frac{d^ny}{dx^n}$$

This notation is more useful for differentiation and integration.

Lagrange's Notation:

$$y', \quad y'', \quad \ldots, \quad y^{(n)}$$

This notation is more useful for representing derivatives of any order compactly.

Newton's Notation:

$$\dot{y}, \quad \ddot{y}, \quad \dddot{y}$$

This notation is often used in physics for representing derivatives of low order with respect to time.

2 First-order Ordinary Differential Equation

2.1 First-order ODE

Definition 2.1. An **ordinary first-order differential equation of the first degree** may be represented as follows:

$$F(x, y, y') = 0 \tag{1}$$

or, solving for the derivative:

$$\frac{dy}{dx} = f(x, y). \tag{2}$$

Example 2.1 (First-order ODE)**.** In the form (1):

$$y' - xy = 0$$

or in the form (2):

$$y' = xy.$$

Here, x is the independent variable, y is the dependent variable, and y' is its derivative.

The **general solution** of a first-order ODE is the function

$$y = \phi(x, C) \tag{3}$$

or the relation

$$\Phi(x, y, C) = 0 \tag{4}$$

which depends on a single arbitrary constant C.

Example 2.2 (General solution of $y' = xy$)**.**

$$y = C \cdot e^{\frac{x^2}{2}}$$

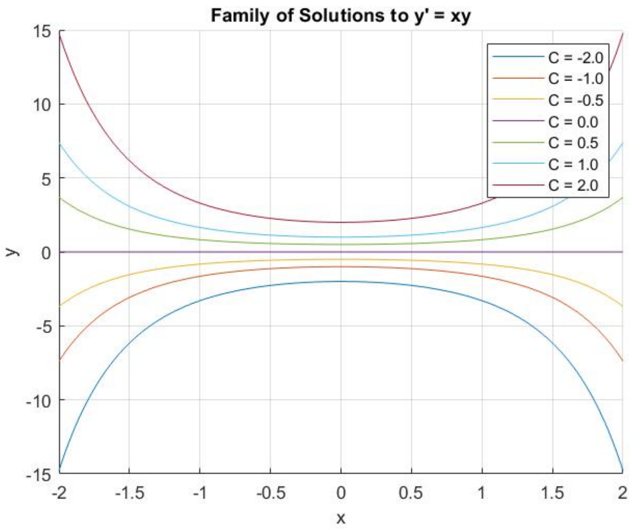

Figure 1: Family of solutions for $y' = xy$.

A **particular solution** is any function

$$y = \phi(x, C_0) \tag{5}$$

or relation

$$\Phi(x, y, C_0) = 0 \tag{6}$$

which is obtained from the general solution by assigning to the arbitrary constant C a definite value $C = C_0$.

In the case of a first-order equation, the **initial value problem** is of the form:

$$y' = f(x, y), \quad y(x_0) = y_0. \tag{7}$$

Example 2.3 (IVP for $y' = xy$)**.** Consider the following IVP:

$$y' = xy, \quad y(1) = 1.$$

The particular solution is:

$$y_p = \frac{1}{\sqrt{e}} \cdot e^{\frac{x^2}{2}}.$$

2.2 Examples of IVPs

Example 2.4 (Unique solution)**.** Consider the IVP:

$$y' = y, \quad y(0) = 2.$$

The general solution is $y(x) = Ce^x$. The particular solution is $y(x) = 2e^x$.

The initial condition $y(0) = 2$ uniquely determines the solution as $y = 2e^x$. This is because the general solution $y = Ce^x$ with C determined by the initial condition results in $C = 2$ when $x = 0$ and $y = 2$. Thus the solution to the IVP exists, is unique, and is defined on all of $\mathbb{R}$.

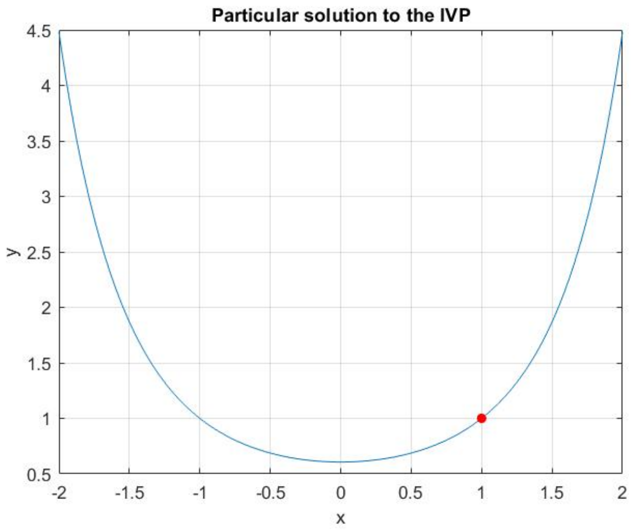

Figure 2: Particular solution to the IVP $y' = xy$, $y(1) = 1$.

Example 2.5 (Solution with finite interval of existence)**.** Consider the IVP:

$$y' = xy^2, \quad y(0) = 3.$$

The general solution is $y(x) = \frac{2}{C - x^2}$. The particular solution is $y(x) = \frac{6}{2-3x^2}$.

The initial condition $y(0) = 3$ uniquely determines the constant $C = \frac{1}{3}$. This particular solution exists and is unique. However, it is only defined on the open interval $\left(-\frac{\sqrt{2}}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}\right)$, since the denominator vanishes at $x = \pm\frac{\sqrt{2}}{\sqrt{3}}$, causing the solution to blow up.

Example 2.6 (Infinitely many solutions)**.** Consider the IVP:

$$y' = \frac{y}{x}, \quad y(0) = 0.$$

The general solution is $y(x) = Cx$. The initial condition $y(0) = 0$ leads to the IVP having infinitely many solutions: $0 = C \cdot 0 \Rightarrow C$ arbitrary.

Example 2.7 (No solution)**.** Consider the IVP:

$$y' = \frac{y}{x}, \quad y(0) = 1.$$

The general solution is $y(x) = Cx$. The initial condition $y(0) = 1$ does not determine a solution because the general solution $y = Cx$ with C being any constant does not satisfy $y(0) = 1$ for any value of C $(1 \neq C \cdot 0)$.

Example 2.8 (Non-uniqueness with piecewise solutions)**.** Consider the IVP:

$$y' = 2\sqrt{y}, \quad y(0) = 0.$$

The general solution is $y(x) = (x - C)^2$. The particular solutions include both $y(x) = x^2$ and $y = 0$.

We can combine both solutions to create infinitely many piecewise-defined solutions like this:

$$y(x) = \begin{cases} 0, & \text{if } x < C, \\ (x - C)^2, & \text{if } x \geq C, \end{cases} \quad \text{for any } C > 0.$$

That means:

- The solution stays at zero up to time $x = C$,
- Then it starts following the parabola $y = (x - C)^2$ from that point onward.

Since $(x - C)^2$ starts at 0 and has zero slope at $x = C$, the function is smooth at that transition point—there is no sudden jump or sharp corner—so it still satisfies the differential equation.

This shows that the initial value problem $y' = 2\sqrt{y}$, $y(0) = 0$ has infinitely many solutions, depending on the value of C.

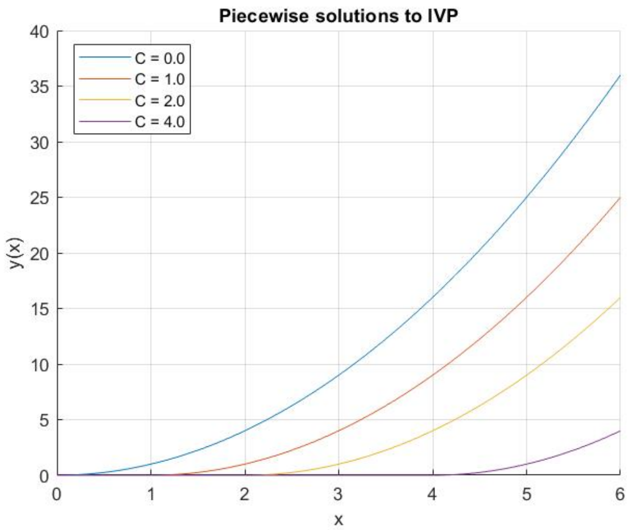

Figure 3: Four solutions for the IVP $y' = 2\sqrt{y}$, $y(0) = 0$.

2.3 Solution to the IVP and Questions of Existence and Uniqueness

A **solution to the IVP** (7) is a function $y(x)$ such that:

1. $y(x)$ is defined on some interval (a, b) containing x_0, and $y(x_0) = y_0$,
2. $y(x)$ satisfies the ODE in (a, b).

The solution to the IVP comes with a domain where it is defined. The largest such interval (a, b) is called the **interval of existence**.

Thus, the following questions naturally arise:

- Given an IVP, does a solution exist? (Question of *existence*)
- If a solution exists, is it unique? (Question of *uniqueness*)
- For which values of x does the solution exist? (The *interval of existence*)

3 General Existence-Uniqueness Theory

3.1 Peano's Theorem

Consider the Initial Value Problem (IVP):

$$y' = f(x, y), \quad y(x_0) = y_0.$$

Theorem 3.1 (Peano Theorem on existence)**.** *Suppose $f(x, y)$ is a continuous function on an open rectangle of the form*

$$D = \{(x, y) \mid a < x < b,\, c < y < d\}$$

in the xy-plane. If (x_0, y_0) is a point in this rectangle, then there exists an $\varepsilon > 0$ and a function $y(x)$ defined for all

$$x_0 - \varepsilon < x < x_0 + \varepsilon$$

that solves the given initial value problem.

In other words, as long as the function $f(x, y)$ is continuous in a rectangle $D = \{(x, y) \mid a < x < b,\, c < y < d\}$, even a tiny rectangle $|x - x_0| < \varepsilon$ that contains the point (x_0, y_0) in its interior, then there exists a function $y(x)$ (many more might exist) that solves a differential equation that passes through the point (x_0, y_0) and this function is defined in at least a tiny interval around this point.

Remark 3.1.

- The solution is guaranteed to exist only on some neighborhood and we don't know how large;
- It does not guarantee the uniqueness of the solution.

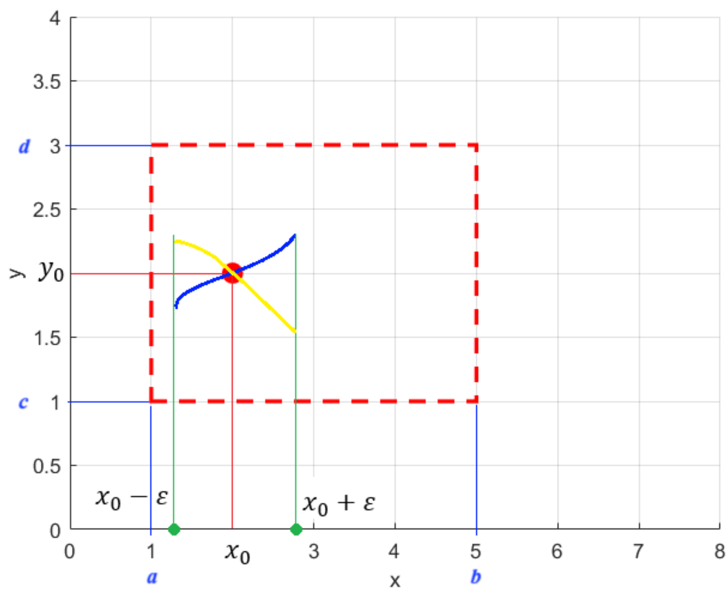

Figure 4: Illustration of Peano's theorem.

Example 3.1 (Applying Peano's theorem)**.** Consider the IVP:

$$\frac{dy}{dx} = \frac{y}{x}, \quad y(x_0) = y_0.$$

The function $f(x, y) = \frac{y}{x}$ is not defined at $x = 0$, and hence is not continuous in any rectangle that includes $x = 0$.

If $x_0 \neq 0$, then f is continuous in a neighborhood around (x_0, y_0), and Peano's theorem guarantees the existence of at least one solution.

However, if $x_0 = 0$, Peano's theorem cannot be applied, and we cannot conclude whether a solution exists from the theorem.

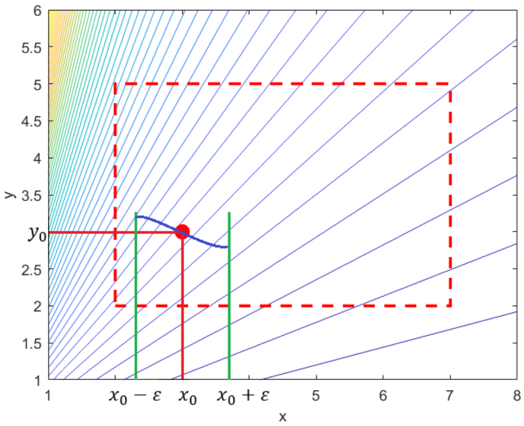

Figure 5: Solutions of $y' = y/x$.

3.2 Lipschitz Continuity

Definition 3.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. We say that:

- f is **locally Lipschitz continuous in the variable** y if for every open rectangle $D = \{(x, y) \mid a < x < b,\ c < y < d\}$, there exists a constant $L > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2| \quad \text{for all } (x, y_1), (x, y_2) \in D.$$

- f is **globally Lipschitz continuous in** y if there exists a constant $L > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2| \quad \text{for all } (x, y_1), (x, y_2) \in \mathbb{R}^2.$$

The constant L is called a **Lipschitz constant** for f in the variable y.

Example 3.2 (Lipschitz condition holds)**.** Consider the IVP $y' = y$, $y(0) = 2$. We show that $f(x, y) = y$ satisfies a Lipschitz condition in y on some rectangle containing $(0, 2)$.

Choose the rectangle

$$D = \{(x, y) \mid -1 \leq x \leq 1,\ 1 \leq y \leq 3\}$$

that contains the initial point $(0, 2)$.

For arbitrary points (x, y_1) and (x, y_2) in D, we have

$$|f(x, y_1) - f(x, y_2)| = |y_1 - y_2| \leq 1 \cdot |y_1 - y_2|.$$

The inequality holds with Lipschitz constant $L = 1$, so the function is **Lipschitz continuous** in y on the domain D.

Example 3.3 (Lipschitz condition fails)**.** Consider the IVP $y' = 2\sqrt{y}$, $y(0) = 0$. Check whether $f(x, y) = 2\sqrt{y}$ satisfies a Lipschitz condition on

$$D = \{(x, y) \mid -1 \leq x \leq 1,\ 0 \leq y \leq 1\}.$$

For arbitrary points (x, y_1) and (x, y_2) in D, we compute:

$$|f(x, y_1) - f(x, y_2)| = |2\sqrt{y_1} - 2\sqrt{y_2}| = 2 \cdot |\sqrt{y_1} - \sqrt{y_2}|.$$

Using the identity

$$|\sqrt{y_1} - \sqrt{y_2}| = \frac{|y_1 - y_2|}{\sqrt{y_1} + \sqrt{y_2}}, \quad \text{for } y_1, y_2 > 0,$$

we get:

$$|f(x, y_1) - f(x, y_2)| = 2 \cdot \frac{|y_1 - y_2|}{\sqrt{y_1} + \sqrt{y_2}}.$$

To satisfy the Lipschitz condition in y, we would require:

$$\frac{2}{\sqrt{y_1} + \sqrt{y_2}} \leq L.$$

As $y_1, y_2 \to 0$, we have $\sqrt{y_1} + \sqrt{y_2} \to 0$, so

$$\frac{2}{\sqrt{y_1} + \sqrt{y_2}} \to \infty.$$

This shows that no finite constant L can satisfy the Lipschitz condition over the entire domain D. Therefore, the function $f(y) = 2\sqrt{y}$ is **not Lipschitz continuous** on D, although it is continuous.

3.3 Picard-Lindelöf Theorem

Theorem 3.2 (Picard-Lindelöf theorem on existence and uniqueness)**.** *Let $f(x, y)$ be continuous and satisfy a Lipschitz condition on an open rectangle*

$$D = \{(x, y) \mid a < x < b,\ c < y < d\}$$

in the xy-plane. If the point (x_0, y_0) belongs to the interior of this rectangle then there is a unique function $y(x)$, defined on an interval

$$|x - x_0| < \varepsilon$$

for some $\varepsilon > 0$, that solves the IVP:

$$y' = f(x, y), \quad y(x_0) = y_0.$$

Remark 3.2. This theorem guarantees a **local** solution—that is, the solution exists and is unique only in a small interval around x_0.

3.4 Continuously Differentiable Implies Lipschitz in y

Definition 3.2. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is **continuously differentiable in** y on $\mathbb{R}^2$ if its partial derivative

$$\frac{\partial f}{\partial y}$$

exists and is a continuous function.

Proposition 3.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable in y. Then f is locally Lipschitz continuous in y.*

Proof. Let $f(x, y)$ be continuously differentiable in some open rectangle $D \subset \mathbb{R}^2$, let a point $(x_0, y_0) \in D$, and fix $x = x_0$. We treat $f(x_0, y)$ as a function of y only (i.e., we *freeze* x).

Let y_1, y_2 be values close to y_0 such that the segment between them lies entirely inside D. Apply the Mean Value Theorem to the function $f(x_0, y)$:

$$f(x_0, y_1) - f(x_0, y_2) = \frac{\partial f}{\partial y}(x_0, \xi) \cdot (y_1 - y_2),$$

for some ξ between y_1 and y_2.

Taking absolute values:

$$|f(x_0, y_1) - f(x_0, y_2)| = \left| \frac{\partial f}{\partial y}(x_0, \xi) \right| \cdot |y_1 - y_2|.$$

Since $\frac{\partial f}{\partial y}$ is continuous in the rectangle, it is **bounded**. So there exists a constant $M > 0$ such that:

$$\left| \frac{\partial f}{\partial y}(x_0, \xi) \right| \leq M.$$

Hence,

$$|f(x_0, y_1) - f(x_0, y_2)| \leq M|y_1 - y_2|.$$

This proves that $f(x, y)$ is Lipschitz continuous in y around (x_0, y_0) with the Lipschitz constant $L = M$. Since (x_0, y_0) was arbitrary, the result holds locally on D. $\square$

Corollary 3.1. *Let $f(x, y)$ and $\frac{\partial f}{\partial y}$ be* ***continuous on an open rectangle***

$$D = \{(x, y) \mid a < x < b,\ c < y < d\}$$

in the xy-plane. Let (x_0, y_0) belong to the interior of D. Then there is a small $\varepsilon > 0$ and a unique function $y(x)$ continuously differentiable on $|x - x_0| < \varepsilon$ such that $(x, y(x))$ remains in D for $|x - x_0| < \varepsilon$ and $y(x)$ solves the IVP:

$$y' = f(x, y), \quad y(x_0) = y_0.$$

Interpretation of the Corollary:

- **I (Existence):** If $f(x, y)$ is continuous in the rectangle, then there is an open interval $I \subset (a, b)$ (possibly smaller) containing x_0 such that a solution $y(x)$ **exists** for the IVP that is defined on I.

- **II (Uniqueness):** The solution is unique if in addition the Lipschitz condition holds. A sufficient condition for (II) (stronger, but easier to check) is that $\frac{\partial f}{\partial y}$ is continuous in the rectangle.
- **Local Result:** The existence theorem provides only a **local solution**, which means it guarantees a solution only in some interval around x_0 that could be arbitrarily small.

3.5 Applying the Theorem

Example 3.4 (Existence and uniqueness hold)**.** Show that there is a unique solution to the IVP:

$$y' = xy^2, \quad y(0) = 3.$$

For this initial value problem we have:

$$f(x, y) = xy^2 \quad \text{and} \quad \frac{\partial f}{\partial y} = 2xy,$$

which are both continuous on all of $\mathbb{R}^2$.

The Corollary guarantees existence and uniqueness of a solution for x near $x_0 = 0$:

$$y(x) = \frac{3}{1 - 3x^2}, \quad x \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right).$$

But just because f and $\frac{\partial f}{\partial y}$ are continuous on all of $\mathbb{R}^2$ we cannot expect existence or uniqueness of a solution $y(x)$ for all x.

Example 3.5 (Theorem fails to apply)**.** Show that the theorem fails to apply to the IVP:

$$y' = 2\sqrt{y}, \quad y(0) = 0.$$

For this initial value problem we have:

$$f(x, y) = 2\sqrt{y} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{1}{\sqrt{y}}.$$

$f(x, y)$ is continuous for $y \geq 0$, but $\frac{\partial f}{\partial y}$ is **not** continuous at $y = 0$.

The Corollary guarantees (I) (existence) (since f is continuous), but **not** (II) (uniqueness) (since $\partial f / \partial y$ is not continuous at the initial point).

Recall, this IVP has **infinitely many solutions**:

$$y(x) = \begin{cases} 0, & \text{if } x < C, \\ (x - C)^2, & \text{if } x \geq C, \end{cases} \quad \text{for any } C > 0,$$

showing failure of uniqueness in practice.

4 Method of Picard Iteration

4.1 Rewriting the IVP as an Integral Equation

For the initial value problem (IVP)

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0,$$

suppose that f is continuous on some appropriate rectangle

$$D = \{(x,y) \mid x_0 - \varepsilon < x < x_0 + \varepsilon,\ c < y < d\},$$

and that there exists a solution $y(x)$ which is continuous on the interval $|x - x_0| < \varepsilon$.

Then we may integrate both sides of the differential equation. Starting from

$$\frac{dy}{dx} = f(x,y), \quad \text{or} \quad dy = f(x,y)\,dx,$$

we integrate both sides with respect to x, over the interval from x_0 to x:

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} f(s, y(s))\,ds.$$

Evaluating the left-hand side, we obtain:

$$y(x) - y(x_0) = \int_{x_0}^{x} f(s, y(s))\,ds.$$

Since $y(x_0) = y_0$, we have:

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))\,ds.$$

This is the **integral form** of the initial value problem. Thus, under the assumptions of existence and continuity, the IVP is equivalent to the **integral equation**.

4.2 The Iteration Sequence

The integral equation is generally difficult to solve explicitly using analytical methods. An alternative approach is to approximate a solution by constructing a sequence of functions that converges to a solution.

We apply the **method of successive approximations**. Starting from the constant function $y_0(x) = y_0$, we define a sequence of functions $\{y_n(x)\}$ iteratively as follows:

$$y_1(x) = y_0 + \int_{x_0}^{x} f(s, y_0)\,ds,$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(s, y_1(s))\,ds,$$

$$\vdots$$

$$y_n(x) = y_0 + \int_{x_0}^{x} f(s, y_{n-1}(s))\,ds.$$

This defines a sequence of approximations $y_1(x), y_2(x), y_3(x), \ldots, y_n(x)$, which will converge to the actual solution.

Definition 4.1 (Picard Iteration)**.** In general, the Picard iteration is given by the recurrence:

$$y_n(x) = y_0 + \int_{x_0}^{x} f(s, y_{n-1}(s))\,ds.$$

4.3 Convergence

Under the **theorem's assumptions** on the function $f(x, y)$, the sequence of approximate solutions defined by Picard iteration:

$$y_n(x) = y_0 + \int_{x_0}^{x} f(s, y_{n-1}(s))\, ds$$

converges uniformly to a limit function $y(x)$ on a small interval around x_0:

$$\lim_{n\to\infty} y_n(x) = y(x).$$

We now verify that $y(x)$ satisfies the differential equation by first showing it satisfies the integral equation:

$$\begin{aligned}
y(x) &= \lim_{n\to\infty} y_n(x) \\
&= \lim_{n\to\infty} \left(y_0 + \int_{x_0}^{x} f(s, y_{n-1}(s))\, ds \right) \\
&= y_0 + \lim_{n\to\infty} \int_{x_0}^{x} f(s, y_{n-1}(s))\, ds \\
&= y_0 + \int_{x_0}^{x} \lim_{n\to\infty} f(s, y_{n-1}(s))\, ds \\
&= y_0 + \int_{x_0}^{x} f(s, y(s))\, ds.
\end{aligned}$$

Since $y(x)$ is continuous and the right-hand side is differentiable, we conclude:

$$\frac{dy}{dx} = f(x, y(x)).$$

Thus, $y(x)$ is a solution of the IVP.

This procedure of generating a sequence of functions which approximate the solution whose existence we are trying to establish, is called **Picard iteration**.

4.4 Picard Iteration: Example 1

Example 4.1. Use the proof of the Picard-Lindelöf Theorem to find the solution to

$$y' = y, \quad y(0) = 2.$$

We know this IVP satisfies the Picard-Lindelöf theorem and has a unique solution $y = 2e^x$.

Step 1: Rewrite IVP as Integral Equation.

$$\begin{aligned}
& y' = y, \quad y(0) = 2 \\
\Rightarrow \quad & y(x) = 2 + \int_0^x y(s)\, ds.
\end{aligned}$$

Step 2: Define Iteration Sequence. Using the method of successive approximations:

$$
\begin{aligned}
y_0(x) &= 2 \\
y_1(x) &= 2 + \int_0^x 2\,ds = 2 + 2x \\
y_2(x) &= 2 + \int_0^x (2 + 2s)\,ds = 2 + 2x + x^2 \\
y_3(x) &= 2 + \int_0^x (2 + 2s + s^2)\,ds = 2 + 2x + x^2 + \frac{x^3}{3} \\
y_4(x) &= 2 + \int_0^x \left(2 + 2s + s^2 + \frac{s^3}{3}\right)\,ds = 2 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12}.
\end{aligned}
$$

By induction, we find:

$$y_n(x) = 2\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right).$$

This is the nth partial sum of the Maclaurin series for $2e^x$.

Step 3: Uniform Convergence. It can be shown by induction that y_n is the partial sum of $2e^x$. Thus, as $n \to \infty$, $y_n(x) \to 2e^x$ uniformly.

Step 4: The limit is the unique solution. The solution to the IVP is:

$$y(x) = \lim_{n\to\infty} y_n(x) = 2\sum_{k=0}^{\infty} \frac{x^k}{k!} = 2e^x.$$

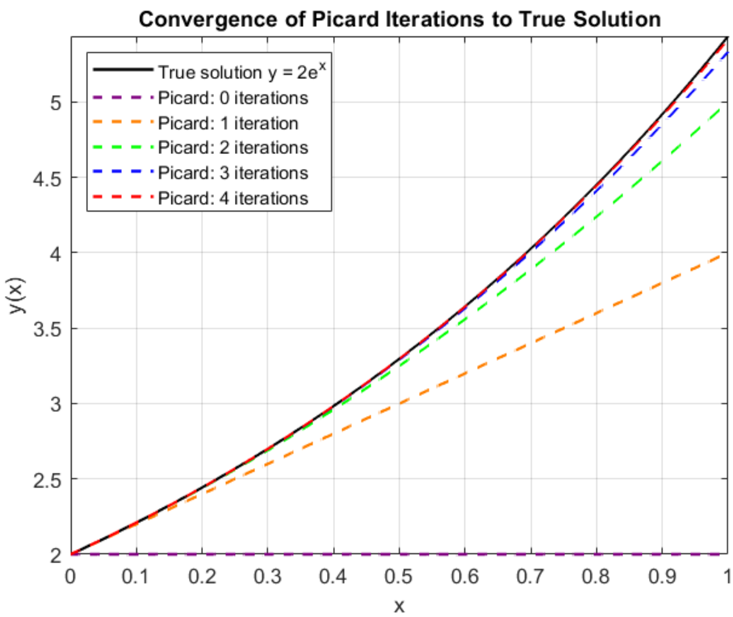

Figure 6: Picard iterations converging to $y = 2e^x$.

4.5 Picard Iteration: Example 2

Example 4.2. Use Picard iteration to study

$$y' = 1 + x\sin(xy), \quad y(0) = 0, \quad 0 \le x \le 2.$$

This is a **nonlinear** initial value problem. Unlike linear problems, this IVP involves the product xy inside a sine function $\sin(xy)$. There is no explicit solution.

Corollary applies. We check conditions for local existence. Define $f(x,y) = 1 + x\sin(xy)$. On some rectangle D:

- the function f is continuous,
- the partial derivative $\frac{\partial f}{\partial y} = x^2\cos(xy)$ is also continuous.

The IVP satisfies the Corollary, so a unique local solution exists.

Step 1: Rewrite IVP as Integral Equation.

$$y' = 1 + x\sin(xy), \quad y(0) = 0$$
$$\Rightarrow \quad y(x) = \int_0^x [1 + s\sin(s\,y(s))]\,ds.$$

Step 2: Define Iteration Sequence.

$$y_0(x) = 0$$
$$y_1(x) = \int_0^x [1 + s\sin(0)]\,ds = \int_0^x 1\,ds = x$$
$$y_2(x) = \int_0^x [1 + s\sin(s\cdot y_1(s))]\,ds = \int_0^x \left[1 + s\sin(s^2)\right]ds = x - \frac{\cos(x^2)}{2}$$
$$y_3(x) = \int_0^x [1 + s\sin(s\cdot y_2(s))]\,ds = \int_0^x \left[1 + s\sin\left(s\cdot\left(s - \frac{\cos(s^2)}{2}\right)\right)\right]ds.$$

No closed-form formula appears possible from this step onward. Numerical integration is used.

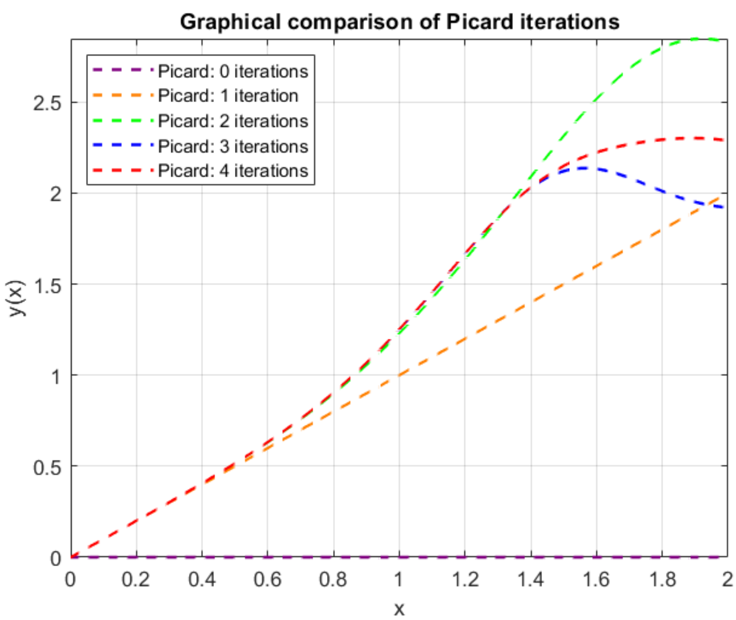

Figure 7: Picard iterations for $y' = 1 + x\sin(xy)$, $y(0) = 0$.