Differential Equations — Practice Exercises

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Contents

1 Practice 1. Introduction

Topics: Instantaneous velocity, equations of motion, radioactive decay, population dynamics. Differential equations, direction fields, initial value problems, directly integrable equations, separable equations, word problems leading to separable equations.

The concept of velocity is illustrated by the formula

$$\dot{x}(t) = \lim_{h\to 0} \frac{x(t+h)-x(t)}{h},$$

where t denotes time and x denotes displacement. Based on this, one can write down the simplest differential equations. For instance, Newton's law: $m\ddot{x}(t) = F$, or the law describing radioactive decay: $\dot{x}(t) = ax(t)$ (this is also the cooling law, and the simplest population dynamics model, where x denotes the size of the population — number of individuals or mass).

1. Find the integral curve of the differential equation $\dot{x}(t) = 2\sin t$ that passes through the origin!

2. Solve the following differential equations! Sketch the direction field, the solutions, and illustrate the initial condition.

   (a) $\dot{x}(t) = 1$

   (b) $\dot{x}(t) = x(t)$ [The solution "by inspection" is visible. Is there a solution other than the exponential function? Multiply the equation by e^{-t}; then we get $(e^{-t}x(t))' = 0$, i.e. $e^{-t}x(t) = C$.]

Definition 1.1. Let $D \subset \mathbb{R}^2$ be a connected open set (domain), $f : D \to \mathbb{R}$ a continuous function. The general form of a **first-order ordinary differential equation (ODE)** is:

$$\dot{x}(t) = f(t, x(t)) \tag{1}$$

The solutions of the equation cannot be given in general. (Differential algebra deals with determining which equations have solutions expressible by formulas.) Solvable types are collected in several references, e.g. Kamke's book, Mathematica, or Maple packages.

3. Prove that every solution of the equation $\dot{x}(t) = kx(t)$ $(k \in \mathbb{R})$ is of the form $x(t) = Ce^{kt}$! [The idea from the previous problem works, but now multiply by e^{-kt}.]

4. $(t+1)\dot{x}(t) = tx(t)$ [Multiply again by e^{-t}.]

5. $\dot{x}(t) = tx(t)$ [The multiplication trick works here too, but instead of $-t$, something else is needed in the exponent of the exponential function.]

In the following, we will deal with several easily solvable types, starting with the

$$\dot{x}(t) = g(t)h(x(t)) \qquad \text{(Separable ODE)}$$

so-called separable (separable-variable) equations. Some of the introductory examples were of this type.

Solution method: Separating the factors depending only on t and on x, respectively:

$$\dot{x}(t) = g(t)h(x(t)) \iff \frac{\dot{x}(t)}{h(x(t))} = g(t) \iff H(x(t)) = G(t) + C,$$

where $\dot{G} = g$ and $\dot{H} = 1/h$.

6. $\dot{x}(t) = \dfrac{1}{x(t)\,(9+4t^2)}$

7. A tank contains 100 litres of solution with 10 kg of salt. Water flows into the tank at a rate of 5 l/min, mixing with the solution inside. The mixture flows out from the bottom of the tank at the same rate. How much salt remains in the tank after 1 hour?

Homework:
$\dot{x}(t) = \left(1 + x^2(t)\right)\left(1 + t^2\right)$
$\dot{x}(t) = 2x(t)\cot t$

A jar of jam at 100 °C is placed in the open air at 20 °C. The temperature of the jam is 30 °C at 10 o'clock and 25 °C at 11 o'clock. When was the jam taken out? (Recall that the rate of cooling is proportional to the difference between the temperature of the body and the ambient temperature.)

2 Practice 2. First-Order ODEs: Solvability, Uniqueness, Linear Equations

Topics: First-order ordinary differential equations, solvability, uniqueness. First-order linear differential equations, solution formula, method of variation of constants.

Theorem 2.1 (Picard–Lindelöf)**.** *If the function f is locally Lipschitz continuous in its second variable (i.e. there exists $L \in \mathbb{R}^+$ such that $|f(t,p_1) - f(t,p_2)| \le L|p_1 - p_2|$), then for every $(t_0, p_0) \in D$ there exists a local (i.e. defined in some neighbourhood of t_0) solution of* (1) *satisfying $x(t_0) = p_0$.*

In addition to the local existence, the above conditions also imply uniqueness of the solution (this follows from the Picard–Lindelöf theorem, see lecture). The existence of the solution can already be guaranteed if the right-hand side f is only continuous (Peano's existence theorem, see lecture).

After separable equations, we turn to a new type, the first-order linear ODE:

$$\dot{x}(t) = a(t)x(t) + b(t), \qquad \text{(First-order linear ODE)}$$

where $a, b : I \to \mathbb{R}$ are given continuous functions on the interval I.

Two solution methods:

(a) Generalization of the earlier multiplication trick. Multiply the equation by $e^{-A(t)}$, where $A' = a$. This leads to the general solution formula!

(b) If we know one solution (call it x_0), then substituting $x = x_0 + y$ for any other solution x, we get a separable equation for y whose solution is $y(t) = Ce^{A(t)}$. The key observation is that the solution x_0 can be written as $x_0(t) = C(t)e^{A(t)}$, and for the function C we always obtain an integrable differential equation. This leads to the general solution formula as well!

It is not worth memorising the solution formula; rather, one should apply one of the solution methods to the concrete example.

1. Solve the following equations!

   (a) $t\dot{x}(t) = t + 2x(t)$

   (b) $\dot{x}(t) = \dfrac{x(t)}{t} + t^2 + 3t - 2$

   (c) $\dot{x}(t) = 3t^2 x(t) + t^2$

2. Let $x : [0, \infty) \to (0, \infty)$ be a differentiable function such that for every $\tau > 0$ the tangent line drawn to the point $(\tau, x(\tau))$, the line $t = \tau$, and the coordinate axes determine a trapezoid of constant area. Find $x(t)$.

3. Solve the following equations and sketch the solutions! Is the solution unique?

   (a) $\dot{x}(t) = \sqrt{|x(t)|}$ [At points on the line $x = 0$, the solution is not locally unique.]

   (b) $\dot{x}(t) = x(t) \cdot \ln |x(t)|$ [The right-hand side is continuous everywhere and the solution is locally unique, despite the fact that the local Lipschitz continuity does not hold at zero.]

Homework:

$t\dot{x}(t) - 2x(t) = 2t^4$

$\dot{x}(t) = \dfrac{\sin t}{\cos t + 1}\left(e^{x(t)} + 1\right)$

3 Practice 3. Homogeneous, Bernoulli, and Exact Equations

Topics: Homogeneous equations, Bernoulli equations, problems reducible to linear equations via substitution, exact differential equations.

A function f is called homogeneous (more precisely, homogeneous of degree 0) if $f(\alpha t, \alpha p) = f(t, p)$ for all $\alpha \in \mathbb{R}$ (for degree r: $f(\alpha t, \alpha p) = \alpha^r f(t, p)$). The equation

$$\dot{x}(t) = g\left(\frac{x(t)}{t}\right) \qquad \text{(Homogeneous ODE)}$$

is called a homogeneous-degree equation.

Solution method: The substitution $y(t) = x(t)/t$ (introducing a new unknown function) reduces it to a separable equation.

The equation

$$\dot{x}(t) = g(at + bx(t) + c)$$

can be reduced to a separable equation by the substitution $y(t) = at + bx(t) + c$.

The following type is called a **Bernoulli equation**:

$$\dot{x}(t) = a(t)x(t) + b(t)x^{\alpha}(t), \qquad \text{(Bernoulli ODE)}$$

where $a, b : I \to \mathbb{R}$ are given continuous functions on the interval I, and $\alpha \in \mathbb{R}$ is a given constant.

Solution method: The substitution $y(t) = x^{1-\alpha}(t)$ transforms it into a linear equation in y.

Finally, we deal with the so-called exact equations:

$$M(t, x(t)) + N(t, x(t))\dot{x}(t) = 0, \quad (\text{where } \partial_2 M = \partial_1 N) \tag{Exact ODE}$$

$M, N : \mathbb{R}^2 \to \mathbb{R}$ are given differentiable functions.

Solution method: Find the differentiable function $F : \mathbb{R}^2 \to \mathbb{R}$ such that $\partial_1 F = M$ and $\partial_2 F = N$. Then the solution in implicit form is $F(t, x(t)) = c$, where $c \in \mathbb{R}$ is an arbitrary constant. For concrete problems, x may not always be expressible explicitly. (Note that if the equation is not exact, it can sometimes be made exact by multiplying by a suitably chosen function $\mu(t, x)$, called an integrating factor, but we will not deal with this here.)

1. $\Big(t^3 + x^3(t)\Big) - 3tx^2(t)\dot{x}(t) = 0$
2. $\dot{x}(t) = \sqrt{x(t)} - 2t$
3. $\dot{x}(t) = 2x(t) + t + 1$
4. $\dot{x}(t) + 2x(t) = x^2(t)e^t$
5. $t + \sin x(t) + \Big(x^2(t) + t\cos x(t)\Big)\dot{x}(t) = 0$
6. $t + \dfrac{2t}{x^3(t)} + \dfrac{x^2(t) - 3t^2}{x^4(t)} \cdot \dot{x}(t) = 0$

Homework:
$t^2 - x(t) - t\dot{x}(t) = 0$
$\dot{x}(t) = \dfrac{2tx(t)}{t^2 + x^2(t)}$
$t\dot{x}(t) - 2t^2\sqrt{x(t)} = 4x(t)$

Homework: An ant starts from the right endpoint of a 10 cm long rubber band towards its left (fixed) endpoint at a speed of 1 cm/s. At the same time, an evil gnome stretches the right endpoint of the band at 100 cm/s. Does the ant ever reach the other end of the band (and if so, when)?

4 Practice 4. Second-Order Linear Equations, Higher-Order Equations

Topics: Second-order linear equations, higher-order equations.

$$\ddot{x}(t) + p(t)\dot{x}(t) + q(t)x(t) = f(t) \tag{2}$$

where $p, q, f : I \to \mathbb{R}$ are given continuous functions on the interval I. The equation is called homogeneous if $f \equiv 0$; otherwise it is called inhomogeneous.

Theorem 4.1. *Every solution of the above differential equation can be written as $x(t) = x_0(t) + c_1x_1(t) + c_2x_2(t)$, where x_0 is a particular solution of the inhomogeneous equation, x_1 and x_2 are linearly independent solutions of the homogeneous equation, and $c_1, c_2 \in \mathbb{R}$ are arbitrary constants.*

Finding the solutions thus consists of two steps: first, determining two linearly independent solutions of the homogeneous equation (the fundamental system), and second, finding a particular solution of the inhomogeneous equation (i.e. $x_0(t)$).

Homogeneous equation: There is no general method for finding solutions; we discuss two special cases.

(a) If the equation has constant coefficients, i.e. $\ddot{x}(t) + p\dot{x}(t) + qx(t) = 0$, where $p, q \in \mathbb{R}$, then we seek solutions of the form $x(t) = e^{\lambda t}$. Substituting for λ, we get the **characteristic equation**:

$$\lambda^2 + p\lambda + q = 0 \tag{3}$$

If its roots are real and distinct (λ_1, λ_2), then the two linearly independent solutions are: $x_1(t) = e^{\lambda_1 t}$, $x_2(t) = e^{\lambda_2 t}$. If the equation has a double real root (λ), then the two linearly independent solutions are: $x_1(t) = e^{\lambda t}$, $x_2(t) = te^{\lambda t}$. If the roots are complex, i.e. $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$, then the two linearly independent solutions are: $x_1(t) = e^{\alpha t}\cos\beta t$, $x_2(t) = e^{\alpha t}\sin\beta t$.

The procedure extends analogously to higher-order equations; then (3) also becomes higher degree.

(b) If we know one solution $x_1(t)$, then we can seek another in the form $x_2(t) = x_1(t)z(t)$, which leads to a first-order equation for $\dot{z}(t)$. There is no general method for finding $x_1(t)$; it is often worth trying special forms (e.g. polynomials or power series).

Inhomogeneous equation:

(a) If equation (2) has constant coefficients and the inhomogeneous term is of special form, one can relatively easily determine a particular solution $x_0(t)$:

Theorem 4.2. *Let p and q in* (2) *be constants, and let*

$$f(t) = e^{\alpha t}(P_1(t)\cos\beta t + P_2(t)\sin\beta t),$$

where $\alpha, \beta \in \mathbb{R}$ and P_1, P_2 are polynomials. Let k denote the multiplicity of $\alpha + \beta i$ in the characteristic equation (3) *(it can be* 0*). Then a particular solution of* (2) *is of the form*

$$x_0(t) = t^k e^{\alpha t}(Q_1(t)\cos\beta t + Q_2(t)\sin\beta t),$$

where Q_1 and Q_2 are polynomials whose degree equals the maximum of the degrees of P_1 and P_2.

(b) The particular solution can be obtained from the fundamental system of the homogeneous equation by the method of **variation of parameters**. This method — unlike the previous one — works in every case, but requires much more computation.

We seek the particular solution of the inhomogeneous equation (2) in the form $x_0(t) = c_1(t)x_1(t) + c_2(t)x_2(t)$, where x_1, x_2 are two linearly independent solutions of the homogeneous equation. Let

$$W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ \dot{x}_1(t) & \dot{x}_2(t) \end{vmatrix},$$

the so-called Wronskian determinant, which is never zero due to the linear independence of the fundamental solutions. Then the sought functions c_1, c_2 can be computed by the following formulas:

$$c_1(t) = \int \frac{\begin{vmatrix} 0 & x_2(t) \\ f(t) & \dot{x}_2(t) \end{vmatrix}}{W(t)}, \qquad c_2(t) = \int \frac{\begin{vmatrix} x_1(t) & 0 \\ \dot{x}_1(t) & f(t) \end{vmatrix}}{W(t)}.$$

1. Solve the following second-order, constant-coefficient homogeneous equations!

   (a) $\ddot{x}(t) - \dot{x}(t) - 6x(t) = 0$

   (b) $\ddot{x}(t) - 8\dot{x}(t) + 16x(t) = 0$

   (c) $4\ddot{x}(t) + 4\dot{x}(t) + 37x(t) = 0$

   [Fundamental systems: (a): e^{3t}, e^{-2t}; (b): e^{4t}, te^{4t}; (c): $e^{-(1/2)t}\cos 3t$, $e^{-(1/2)t}\sin 3t$.]

2. Solve the following second-order, constant-coefficient inhomogeneous equations, using the method of undetermined coefficients or variation of parameters.

   (a) $\ddot{x}(t) + 5\dot{x}(t) + 4x(t) = 3 - 2t - t^2$

   (b) $\ddot{x}(t) + 2\dot{x}(t) - 3x(t) = t^2 e^t$

   Homework: $\ddot{x}(t) + 4x(t) = \cos 2t$

   [Solutions: (a): $C_1 e^{-4t} + C_2 e^{-t} - t^2/4 + t/8 + 23/32$; (b): $C_1 e^t + C_2 e^{-3t} + (t^3/12 - t^2/16 + t/32)e^t$; HF: $C_1 \cos 2t + C_2 \sin 2t + (t/4)\sin 2t$.]

3. For the following second-order, variable-coefficient homogeneous equations, guess one solution $x_1(t)$ and find the other solution in the form $x_2(t) = x_1(t)z(t)$!

   (a) $(t^2 + 1)\ddot{x}(t) - 2t\dot{x}(t) + 2x(t) = 0$

   (b) $(2t + 1)\ddot{x}(t) + 4t\dot{x}(t) - 4x(t) = 0$

   Homework: $t\ddot{x}(t) - (2t + 1)\dot{x}(t) + (t + 1)x(t) = 0$

   [Fundamental systems: (a): t, $t^2 - 1$; (b): t, e^{-2t}; HF: e^t, $t^2 e^t$.]

4. Solve the following higher-order, constant-coefficient homogeneous equations!

   (a) $\dddot{x}(t) - 2\ddot{x}(t) - 3\dot{x}(t) = 0$

   Homework: $x^{(5)}(t) - 2x^{(4)}(t) + 2\ddot{x}(t) - 2\dot{x}(t) + x(t) = 0$

   [Fundamental systems: (a): 1, e^{3t}, e^{-t}; HF: e^t, te^t, $\cos t$, $\sin t$.]

5 Practice 5. Boundary Value Problems, Eigenvalue Problems

Topics: Boundary value problems, eigenvalue problems.

Let $a, b, p, q \in \mathbb{R}$, $a < b$, and let $\eta_1, \eta_2 \in \mathbb{R}$ be given, and $f \in C[a,b]$. The problem

$$\begin{cases} \ddot{x}(t) + p\dot{x}(t) + qx(t) = f(t) \\ H_1 x = \eta_1, \quad H_2 x = \eta_2 \end{cases} \tag{4}$$

is called a **boundary value problem** (BVP), where $H_1, H_2 : C^1[a,b] \to \mathbb{R}$ are linear maps that fall into the following three types:

(a) $H_1 x := x(a) = \eta_1$, $H_2 x := x(b) = \eta_2$ (Dirichlet boundary conditions)

(b) $H_1 x := \dot{x}(a) = \eta_1$, $H_2 x := \dot{x}(b) = \eta_2$ (Neumann boundary conditions)

(c) $H_1 x := \alpha_1 x(a) + \beta_1 \dot{x}(a) = \eta_1$, $H_2 x := \alpha_2 x(b) + \beta_2 \dot{x}(b) = \eta_2$, where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ and $\alpha_i^2 + \beta_i^2 \neq 0$ $(i = 1, 2)$ (mixed boundary conditions)

(The mixed boundary condition case includes the other two as special cases. However, in computations we mainly deal with the first two types here; for the general case see the lecture.)

From the theory of second-order equations, every solution of the differential equation (4) can be written as $x(t) = x_0(t) + c_1 x_1(t) + c_2 x_2(t)$, where x_0 is a particular solution of the inhomogeneous equation, x_1 and x_2 are solutions of the homogeneous equation, and $c_1, c_2 \in \mathbb{R}$ are arbitrary constants. Once x_0, x_1, x_2 have been found, solving the BVP means determining the constants c_1, c_2 so that x also satisfies the boundary conditions. Substituting x into the boundary conditions, we obtain the following:

Theorem 5.1. *The BVP has exactly one solution for every $f \in C[a,b]$ and every $\eta_1, \eta_2 \in \mathbb{R}$ if and only if there exist linearly independent solutions x_1, x_2 of the homogeneous equation for which*

$$\det A := \begin{vmatrix} H_1 x_1 & H_1 x_2 \\ H_2 x_1 & H_2 x_2 \end{vmatrix} \neq 0.$$

If this determinant is zero, then the BVP either has infinitely many solutions or no solution.

Since $\det A = 0$ means that A maps some nonzero vector to the zero vector, the following is a consequence:

Corollary 5.1. *The BVP has exactly one solution for every $f \in C[a,b]$ and every $\eta_1, \eta_2 \in \mathbb{R}$ if and only if the homogeneous BVP (where $f \equiv 0$ and $\eta_1 = \eta_2 = 0$) has only the identically zero solution.*

The investigation of the eigenvalue problem helps decide this.

Definition 5.1. The number λ is called an **eigenvalue** of the BVP *(4)* if the problem

$$\begin{cases} \ddot{x} + p\dot{x} = \lambda x \\ H_1 x = H_2 x = 0 \end{cases} \tag{5}$$

has a non-identically-zero solution.

Since $-q$ is an eigenvalue if and only if $\det A = 0$, the following holds:

Theorem 5.2. *The BVP* (4) *has exactly one solution for every* $f \in C[a,b]$ *and every* $\eta_1, \eta_2 \in \mathbb{R}$ *if* $-q$ *is not an eigenvalue of the eigenvalue problem* (5) *associated with the BVP.*

1. Are there nontrivial (i.e. other than the identically zero function) solutions to the following boundary value problems? What does the result mean in terms of eigenvalues?

   (a) $\begin{cases} \ddot{x} + x = 0 \\ x(0) = 0, \ x(2\pi) = 0 \end{cases}$

   (b) $\begin{cases} \ddot{x} - x = 0 \\ x(0) = 0, \ x(2\pi) = 0 \end{cases}$

   [(a): $x(t) = C \sin t$ is a solution for every $C \in \mathbb{R}$ (i.e. infinitely many solutions exist); (b): only the trivial solution. This means that $\lambda = -1$ is an eigenvalue of the corresponding eigenvalue problem, but $\lambda = 1$ is not.]

2. Which of the following homogeneous boundary value problems has a unique solution, which has no solution? For those that have a solution, compute it.

   (a) $\begin{cases} \ddot{x} + x = 0 \\ x(0) = 0, \ x(\pi/2) = 2 \end{cases}$

   (b) $\begin{cases} \ddot{x} + x = 0 \\ x(0) = 0, \ x(\pi) = 1 \end{cases}$

   Homework: $\begin{cases} \ddot{x} - 2\dot{x} + 2x = 0 \\ x(0) = 0, \ \dot{x}(\pi) = e^{\pi} \end{cases}$

   [(a): $x(t) = 2 \sin t$; (b): no solution; HF: $x(t) = -e^t \sin t$.]

3. Which of the following inhomogeneous boundary value problems has a unique solution, which has no solution? For those that have a solution, compute it.

   (a) $\begin{cases} \ddot{x} + x = 1 \\ x(0) = x(\pi/2) = 0 \end{cases}$

   (b) (HF) $\begin{cases} \ddot{x} + x = 1 \\ x(0) = x(\pi) = 0 \end{cases}$

   (c) $\begin{cases} \ddot{x}(t) + x(t) = 2t - \pi \\ x(0) = x(\pi) = 0 \end{cases}$

   [(a): $x(t) = -\cos t - \sin t + 1$; HF: no solution; (c): $x(t) = \pi \cos t + C \sin t + 2t - \pi$ for every $C \in \mathbb{R}$ (i.e. infinitely many solutions).]

4. Find the eigenvalues of the following eigenvalue problem.

   $$\begin{cases} \ddot{x} = \lambda x \\ x(0) = x(2\pi) = 0 \end{cases}$$

   [Eigenvalues: $\lambda = -k^2/4$, $k \in \mathbb{N}^+$.]

Homework: When does the following boundary value problem have a unique solution, for $b = \pi/4$ or $b = \pi/2$? If a solution exists, compute it.

$$\begin{cases} \ddot{x}(t) + 4x(t) = e^t \\ x(0) = 1, \quad x(b) = 2 \end{cases}$$

[By the corollary, for $b = \pi/2$ the solution is certainly not unique (in fact, there is no solution), while for $b = \pi/4$: $x(t) = \frac{4}{5}\cos 2t + (2 - e^{\pi/4}/5)\sin 2t + \frac{1}{5}e^t$.]

6 Practice 6

1st Midterm Exam.

7 Practice 7. Linear ODE Systems, Matrix Exponential

Topics: Linear differential equation systems, computing the solution via e^{At}.

In the following, we deal with the constant-coefficient linear system

$$\dot{x}(t) = Ax(t), \tag{6}$$

where $A \in \mathbb{R}^{n\times n}$ is a matrix. We will need the concept of the matrix exponential function:

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$$

1. Compute the matrix exponential $t \mapsto e^{At}$ for the following matrices A, using the definition!

   (a) $\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$

The fundamental system of equation (6) can be obtained using the matrix exponential, analogously to the one-dimensional case:

$$x(t) = e^{At}C, \quad C \in \mathbb{R}^n.$$

We mention that computing the exponential function by definition is rarely done. Two methods are well known. One uses the Jordan normal form of the matrix (see lecture); the other uses Hermite interpolation polynomials. We present the latter.

Theorem 7.1 (Computing e^{At} via the Hermite interpolation polynomial)**.** *Let m denote the degree of the minimal polynomial of A, let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of A, and let $m_1, \ldots, m_k$ be their multiplicities in the minimal polynomial. There exists a polynomial p of degree at most $m-1$ (the Hermite interpolation polynomial) such that $e^{At} = p(A)$. This polynomial is determined by the following equations:*

$$p^{(i)}(\lambda_j) = t^i e^{\lambda_j t} \quad (j = 1, \ldots, k;\ i = 0, 1, \ldots, m_j - 1)$$

Solve the following linear differential equation systems!

2. $\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = -x_2 \end{cases}$ $[x_1(t) = C_1 e^t,\ x_2(t) = C_2 e^{-t}]$

3. $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \end{cases}$ $[x_1(t) = C_1 \cos t + C_2 \sin t,\ x_2(t) = -C_1 \sin t + C_2 \cos t]$

4. $\begin{cases} \dot{x}_1 = x_1 - x_2 \\ \dot{x}_2 = -4x_1 + x_2 \end{cases}$ $[x_1(t) = C_1 e^{3t} + C_2 e^{-t},\ x_2(t) = -2C_1 e^{3t} + 2C_2 e^{-t}]$

5. $\begin{cases} \dot{x}_1 = x_1 + x_2 \\ \dot{x}_2 = 3x_2 - 2x_1 \end{cases}$ $[x_1(t) = e^{2t}(C_1 \cos t + (C_2 - C_1)\sin t),\ x_2(t) = e^{2t}((-2C_1 + C_2)\sin t + C_2 \cos t)]$

6. $\begin{cases} \dot{x}_1 = 2x_1 + x_2 \\ \dot{x}_2 = -x_1 + 4x_2 \end{cases}$ $[x_1(t) = e^{3t}(C_1(1-t) + C_2 t),\ x_2(t) = e^{3t}(-C_1 t + C_2(1+t))]$

7. $\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = x_1 + x_2 \end{cases}$ $[x_1(t) = C_1 e^t,\ x_2(t) = (C_1 t + C_2)e^t]$

Homework:

$\begin{cases} \dot{x}_1 = x_1 + x_2 \\ \dot{x}_2 = x_1 + x_2 \end{cases}$ $[x_1(t) = C_1 e^{2t} + C_2,\ x_2(t) = C_1 e^{2t} - C_2]$

$\begin{cases} \dot{x}_1 = -x_1 + 8x_2 \\ \dot{x}_2 = x_1 + x_2 \end{cases}$ $[x_1(t) = 2C_1 e^{3t} - 4C_2 e^{-3t},\ x_2(t) = C_1 e^{3t} + C_2 e^{-3t}]$

8 Practice 8. Phase Portraits of Linear Systems, Geometric Classification

Topics: Linear differential equation systems, phase portraits, geometric classification.

Two-dimensional systems are conveniently represented by plotting their solutions in the phase plane (the (x_1, x_2) plane), which requires one fewer dimension. As motivation, let us compute the solutions and then draw the phase portrait (also called trajectory portrait) of the following three systems!

1. Solve the following linear systems and plot the solutions in the phase plane!

   (a) $\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = x_2 \end{cases}$ (b) $\begin{cases} \dot{x}_1 = 2x_1 \\ \dot{x}_2 = 2x_2 \end{cases}$ (c) $\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = -x_2 \end{cases}$

Goal: Determine the possible phase portraits of linear systems up to some notion of equivalence. We accept (without going into precise details; see the lecture) that if two systems can be transformed into each other via a linear coordinate transformation, then we consider their phase portraits equivalent. We decide when two phase portraits within the same equivalence class are distinguished along the way, as it will become apparent.

Starting from the 2D linear system $\dot{x} = Ax$ ($A \in \mathbb{R}^{2\times 2}$), let $P \in \mathbb{R}^{2\times 2}$ be an invertible matrix, and introduce the new variable $y = Px$. Then $\dot{y} = PAP^{-1}y$. Hence it suffices to determine the phase portraits for the following three Jordan normal forms (see lecture):

$$\text{(I)} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \qquad \text{(II)} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad \text{(III)} \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

We first draw the phase portraits in those cases where the origin is the only equilibrium point (i.e. the determinant of the matrix is nonzero).

Case I. The solution is $x_1(t) = C_1 e^{\lambda t}$, $x_2(t) = C_2 e^{\mu t}$. The trajectory equation is:

$$x_2 = C_2 \left(\frac{x_1}{C_1}\right)^{\mu/\lambda} = C x_1^{\mu/\lambda}.$$

For $\lambda \neq 0 \neq \mu$, we classify phase portraits into 3 classes:

1. $\lambda, \mu > 0$: Unstable node. Sketch trajectories for (a) $0 < \mu < \lambda$; (b) $0 < \lambda < \mu$; (c) $0 < \mu = \lambda$!
2. $\lambda, \mu < 0$: Stable node. Sketch trajectories for (a) $\mu < \lambda < 0$; (b) $\lambda < \mu < 0$; (c) $\mu = \lambda < 0$!
3. $\lambda < 0 < \mu$: Saddle. Sketch the trajectories!

Case II. The solution is $x_1(t) = (C_1 + C_2 t)e^{\lambda t}$, $x_2(t) = C_2 e^{\lambda t}$. The trajectory equation is:

$$x_1 = x_2 \left(\frac{C_1}{C_2} + \frac{1}{\lambda} \ln \frac{x_2}{C_2}\right).$$

For $\lambda \neq 0$, we classify phase portraits into 2 classes:

1. $\lambda > 0$: Unstable degenerate node. Sketch the trajectories!
2. $\lambda < 0$: Stable degenerate node. Sketch the trajectories!

Case III. Introducing polar coordinates: $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$. The new variables satisfy:

$$\begin{cases} \dot{r} = \alpha r \\ \dot{\varphi} = \beta \end{cases}$$

whose solution is $r(t) = e^{\alpha t} r_0$, $\varphi(t) = \beta t + \varphi_0$. For $\beta \neq 0$, we classify phase portraits into 3 classes:

1. $\alpha > 0$: Unstable focus. Sketch the trajectories!
2. $\alpha < 0$: Stable focus. Sketch the trajectories!
3. $\alpha = 0$: Center. Sketch the trajectories!

There are four degenerate phase portraits where the equilibrium point is not only the origin (i.e. the determinant of the matrix is zero): Case I with $\lambda = 0$ and $\mu > 0$; Case I with $\lambda = 0$ and $\mu < 0$; Case I with $\lambda = 0$ and $\mu = 0$; Case II with $\lambda = 0$. If time permits, sketch these phase portraits as well.

Returning to the general case (before transforming to Jordan normal form), the phase portrait corresponding to the appropriate Jordan form can be recovered by an inverse transformation. It is easy to verify that in Cases I and II the axes of the nodes, and the axes of the saddle, coincide with the eigenvectors of the matrix A.

It is worth noting that the type of the phase portrait can be determined without computing the eigenvalues, using the determinant (D) and trace (Tr) of the matrix, as shown in the diagram below. (The equation of the parabola is $\mathrm{Tr}^2 = 4D$.)

[Phase portrait classification diagram: D vs Tr]
Stable focus | Unstable focus
Center
Stable node | Unstable node
Saddle | Saddle

2. Solve the following linear systems and plot the trajectories in the phase plane!

   (a) $\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 \end{cases}$ [center]

   For the following system, switch to polar coordinates. The case $\omega = 0$ gives the previous problem; what happens if $\omega > 0$ or $\omega < 0$?

   (b) $\begin{cases} \dot{x}_1 = \omega x_1 - x_2 \\ \dot{x}_2 = x_1 + \omega x_2 \end{cases}$ [$\omega > 0$: unstable focus; $\omega < 0$: stable focus]

   Determine the type of the phase portrait and sketch the phase portraits!

3. $\begin{cases} \dot{x}_1 = x_1 - x_2 \\ \dot{x}_2 = -4x_1 + x_2 \end{cases}$ [saddle]

4. $\begin{cases} \dot{x}_1 = 2x_1 + x_2 \\ \dot{x}_2 = -x_1 + 4x_2 \end{cases}$ [unstable degenerate node]

Homework:

$\begin{cases} \dot{x}_1 = 8x_1 + x_2 \\ \dot{x}_2 = -2x_1 + 5x_2 \end{cases}$ [unstable focus]

$\begin{cases} \dot{x}_1 = 3x_1 + 2x_2 \\ \dot{x}_2 = 2x_1 \end{cases}$ [saddle]

$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = x_1 - 2x_2 \end{cases}$ [stable node]

9 Practice 9. Autonomous and Nonlinear Systems

Topics: Autonomous equations/systems, nonlinear differential equation systems, the phase portrait near an equilibrium point is locally the same as that of the linearised system (if the eigenvalues of $f'(p)$ satisfy $\mathrm{Re}\,\lambda \neq 0$), sketching the full phase portrait.

Consider first an n-dimensional autonomous system:

$$\dot{x}(t) = f(x(t)) \qquad (f : \mathbb{R}^n \to \mathbb{R}^n) \tag{7}$$

In general, it cannot be solved by formulas, so most information about the solutions is provided by the phase portrait. The constant solutions $x(t) \equiv p$ are obtained by solving the algebraic system $f(p) = 0$. These points p are called **equilibrium points** or **stationary points**. The behaviour of trajectories near equilibrium points can be determined by linearisation. Heuristically, this means the following. For the new function $y(t) = x(t) - p$:

$$\dot{y}(t) = \dot{x}(t) = f(x(t)) = f(p + y(t)) = f(p) + f'(p)y(t) + r(y(t)) = f'(p)y(t) + r(y(t)),$$

where r denotes the remainder term. Since for small y this is of smaller order than the linear term (provided the linear term is not zero, i.e. nonzero), one expects that near the equilibrium point p the phase portrait is determined by the

$$\dot{y}(t) = f'(p)y(t) \tag{8}$$

so-called linearised equation. Two things need to be made precise here: first, what "not too small linear term" means, and second, in what sense the linearised equation determines the phase portrait (see lecture).

The following theorem applies only to two-dimensional systems, so from now on let $n = 2$ in (7). We first need to define what it means for an equilibrium point to be a focus, node, or saddle.

Definition 9.1. Write the solutions in a neighbourhood U of the equilibrium point p in polar coordinates. The point p is called

- a *stable focus* if $\lim_{t\to+\infty} r = 0$, $\lim_{t\to+\infty} |\varphi| = \infty$;
- an *unstable focus* if $\lim_{t\to-\infty} r = 0$, $\lim_{t\to-\infty} |\varphi| = \infty$;
- a *stable node* if $\lim_{t\to+\infty} r = 0$, $\lim_{t\to+\infty} |\varphi| < \infty$;
- an *unstable node* if $\lim_{t\to-\infty} r = 0$, $\lim_{t\to-\infty} |\varphi| < \infty$;
- a *saddle* if there exist 2 trajectories in U approaching p as $t \to +\infty$, 2 trajectories in U approaching p as $t \to -\infty$, and all other trajectories starting in U leave U as $t \to +\infty$ and $t \to -\infty$.

Theorem 9.1. *Let $n = 2$. If $f \in C^2$ and $\operatorname{Re}\lambda \neq 0$ for every eigenvalue λ of the matrix $f'(p)$, then the equilibrium point p of system* (7) *is of the same type as the origin in the system* (8)*.*

In the following two-dimensional systems, find the equilibrium points and determine their types. Try to sketch the full phase portrait. To this end, use the direction field, i.e. at each point the vector $f(x)$ is the tangent vector of the trajectory. For drawing, it often suffices to know whether the trajectory at various points goes up or down, or left or right. The **nullclines** — i.e. the curves $\dot{x} = 0$ and $\dot{y} = 0$ — help with this: they divide the phase plane into regions within which the direction vector cannot change from up-left to down-right; it suffices to draw the direction vector at a single point of each region from the left-right and up-down point of view.

1. $\begin{cases} \dot{x} = x - xy \\ \dot{y} = x^2 - y \end{cases}$ [equilibrium points: $(0,0)$, saddle; $(1,1)$, stable focus; $(-1,-1)$, stable focus]

2. $\begin{cases} \dot{x} = x(y-1) \\ \dot{y} = y(x-1) \end{cases}$ [equilibrium points: $(0,0)$, stable node; $(1,1)$, saddle]

3. $\begin{cases} \dot{x} = y^2 - 4x^2 \\ \dot{y} = 8 - 4y \end{cases}$ [equilibrium points: $(1,2)$, stable node; $(-1,2)$, saddle]

Homework:

$\begin{cases} \dot{x} = x - x^2 - xy \\ \dot{y} = 3y - xy - 2y^2 \end{cases}$ [equilibrium points: $(0,0)$, unstable node; $(0,3/2)$, stable node; $(1,0)$, saddle; $(-1,2)$, saddle]

$\begin{cases} \dot{x} = x + y^2 \\ \dot{y} = x + y \end{cases}$ [equilibrium points: $(0,0)$, unstable degenerate node; $(-1,1)$, saddle]

10 Practice 10. Lyapunov Functions, Lie Derivatives

Topics: Lyapunov functions, derivative along the system (Lie derivative).

Consider the system $\dot{x} = -y - x^3$, $\dot{y} = x - y^3$. At the equilibrium point (origin), the linearisation does not reveal the local phase portrait, since the eigenvalues of the derivative matrix are $\pm i$. From the direction field, one can see that the trajectories spiral around the origin, but it is not clear whether they approach or move away from it. Take the function $V(x,y) = x^2 + y^2$ (Lyapunov function), and investigate whether V decreases or increases along trajectories. Let $(x(t), y(t))$ be any trajectory, and let $V^*(t) = V(x(t), y(t))$. Its derivative gives $V^{*\prime}(t) = -2(x^4(t) + y^4(t)) < 0$, so the trajectories approach the origin. Thus we obtained the phase portrait not only in a neighbourhood of the origin but globally, on the entire phase plane. (The asymptotic stability of the origin follows from Lyapunov's stability theorem; see lecture.)

Definition 10.1. Let $M \subset \mathbb{R}^n$ be a domain, $f : M \to \mathbb{R}^n$ a differentiable function, and consider the autonomous equation (system)

$$\dot{x}(t) = f(x(t)). \tag{9}$$

Let $U \subset M$ be an open set and $V : U \to \mathbb{R}$ a continuously differentiable function. Then the **derivative of V along the system *(9)*** (or the **derivative of V along the vector field f**, or the **Lie derivative of V**) is the function:

$$L_f V := \langle \operatorname{grad} V, f \rangle, \quad \text{i.e.} \quad (L_f V)(p) = \partial_1 V(p) f_1(p) + \cdots + \partial_n V(p) f_n(p), \quad (p \in \mathbb{R}^n).$$

If $x(t)$ is a solution of equation (9), then for $V^*(t) = V(x(t))$ we have $V^{*\prime}(t) = (L_f V)(x(t))$. If $V^{*\prime} \equiv 0$, then V is called a **first integral** of the system. This means that the value of V is constant along solutions.

1. Compute the Lie derivative of the following functions for the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - y \end{cases}$$

(a) $V(x,y) = x^2 + y^2$;

(b) $V(x,y) = \frac{3}{2}x^2 + xy + y^2$.

2. Consider the harmonic oscillator equation $m\ddot{x} = -kx$, where m is the mass of a body attached to a spring and $k > 0$ is the spring constant. Written as a two-dimensional system:

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\frac{k}{m}x \end{cases}$$

Show that $E(x,v) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$ (the total energy) is a first integral of the system (i.e. the energy is constant along solutions). Sketch the phase portrait using this.

(a) Damped oscillation equation: $ma = -kx + s(v)$, where $s(v)$ is the velocity-dependent drag; we assume $vs(v) < 0$, i.e. the drag force opposes the direction of the velocity. Written as a system: $\dot{x} = v$, $\dot{v} = -\frac{k}{m}x + \frac{s(v)}{m}v$. Show that in this case the total energy $E(x,v) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$ decreases over time, i.e. the origin is asymptotically stable (the oscillation gradually stops). Sketch the phase portrait!

(b) Lotka–Volterra system: $\dot{x} = x - xy$, $\dot{y} = xy - y$, with $x, y \geq 0$. The first integral can in principle be found by the method used for first-order partial differential equations, which in this case amounts essentially to dividing the two equations by each other: $dy/dx = y(x-1)/[x(1-y)]$; separating variables and integrating, we get the first integral: $V(x,y) = x - \ln x + y - \ln y$. Sketch the phase portrait in the non-negative quadrant using this!

For definitions of stability and the statements of the Lyapunov stability and instability theorems for equation (7), see the lecture notes. In practice, demonstrating these in the two-dimensional case is quite time-consuming, because stating the theorems takes a lot of time.

3. Using a Lyapunov function of the form $V(x,y) = ax^2 + by^2$, show that:

(a) the equilibrium point $(0,0)$ of the system $\begin{cases} \dot{x} = -x + 5y^2 \\ \dot{y} = -3xy \end{cases}$ is stable. $[V(x,y) = 3x^2 + 5y^2]$

(b) the equilibrium point $(0,0)$ of the system $\begin{cases} \dot{x} = -x^3 - 2xy \\ \dot{y} = 2x^2 - 6y \end{cases}$ is asymptotically stable. $[V(x,y) = x^2 + y^2]$

4. Sketch the global phase portrait of the system

$$\begin{cases} \dot{x} = -xy^4 \\ \dot{y} = x^6 y \end{cases}$$

using a suitably chosen Lyapunov function of the form $V(x,y) = ax^\alpha + by^\beta$. $[V(x,y) = x^6/6 + y^4/4]$

Homework: Sketch the global phase portrait of the system

$$\begin{cases} \dot{x} = xy^2 - x^3 \\ \dot{y} = -y^3 - 2x^2 y \end{cases}$$

using a suitably chosen Lyapunov function of the form $V(x,y) = ax^2 + by^2$. $[V(x,y) = 2x^2 + y^2]$

Homework: Determine the stability of the origin in the following systems, using a suitably chosen Lyapunov function.

(a) $\begin{cases} \dot{x} = xy + x^3 \\ \dot{y} = -y + y^2 - x^3 + x^4 \end{cases}$

(b) $\begin{cases} \dot{x} = 2y^5 - x^3 \\ \dot{y} = -2xy^2 \end{cases}$

[(a): $V(x,y) = 2y^2 - x^4$; (b): $V(x,y) = 2x^2 + y^4$.]

11 Practice 11. Approximate Solutions

Topics: Approximate solutions: successive approximation, power series method, Taylor series method, Euler polygonal method.

Let $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}$, $f : \mathbb{R}^2 \to \mathbb{R}$ a continuous function defined in a neighbourhood of the point (t_0, x_0). We seek approximate solutions of the equation

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = p. \tag{10}$$

We present a few simple methods here.

Successive approximation (Picard iteration). The proof of the Picard–Lindelöf theorem is based on the Banach fixed-point theorem, so a sequence of terms converging to the fixed point (hence to the solution) yields a sequence of approximations converging to the (local) solution. Rewriting equation (10) as the equivalent integral equation, we get the following recurrence:

$$x_0(t) \equiv p, \qquad x_{n+1}(t) = p + \int_{t_0}^{t} f(s, x_n(s))\,\mathrm{d}s.$$

Power series method. If the function f is analytic in a neighbourhood of the point (t_0, p), then we seek the solution of equation (10) — defined locally in a neighbourhood of t_0 — in the form of a power series. We substitute

$$x(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n$$

into the equation. Using the initial condition and matching coefficients, the coefficients can be successively determined.

Taylor series method. If the function f is analytic in a neighbourhood of the point (t_0, p), then we seek the solution of equation (10) — defined locally — in Taylor series form:

$$x(t) = \sum_{n=0}^{\infty} \frac{x^{(n)}(t_0)}{n!} (t - t_0)^n.$$

The values of the derivative functions $x, \dot{x}, \ddot{x}, \ldots$ at the point t_0 are needed. Here $x(t_0)$ comes from the initial condition, $\dot{x}(t_0)$ from the equation itself, and the higher derivatives can be computed by (implicit) differentiation of equation (10).

Euler polygonal method. Choose a step size h, then generate the time points $t_0, t_1, t_2, t_3, \ldots$ and the approximate values $x_0, x_1, x_2, x_3, \ldots$ using the rule $t_k = t_0 + kh$ and $x_k \approx x(t_k)$. Starting from t_0 and $x_0 = p$, let $x_{n+1} = x_n + hf(t_n, x_n)$.

1. Solve the differential equation $\dot{x}(t) = x(t)$ by series expansion with the initial condition $x(0) = 1$.

   $[x(t) = e^t]$

2. Compute the approximate solution of the IVP

$$\begin{cases} \dot{x}(t) = t + x(t) \\ x(0) = 0 \end{cases}$$

   using successive approximation, and using the power series method.

   $[x(t) = e^t - t - 1]$

3. Consider the initial value problem

$$\begin{cases} \dot{x}(t) = \dfrac{2x(t)}{t} \\ x(1) = 1 \end{cases}$$

   (a) Determine the derivatives of $x(t)$ and write the Taylor polynomial of the solution to as high a degree as possible. What approximate values do the polynomials $T_1(t)$, $T_2(t)$, $T_3(t), \ldots$ give at $t = 2$? $[x(t) = t^2$, $T_1(2) = 3$, $T_2(2) = 4$, $T_3(2) = 4]$

   (b) Compute the approximate value of the solution at $t = 2$ using the Euler method with step sizes $h = 1$, $h = 1/2$, and $h = 1/3$. $[h = 1$: $x(2) \approx x_1 = 3$; $h = 1/2$: $x(2) \approx x_2 = 3.\overline{3}$; $h = 1/3$: $x(2) \approx x_3 = 3.5]$

4. Determine the first four terms of the Taylor series of the solution of the differential equation $\dot{x}(t) = x^2(t) - t$ satisfying the initial condition $x(0) = 1$. $[T_3(t) = 1 + t + (1/2)t^2 + (2/3)t^3]$

Homework: Find the solutions of the differential equation $(1-t^2)\ddot{x}(t) - 4t\dot{x}(t) - 2x(t) = 0$ satisfying the initial conditions $x(0) = 1$, $\dot{x}(0) = 0$ and $x(0) = 0$, $\dot{x}(0) = 1$, respectively.
$\left[x(t) = \dfrac{1}{1+t^2}, \text{ and } x(t) = \dfrac{t}{1+t^2}\right]$

12 Practice 12

2nd Midterm Exam.