Lesson 3. Second-order Linear Ordinary Differential Equations

ELTE Eötvös Loránd University
Faculty of Informatics, Department of Numerical Analysis

2025

Contents

1 General Theory. Homogeneous Constant Coefficient Equations

1.1 Second-order Linear ODEs

Definition 1.1 (Second-order Linear ODE). A **second-order linear ordinary differential equation** for the function y is

$$y'' + a_1(x)y' + a_0(x)y = b(x),$$

where a_1, a_0, b are given functions on the interval $I \subset \mathbb{R}$. The equation:

- is **homogeneous** if the source $b(x) = 0$ for all $x \in \mathbb{R}$;
- has **constant coefficients** if a_1 and a_0 are constants;
- has **variable coefficients** if either a_1 or a_0 is not constant.

Definition 1.2 (Initial-Value Problem). An **initial-value problem** for the second-order differential equation consists of finding a solution of the differential equation that also satisfies initial conditions of the form

$$y(x_0) = y_0, \quad y'(x_0) = y_1,$$

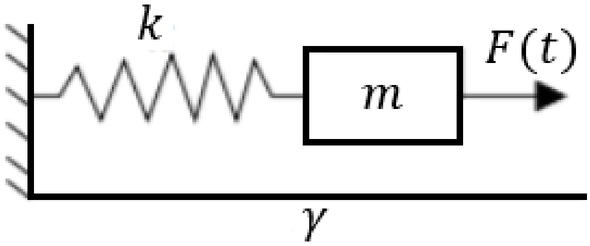
where y_0 and y_1 are given constants.

Remark 1.1. The form of the initial conditions involves the specification of both $y(x)$ and its derivative $y'(x)$ at an initial point x_0.

1.2 Classic Examples: Mechanical Vibrations

Spring-mass system. With mass m, position $y(t)$, spring constant k, viscous damping γ, and external force $F(t)$:

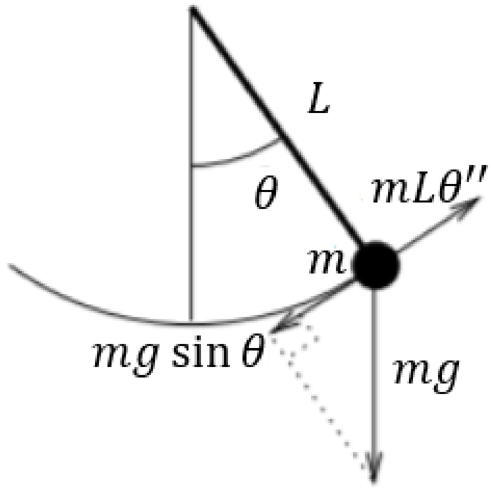
- Unforced, undamped oscillator: $my'' + ky = 0$
- Unforced, damped oscillator: $my'' + \gamma y' + ky = 0$
- Forced, undamped oscillator: $my'' + ky = F(t)$
- Forced, damped oscillator: $my'' + \gamma y' + ky = F(t)$



Pendulum motion. With mass m, drag c, length L, $\gamma = \frac{c}{mL}$, $\omega^2 = \frac{g}{L}$, angle $\theta(t)$:

- Nonlinear pendulum: $L\ddot{\theta} + g\sin\theta = 0$

- Damped nonlinear pendulum: $L\ddot{\theta} + b\dot{\theta} + g\sin\theta = 0$
- Linear pendulum: $L\ddot{\theta} + g\theta = 0$
- Damped linear pendulum: $L\ddot{\theta} + b\dot{\theta} + g\theta = 0$
- Forced damped nonlinear pendulum: $L\ddot{\theta} + b\dot{\theta} + g\sin\theta = F\cos\omega t$
- Forced damped linear pendulum: $L\ddot{\theta} + b\dot{\theta} + g\theta = F\cos\omega t$



1.3 Existence and Uniqueness

Theorem 1.1 (Existence and Uniqueness). *If the functions a_1, a_0, b are continuous on a closed interval $I \subset \mathbb{R}$, the constant $x_0 \in I$, and $y_0, y_1 \in \mathbb{R}$ are arbitrary constants, then there is a unique solution y, defined on I, of the initial value problem*

$$y'' + a_1(x)y' + a_0(x)y = b(x), \quad y(x_0) = y_0, \quad y'(x_0) = y_1.$$

Example 1.1 (Domain of the Solution). Find the domain of the solution to the initial value problem

$$(t-1)y'' - 3ty' + \frac{4(t-1)}{(t-3)}y = t(t-1), \quad y(2) = 1, \quad y'(2) = 0.$$

Solution: We first write the equation in standard form:

$$y'' - \frac{3t}{(t-1)}y' + \frac{4}{(t-3)}y = t.$$

The equation coefficients are defined on the domain

$$(-\infty, 1) \cup (1, 3) \cup (3, \infty).$$

So the solution may not be defined at $t = 1$ or $t = 3$. Since the initial condition is at $t_0 = 2 \in (1, 3)$, the domain of the solution is

$$D = (1, 3).$$

1.4 Properties of Homogeneous Equations

Theorem 1.2 (Superposition Principle)**.** *If $y = y_1(x)$ and $y = y_2(x)$ are two solutions of the homogeneous differential equation, then any linear combination*

$$y = C_1 y_1(x) + C_2 y_2(x),$$

where C_1 and C_2 are constants, is also a solution of the differential equation.

Remark 1.2. This result is *not true* for nonhomogeneous equations.

Definition 1.3 (Linear Independence)**.** The functions $y_1(x)$ and $y_2(x)$ are **linearly independent** on the interval $I = (\alpha, \beta)$ if their quotient on this interval is not identically equal to a constant:

$$\frac{y_1(x)}{y_2(x)} \neq \text{const}.$$

Otherwise, these functions are **linearly dependent**.

Theorem 1.3 (General Solution)**.** *If $y_1(x)$ and $y_2(x)$ are linearly independent solutions of the differential equation*

$$y'' + a_1(x)y' + a_0(x)y = 0 \quad \text{on an interval } I = (\alpha, \beta),$$

where a_1, a_0 are continuous functions on I, then there are unique constants C_1, C_2 such that every solution y of the differential equation can be written as

$$y(x) = C_1 y_1(x) + C_2 y_2(x).$$

Definition 1.4 (Fundamental Solutions)**.** Two solutions $y_1(x)$ and $y_2(x)$ form a **fundamental set of solutions** to the differential equation if every solution of

$$y'' + a_1(x)y' + a_0(x)y = 0$$

can be expressed as a linear combination of $y_1(x)$ and $y_2(x)$.

1.5 Homogeneous Constant Coefficient Equations

Consider the second-order homogeneous linear differential equation with constant coefficients:

$$y'' + a_1 y' + a_0 y = 0,$$

where a_1 and a_0 are constants.

Assume a solution of the form $y(x) = e^{\lambda x}$, where λ is a constant to be determined. Substituting into the differential equation and factoring out $e^{\lambda x} \neq 0$, we get the **auxiliary equation** (or **characteristic equation**):

$$\lambda^2 + a_1 \lambda + a_0 = 0.$$

The roots are given by:

$$\lambda = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}.$$

Let λ_1 and λ_2 be the roots of the characteristic equation. Then the general solution is:

Roots	General Solution
$\lambda_1 \neq \lambda_2$ (real)	$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$
$\lambda_1 = \lambda_2$	$y(x) = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x}$
$\lambda_{1,2} = \alpha \pm i\beta$ (complex)	$y(x) = C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x)$

1.6 Homogeneous Constant Coefficient Equations: Examples

Example 1.2 (Distinct Real Roots)**.** Solve the equation

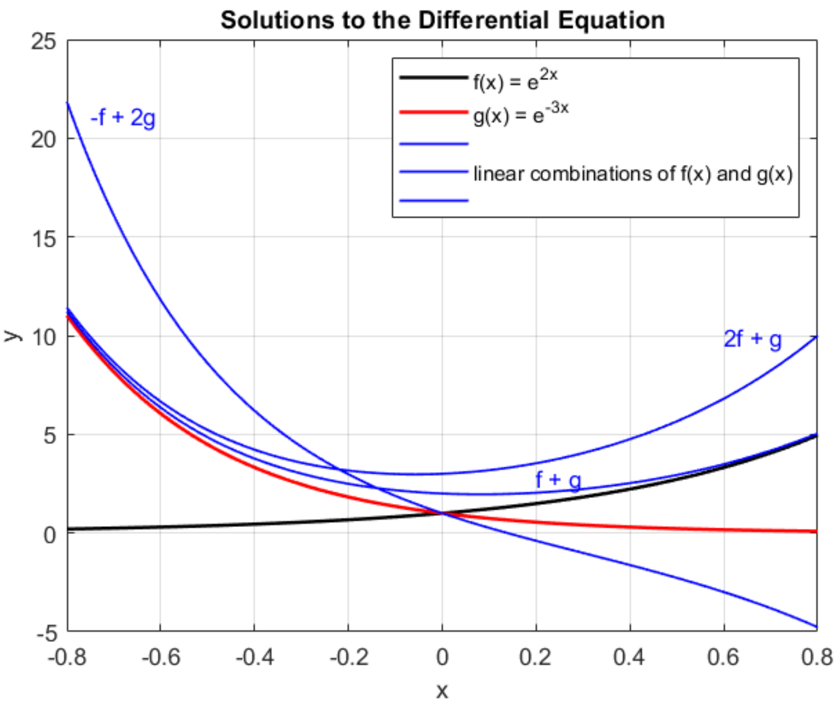
$$y'' + y' - 6y = 0.$$

Solution: The auxiliary equation is

$$\lambda^2 + \lambda - 6 = 0,$$

whose roots are 2 and -3. Therefore, the general solution is

$$y = C_1 e^{2x} + C_2 e^{-3x}.$$



Example 1.3 (Repeated Roots)**.** Solve the equation

$$4y'' + 12y' + 9y = 0.$$

Solution: The auxiliary equation is

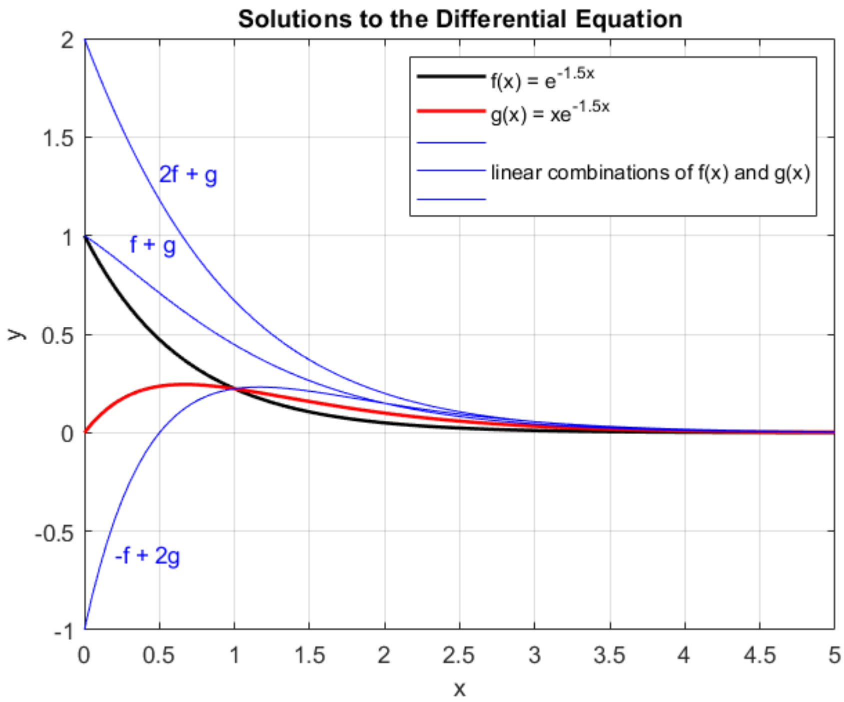
$$4\lambda^2 + 12\lambda + 9 = 0,$$

which can be factored as $(2\lambda + 3)^2 = 0$. The only root is $-\frac{3}{2}$. Therefore, the general solution is

$$y = C_1 e^{-\frac{3}{2}x} + C_2 x e^{-\frac{3}{2}x}.$$

Example 1.4 (Complex Roots)**.** Solve the equation

$$y'' - 6y' + 13y = 0.$$



Solution: The auxiliary equation is

$$\lambda^2 - 6\lambda + 13 = 0.$$

By the quadratic formula, the roots are

$$\lambda = \frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i.$$

Therefore, the general solution is

$$y(x) = C_1 e^{3x} \cos(2x) + C_2 e^{3x} \sin(2x).$$

2 The Method of Variation of Parameters

2.1 The Wronskian

Definition 2.1 (Wronskian)**.** Given two functions $y_1(x)$ and $y_2(x)$, their Wronskian is defined as:

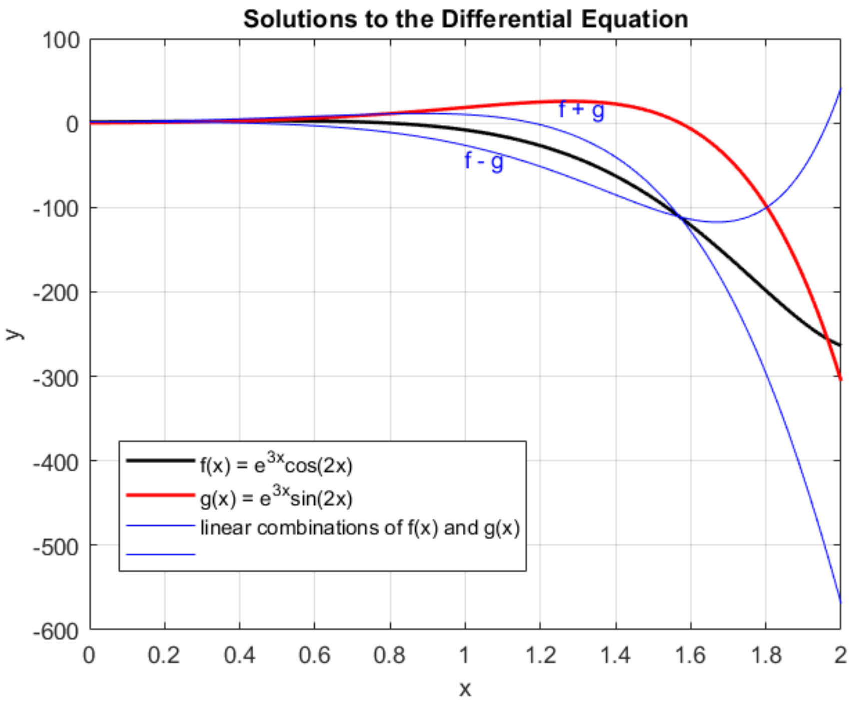
$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x).$$

Two functions $y_1(x)$ and $y_2(x)$ are **linearly independent** on an interval I if and only if their Wronskian is not identically zero on I. If the functions $y_1(x)$ and $y_2(x)$ are **linearly dependent** on the interval I, then the Wronskian vanishes on this interval.

Example 2.1 (Wronskian Computation)**.** Consider the functions $y_1(x) = e^x$ and $y_2(x) = e^{-x}$. Their Wronskian is:

$$W(y_1, y_2)(x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = e^x(-e^{-x}) - e^{-x}(e^x) = -1 - 1 = -2.$$

Since the Wronskian is non-zero, $y_1(x)$ and $y_2(x)$ are linearly independent.



2.2 General Solution of Nonhomogeneous Equations

Consider the differential equation

$$y'' + a_1 y' + a_0 y = f(x). \tag{1}$$

The related homogeneous equation

$$y'' + a_1 y' + a_0 y = 0 \tag{2}$$

is called the **complementary equation**.

Theorem 2.1 (General Solution of the Nonhomogeneous Equation)**.** *The general solution of the nonhomogeneous differential equation (1) is represented as the sum of some particular solution and the general solution of the complementary equation:*

$$y(x) = y_p(x) + y_c(x), \tag{3}$$

where $y_p(x)$ is a particular solution of (1) and $y_c(x)$ is the general solution of the complementary equation (2).

2.3 The Method of Variation of Parameters

If we know the general solution of the homogeneous equation (2), the basic difficulty lies in finding some particular solution. We give a general method for finding the particular solution of the nonhomogeneous equation (1).

We write the general solution of the complementary equation (2):

$$y_c(x) = C_1 y_1(x) + C_2 y_2(x), \tag{4}$$

where C_1, C_2 are arbitrary constants.

We seek a particular solution to (1) in the form (4), considering the constants C_1, C_2 as undetermined functions $C_1(x), C_2(x)$.

The function (4) will be a solution of the nonhomogeneous equation (1) provided the functions $C_1(x), C_2(x)$ satisfy the system:

$$\begin{cases} C_1'(x)y_1(x) + C_2'(x)y_2(x) = 0 \\ C_1'(x)y_1'(x) + C_2'(x)y_2'(x) = f(x) \end{cases}$$

The main determinant of this system is the Wronskian of the functions $y_1(x)$ and $y_2(x)$, which is not equal to zero due to the linear independence of the solutions. Therefore, this system always has a unique solution.

The formulas for $C_1'(x)$ and $C_2'(x)$ are:

$$C_1'(x) = -\frac{y_2(x)f(x)}{W(y_1, y_2)}, \qquad C_2'(x) = \frac{y_1(x)f(x)}{W(y_1, y_2)}. \tag{5}$$

Knowing the derivatives, one can find the functions $C_1(x), C_2(x)$:

$$C_1(x) = -\int \frac{y_2(x)f(x)}{W(y_1, y_2)}\,dx + \tilde{C}_1, \qquad C_2(x) = \int \frac{y_1(x)f(x)}{W(y_1, y_2)}\,dx + \tilde{C}_2, \tag{6}$$

where $\tilde{C}_1, \tilde{C}_2$ are constants of integration. Setting $\tilde{C}_1 = \tilde{C}_2 = 0$ gives the particular solution of equation (1).

Then the general solution of the original nonhomogeneous equation is:

$$y(x) = y_2(x)\int \frac{y_1(x)f(x)}{W(y_1, y_2)}\,dx - y_1(x)\int \frac{y_2(x)f(x)}{W(y_1, y_2)}\,dx + C_1y_1(x) + C_2y_2(x). \tag{7}$$

2.4 Variation of Parameters: Examples

Example 2.2 (Variation of Parameters – Example 1)**.** Solve the equation

$$y'' + y = \tan(x), \quad 0 < x < \frac{\pi}{2},$$

using the method of variation of parameters.

Step 1: Complementary solution. The complementary equation $y'' + y = 0$ has the characteristic equation $\lambda^2 + 1 = 0$, giving $\lambda = \pm i$. Thus:

$$y_c(x) = C_1 \sin(x) + C_2 \cos(x).$$

We identify: $y_1(x) = \sin(x)$, $y_2(x) = \cos(x)$.

Step 2: Wronskian.

$$W(y_1, y_2)(x) = \begin{vmatrix} \sin(x) & \cos(x) \\ \cos(x) & -\sin(x) \end{vmatrix} = -\sin^2(x) - \cos^2(x) = -1.$$

Step 3: Particular solution. Using the variation of parameters formula:

$$y_p(x) = \cos(x)\int \frac{\sin(x)\tan(x)}{-1}\,dx - \sin(x)\int \frac{\cos(x)\tan(x)}{-1}\,dx.$$

$$y_p(x) = -\cos(x)\int \sin(x)\tan(x)\,dx + \sin(x)\int \cos(x)\tan(x)\,dx.$$

First integral:

$$-\cos(x)\int \sin(x)\tan(x)\,dx = -\cos(x)\int \frac{\sin^2(x)}{\cos(x)}\,dx = -\cos(x)\int(\sec(x)-\cos(x))\,dx$$

$$= -\cos(x)(\ln(\tan(x)+\sec(x)) - \sin(x)) = -\cos(x)\ln(\tan(x)+\sec(x)) + \cos(x)\sin(x).$$

Second integral:

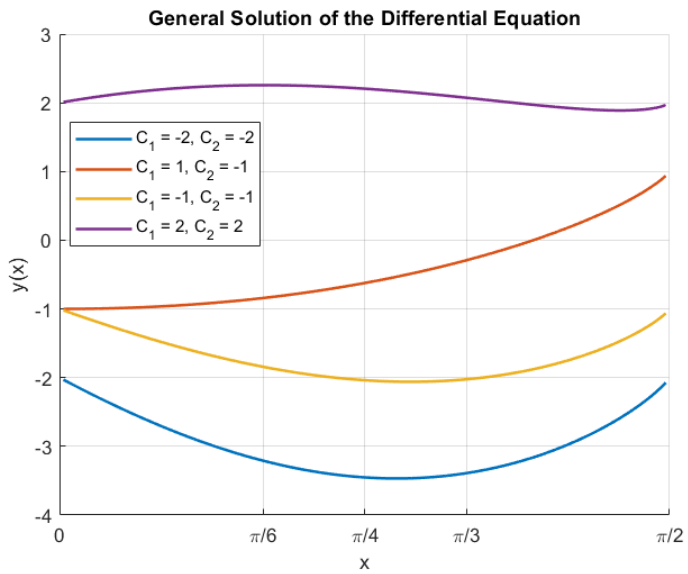
$$\sin(x)\int \sin(x)\,dx = -\sin(x)\cos(x).$$

Combining:

$$y_p(x) = -\cos(x)\ln(\tan(x)+\sec(x)) + \cos(x)\sin(x) - \sin(x)\cos(x) = -\cos(x)\ln(\tan(x)+\sec(x)).$$

Step 4: General solution.

$$y(x) = -\cos(x)\ln(\tan(x)+\sec(x)) + C_1\sin(x) + C_2\cos(x).$$



Example 2.3 (Variation of Parameters – Example 2)**.** Solve the equation

$$y'' + 3y' + 2y = \sin(e^x),$$

using the method of variation of parameters.

Step 1: Complementary solution. The characteristic equation $\lambda^2 + 3\lambda + 2 = 0$ gives $\lambda_{1,2} = -1, -2$. Thus:

$$y_c(x) = C_1 e^{-x} + C_2 e^{-2x}.$$

We identify: $y_1(x) = e^{-x}$, $y_2(x) = e^{-2x}$.

Step 2: Wronskian.

$$W(y_1, y_2)(x) = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x} = -e^{-3x}.$$

Step 3: Particular solution.

$$y_p(x) = -e^{-2x}\int e^{2x}\sin(e^x)\,dx + e^{-x}\int e^x \sin(e^x)\,dx.$$

First integral (substituting $u = e^x$, $du = e^x dx$):

$$-e^{-2x}\int u\sin(u)\,du = -e^{-2x}(-u\cos(u)+\sin(u)) = -e^{-2x}(-e^x\cos(e^x)+\sin(e^x)).$$

Second integral (substituting $u = e^x$):

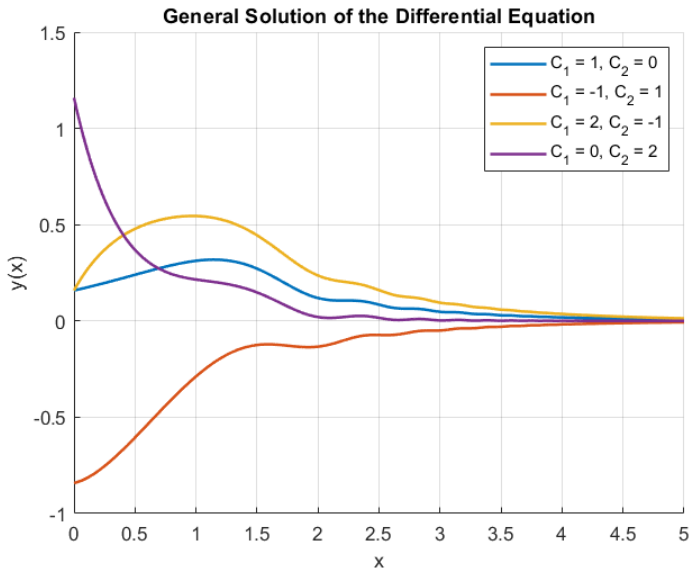
$$e^{-x}\int \sin(u)\,du = e^{-x}(-\cos(e^x)) = -e^{-x}\cos(e^x).$$

Combining:

$$y_p(x) = -e^{-2x}\sin(e^x) + e^{-2x}e^x\cos(e^x) - e^{-x}\cos(e^x) = -e^{-2x}\sin(e^x).$$

Step 4: General solution.

$$y(x) = C_1e^{-x} + C_2e^{-2x} - e^{-2x}\sin(e^x).$$



Example 2.4 (Variation of Parameters – Example 3)**.** Solve the equation

$$ty'' - (t+1)y' + y = t^2,$$

using the method of variation of parameters, given that $y_1(t) = e^t$ and $y_2(t) = t+1$ form a fundamental set of solutions.

Step 0: Standard form. Dividing by t:

$$y'' - \left(1+\frac{1}{t}\right)y' + \frac{1}{t}y = t.$$

Step 1: Complementary solution. Given that $y_1(t) = e^t$ and $y_2(t) = t+1$ form a fundamental set, the general solution to the homogeneous equation is:

$$y_c(t) = C_1e^t + C_2(t+1).$$

Step 2: Wronskian.

$$W(y_1, y_2)(t) = \begin{vmatrix} e^t & t+1 \\ e^t & 1 \end{vmatrix} = e^t - (t+1)e^t = -te^t.$$

Step 3: Particular solution.

$$y_p(t) = (t+1)\int \frac{e^t \cdot t}{-te^t}\,dt - e^t \int \frac{(t+1)t}{-te^t}\,dt = -(t+1)\int dt + e^t \int (t+1)e^{-t}\,dt.$$

First integral:

$$-(t+1)\int dt = -(t+1)t.$$

Second integral (integration by parts with $u = t+1$, $dv = e^{-t}dt$):

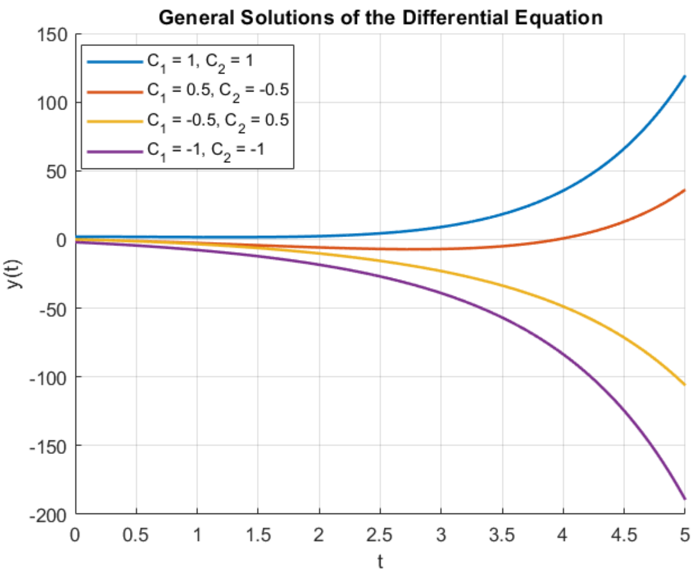
$$e^t \int (t+1)e^{-t}\,dt = e^t\left[-(t+1)e^{-t} + \int e^{-t}\,dt\right] = -(t+1) - e^t e^{-t} = -t-2.$$

Combining:

$$y_p(t) = -(t+1)t - t - 2 = -t^2 - 2t - 2.$$

Step 4: General solution.

$$y(t) = C_1 e^t + C_2(t+1) - t^2 - 2t - 2.$$



3 Ordinary Points, Singularities, and Euler Equations

3.1 Regular and Singular Points

Consider an inhomogeneous second-order linear ODE with variable coefficients:

$$y'' + a_1(x)y' + a_0(x)y = f(x). \tag{1}$$

Definition 3.1 (Regular Point)**.** A point $x = x_0$ is called a **regular point** of the DE (1) if the functions $a_1(x)$, $a_0(x)$ and $f(x)$ are analytic at $x = x_0$ (i.e., each function has a Taylor series expansion in powers of $x - x_0$ valid in a neighborhood of x_0).

Definition 3.2 (Singular Point)**.** A point $x = x_0$ is called a **singular point** of DE (1) if one or more of the functions $a_1(x)$, $a_0(x)$ and $f(x)$ is not analytic at $x = x_0$.

Singular points can be further classified:

- **Regular Singular Point**: If $x = x_0$ is a singular point and if $a_1(x)(x - x_0)$ and $a_0(x)(x-x_0)^2$ are both analytic at $x = x_0$, then $x = x_0$ is called a **regular singular point**.
- **Irregular Singular Point**: If the above conditions for a regular singular point are not met, then $x = x_0$ is called an **irregular singular point**.

An equivalent characterization: x_0 is a **regular singular point** if

$$\lim_{x\to x_0} a_1(x)(x - x_0) \quad \text{and} \quad \lim_{x\to x_0} a_0(x)(x - x_0)^2$$

are both finite. A singular point that is not a regular singular point is called an **irregular singular point**.

3.2 Regular and Singular Points: Examples

Example 3.1 (Regular Singular Points)**.** Find all singular points of the equation and determine whether each one is regular or irregular:

$$(x - 1)y'' + \frac{1}{x}y' - 2y = 0.$$

Solution: Dividing by $(x - 1)$:

$$y'' + \frac{1}{x(x-1)}y' - \frac{2}{(x-1)}y = 0.$$

We identify the coefficients:

$$a_1(x) = \frac{1}{x(x-1)}, \qquad a_0(x) = -\frac{2}{(x-1)}.$$

The singular points are $x = 0$ and $x = 1$.

At $x = 0$:

$$a_1(x)(x - 0) = \frac{1}{x-1}, \qquad a_0(x)(x - 0)^2 = -\frac{2x^2}{x-1}.$$

The function $\frac{1}{x-1} = -(1+x+x^2+\cdots)$ for $|x| < 1$ is analytic around $x = 0$. The function $-\frac{2x^2}{x-1} = 2(x^2 + x^3 + \cdots)$ for $|x| < 1$ is also analytic around $x = 0$. Thus, $x = 0$ is a **regular singular point**.

At $x = 1$:

$$a_1(x)(x - 1) = \frac{1}{x}, \qquad a_0(x)(x - 1)^2 = -2(x - 1).$$

The function $\frac{1}{x} = \sum_{n=0}^{\infty}(-1)^n(x - 1)^n$ for $|x - 1| < 1$ is analytic around $x = 1$. The function $-2(x - 1)$ is a polynomial, hence analytic everywhere. Thus, $x = 1$ is a **regular singular point**.

Example 3.2 (Irregular Singular Points)**.** Find all singular points of the equation and determine whether each one is regular or irregular:

$$(x-1)^2y'' + \frac{1}{x^2}y' + 2y = 0.$$

Solution: Dividing by $(x-1)^2$:

$$y'' + \frac{1}{x^2(x-1)^2}y' + \frac{2}{(x-1)^2}y = 0.$$

We identify the coefficients:

$$a_1(x) = \frac{1}{x^2(x-1)^2}, \qquad a_0(x) = \frac{2}{(x-1)^2}.$$

The singular points are $x = 0$ and $x = 1$.

At $x = 0$:

$$a_1(x)(x-0) = \frac{1}{x(x-1)^2}.$$

We compute:

$$\lim_{x\to 0} \frac{1}{x(x-1)^2} = \infty.$$

Since the limit is not finite, $x = 0$ is an **irregular singular point**.

At $x = 1$:

$$a_1(x)(x-1) = \frac{1}{x^2(x-1)}.$$

We compute:

$$\lim_{x\to 1} \frac{1}{x^2(x-1)} = \infty.$$

Since the limit is not finite, $x = 1$ is an **irregular singular point**.

3.3 Classic Equations

Several important differential equations from mathematical physics feature regular singular points:

- **Euler–Cauchy equation**: $ax^2y'' + bxy' + cy = 0$. Regular singular point at $x = 0$.
- **Legendre's equation**: $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$. Regular singular points at $x = \pm 1$.
- **Chebyshev's equation**: $(1-x^2)y'' - xy' + \alpha^2 y = 0$. Regular singular points at $x = \pm 1$.
- **Bessel's equation**: $x^2y'' + xy' + (x^2 - \nu^2)y = 0$. Regular singular points at $x = 0$ and $x = \infty$.
- **Hermite equation**: $y'' - 2xy' + \lambda y = 0$, $-\infty < x < \infty$. Regular singular point at $x = \infty$.
- **Airy equation**: $y'' - xy = 0$. Irregular singular point at $x = \infty$.

3.4 Euler Equations

Consider the Euler equation:

$$ax^2y'' + bxy' + cy = 0, \tag{1}$$

where a, b, and c are real constants and $a \neq 0$.

The Euler equation has solutions defined on $(0, \infty)$ and $(-\infty, 0)$, since it can be rewritten as

$$ay'' + \frac{b}{x}y' + \frac{c}{x^2}y = 0.$$

On the interval $(0, \infty)$, we look for solutions of the form $y = x^r$, $x > 0$. Substituting into equation (1):

$$ax^2r(r-1)x^{r-2} + bx \cdot rx^{r-1} + cx^r = (ar(r-1) + br + c)x^r.$$

Definition 3.3 (Indicial Polynomial)**.** The polynomial

$$p(r) = ar(r-1) + br + c$$

is called the **indicial polynomial** of equation (1), and $p(r) = 0$ is its **indicial equation**.

Let r_1 and r_2 be the roots of the indicial equation. Then the general solution of the Euler equation on $(0, \infty)$ is:

Roots	General Solution
$r_1 \neq r_2$ (real)	$y(x) = C_1x^{r_1} + C_2x^{r_2}$
$r_1 = r_2 = r$	$y(x) = x^r(C_1 + C_2\ln(x))$
$r_{1,2} = \alpha \pm i\beta$ (complex)	$y(x) = x^\alpha(C_1\cos(\beta\ln(x)) + C_2\sin(\beta\ln(x)))$

Remark 3.1. If r is not a rational number, then x^r is defined by $x^r = e^{r\ln x}$.

For the shifted Euler equation

$$a(x-x_0)^2y'' + b(x-x_0)y' + cy = 0,$$

we look for solutions of the form $y = (x-x_0)^r$. The general solution on intervals not containing x_0 is:

Roots	General Solution
$r_1 \neq r_2$ (real)	$y = C_1\lvert x-x_0\rvert^{r_1} + C_2\lvert x-x_0\rvert^{r_2}$
$r_1 = r_2 = r$	$y = (C_1 + C_2\ln\lvert x-x_0\rvert)\lvert x-x_0\rvert^r$
$r_{1,2} = \alpha \pm i\beta$ (complex)	$y = \lvert x-x_0\rvert^\alpha(C_1\cos(\beta\ln\lvert x-x_0\rvert) + C_2\sin(\beta\ln\lvert x-x_0\rvert))$

3.5 Euler Equations: Examples

Example 3.3 (Euler Equation – Distinct Real Roots)**.** Find the general solution of

$$6x^2y'' + 5xy' - y = 0$$

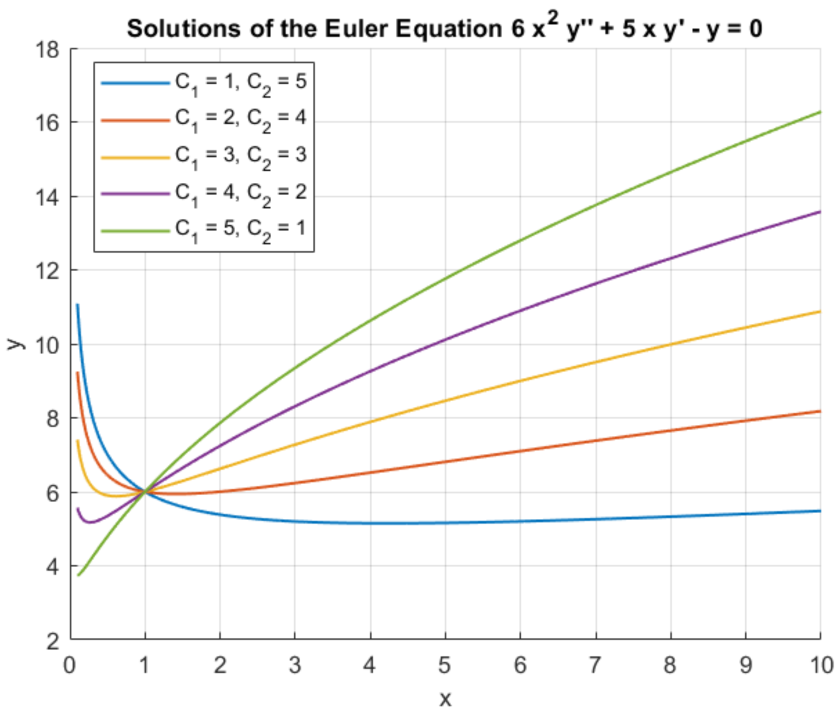
on $(0, \infty)$.

Solution: The indicial polynomial is

$$p(r) = 6r(r-1) + 5r - 1 = (2r-1)(3r+1),$$

whose roots are $\frac{1}{2}$ and $-\frac{1}{3}$. Therefore, the general solution is

$$y = C_1x^{1/2} + C_2x^{-1/3}.$$



Example 3.4 (Euler Equation – Repeated Roots)**.** Find the general solution of

$$x^2y'' - 5xy' + 9y = 0$$

on $(0, \infty)$.

Solution: The indicial polynomial is

$$p(r) = r(r-1) - 5r + 9 = (r-3)^2,$$

whose root is 3. Therefore, the general solution is

$$y = x^3(C_1 + C_2 \ln x).$$

Example 3.5 (Euler Equation – Complex Roots)**.** Find the general solution of

$$x^2y'' + 3xy' + 2y = 0$$

on $(0, \infty)$.

Solution: The indicial polynomial is

$$p(r) = r(r-1) + 3r + 2 = (r+1)^2 + 1.$$

The roots of the indicial equation are $r = -1 \pm i$. Therefore, the general solution is

$$y = x^{-1}\left[C_1 \cos(\ln x) + C_2 \sin(\ln x)\right].$$

Example 3.6 (Shifted Euler Equation)**.** Find the solution to the following differential equation on any interval not containing $x_0 = -6$:

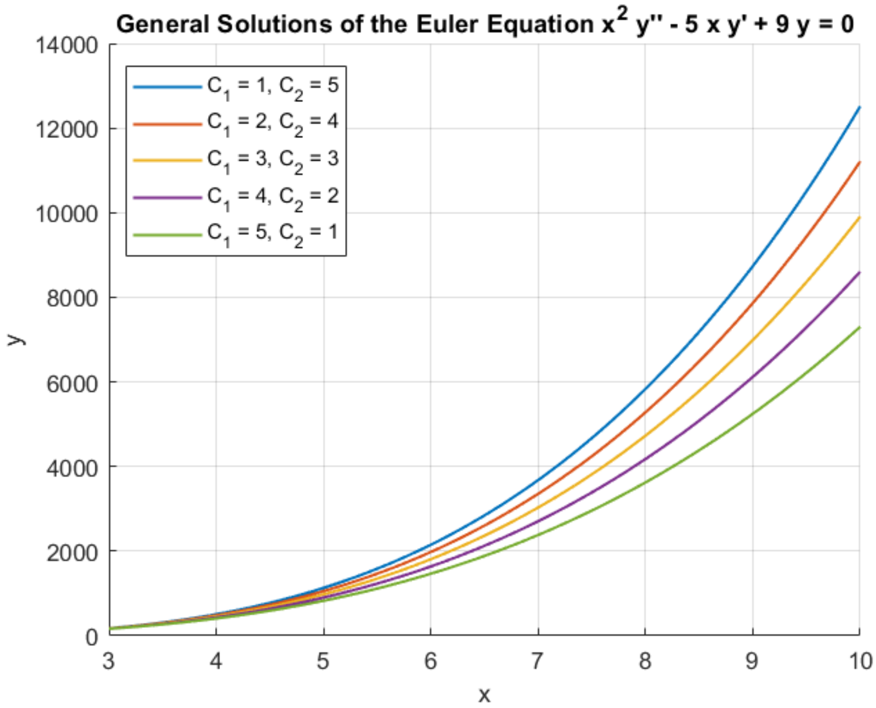
$$3(x+6)^2y'' + 25(x+6)y' - 16y = 0.$$

Solution: The indicial polynomial is

$$3r(r-1) + 25r - 16 = 0 \implies 3r^2 + 22r - 16 = 0 \implies (3r-2)(r+8) = 0.$$

Thus, the roots are $r_1 = \frac{2}{3}$ and $r_2 = -8$. The general solution is

$$y(x) = C_1|x+6|^{2/3} + C_2|x+6|^{-8}.$$



4 Initial Value Problems and Taylor Series Solutions

4.1 IVP Examples

Example 4.1 (IVP – Constant Coefficients)**.** Solve the following initial-value problem:

$$y'' + 6y' + 13y = 0, \quad y(0) = 0, \quad y'(0) = 2.$$

Solution: The general solution is:

$$y(x) = e^{-3x}(c_1 \cos 2x + c_2 \sin 2x).$$

Differentiating:

$$y'(x) = e^{-3x}(-2c_1 \sin 2x + 2c_2 \cos 2x) - 3e^{-3x}(c_1 \cos 2x + c_2 \sin 2x).$$

Applying the initial conditions:

$$\begin{cases} y(0) = c_1 = 0, \\ y'(0) = 2c_2 - 3c_1 = 2. \end{cases}$$

Therefore, $c_1 = 0$ and $c_2 = 1$, and the solution is

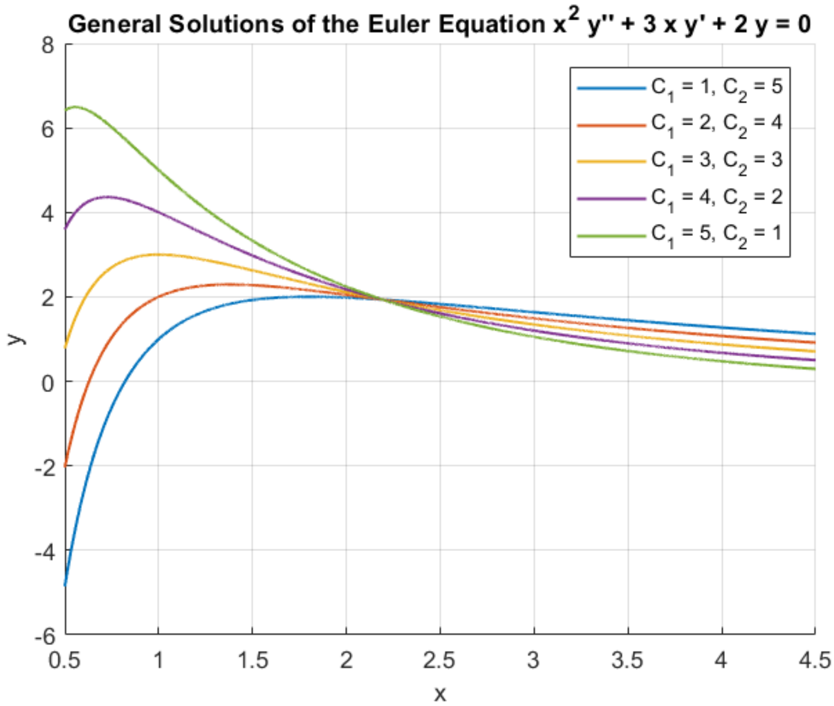
$$y = e^{-3x} \sin 2x.$$

Example 4.2 (IVP – Euler Equation)**.** Solve the following initial-value problem:

$$2x^2y'' + 3xy' - 15y = 0, \quad y(1) = 0, \quad y'(1) = 1.$$

Solution: The indicial equation is:

$$2r(r-1) + 3r - 15 = 0 \implies 2r^2 + r - 15 = 0 \implies (2r-5)(r+3) = 0.$$



The roots are $r_1 = \frac{5}{2}$ and $r_2 = -3$. The general solution is:

$$y(x) = c_1 x^{5/2} + c_2 x^{-3}.$$

Differentiating:

$$y'(x) = \frac{5}{2} c_1 x^{3/2} - 3c_2 x^{-4}.$$

Applying the initial conditions:

$$\begin{cases} y(1) = c_1 + c_2 = 0, \\ y'(1) = \frac{5}{2}c_1 - 3c_2 = 1. \end{cases}$$

Solving: $c_1 = \frac{2}{11}$, $c_2 = -\frac{2}{11}$. The particular solution is:

$$y(x) = \frac{2}{11} x^{5/2} - \frac{2}{11} x^{-3}.$$

4.2 Taylor Series Solutions of Initial Value Problems

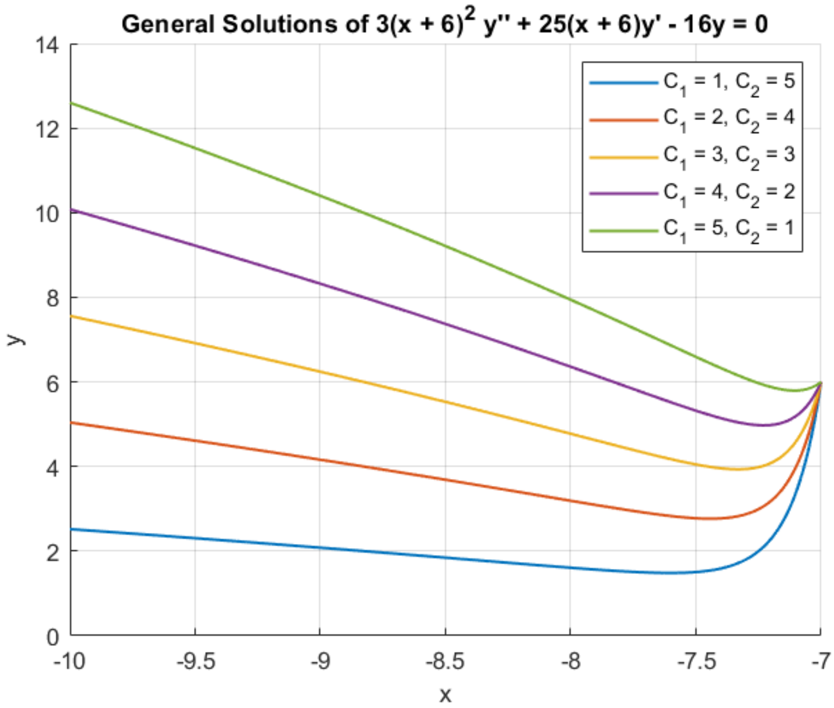
Consider the initial value problem:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \quad y(x_0) = y_0, \quad y'(x_0) = y_0'.$$

Assume that the solution $y(x)$ can be written as a Taylor series expanded about x_0:

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \frac{(x - x_0)^3}{3!} y_0''' + \cdots$$

The initial conditions give the first two coefficients. To find the higher-order derivatives $y_0^{(n)}$, differentiate the given differential equation successively and evaluate at $x = x_0$.



Example 4.3 (Taylor Series Solution – Example 1)**.** Solve the IVP using a Taylor series of degree 4:

$$y'' - 2xy' + x^2y = 0, \quad y(0) = 1, \quad y'(0) = -1.$$

Solution: The Taylor series expansion around $x = 0$:

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{(4)}(0) + \cdots$$

From the ODE: $y'' = 2xy' - x^2y$. At $x = 0$:

$$y''(0) = 2(0)(-1) - (0)^2(1) = 0.$$

Differentiate: $y''' = 2y' + 2xy'' - 2xy - x^2y'$. At $x = 0$:

$$y'''(0) = 2(-1) + 0 - 0 - 0 = -2.$$

Differentiate: $y^{(4)} = 4y'' + 2xy''' - 2y - 4xy' - x^2y''$. At $x = 0$:

$$y^{(4)}(0) = 4(0) + 0 - 2(1) - 0 - 0 = -2.$$

The degree 4 Taylor polynomial solution is:

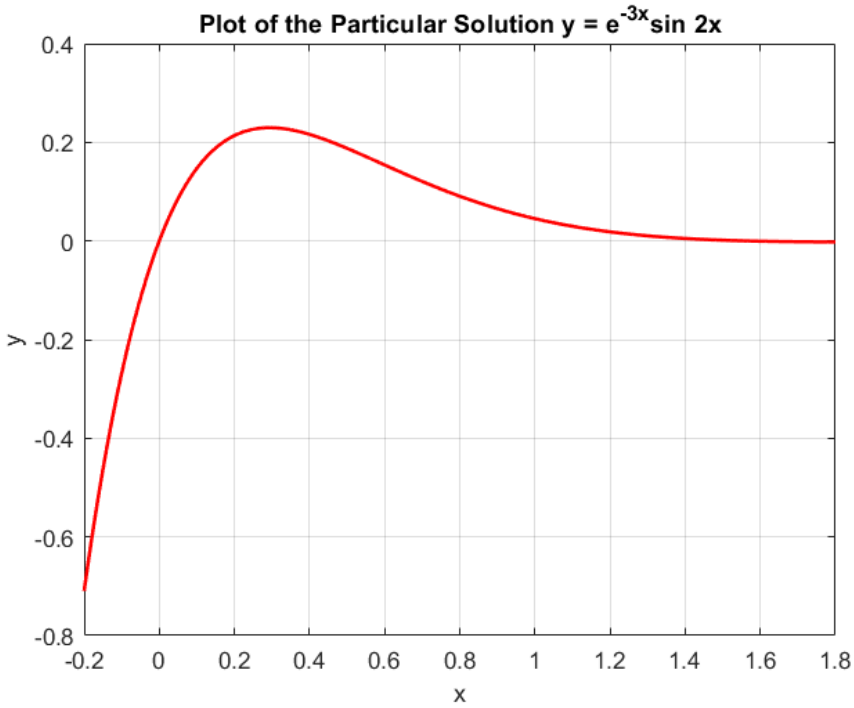
$$y(x) = 1 - x - \frac{x^3}{3} - \frac{x^4}{12}.$$

Example 4.4 (Taylor Series Solution – Example 2)**.** Solve the IVP using a Taylor series of degree 5:

$$y'' + xy = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Solution: The Taylor series expansion around $x = 0$:

$$y(x) = x + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{(4)}(0) + \frac{x^5}{5!}y^{(5)}(0) + \cdots$$



From the ODE: $y'' = -xy$. At $x = 0$: $y''(0) = 0$.
Differentiate: $y''' = -y - xy'$. At $x = 0$: $y'''(0) = -y(0) = 0$.
Differentiate: $y^{(4)} = -2y' - xy''$. At $x = 0$: $y^{(4)}(0) = -2(1) = -2$.
Differentiate: $y^{(5)} = -3y'' - xy'''$. At $x = 0$: $y^{(5)}(0) = -3(0) = 0$.
The degree 5 Taylor polynomial solution is:

$$y(x) = x - \frac{x^4}{12}.$$

5 Boundary Value Problems

5.1 Definition

Definition 5.1 (Boundary Value Problem)**.** A **boundary value problem** for the second-order linear ODE is to find a function $y(x)$ that satisfies the differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x), \quad x \in [a, b],$$

subject to the **boundary conditions** (e.g.):

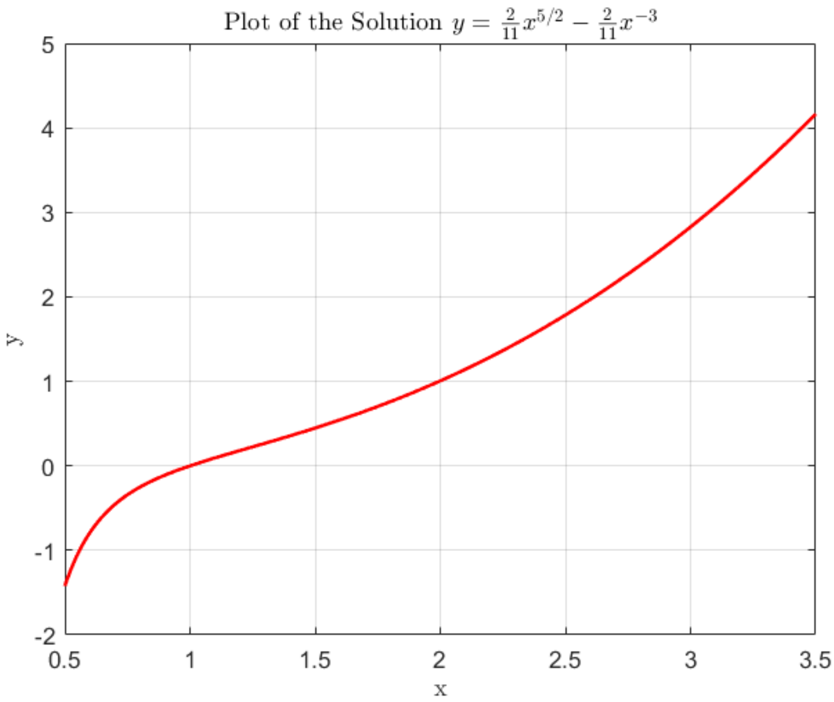
$$y(a) = \alpha, \quad y(b) = \beta,$$

where $a_2(x)$, $a_1(x)$, $a_0(x)$, and $f(x)$ are given functions, and α and β are specified constants. We call a and b **boundary points**.

Other types of boundary conditions include:

$$y'(a) = \alpha,\ y'(b) = \beta; \qquad y'(a) = \alpha,\ y(b) = \beta; \qquad y(a) = \alpha,\ y'(b) = \beta.$$

The question of existence and uniqueness for solutions of boundary value problems is more complicated than for initial value problems, as the following three examples demonstrate.



5.2 Boundary Value Problems: Examples

Example 5.1 (BVP – Unique Solution)**.** Consider the boundary value problem:

$$y'' + y = 1, \quad y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 0.$$

Solution: The general solution of $y'' + y = 1$ is

$$y = 1 + c_1 \sin x + c_2 \cos x.$$

The condition $y(0) = 0$ gives $c_2 = -1$, and $y(\pi/2) = 0$ gives $c_1 = -1$. Therefore,

$$y = 1 - \sin x - \cos x$$

is the **unique solution** of the boundary value problem.

Example 5.2 (BVP – No Solution)**.** Consider the boundary value problem:

$$y'' + y = 1, \quad y(0) = 0, \quad y(\pi) = 0.$$

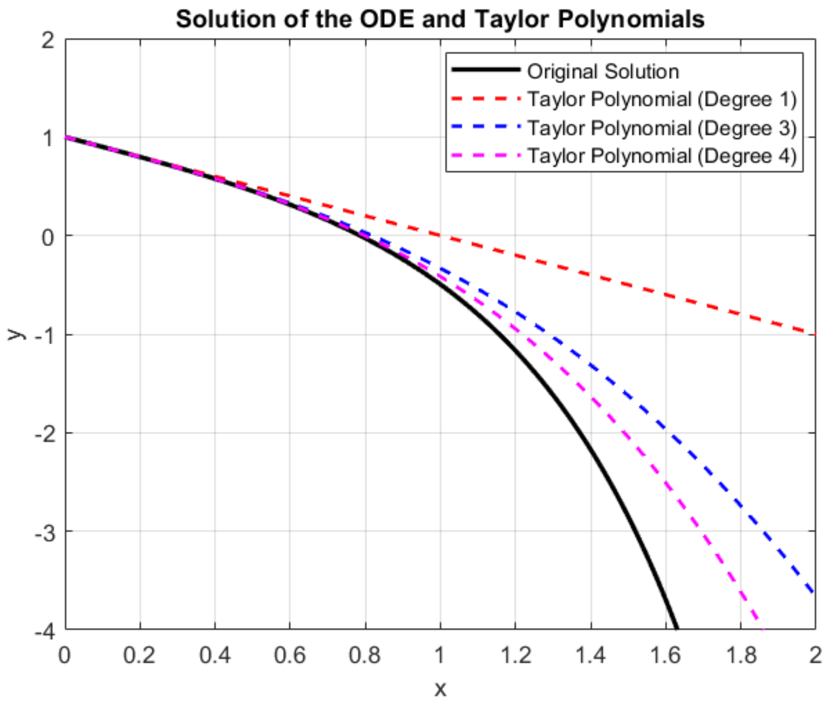
Solution: The general solution of $y'' + y = 1$ is

$$y = 1 + c_1 \sin x + c_2 \cos x.$$

The condition $y(0) = 0$ requires $c_2 = -1$, but $y(\pi) = 0$ requires $c_2 = 1$. Therefore the boundary value problem has **no solution**.

Example 5.3 (BVP – Infinitely Many Solutions)**.** Consider the boundary value problem:

$$y'' + y = \sin 2x, \quad y(0) = 0, \quad y(\pi) = 0.$$



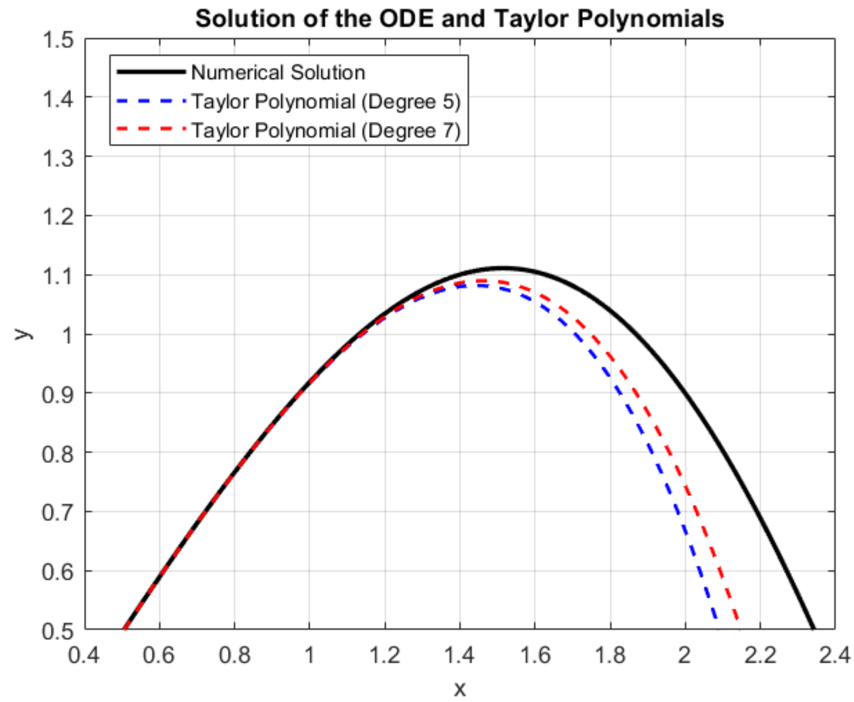
Solution: The general solution of $y'' + y = \sin 2x$ is

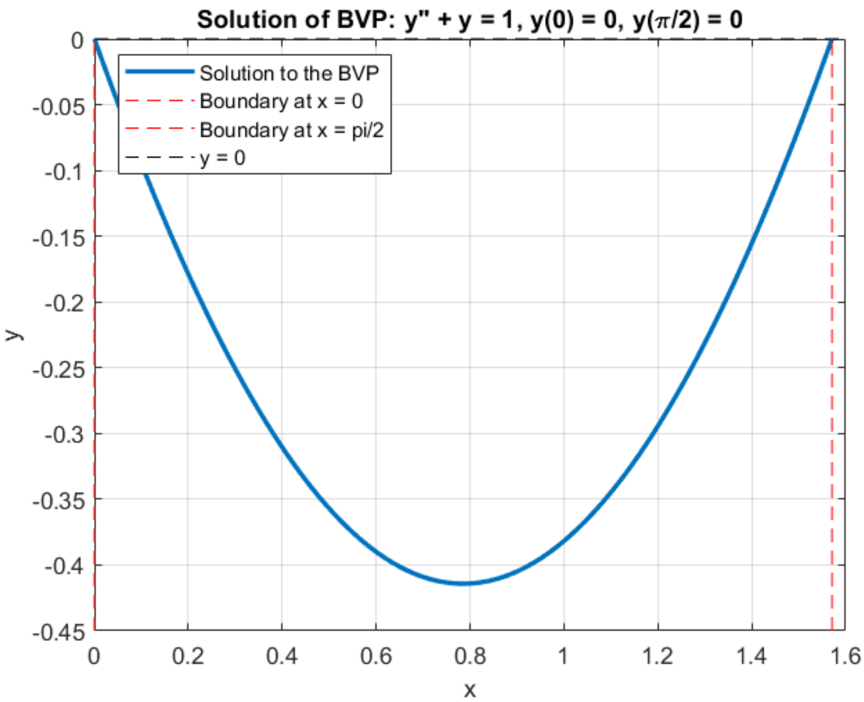
$$y = -\frac{\sin 2x}{3} + c_1 \sin x + c_2 \cos x.$$

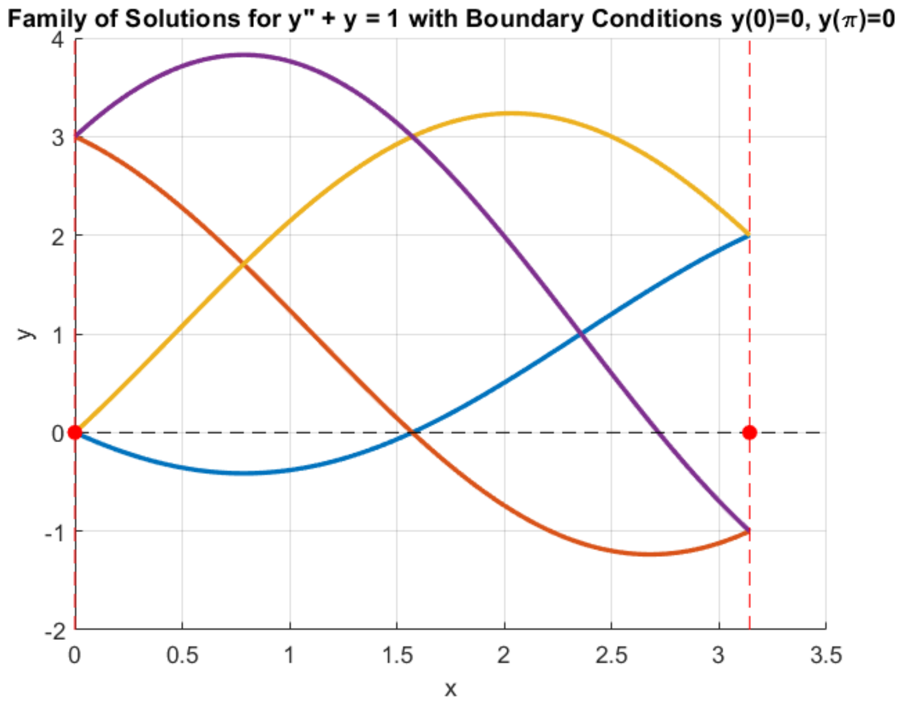
The boundary conditions $y(0) = 0$ and $y(\pi) = 0$ both require $c_2 = 0$, but they do not restrict c_1. Therefore the boundary value problem has **infinitely many solutions**:

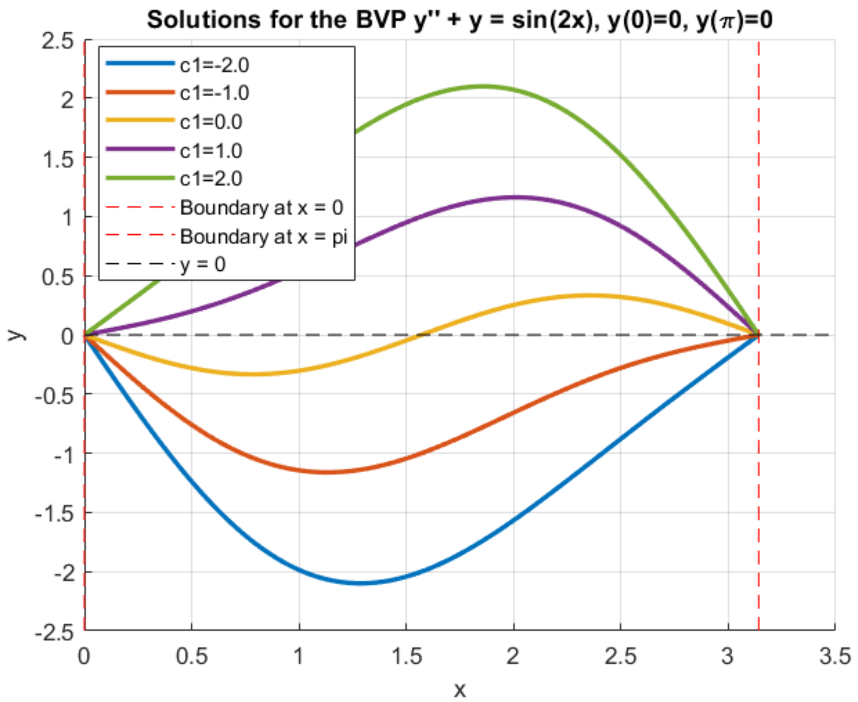
$$y = -\frac{\sin 2x}{3} + c_1 \sin x,$$

where c_1 is arbitrary.


Solution of the ODE and Taylor Polynomials
Numerical Solution
Taylor Polynomial (Degree 5)
Taylor Polynomial (Degree 7)
y
x
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4


Solution of BVP: y'' + y = 1, y(0) = 0, y(π/2) = 0
Solution to the BVP
Boundary at x = 0
Boundary at x = pi/2
y = 0
y
x
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6


Family of Solutions for y" + y = 1 with Boundary Conditions y(0)=0, y(π)=0
y
x


Solutions for the BVP y'' + y = sin(2x), y(0)=0, y(π)=0
c1=-2.0
c1=-1.0
c1=0.0
c1=1.0
c1=2.0
Boundary at x = 0
Boundary at x = pi
y = 0
y
x