Lesson 1. First-order Ordinary Differential Equations I

ELTE Eötvös Loránd University

Faculty of Informatics, Department of Numerical Analysis

2025

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Overview of Differential Equations

Equations in which the unknown function or the vector function appears under the sign of the derivative or the differential are called **differential equations**.

Which real-world phenomena can we model with differential equations?

Differential equations are used to model a wide range of real-world phenomena, particularly those involving **rates of change**.

- velocity is the rate of change of displacement: $v(t) = x'(t)$;
- acceleration is the rate of change of velocity: $a(t) = v'(t) = x''(t)$

Example 1: Newton's Second Law of Motion

$$m\frac{d^2x}{dt^2}(t) = f\left(t, x(t), \frac{dx}{dt}(t)\right) \tag{1}$$

This equation governs the motion of the object, and solving it will give the object's position x(t) as a function of time under the influence of the specified force.

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Example 2: Radioactive Decay

The amount u of a radioactive material changes in time as follows,

$$\frac{du}{dt}(t) = -ku(t), \quad k > 0, \tag{2}$$

where k is a positive constant representing the radioactive properties of the material.

Example 3: The Heat Equation

Describes how heat diffuses through a given region over time:

$$\frac{\partial T}{\partial t} = k\nabla^2 T, \tag{3}$$

where:

- $T = T(t, \mathbf{x})$ is the temperature at time t and position $\mathbf{x} = (x, y, z)$,
- k is a positive constant representing the thermal conductivity of the material,
- ∇^2 is the Laplacian operator.

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Example 4: The Wave Equation

Describes how a wave propagates through a medium over time. In this case, $u = u(t, \mathbf{x})$ represents the wave disturbance at time t and position $\mathbf{x} = (x, y, z)$, and c is the speed of the wave:

$$\frac{\partial^2 u(t, \mathbf{x})}{\partial t^2} = c^2 \nabla^2 u(t, \mathbf{x}) \tag{4}$$

The wave equation can be used to model different types of wave propagation: Vibrating String (1D Wave Equation); Sound Waves (3D Wave Equation); ...

Overview of Differential Equations

Example 5: Schrödinger equation

Schrödinger equation in Quantum Mechanics, in one space dimension, stationary, is

$$-\frac{\hbar^2}{2m}\psi'' + V(x)\psi = E\psi; \tag{5}$$

where $\psi(x)$ is the probability density of finding a particle of mass m at the position x having energy E under a potential $V(x)$, and $\hbar$ is Planck's constant divided by 2π.

Overview of Differential Equations

If there are several equations working together, we have a so-called **system of differential equations**.

Maxwell's equations for electromagnetics

Maxwell's equations are comprised of four partial differential equations, each representing a different aspect of electromagnetism. These equations are:

$$\operatorname{div}\mathbf{E} = \frac{\rho}{\varepsilon_0} \quad (1) \qquad \operatorname{div}\mathbf{B} = 0 \quad (2)$$

$$\operatorname{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (3) \qquad \operatorname{curl}\mathbf{B} = \mu_0\mathbf{J} + \mu_0\varepsilon_0\frac{\partial \mathbf{E}}{\partial t} \quad (4)$$

Gauss's law for electricity (1) describes how electric charges produce electric fields. **Gauss's law for magnetism** (2) states that there are no magnetic monopoles in nature. **Faraday's law** (3) describes how a changing magnetic field creates an electric field. **Ampere's law with Maxwell's addition** (4) combines the effects of electric currents (Ampere's law) and changing electric fields (Maxwell's addition).

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Classification of Differential Equations

If an equation involves the derivative of one variable with respect to another, then the former is called a **dependent variable** and the latter an **independent variable**.

If in a differential equation the dependent variables are functions of one independent variable, then the differential equation is called **ordinary (ODE)**. But if the unknown function appearing in the differential equation is a function of two or more independent variables, the differential equation is called a **partial differential equation (PDE)**.

- The equations in examples (1), (2) and (5) are (ODE) – the unknown function depends on a single independent variable, t or x.
- The equations in examples (3) and (4) are (PDE) – the unknown function depends on two or more independent variables, t, x, y, and z, and their partial derivatives appear in the equations.

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Classification of Differential Equations

The **order** of a differential equation (or system) is the order of the highest derivative in the equation (system).

- Newton's equation in example (1) is second order;
- The time decay equation in example (2) is first order;
- Schrödinger equation in example (5) is second order;
- The heat equation in example (3) is first order in time and second order in space variables;
- The wave equation in example (4) is second order in time and space variables

The **degree** of a differential equation (system) is the highest power of the highest-order derivative in the equation (system).

Second degree first-order ODE:

$$(y')^2 = x - y^3$$

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Classification of Differential Equations

We also distinguish how the **dependent variables** appear in the equation (or system).

An equation is **linear** if the dependent variable (or variables) and their derivatives appear linearly, that is only as first powers, they are not multiplied together, and no other functions of the dependent variables appear. In other words, the equation is a sum of terms, where each term is some function of the independent variables or some function of the independent variables multiplied by a dependent variable or its derivative.
Otherwise, the equation is called **nonlinear**.

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Otherwise, the equation is called **nonlinear**.

Classification of Differential Equations

An ordinary differential equation is linear if it can be put into the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x) \tag{1.1}$$

The functions $a_0(x), a_1(x), ..., a_n(x)$ of eq.(1) are called **the coefficients**.

For a function $u(x, y)$, a second order linear PDE is of the form

$$\begin{aligned} &a_1(x,y)\,u_{xx} + a_2(x,y)\,u_{xy} + a_3(x,y)\,u_{yx} + a_4(x,y)\,u_{yy} \\ &+\ a_5(x,y)\,u_x + a_6(x,y)\,u_y + a_7(x,y)\,u = f(x,y) \end{aligned} \tag{1.2}$$

where a_i and f are functions of the independent variables x and y only.

Classification of Differential Equations

An ordinary differential equation is linear if it can be put into the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x) \tag{1.1}$$

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Classification of Differential Equations

When the coefficients of a linear DE do not depend on x, the DE is said to have **constant coefficients**.

A linear equations (1.1)-(1.2) may further be called **homogeneous** if all terms depend on the dependent variable. That is, if no term is a function of the independent variables alone. Otherwise, the equation is called **nonhomogeneous** or **inhomogeneous**.

Examples

- Mass-Spring-Damper System:

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

- Simple Pendulum:

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0$$

- Transport equation:

$$u_x + u_y = 0$$

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Classification of Differential Equations

Finally, an equation (or system) is called **autonomous** if the equation does not depend on the independent variable.

Autonomous equation (system)

- Newton's Law of Cooling:

$$\frac{dT}{dt} = -k(T - T_{out})$$

- Logistic Growth Model

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

- Predator-Prey Model (Lotka-Volterra Equations)

$$\frac{dx}{dt} = ax - bxy$$

$$\frac{dy}{dt} = -cy + dxy$$

where x and y represent the prey and predator populations, respectively.

Fundamentals of Differential Equations

A **solution** of a differential equation is a function which, when substituted into the differential equation, reduces it to an identity.

Initial Condition(s) are a condition, or set of conditions, on the solution that will allow us to determine which solution that we are after. Initial conditions are of the form,

$$y(t_0) = y_0 \text{ and/or } y^{(k)}(t_0) = y_k$$

So, in other words, initial conditions are values of the solution and/or its derivative(s) at specific points.
The number of initial conditions that are required for a given differential equation will depend upon the order of the differential equation.

The general solution to a differential equation is the most general form that the solution can take and doesn't take any initial conditions into account.

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Fundamentals of Differential Equations

The particular solution to a differential equation is the specific solution that not only satisfies the differential equation, but also satisfies the given initial condition(s).

An **Initial Value Problem** (or **IVP**) is a differential equation along with an appropriate number of initial conditions.

A **singular solution** is the solution of a differential equation that cannot be obtained from the general solution gotten by the usual method of solving the differential equation.

An **equilibrium solution** is a solution to a DE whose derivative is zero everywhere. On a graph an equilibrium solution looks like a horizontal line.

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An equilibrium solution is said to be **Asymptotically Stable** if on both sides of this equilibrium solution, there exists other solutions which approach this equilibrium solution.

An equilibrium solution is said to be **Semi-Stable** if one one side of this equilibrium solution there exists other solutions which approach this equilibrium solution, and on the other side of the equilibrium solution other solutions diverge from this equilibrium solution.

An equilibrium solution is said to be **Unstable** if on both sides of this equilibrium solution other solutions diverge from this equilibrium solution.

Fundamentals of Differential Equations

Leibniz's Notation

$$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \ldots, \quad \frac{d^ny}{dx^n}$$

This notation is more useful for differentiation and integration.

Lagrange's Notation

$$y', \quad y'', \quad \ldots, \quad y^{(n)}$$

This notation is more useful for representing derivatives of any order compactly.

Newton's Notation

$$\dot{y}, \quad \ddot{y}, \quad \dddot{y}$$

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$$F(x, y, y') = 0 \tag{1}$$

or, solving for the derivative:

$$\frac{dy}{dx} = f(x, y) \tag{2}$$

First-order ODE:

In the form (1):

$$y' - xy = 0$$

or in the form (2):

$$y' = xy$$

Here, x is the independent variable, y is the dependent variable, and y' is its derivative.

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or in the form (2):

$$y' = xy$$

Here, x is the independent variable, y is the dependent variable, and y' is its derivative.

First-order ODE

The **general solution** of a first-order ODE is the function

$$y = \phi(x, C) \tag{3}$$

or the relation

$$\Phi(x, y, C) = 0 \tag{4}$$

which depends on a single arbitrary constant C.

General solution of $y' = xy$

$$y = C \cdot e^{\frac{x^2}{2}}$$

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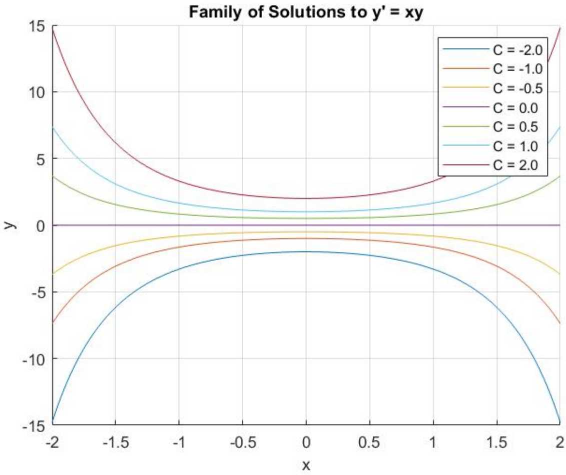
Family of solutions



First-order ODE

A **particular solution** is any function

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which is obtained from the general solution by assigning to the arbitrary constant C a definite value $C = C_0$.

In the case of a first-order equation, the **initial value problem** is of the form:

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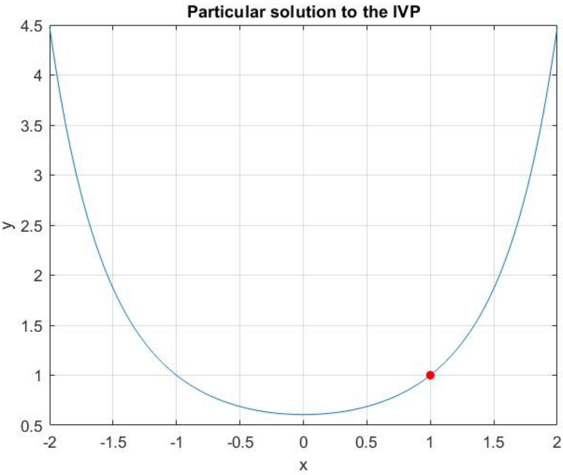

Examples of IVPs

Example 1

Consider the following IVP:

$$y' = y, \quad y(0) = 2$$

The general solution

$$y(x) = Ce^x$$

The particular solution

$$y(x) = 2e^x$$

The initial condition $y(0) = 2$ uniquely determines the solution as $y = 2e^x$. This is because the general solution $y = Ce^x$ with C determined by the initial condition results in $C = 2$ when $x = 0$ and $y = 2$. Thus the solution to the IVP exists, is unique, and is defined on all of $\mathbb{R}$.

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Examples of IVPs

Example 2

Consider the following IVP:

$$y' = xy^2, \quad y(0) = 3$$

The general solution

$$y(x) = \frac{2}{C - x^2}$$

The particular solution

$$y(x) = \frac{6}{2 - 3x^2}$$

The initial condition $y(0) = 3$ uniquely determines the constant $C = \frac{1}{3}$. This particular solution exists and is unique. However, it is only defined on the open interval $\left(-\frac{\sqrt{2}}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}\right)$, since the denominator vanishes at $x = \pm\frac{\sqrt{2}}{\sqrt{3}}$, causing the solution to blow up.

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Examples of IVPs

Example 3

Consider the following IVP:

$$y' = \frac{y}{x}, \quad y(0) = 0$$

The general solution

$$y(x) = Cx$$

The particular solution

Infinitely many solutions

The initial condition $y(0) = 0$ leads IVP having infinitely many solutions:

$$0 = C \cdot 0^2 \Rightarrow C - \forall$$

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Example 4

Consider the following IVP:

$$y' = \frac{y}{x}, \quad y(0) = 1$$

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No solution

The initial condition $y(0) = 1$ does not determine a solution because the general solution $y = Cx$ with C being any constant does not satisfy $y(0) = 1$ for any value of C $(1 \neq C \cdot 0)$.

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Examples of IVPs

Example 5

Consider the following IVP:

$$y' = 2\sqrt{y}, \quad y(0) = 0$$

The general solution

$$y(x) = (x - C)^2$$

The particular solution

$$y(x) = x^2$$

and

$$y = 0$$

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Examples of IVPs

Example 5 (Cont.)

We can combine both solutions to create infinitely many piecewise-defined solutions like this:

$$y(x) = \begin{cases} 0, & \text{if } x < C, \\ (x-C)^2, & \text{if } x \geq C, \end{cases} \quad \text{for any } C > 0.$$

That means:

- The solution stays at zero up to time $x = C$,
- Then it starts following the parabola $y = (x-C)^2$ from that point onward.

Since $(x-C)^2$ starts at 0 and has zero slope at $x = C$, the function is smooth at that transition point - there is no sudden jump or sharp corner - so it still satisfies the differential equation.

This shows that the initial value problem

$$y' = 2\sqrt{y}, \quad y(0) = 0$$

has infinitely many solutions, depending on the value of C.

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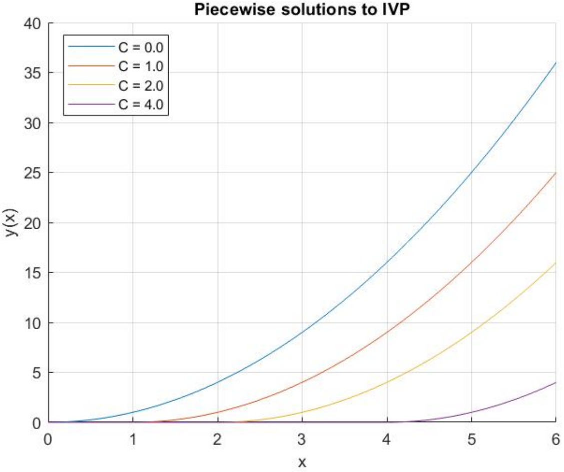
Graph representing four solutions for IVP(5)



A **solution to the IVP** (7) is a function $y(x)$ such that:

1. $y(x)$ is defined on some interval (a, b) containing x_0, and $y(x_0) = y_0$,
2. $y(x)$ satisfies the ODE in (a, b).

The solution to the IVP comes with a domain where it is defined. The largest such interval (a, b) is called the **interval of existence**.

Thus, the following questions are naturally arising:

- Given an IVP, does a solution exist? (Question of *existence*)
- If a solution exists, is it unique? (Question of *uniqueness*)
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Peano's Theorem

Consider the Initial Value Problem (IVP)

$$y' = f(x, y), \quad y(x_0) = y_0$$

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$$D = \{(x, y) \mid a < x < b,\ c < y < d\}$$

in the xy-plane. If (x_0, y_0) is a point in this rectangle, then there exists an $\varepsilon > 0$ and a function $y(x)$ defined for all

$$x_0 - \varepsilon < x < x_0 + \varepsilon$$

that solves the given initial value problem.

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Peano's Theorem

In other words, as long as the function $f(x, y)$ is continuous in a rectangle

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even a tiny rectangle

$$|x - x_0| < \varepsilon$$

that contains the point (x_0, y_0) in its interior, then there exists a function $y(x)$ (many more might exist) that solves a differential equation that passes through the point (x_0, y_0) and this function is defined in at least a tiny interval around this point.

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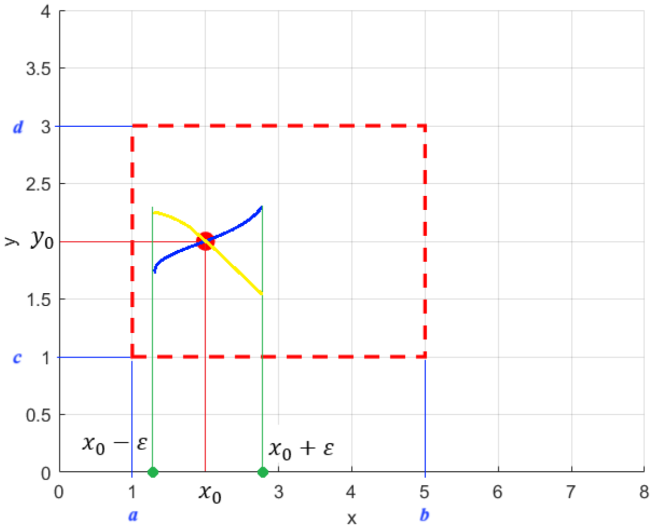


Peano's Theorem

Consider the IVP

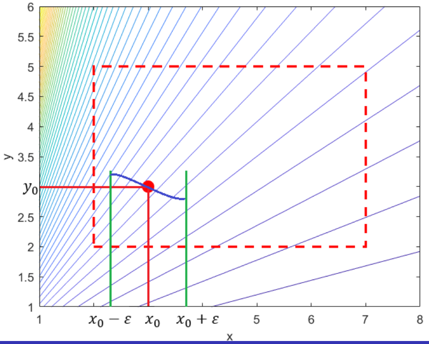
$$\frac{dy}{dx} = \frac{y}{x}, \quad y(x_0) = y_0, (a, b)$$

Discussion

The function $f(x, y) = \frac{y}{x}$ is not defined at $x = 0$, and hence is not continuous in any rectangle that includes $x = 0$.

If $x_0 \neq 0$, then f is continuous in a neighborhood around (x_0, y_0), and Peano's theorem guarantees the existence of at least one solution.

However, if $x_0 = 0$, Peano's theorem cannot be applied, and we cannot conclude whether a solution exists from the theorem.



Lipschitz Continuity

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. We say that:

- f is **locally Lipschitz continuous in the variable y** if for every open rectangle $D = \{(x, y) \mid a < x < b,\ c < y < d\}$, there exists a constant $L > 0$ such that
$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2| \quad \text{for all } (x, y_1), (x, y_2) \in D.$$
- f is **globally Lipschitz continuous in y** if there exists a constant $L > 0$ such that
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Lipschitz Continuity

Example 1: Consider the IVP from Example 1:

$$y' = y, \quad y(0) = 2.$$

Show that the function $f(x, y) = y$ satisfies a Lipschitz condition in y on some rectangle containing the point $(0, 2)$.

Solution:

We begin by choosing a rectangle

$$D = \{(x, y) \mid -1 \leq x \leq 1,\ 1 \leq y \leq 3\}$$

that contains the initial point $(0, 2)$.

The bounds on x and y are selected to include a reasonable neighborhood around the initial value.

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Example 1 continued:

For arbitrary points (x, y_1) and (x, y_2) in D, we have

$$|f(x, y_1) - f(x, y_2)| = |y_1 - y_2| \leq 1 \cdot |y_1 - y_2|.$$

The inequality holds with Lipschitz constant $L = 1$, so the function is **Lipschitz continuous** in y on the domain D.

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Lipschitz Continuity

Example 2: Consider the IVP from Example 5:

$$y' = 2\sqrt{y}, \quad y(0) = 0$$

Check whether the function $f(x, y) = 2\sqrt{y}$ satisfies a Lipschitz condition on the domain

$$D = \{(x, y) \mid -1 \leq x \leq 1,\ 0 \leq y \leq 1\}.$$

Lipschitz Continuity

Solution:

For arbitrary points (x, y_1) and (x, y_2) in D, we compute:

$$|f(x, y_1) - f(x, y_2)| = |2\sqrt{y_1} - 2\sqrt{y_2}| = 2 \cdot |\sqrt{y_1} - \sqrt{y_2}|.$$

Using the identity:

$$|\sqrt{y_1} - \sqrt{y_2}| = \frac{|y_1 - y_2|}{\sqrt{y_1} + \sqrt{y_2}}, \quad \text{for } y_1, y_2 > 0,$$

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Lipschitz Continuity

Example 2 continued:

To satisfy the Lipschitz condition in y, we would require:

$$|f(y_1) - f(y_2)| \leq L|y_1 - y_2| \quad \text{for all } y_1, y_2 \in [0, 1].$$

Substituting in the earlier expression, this becomes:

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Dividing both sides by $|y_1 - y_2|$ (assuming $y_1 \neq y_2$):

$$\frac{2}{\sqrt{y_1} + \sqrt{y_2}} \leq L.$$

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$$\frac{2}{\sqrt{y_1} + \sqrt{y_2}} \leq L.$$

Lipschitz Continuity

Example 2 continued:

To satisfy the Lipschitz condition in y, we would require:

$$|f(y_1) - f(y_2)| \leq L|y_1 - y_2| \quad \text{for all } y_1, y_2 \in [0,1].$$

Substituting in the earlier expression, this becomes:

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As $y_1, y_2 \to 0$, we have $\sqrt{y_1} + \sqrt{y_2} \to 0$, so

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This shows that no finite constant L can satisfy the Lipschitz condition over the entire domain D.

Therefore, the function $f(y) = 2\sqrt{y}$ is **not Lipschitz continuous** on D, although it is continuous.

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Picard-Lindelöf Theorem

Theorem 2 (Picard-Lindelöf theorem on existence and uniqueness)
Let $f(x, y)$ be continuous and satisfy a Lipschitz condition on an open rectangle

$$D = \{(x, y) \mid a < x < b,\ c < y < d\}$$

in the xy-plane. If the point (x_0, y_0) belongs to the interior of this rectangle then there is a unique function $y(x)$, defined on an interval

$$|x - x_0| < \varepsilon$$

for some $\varepsilon > 0$, that solves the IVP:

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This theorem guarantees a **local** solution - that is, the solution exists and is unique only in a small interval around x_0.

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A function $f : \mathbb{R}^2 \to \mathbb{R}$ is **continuously differentiable in y** on $\mathbb{R}^2$ if its partial derivative

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Proposition: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable in y. Then f is locally Lipschitz continuous in y.

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Proposition: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable in y. Then f is locally Lipschitz continuous in y.

Proof the proposition

Let $f(x, y)$ be continuously differentiable in some open rectangle $D \subset \mathbb{R}^2$, let a point $(x_0, y_0) \in D$, and fix $x = x_0$.
We treat $f(x_0, y)$ as a function of y only (i.e., we *freeze* x).

Let y_1, y_2 be values close to y_0 such that the segment between them lies entirely inside D. Apply the Mean Value Theorem to the function $f(x_0, y)$:

$$f(x_0, y_1) - f(x_0, y_2) = \frac{\partial f}{\partial y}(x_0, \xi) \cdot (y_1 - y_2),$$

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$$|f(x_0, y_1) - f(x_0, y_2)| = \left|\frac{\partial f}{\partial y}(x_0, \xi)\right| \cdot |y_1 - y_2|.$$

Since $\frac{\partial f}{\partial y}$ is continuous in the rectangle, it is **bounded**. So there exists a constant $M > 0$ such that:

$$\left|\frac{\partial f}{\partial y}(x_0, \xi)\right| \leq M.$$

Hence,

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Corollary

Corollary: Let

$$f(x,y) \quad \text{and} \quad \frac{\partial f}{\partial y}$$

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$$y' = f(x,y), \quad y(x_0) = y_0 \tag{1}$$

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Interpretation of the Corollary:

- **I (Existence):** If
$$f(x,y) \quad \text{is continuous in } \mathbb{R}$$
then there is an open interval $I \subset (a,b)$ (possibly smaller) containing x_0 such that a solution $y(x)$ **exists** for the IVP (1) that is defined on I.
- **II (Uniqueness):** The solution is unique if in addition Lipschitz condition holds in $\mathbb{R}$.
A sufficient condition for (II) (stronger, but easier to check) is that
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Example 1: Applying the theorem

Show that there is a unique solution to the IVP from Example 2

$$y' = xy^2, \quad y(0) = 3$$

Solution:

For this initial value problem we have:

$$f(x, y) = xy^2 \quad \text{and} \quad \frac{\partial f}{\partial y} = 2xy$$

which are both continuous on all of $\mathbb{R}^2$. The Corollary guarantees existence and uniqueness of a solution for x near $x_0 = 0$:

$$y(x) = \frac{3}{1 - 3x^2}, \quad x \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

But just because f and $\frac{\partial f}{\partial y}$ are continuous on all of $\mathbb{R}^2$ we cannot expect existence or uniqueness of a solution $y(x)$ for all x.

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$f(x, y)$ is continuous for $y \geq 0$, but $\frac{\partial f}{\partial y}$ is **not** continuous at $y = 0$. The Corollary guarantees (I) (existence) (since f is continuous), but **not** (II) (uniqueness) (since $\partial f/\partial y$ is not continuous at the initial point). Recall, this IVP has **infinitely many solutions**:

$$y(x) = \begin{cases} 0, & \text{if } x < C, \\ (x - C)^2, & \text{if } x \geq C, \end{cases} \quad \text{for any } C > 0.$$

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$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0,$$

suppose that f is continuous on some appropriate rectangle

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and that there exists a solution $y(x)$ which is continuous on the interval $|x - x_0| < \varepsilon$.

Then we may integrate both sides of the differential equation to obtain the corresponding integral equation:

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$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))\, ds.$$

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Remarks on the proof of the theorem: Step 2: Define the iteration sequence

The integral equation is generally difficult to solve explicitly using analytical methods. An alternative approach is to approximate a solution by constructing a sequence of functions that converges to a solution.

We apply the **method of successive approximations**. Starting from the constant function $y_0(x) = y_0$, we define a sequence of functions $\{y_n(x)\}$ iteratively as follows:

$$y_1(x) = y_0 + \int_{x_0}^{x} f(s, y_0)\, ds,$$

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This defines a sequence of approximations $y_1(x), y_2(x), y_3(x), \ldots, y_n(x)$, which will converge to the actual solution.

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Remarks on the proof of the theorem: Step 3: Show that the sequence converges uniformly

Convergence

Under **theorem's assumptions** on the function $f(x, y)$, the sequence of approximate solutions defined by Picard iteration:

$$y_n(x) = y_0 + \int_{x_0}^{x} f(s, y_{n-1}(s))\, ds$$

converges uniformly to a limit function $y(x)$ on a small interval around x_0:

$$\lim_{n\to\infty} y_n(x) = y(x).$$

Remarks on the proof of the theorem: Step 4: Show that the limit is the unique solution to the IVP

We now verify that $y(x)$ satisfies the differential equation by first showing it satisfies the integral equation:

$$
\begin{aligned}
y(x) &= \lim_{n\to\infty} y_n(x) \\
&= \lim_{n\to\infty} \left(y_0 + \int_{x_0}^{x} f(s, y_{n-1}(s))\, ds \right) \\
&= y_0 + \lim_{n\to\infty} \int_{x_0}^{x} f(s, y_{n-1}(s))\, ds \\
&= y_0 + \int_{x_0}^{x} \lim_{n\to\infty} f(s, y_{n-1}(s))\, ds \\
&= y_0 + \int_{x_0}^{x} f(s, y(s))\, ds
\end{aligned}
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Remarks on the proof of the theorem: Step 4: Show that the limit is the unique solution to the IVP

Since $y(x)$ is continuous and the right-hand side is differentiable, we conclude:

$$\frac{dy}{dx} = f(x, y(x))$$

Thus, $y(x)$ is a solution of the IVP.

This procedure of generating a sequence of functions which approximate the solution whose existence we are trying to establish, is called **Picard iteration**.

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Picard iteration: Example 1

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$$y' = y, \quad y(0) = 2$$

We know this IVP satisfies the Picard-Lindelöf theorem and has a unique solution $y = 2e^x$.

Step 1: Rewrite IVP as Integral Equation

$$\begin{aligned} & y' = y, \quad y(0) = 2 \\ \Rightarrow \quad & y(x) = y_0 + \int_0^x f(s, y(s))\, ds \\ & y(x) = 2 + \int_0^x y(s)\, ds \end{aligned}$$

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$$\begin{aligned} & y' = y, \quad y(0) = 2 \\ \Rightarrow \quad & y(x) = y_0 + \int_0^x f(s, y(s))\, ds \\ & y(x) = 2 + \int_0^x y(s)\, ds \end{aligned}$$

Picard iteration: Example 1

Example 1: Use the proof of Picard-Lindelöf's Theorem to find the solution to

$$y' = y, \quad y(0) = 2$$

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Picard iteration: Example 1

Step 2: Define Iteration Sequence

Using method of successive approximations:

$$
\begin{aligned}
y_0(x) &= 2 \\
y_1(x) &= 2 + \int_0^x 2\,ds = 2 + 2x \\
y_2(x) &= 2 + \int_0^x (2+2s)\,ds = 2 + 2x + x^2 \\
y_3(x) &= 2 + \int_0^x (2+2s+s^2)\,ds = 2 + 2x + x^2 + \frac{x^3}{3} \\
y_4(x) &= 2 + \int_0^x \left(2 + 2s + s^2 + \frac{s^3}{3}\right) ds \\
&= 2 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12}
\end{aligned}
$$

Picard iteration: Example 1

Step 2: Define Iteration Sequence

By induction, we find:

$$y_n(x) = 2\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right)$$

This is the nth partial sum of the Maclaurin series for $2e^x$.

$$\Rightarrow y_n(x) \to 2e^x \quad \text{as } n \to \infty$$

Picard iteration: Example 1

Step 3: Uniform Convergence

We show that $y_n(x)$ converges uniformly to the true solution:

- It can be shown by induction that y_n is the partial sum of $2e^x$
- Thus, as $n \to \infty$, $y_n(x) \to 2e^x$ uniformly

Step 4: Show that the limit is the unique solution to the IVP

The solution to the IVP is:

$$y(x) = \lim_{n\to\infty} y_n(x) = 2\sum_{k=0}^{\infty} \frac{x^k}{k!} = 2e^x$$

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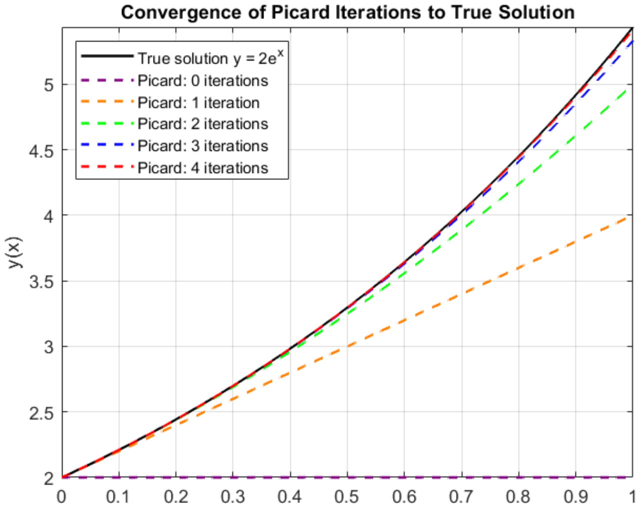

Picard iteration: Example 2

Example 2: Use Picard iteration to study

$$y' = 1 + x\sin(xy), \quad y(0) = 0, \quad 0 \le x \le 2$$

This is a **nonlinear** initial value problem.

- Unlike linear problems we have studied, this IVP involves the product xy inside a sine function $\sin(xy)$.
- There is no explicit solution.

Picard iteration: Example 2

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Picard iteration: Example 2

Corollary applies

We check conditions for local existence:

- Define $f(x, y) = 1 + x\sin(xy)$.
- On some rectangle D:
    - the function f is continuous.
    - the partial derivative:

$$\frac{\partial f}{\partial y} = x^2 \cos(xy)$$

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Picard iteration: Example 2

Step 1: Rewrite IVP as Integral Equation

$$y' = 1 + x\sin(xy), \quad y(0) = 0$$
$$\Rightarrow \quad y(x) = \int_0^x [1 + s\sin(sy(s))]\, ds$$

Picard iteration: Example 2

Step 2: Define Iteration Sequence

Using method of successive approximations:

$$\begin{aligned}
y_0(x) &= 0 \\
y_1(x) &= \int_0^x [1 + s\sin(0)]\,ds = \int_0^x 1\,ds = x \\
y_2(x) &= \int_0^x [1 + s\sin(s\cdot y_1(s))]\,ds \\
&= \int_0^x \left[1 + s\sin(s^2)\right] ds = x - \frac{\cos(x^2)}{2} \\
y_3(x) &= \int_0^x [1 + s\sin\left(s\cdot y_2(s)\right)]\,ds \\
&= \int_0^x \left[1 + s\sin\left(s\cdot(s - \frac{\cos(x^2)}{2})\right)\right] ds
\end{aligned}$$

- No closed-form formula appears possible
- Numerical integration is used from this step

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