# Lesson 2. First-order Ordinary Differential Equations II

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# 1 Well-Posed Problems

## 1.1 Continuous Dependence on Initial Data

**Question:** How do we determine whether a particular problem has the property that small changes, or perturbations, in the statement of the problem introduce correspondingly small changes in the solution?

**Definition 1.1** (Continuous Dependence on Initial Data)**.** For the unique solution $y(x)$ of the IVP

$$y' = f(x, y), \quad y(x_0) = y_0,$$

the **continuous dependence on the initial data** means that small changes in the initial value $y_0$ lead to small changes in the solution $y(x)$ (at least on some interval around $x_0$).

In essence, continuous dependence on initial data means that the problem is stable in the sense that small errors in the initial data do not lead to large errors in the solution, at least initially.

**Theorem 1.1** (Continuous Dependence on Initial Data)**.** *Let $f(x, y)$ be continuous on an open rectangle*

$$D = \{(x, y) \mid a < x < b,\ c < y < d\}$$

*in the xy-plane. Let $(x_0, y_0) \in D$, and suppose that $f$ satisfies a* ***Lipschitz condition with respect to*** *$y$ on $D$.*

*Let $y(x)$ be the unique solution to the IVP*

$$y' = f(x, y), \quad y(x_0) = y_0,$$

*and let $z(x)$ be the solution to the nearby IVP*

$$z' = f(x, z), \quad z(x_0) = y_0 + \delta.$$

*Then there exists an interval $I = [x_0 - \varepsilon, x_0 + \varepsilon] \subset (a, b)$, and a constant $C > 0$, such that*

$$|y(x) - z(x)| \leq C|\delta| \quad \text{for all } x \in I.$$

*Proof.* **Step 1.** Let $y(x)$ be the unique solution to the IVP

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0,$$

and let $z(x)$ be the unique solution to the IVP

$$z'(x) = f(x, z(x)), \quad z(x_0) = y_0 + \delta.$$

Then, using the integral form of the solution, we have:

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))\, ds, \quad z(x) = y_0 + \delta + \int_{x_0}^{x} f(s, z(s))\, ds.$$

**Step 2.** Subtract the two equations:

$$|y(x) - z(x)| = \left| -\delta + \int_{x_0}^{x} [f(s, y(s)) - f(s, z(s))]\, ds \right|.$$

**Step 3.** Apply the triangle inequality:

$$|y(x) - z(x)| \leq |\delta| + \left| \int_{x_0}^{x} [f(s, y(s)) - f(s, z(s))] \, ds \right|.$$

**Step 4.** Use the inequality:

$$\left| \int_{x_0}^{x} h(s) \, ds \right| \leq \int_{x_0}^{x} |h(s)| \, ds,$$

to get:

$$|y(x) - z(x)| \leq |\delta| + \int_{x_0}^{x} |f(s, y(s)) - f(s, z(s))| \, ds.$$

**Step 5.** Since $f(x, y)$ satisfies a Lipschitz condition in $y$, i.e.,

$$|f(s, y_1) - f(s, y_2)| \leq L|y_1 - y_2|,$$

we get:

$$|y(x) - z(x)| \leq |\delta| + L \int_{x_0}^{x} |y(s) - z(s)| \, ds.$$

**Step 6.** Applying successive approximation, and in general, we can show by induction that

$$|y(x) - z(x)| \leq |\delta| \left( 1 + L|x - x_0| + \frac{L^2|x - x_0|^2}{2!} + \cdots + \frac{L^n|x - x_0|^n}{n!} + \cdots \right) \leq C|\delta|,$$

where $C = e^{L|x - x_0|}$.

Thus

$$|y(x) - z(x)| \leq |\delta| e^{L|x - x_0|}$$

which implies that $z(x) \to y(x)$ as $\delta \to 0$. □

## 1.2 Continuous Dependence on Initial Data: Example

**Example 1.1** (Continuous Dependence)**.** Consider the IVP

$$y' = 5xy; \quad y(0) = 1.$$

The general solution (analytical):

$$y(x) = Ce^{\frac{5}{2}x^2},$$

where $C$ is a constant determined by the initial condition.

The particular solution (analytical):

$$y(x) = e^{\frac{5}{2}x^2}.$$

The value of the function at $x = 1$:

$$y(1) = e^{\frac{5}{2}1^2} = 12.1825.$$

**Example 1.2** (Perturbed Initial Data)**.** Consider the IVP

$$y' = 5xy; \quad y(0) = 1.0002.$$

The general solution (analytical):

$$y(x) = Ce^{\frac{5}{2}x^2},$$

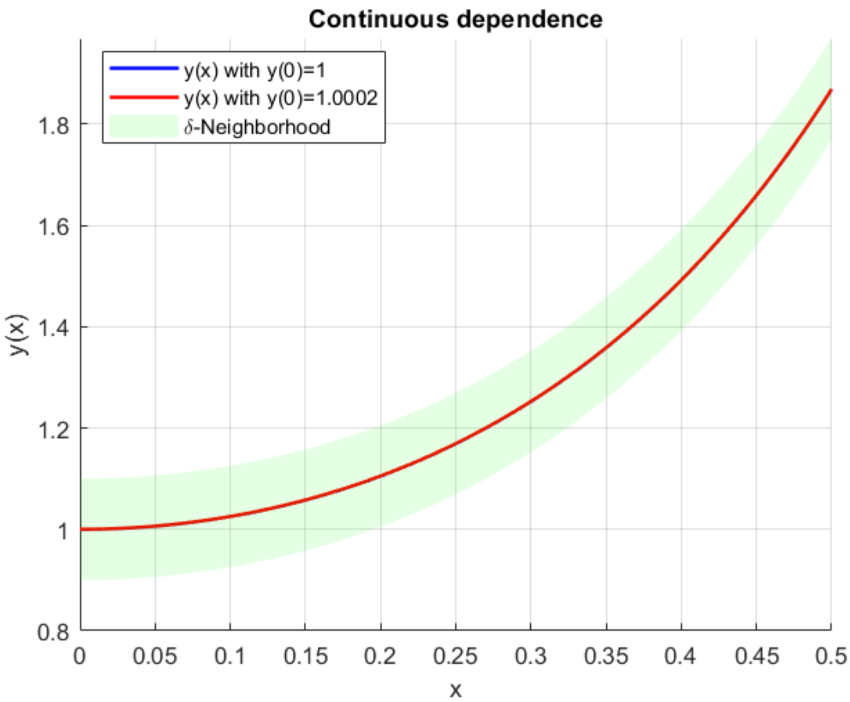
where $C$ is a constant determined by the initial condition.

The particular solution (analytical):

$$y(x) = 5001 \cdot \frac{e^{\frac{5}{2}x^2}}{5000}.$$

The value of the function at $x = 1$:

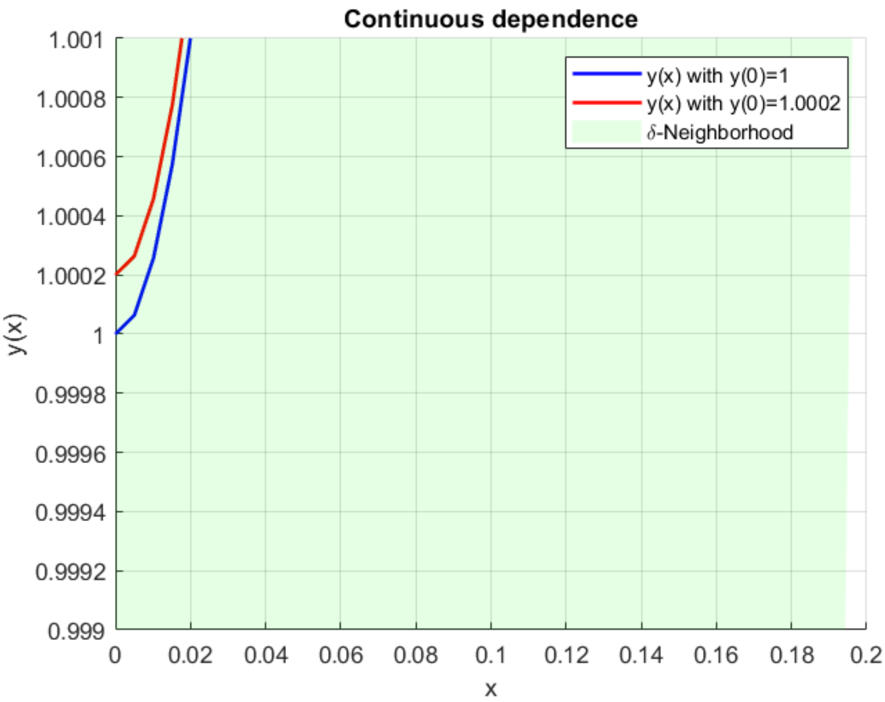
$$y(1) = 5001 \cdot \frac{e^{\frac{5}{2}1^2}}{5000} = 12.1849.$$




## 1.3 Well-Posedness in the Sense of Hadamard

The second problem in the example is a **perturbed problem** associated with the original one, differing only slightly in the initial condition. In practice, most numerical methods deal with such perturbed problems due to **round-off errors**. Thus, it is important to understand when a problem remains stable under small changes, which leads us to the following definition of a well-posed problem.

**Definition 1.2** (Well-Posed Problem)**.** An ODE is said to be **well-posed** if it has the EUC property (Hadamard criteria):




1. **Existence**: The problem should have at least one solution.

2. **Uniqueness**: The problem has at most one solution.

3. **Continuous dependence**: The solution depends continuously on the data that are present in the problem.

**Definition 1.3** (Ill-Posed Problem)**.** An **ill-posed problem** is one which doesn't meet the three Hadamard criteria for being well posed.

**Theorem 1.2** (Well-Posedness of the Initial Value Problem)**.** *The initial value problem for an ordinary differential equation $y' = f(x, y)$, where $f$ is continuous and satisfies a Lipschitz condition in the variable $y$ on $D = \{(x, y) \mid a < x < b,\ c < y < d\}$ containing the initial data $(x_0, y_0)$, is well-posed.*

**Example 1.3** (Well-Posedness)**.** Show that the initial value problem

$$y' = x^2 y + 1, \quad y(0) = 1$$

is well-posed on $D = \{(x, y) \mid 0 \leq x \leq 1,\ -\infty < y < +\infty\}$.

**Solution:**

- **Existence:** The function $f(x, y) = x^2 y + 1$ is continuous, so a solution exists (Peano Theorem).

- **Uniqueness:** The function $f(x, y)$ satisfies a Lipschitz condition in $y$ on any closed rectangle, so the solution is unique:

$$|f(x, y_1) - f(x, y_2)| = |x^2(y_1 - y_2)| = x^2|y_1 - y_2| \leq 1 \cdot |y_1 - y_2| \quad \text{on}$$

$$D = \{(x, y) \mid 0 \leq x \leq 1, -\infty < y < +\infty\}$$

Since $f$ is continuous and satisfies a Lipschitz condition in $y$ on $D$, the Theorem implies that the problem is **well-posed**.

Thus,

- **Continuous dependence:** Small changes in the initial value $y(0)$ lead to small changes in the solution, which follows from the Lipschitz condition.

# 2 Numerical Solution of a First-Order ODE: Euler's Methods

## 2.1 Introduction

**Definition 2.1** (Numerical Solution)**.** A **numerical solution** to an initial value problem is a set of discrete points that approximate the solution function $y(x)$.

For the given IVP

$$y' = f(x, y), \quad y(x_0) = y_0, \quad a \le x \le b \tag{1}$$

The analytical solution – if it exists – might be of the form

$$y = \varphi(x),$$

such as $y = \ln x + 3$.

A numerical solution produces a table of approximate values:

| $x_i$ | $y_i$ |
|---|---|
| 0.23 | 0.1829 |
| 0.24 | 0.1731 |
| 0.25 | 0.1628 |
| 0.26 | 0.1588 |
| 0.27 | 0.1400 |
| $\vdots$ | $\vdots$ |

## 2.2 Overview of Numerical Methods

There exist various numerical methods for solving IVPs, and they are commonly classified into two broad categories:

- **Single-step methods**, where the solution at the next point $x_{i+1}$ is computed using only the information from the current point $x_i$.
- **Multi-step methods**, where the solution at $x_{i+1}$ depends on the computed values at several previous points $x_i, x_{i-1}, \ldots$.

Additionally, numerical methods can be classified as either:

- **Explicit methods**, which provide a direct formula for computing $y_{i+1}$, such as:

$$y_{i+1} = F(x_i, y_i)$$

- **Implicit methods**, where $y_{i+1}$ appears on both sides of the formula and must be computed by solving an equation like:

$$y_{i+1} = F(x_i, x_{i+1}, y_i, y_{i+1})$$

To compute $y_{i+1}$ in implicit methods, we often need to solve nonlinear equations numerically. Standard techniques for this include the *Bisection method* and the *Newton-Raphson method*, among others.

## 2.3 Euler's Explicit Method

Consider the well-posed initial value problem (IVP):

$$y' = f(x, y), \quad y(x_0) = y_0, \quad \text{for } a \leq x \leq b.$$

**Discretization of the Interval.** We divide the interval $[a, b]$ into $N$ equal subintervals using a uniform grid of **mesh points**. Let

$$h = \frac{b-a}{N}$$

be the *step size*, and define these points as:

$$x_i = a + ih, \quad \text{for } i = 0, 1, \ldots, N-1.$$

**Integral Form of the IVP.** Since $f(x, y)$ is continuous, the differential equation can be rewritten in integral form over each subinterval $[x_i, x_{i+1}]$ as:

$$y(x_{i+1}) = y(x_i) + \int_{x_i}^{x_{i+1}} f(s, y(s))\, ds.$$

**Numerical Approximation of the Integral.** We now approximate the integral using the *rectangle rule*:

$$\int_a^b F(x)\, dx \approx (b-a) \cdot F(a).$$

Substituting this approximation into the integral form gives:

$$y(x_{i+1}) \approx y(x_i) + (x_{i+1} - x_i) \cdot f(x_i, y_i) = y(x_i) + h \cdot f(x_i, y_i)$$

**Definition 2.2** (Euler's Method)**.** Given the well-posed IVP

$$y' = f(x, y), \quad y(x_0) = y_0, \quad a \leq x \leq b$$

let the interval $[a, b]$ be divided into $N$ equal subintervals of **step size** $h = \frac{b-a}{N}$, and let the **mesh points** be $x_i = x_0 + ih, \quad \text{for } i = 0, 1, 2, \ldots, N-1$.

Then the Euler Method is a **recursive algorithm** for approximating the solution $y(x)$ at these mesh points using the formula:

$$y_{i+1} = y_i + h \cdot f(x_i, y_i), \quad x_{i+1} = x_i + h. \tag{1}$$

Formula (1) is called the **explicit Euler's method** or **forward Euler's method**. If, instead of (1), we use

$$y_{i+1} = y_i + h \cdot f(x_{i+1}, y_{i+1}), \quad x_{i+1} = x_i + h, \tag{2}$$

then (2) is called the **implicit Euler's method** or **backward Euler's method**.

**Example 2.1** (Explicit Euler – Example 1)**.** Consider the well-posed IVP:

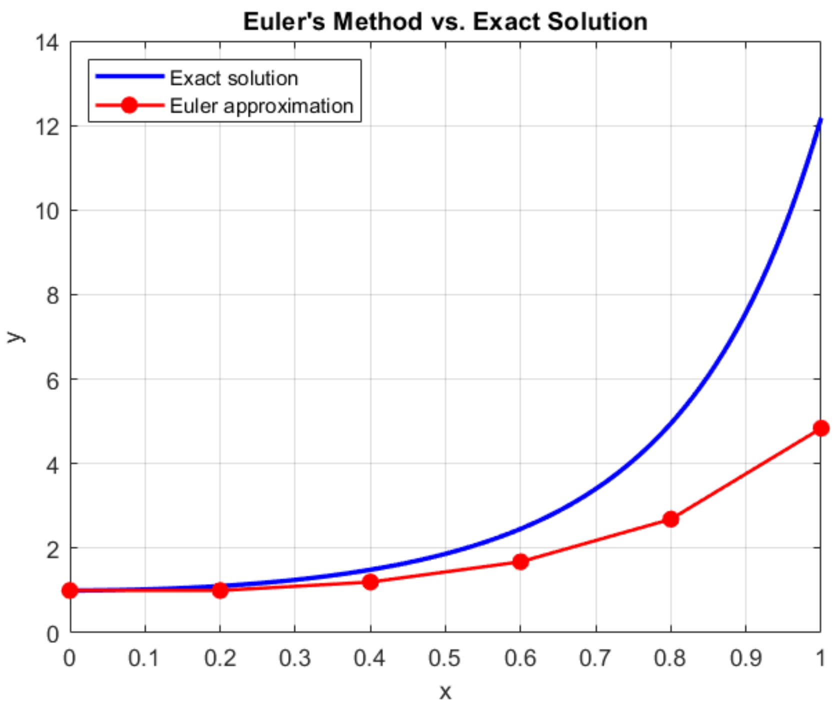
$$y' = 5xy; \quad y(0) = 1, \quad 0 \le x \le 1.$$

Using Euler's explicit method, find an approximate value of $y$ corresponding to $x = 1$.

**Solution:** for $N = 5$ and $h = \frac{b-a}{N} = \frac{1-0}{5} = 0.2$:

| $i$ | $x_i$ | $y_i$ | $y_{i+1} = y_i + 0.2 \cdot 5x_i y_i$ |
|---|---|---|---|
| 0 | 0.0 | 1.0000 | $y_1 = 1 + 0.2 \cdot 0 = 1.0000$ |
| 1 | 0.2 | 1.0000 | $y_2 = 1 + 0.2 \cdot 1.0 = 1.2000$ |
| 2 | 0.4 | 1.2000 | $y_3 = 1.2 + 0.2 \cdot 2.4 = 1.6800$ |
| 3 | 0.6 | 1.6800 | $y_4 = 1.68 + 0.2 \cdot 5.04 = 2.6880$ |
| 4 | 0.8 | 2.6880 | $y_5 = 2.688 + 0.2 \cdot 10.752 = 4.8384$ |

The approximate value is $y(1) \approx \boxed{4.8384}$.




**Example 2.2** (Explicit Euler – Example 2)**.** Consider the well-posed IVP:

$$y' = x^2 y + 1; \quad y(0) = 1, \quad 0 \le x \le 1.$$

Using Euler's explicit method, find an approximate value of $y$ corresponding to $x = 1$.

**Solution:** for $N = 10$ and $h = 0.1$:

| $i$ | $x_i$ | $y_i$ | $y_{i+1} = y_i + 0.1 \cdot (x_i^2 y_i + 1)$ |
|---|---|---|---|
| 0 | 0.0 | 1.000000 | $1 + 0.1 \cdot 1 = 1.100000$ |
| 1 | 0.1 | 1.100000 | $1.1 + 0.1 \cdot (0.01 \cdot 1.1 + 1) = 1.201100$ |
| 2 | 0.2 | 1.201100 | $1.2011 + 0.1 \cdot (0.04 \cdot 1.2011 + 1) = 1.305904$ |
| 3 | 0.3 | 1.305904 | $1.3059 + 0.1 \cdot (0.09 \cdot 1.3059 + 1) = 1.417657$ |
| 4 | 0.4 | 1.417657 | $1.4177 + 0.1 \cdot (0.16 \cdot 1.4177 + 1) = 1.540339$ |
| 5 | 0.5 | 1.540339 | $1.5403 + 0.1 \cdot (0.25 \cdot 1.5403 + 1) = 1.678848$ |
| 6 | 0.6 | 1.678848 | $1.6788 + 0.1 \cdot (0.36 \cdot 1.6788 + 1) = 1.839287$ |
| 7 | 0.7 | 1.839287 | $1.8393 + 0.1 \cdot (0.49 \cdot 1.8393 + 1) = 2.029312$ |
| 8 | 0.8 | 2.029312 | $2.0293 + 0.1 \cdot (0.64 \cdot 2.0293 + 1) = 2.259188$ |
| 9 | 0.9 | 2.259188 | $2.2592 + 0.1 \cdot (0.81 \cdot 2.2592 + 1) = 2.542182$ |

The approximate value is $y(1) \approx \boxed{2.5422}$.

**Example 2.3** (Explicit Euler – Example 3)**.** Consider the well-posed IVP:

$$y' = 1 + x\sin(xy); \quad y(0) = 0, \quad 0 \leq x \leq 2.$$

Using Euler's explicit method, find an approximate value of $y$ corresponding to $x = 2$.

**Solution:** for $N = 5$ and $h = \frac{2-0}{5} = 0.4$.

We apply Euler's method: $y_{i+1} = y_i + h \cdot f(x_i, y_i)$ with $f(x, y) = 1 + x\sin(xy)$.

| $i$ | $x_i$ | $y_i$ | $f(x_i, y_i)$ | $y_{i+1}$ |
|---|---|---|---|---|
| 0 | 0.0 | 0.0000 | $1 + 0 \cdot \sin(0) = 1.0000$ | $0.0000 + 0.4 \cdot 1 = 0.4000$ |
| 1 | 0.4 | 0.4000 | $1 + 0.4 \cdot \sin(0.16) \approx 1.0638$ | $0.4000 + 0.4 \cdot 1.0638 \approx 0.8255$ |
| 2 | 0.8 | 0.8255 | $1 + 0.8 \cdot \sin(0.6604) \approx 1.4645$ | $0.8255 + 0.4 \cdot 1.4645 \approx 1.4113$ |
| 3 | 1.2 | 1.4113 | $1 + 1.2 \cdot \sin(1.6936) \approx 1.9481$ | $1.4113 + 0.4 \cdot 1.9481 \approx 2.1906$ |
| 4 | 1.6 | 2.1906 | $1 + 1.6 \cdot \sin(3.5049) \approx 2.4783$ | $2.1906 + 0.4 \cdot 2.4783 \approx 3.1829$ |

The approximate value is $y(2) \approx \boxed{3.1829}$.

## 2.4 Implicit (Backward) Euler's Method

**Example 2.4** (Implicit Euler – Example 4)**.** Consider the well-posed IVP:

$$y' = xy + 1, \quad y(0) = 1.$$

Use Euler's implicit method with a step size of $h = 0.25$ to find approximate values of the solution at $x = 1$.

**Solution:**

- $n = 0$: $y_0 = 1, x_0 = 0, x_1 = x_0 + h = 0 + 0.25 = 0.25$

$$y_1 = y_0 + h \cdot f(x_1, y_1) = 1 + 0.25 \cdot (0.25y_1 + 1)$$

$$y_1 = 1 + 0.0625y_1 + 0.25$$

$$0.9375y_1 = 1.25 \Rightarrow y_1 = \frac{1.25}{0.9375} \approx 1.3333$$

- $n = 1$: $x_1 = 0.25,\ y_1 \approx 1.3333$, so

$$x_2 = x_1 + h = 0.25 + 0.25 = 0.5$$
$$y_2 = y_1 + h \cdot f(x_2, y_2) = 1.3333 + 0.25 \cdot (0.5y_2 + 1)$$
$$y_2 = 1.3333 + 0.125y_2 + 0.25$$
$$0.875y_2 = 1.5833 \Rightarrow y_2 = \frac{1.5833}{0.875} \approx 1.8101$$

- $n = 2$: $x_2 = 0.5,\ y_2 \approx 1.8101$, so

$$x_3 = x_2 + h = 0.5 + 0.25 = 0.75$$
$$y_3 = y_2 + h \cdot f(x_3, y_3) = 1.8101 + 0.25 \cdot (0.75y_3 + 1)$$
$$y_3 = 1.8101 + 0.1875y_3 + 0.25$$
$$0.8125y_3 = 2.0601 \Rightarrow y_3 = \frac{2.0601}{0.8125} \approx 2.5362$$

- $n = 3$: $x_3 = 0.75,\ y_3 \approx 2.5362$, so

$$x_4 = x_3 + h = 0.75 + 0.25 = 1$$
$$y_4 = y_3 + h \cdot f(x_4, y_4) = 2.5362 + 0.25 \cdot (1.0y_4 + 1)$$
$$y_4 = 2.5362 + 0.25y_4 + 0.25$$
$$0.75y_4 = 2.7862 \Rightarrow y_4 = \frac{2.7862}{0.75} \approx 3.7149$$

**Summary of approximate values using Implicit Euler's Method with $h = 0.25$:**

| $n$ | $x_n$ | $y_n$ (approx) |
|---|---|---|
| 0 | 0.00 | 1.0000 |
| 1 | 0.25 | 1.3333 |
| 2 | 0.50 | 1.8101 |
| 3 | 0.75 | 2.5362 |
| 4 | 1.00 | 3.7149 |

## 2.5 Geometric Interpretation of Euler's Method: Direction Fields

Consider the well-posed initial value problem:

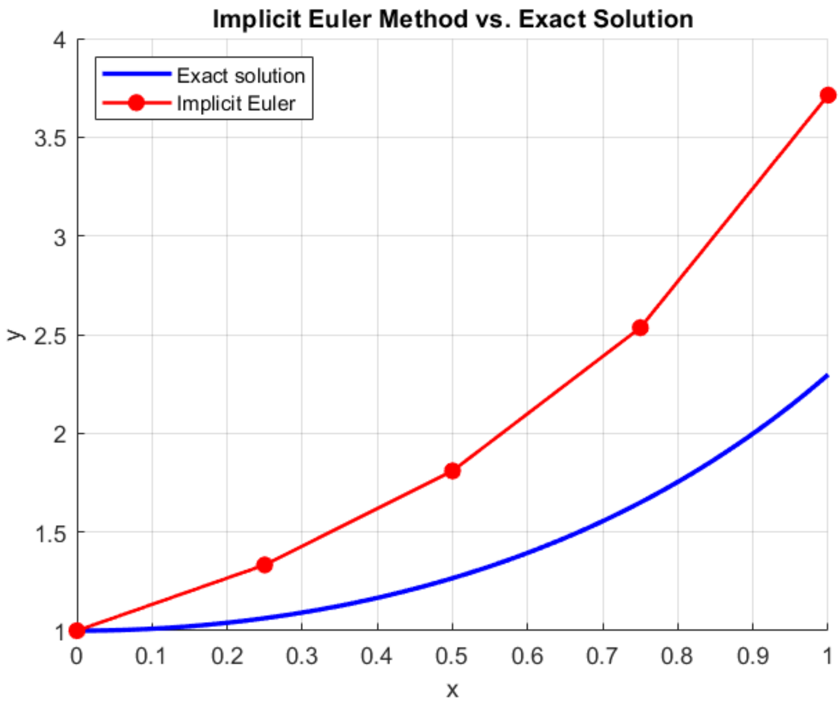
$$y' = f(x, y), \quad y(x_0) = y_0.$$

The derivative $y'$ gives the slope of the tangent to the solution $y(x)$. This slope is determined by the differential equation. At the point $(x_0, y_0)$, it is

$$y' = \frac{dy}{dx} = f(x_0, y_0).$$

This slope can be represented as a *short line segment* or *arrow* indicating the direction of the solution curve.

**Direction Field:** By evaluating $f(x, y)$ at multiple points, we construct a family of such directional arrows – a **direction field** – a visualization of the behavior of solutions. Since solutions do not cross and nearby tangents nearly match (Picard-Lindelöf Theorem), the direction field offers a complete **qualitative** picture of the system's dynamics.




# 3 Taylor Series Methods

## 3.1 Derivation

Consider the initial value problem:

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0.$$

To approximate $y(x)$ at $x = x_0 + h$, we apply the Taylor expansion of $y(x)$ about $x_0$:

$$y(x_0 + h) = y(x_0) + hy'(x_0) + \frac{h^2}{2!}y''(x_0) + \frac{h^3}{3!}y^{(3)}(x_0) + \cdots + \frac{h^p}{p!}y^{(p)}(x_0) + \mathcal{O}(h^{p+1}).$$

Truncating the series after $p$ terms gives the **Taylor method of order** $p$:

$$y_{n+1} = y_n + hy'(x_n) + \frac{h^2}{2!}y''(x_n) + \cdots + \frac{h^p}{p!}y^{(p)}(x_n),$$

where $y_n \approx y(x_n)$, and $x_{n+1} = x_n + h$.

Each derivative $y^{(k)}(x_n)$ can be recursively computed using the chain rule:

$$\begin{aligned}
y' &= f(x, y), \\
y'' &= \frac{d}{dx}f(x, y) = f_x + f_y y', \\
y^{(3)} &= \frac{d}{dx}(f_x + f_y y') = \cdots,
\end{aligned}$$

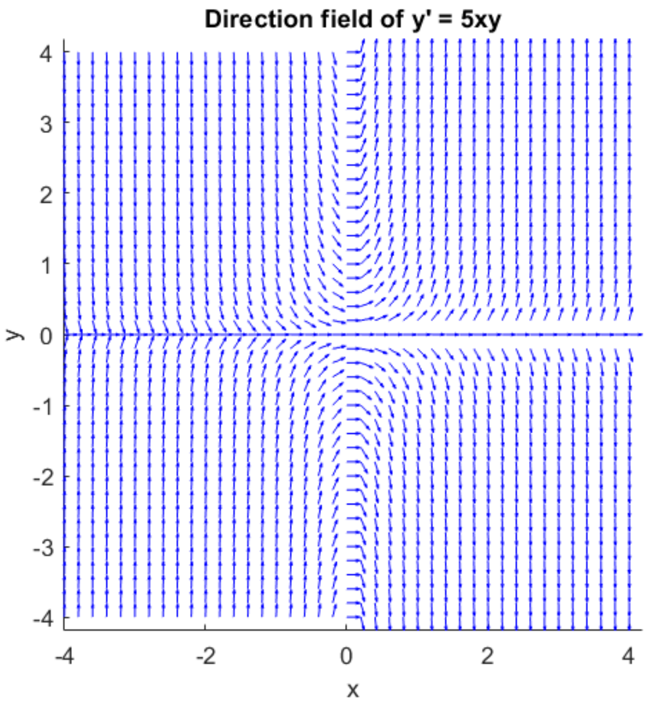
and so on, using repeated applications of the total derivative:

$$\frac{d^k y}{dx^k} = \frac{d^{k-1}}{dx^{k-1}}(f(x, y)).$$

*Remark* 3.1. Euler's method is a first-order Taylor method. For $p = 1$, we get:

$$y_{n+1} = y_n + hf(x_n, y_n),$$

which is **Euler's method**.




## 3.2 Taylor Series Methods: Examples

**Example 3.1** (Taylor Series – Example 1)**.** Solve the well-posed IVP:

$$y' = 5xy, \quad y(0) = 1, \quad 0 \le x \le 1$$

by Taylor's series method.

We compute derivatives:

$$\begin{aligned}
y'(x) &= 5xy \quad \Rightarrow \quad y'(0) = 5 \cdot 0 \cdot y(0) = 0, \\
y''(x) &= \frac{d}{dx}(5xy) = 5y + 5xy' \quad \Rightarrow \quad y''(0) = 5 \cdot 1 + 5 \cdot 0 \cdot 0 = 5, \\
y^{(3)}(x) &= \frac{d}{dx}(5y + 5xy') = 5y' + 5y' + 5xy'' = 10y' + 5xy'' \\
&\Rightarrow \quad y^{(3)}(0) = 10 \cdot 0 + 5 \cdot 0 \cdot 5 = 0, \\
y^{(4)}(x) &= \frac{d}{dx}(10y' + 5xy'') = 10y'' + 5y'' + 5xy^{(3)} = 15y'' + 5xy^{(3)} \\
&\Rightarrow \quad y^{(4)}(0) = 15 \cdot 5 + 5 \cdot 0 \cdot 0 = 75.
\end{aligned}$$

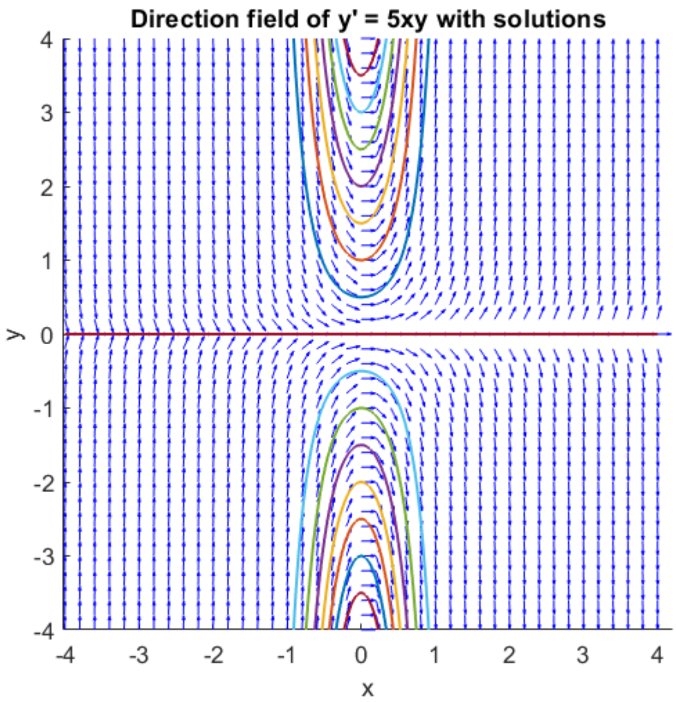
Now we apply the Taylor series expansion around $x = 0$:

$$y(x) = y(0) + \frac{x^1}{1!}y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y^{(3)}(0) + \frac{x^4}{4!}y^{(4)}(0) + \cdots$$

Substitute the known values:

$$y(x) \approx 1 + 0 + \frac{5}{2}x^2 + 0 + \frac{75}{24}x^4 + \cdots = 1 + \frac{5}{2}x^2 + \frac{25}{8}x^4 + \cdots$$

This is the Taylor approximation to the solution of the IVP.




## 3.3 Picard Iteration

**Example 3.2** (Picard Iteration – Example 1)**.** Solve the well-posed IVP:

$$y' = 5xy, \quad y(0) = 1, \quad 0 \le x \le 1$$

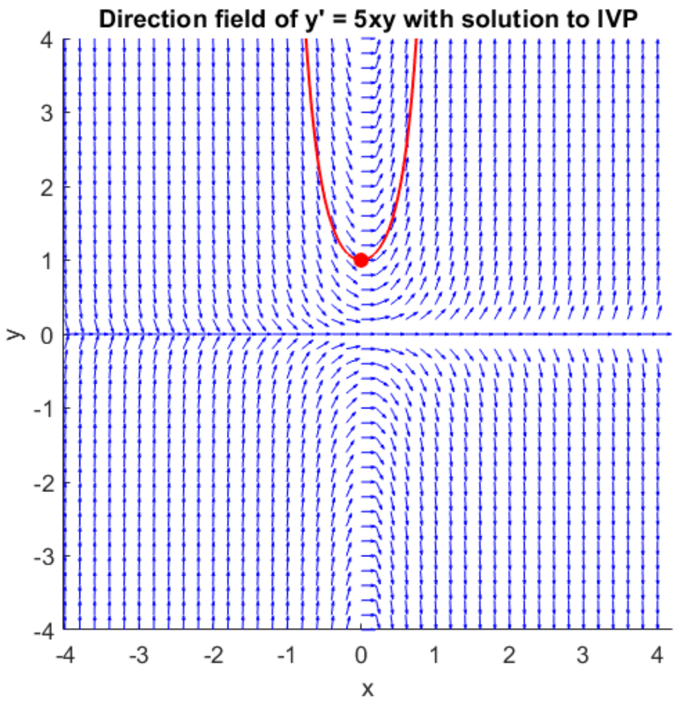
by Picard's iteration.

**Step 1: Rewrite IVP as Integral Equation.**

$$\begin{aligned} & y' = 5xy, \quad y(0) = 1 \\ \Rightarrow \quad & y(x) = y_0 + \int_0^x f(s, y(s))\, ds \\ & y(x) = 1 + \int_0^x 5sy(s)\, ds \end{aligned}$$

**Step 2: Define Iteration Sequence.** Using method of successive approximations:

$$\begin{aligned} y_0(x) &= 1 \\ y_1(x) &= 1 + \int_0^x 5s \cdot y_0(s)\, ds = 1 + \int_0^x 5s \cdot 1\, ds = 1 + \frac{5x^2}{2} \\ y_2(x) &= 1 + \int_0^x 5s \left(1 + \frac{5s^2}{2}\right) ds = 1 + \frac{5x^2}{2} + \frac{25x^4}{8} \\ y_3(x) &= 1 + \int_0^x 5s \left(1 + \frac{5s^2}{2} + \frac{25s^4}{8}\right) ds \\ &= 1 + \frac{5x^2}{2} + \frac{25x^4}{8} + \frac{125x^6}{48} \\ y_4(x) &= 1 + \int_0^x 5s \left(1 + \frac{5s^2}{2} + \frac{25s^4}{8} + \frac{125s^6}{48}\right) ds \\ &= 1 + \frac{5x^2}{2} + \frac{25x^4}{8} + \frac{125x^6}{48} + \frac{625x^8}{384} \end{aligned}$$




### 3.4 Comparison: Taylor vs Picard (Example 1)

**Given IVP:** $y' = 5xy, \quad y(0) = 1$

- **Taylor Series Method:**

$$y(x) = 1 + \frac{5x^2}{2!} + \frac{25x^4}{4!} + \frac{125x^6}{6!} + \cdots$$

- **Picard Iteration:**

$$y_4(x) \approx 1 + \frac{5x^2}{2} + \frac{25x^4}{8} + \frac{125x^6}{48} + \frac{625x^8}{384} + \cdots$$

**Approximations at $x = 1$:**

- **Exact solution:**

$$y(1) = e^{\frac{5}{2}} \approx \boxed{12.1825}$$

- **Euler's method (step size $h = 0.2$):**

$$y(1) \approx \boxed{4.8384}$$

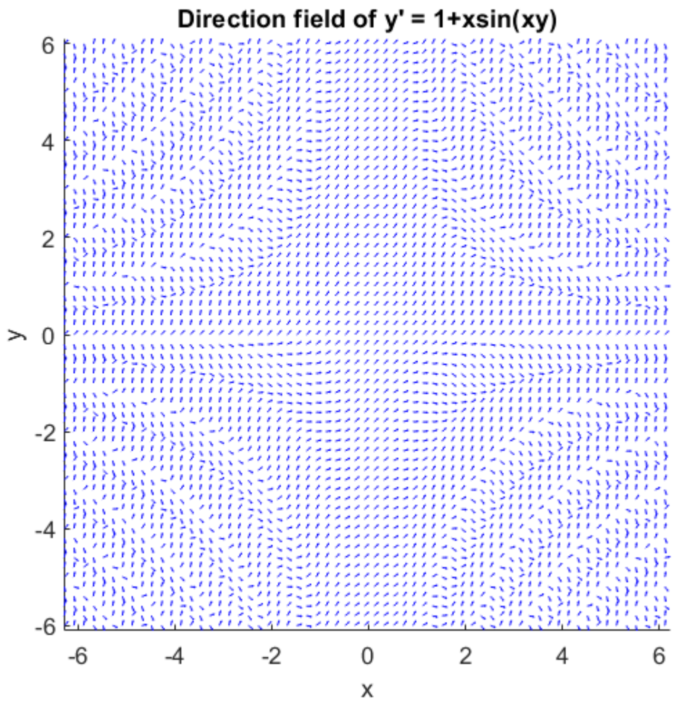
- **Taylor method (4th-degree polynomial):**

$$y(x) \approx 1 + \frac{5x^2}{2!} + \frac{25x^4}{4!} + \frac{125x^6}{6!} + \frac{625x^8}{8!}$$

$$y(1) \approx 1 + \frac{5}{2} + \frac{25}{24} + \frac{125}{720} + \frac{625}{40320} \approx \boxed{11.892}$$

- **Picard iteration (4th step):**

$$y_4(1) = 1 + \frac{5}{2} + \frac{25}{8} + \frac{125}{48} + \frac{625}{384} \approx \boxed{11.9245}$$




## 3.5 Taylor Series Methods: Example 2

**Example 3.3** (Taylor Series – Example 2)**.** Solve the well-posed IVP:

$$y' = x^2 y + 1, \quad y(0) = 1, \quad 0 \le x \le 1$$

by Taylor's series method.

We compute derivatives:

$$\begin{aligned}
y'(x) &= x^2 y + 1, \quad y'(0) = 0^2 \cdot 1 + 1 = 1 \\
y''(x) &= \frac{d}{dx}(x^2 y + 1) = 2xy + x^2 y' \\
y''(0) &= 2 \cdot 0 \cdot 1 + 0^2 \cdot 1 = 0 \\
y'''(x) &= \frac{d}{dx}(2xy + x^2 y') = 2y + 2xy' + 2xy' + x^2 y'' \\
&= 2y + 4xy' + x^2 y'' \\
y'''(0) &= 2 \cdot 1 + 4 \cdot 0 \cdot 1 + 0^2 \cdot 0 = 2 \\
y^{(4)}(x) &= \frac{d}{dx}(2y + 4xy' + x^2 y'') \\
&= 2y' + 4y' + 4xy'' + 2xy'' + x^2 y''' \\
&= 6y' + 6xy'' + x^2 y''' \\
y^{(4)}(0) &= 6 \cdot 1 + 6 \cdot 0 \cdot 0 + 0^2 \cdot 2 = 6
\end{aligned}$$

Taylor expansion around $x = 0$:

$$y(x) \approx y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 =$$

$$= 1 + x + 0 + \frac{2}{6}x^3 + \frac{6}{24}x^4 = 1 + x + \frac{x^3}{3} + \frac{x^4}{4}$$

This is the Taylor approximation to the solution of the IVP.

**Example 3.4** (Picard Iteration – Example 2)**.** Solve the well-posed IVP:

$$y' = x^2y + 1, \quad y(0) = 1, \quad 0 \le x \le 1$$

by Picard's iteration.

**Step 1: Rewrite IVP as Integral Equation.**

$$\begin{aligned} & y' = x^2y + 1, \quad y(0) = 1 \\ \Rightarrow \quad & y(x) = y_0 + \int_0^x f(s, y(s))\, ds \\ & y(x) = 1 + \int_0^x (s^2y(s) + 1)\, ds \end{aligned}$$

**Step 2: Define Iteration Sequence.** Using method of successive approximations:

$$\begin{aligned} y_0(x) &= 1 \\ y_1(x) &= 1 + \int_0^x (s^2 \cdot y_0(s) + 1)\, ds = 1 + \int_0^x (s^2 + 1)\, ds = 1 + \frac{x^3}{3} + x \\ y_2(x) &= 1 + \int_0^x (s^2 \cdot y_1(s) + 1)\, ds \\ &= 1 + \int_0^x s^2 \left(1 + s + \frac{s^3}{3}\right) + 1\, ds \\ &= 1 + \int_0^x \left(s^2 + s^3 + \frac{s^5}{3} + 1\right) ds \\ &= 1 + x + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^6}{18} \\ y_3(x) &= 1 + \int_0^x s^2 \left(1 + s + \frac{s^3}{3} + \frac{s^4}{4} + \frac{s^6}{18}\right) + 1\, ds \\ &= 1 + \int_0^x \left(s^2 + s^3 + \frac{s^5}{3} + \frac{s^6}{4} + \frac{s^8}{18} + 1\right) ds \\ &= 1 + x + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^6}{18} + \frac{x^7}{28} + \frac{x^9}{162} \\ y_4(x) &= 1 + \int_0^x s^2 \left(1 + s + \frac{s^3}{3} + \frac{s^4}{4} + \frac{s^6}{18} + \frac{s^7}{28} + \frac{s^9}{162}\right) + 1\, ds \\ &= 1 + x + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^6}{18} + \frac{x^7}{28} + \frac{x^9}{162} + \frac{x^5}{5} + \frac{x^8}{8} + \frac{x^{10}}{90} + \frac{x^{11}}{308} + \frac{x^{13}}{2106} \end{aligned}$$

## 3.6 Comparison: Taylor vs Picard (Example 2)

**Given IVP:** $\quad y' = x^2y + 1, \quad y(0) = 1$

- **Taylor Series Method:**

$$y(x) = 1 + x + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

- **Picard Iteration:**

$$y_4(x) \approx 1 + x + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^6}{18} + \frac{x^7}{28} + \frac{x^9}{162} + \frac{x^5}{5} + \frac{x^8}{8} + \frac{x^{10}}{90} + \frac{x^{11}}{308} + \frac{x^{13}}{2106}$$

**Approximations at $x = 1$:**

- **Euler's method (step size $h = 0.1$):**

$$y(1) \approx \boxed{2.542182}$$

- **Taylor method (4th-degree polynomial):**

$$y(1) \approx 1 + 1 + \frac{1^3}{3} + \frac{1^4}{4} = 1 + 1 + \frac{1}{3} + \frac{1}{4} = \frac{31}{12} \approx \boxed{2.5833}$$

- **Picard iteration (4th step):**

$$y_4(1) \approx 1 + 1 + \frac{1}{3} + \frac{1}{4} + \frac{1}{18} + \frac{1}{28} + \frac{1}{162} + \frac{1}{5} + \frac{1}{8} + \frac{1}{90} + \frac{1}{308} + \frac{1}{2106} \approx \boxed{2.5930}$$