Lesson 2. First-order Ordinary Differential Equations II

ELTE Eötvös Loránd University

Faculty of Informatics, Department of Numerical Analysis

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Table of Contents

Continuous Dependence on Initial Data

Question:

How do we determine whether a particular problem has the property that small changes, or perturbations, in the statement of the problem introduce correspondingly small changes in the solution?

For the unique solution $y(x)$ of the IVP

$$y' = f(x, y), \quad y(x_0) = y_0,$$

the **continuous dependence on the initial data** means that small changes in the initial value y_0 lead to small changes in the solution $y(x)$ (at least on some interval around x_0).

In essence:

continuous dependence on initial data means that the problem is stable in the sense that small errors in the initial data do not lead to large errors in the solution, at least initially.

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Continuous Dependence on Initial Data

Theorem (Continuous dependence on Initial Data)

Let $f(x, y)$ be continuous on an open rectangle

$$D = \{(x, y) \mid a < x < b,\ c < y < d\}$$

in the xy-plane. Let $(x_0, y_0) \in D$, and suppose that f satisfies a **Lipschitz condition with respect to** y on D.

Let $y(x)$ be the unique solution to the IVP

$$y' = f(x, y), \quad y(x_0) = y_0,$$

and let $z(x)$ be the solution to the nearby IVP

$$z' = f(x, z), \quad z(x_0) = y_0 + \delta.$$

Then there exists an interval $I = [x_0 - \varepsilon, x_0 + \varepsilon] \subset (a, b)$, and a constant $C > 0$, such that

$$|y(x) - z(x)| \leq C|\delta| \quad \text{for all } x \in I.$$

Proof of Continuous Dependence on Initial Data

Step 1

Let $y(x)$ be the unique solution to the initial value problem (IVP)

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0,$$

and let $z(x)$ be the unique solution to the IVP

$$z'(x) = f(x, z(x)), \quad z(x_0) = y_0 + \delta.$$

Then, using the integral form of the solution, we have:

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))\, ds, \quad z(x) = y_0 + \delta + \int_{x_0}^{x} f(s, z(s))\, ds.$$

Proof of Continuous Dependence on Initial Data

Step 2

Subtract the two equations:

$$|y(x) - z(x)| = \left| -\delta + \int_{x_0}^{x} [f(s, y(s)) - f(s, z(s))] \, ds \right|.$$

Step 3

Apply the triangle inequality:

$$|y(x) - z(x)| \leq |\delta| + \left| \int_{x_0}^{x} [f(s, y(s)) - f(s, z(s))] \, ds \right|.$$

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Proof of Continuous Dependence on Initial Data

Step 4

Use the inequality:

$$\left|\int_{x_0}^{x} h(s)\,ds\right| \leq \int_{x_0}^{x} |h(s)|\,ds,$$

to get:

$$|y(x) - z(x)| \leq |\delta| + \int_{x_0}^{x} |f(s, y(s)) - f(s, z(s))|\,ds.$$

Proof of Continuous Dependence on Initial Data

Step 5

Since $f(x, y)$ satisfies a Lipschitz condition in y, i.e.,

$$|f(s, y_1) - f(s, y_2)| \leq L|y_1 - y_2|,$$

we get:

$$|y(x) - z(x)| \leq |\delta| + L \int_{x_0}^{x} |y(s) - z(s)|\, ds.$$

Proof of Continuous Dependence on Initial Data

Step 6

Applying successive approximation, and in general, we can show, by induction, that

$$|y(x) - z(x)| \leq |\delta| \left(1 + L|x - x_0| + \frac{L^2|x - x_0|^2}{2!} + \cdots + \frac{L^n|x - x_0|^n}{n!} + \cdots \right) \leq$$

$$\leq C|\delta|,$$

where $C = e^{L|x-x_0|}$.

Thus

$$|y(x) - z(x)| \leq |\delta| e^{L|x-x_0|}$$

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Continuous Dependence on Initial Data: Example

Example: Consider the IVP

$$y' = 5xy; \quad y(0) = 1$$

The general solution (analitical):

$$y(x) = Ce^{\frac{5}{2}x^2},$$

where C is a constant determined by the initial condition.

The particular solution (analitical):

$$y(x) = e^{\frac{5}{2}x^2}.$$

The value of the function at $x = 1$

$$y(1) = e^{\frac{5}{2}1^2} = 12.1825$$

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Continuous Dependence on Initial Data: Example

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$$y' = 5xy; \quad y(0) = 1.0002$$

The general solution (analitical):

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where C is a constant determined by the initial condition.

The particular solution (analitical):

$$y(x) = 5001 \cdot \frac{e^{\frac{5}{2}x^2}}{5000}.$$

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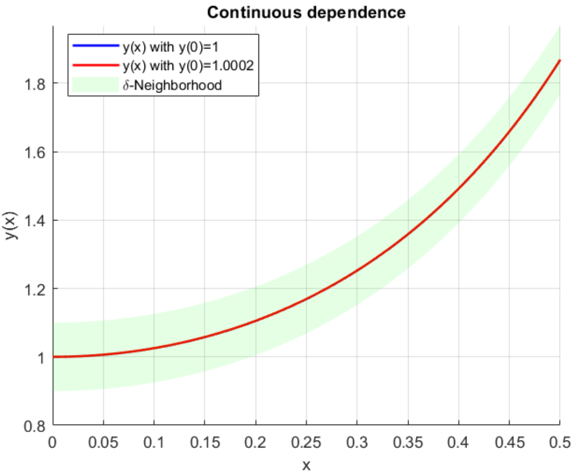

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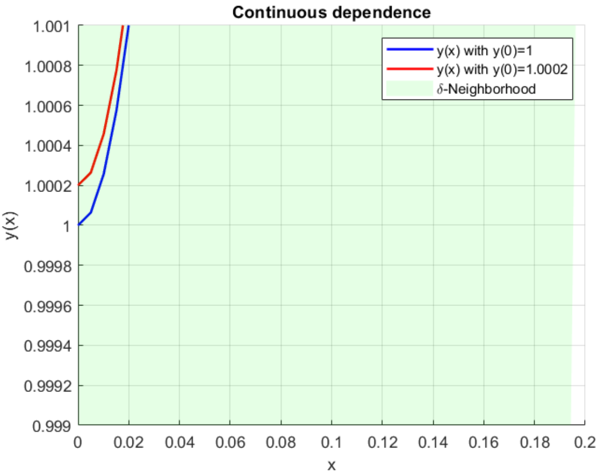

The "Well-Posedness" of Differential Equations: the Sense of Hadamard

Perturbed problem

The second problem in the example is a **perturbed problem** associated with the original one, differing only slightly in the initial condition. In practice, most numerical methods deal with such perturbed problems due to **round-off errors**. Thus, it is important to understand when a problem remains stable under small changes, which leads us to the following definition of a well-posed problem.

An ODE is said to be **well-posed** if it has the EUC property (Hadamard criteria):

1. **Existence**: The problem should have at least one solution.
2. **Uniqueness**: The problem has at most one solution.
3. **Continuous dependence**: The solution depends continuously on the data that are present in the problem.

An **ill-posed problem** is one which doesn't meet the three Hadamard criteria for being well posed.

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The "Well-Posedness" of Differential Equations: the Sense of Hadamard

Theorem (Well-posedness of the Initial Value Problem)

The initial value problem for an ordinary differential equation $y' = f(x, y)$, where f is continuous and satisfies a Lipschitz condition in the variable y on $D = \{(x, y) \mid a < x < b,\ c < y < d\}$ containing the initial data (x_0, y_0), is well-posed.

Show that the initial value problem

$$y' = x^2 y + 1, \quad y(0) = 1$$

is well-posed on $D = \{(x, y) \mid 0 \leq x \leq 1,\ -\infty < y < +\infty\}$

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Solution:

- **Existence:** The function $f(x, y) = x^2y + 1$ is continuous, so a solution exists (Peano Theorem).
- **Uniqueness:** The function $f(x, y)$ satisfies a Lipschitz condition in y on any closed rectangle, so the solution is unique:

$$|f(x, y_1) - f(x, y_2)| = |x^2(y_1 - y_2)| = x^2|y_1 - y_2| \leq 1 \cdot |y_1 - y_2| \quad \text{on}$$

$$D = (x, y)|0 \leq x \leq 1, -\infty < y < +\infty$$

Since f is continuous and satisfies a Lipschitz condition in y on D, Theorem implies that the problem is **well-posed**.
Thus,

- **Continuous dependence:** Small changes in the initial value $y(0)$ lead to small changes in the solution, which follows from the Lipschitz condition.

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Introduction

A **numerical solution** to an initial value problem (1) is a set of discrete points that approximate the solution function $y(x)$.

For the given IVP

$$y' = f(x, y), \quad y(x_0) = y_0, \quad a \le x \le b \tag{1}$$

Analytical Solution

The analytical solution - if it exists - might be of the form

$$y = \varphi(x),$$

such as

$$y = \ln x + 3.$$

Numerical Solution

A numerical solution produces a table of approximate values:

x_i	y_i
0.23	0.1829
0.24	0.1731
0.25	0.1628
0.26	0.1588
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Overview of numerical methods used/or solving a first-order ODE

Single vs Multi-steps Methods

There exist various numerical methods for solving IVPs, and they are commonly classified into two broad categories:

- **Single-step methods**, where the solution at the next point x_{i+1} is computed using only the information from the current point x_i.
- **Multi-step methods**, where the solution at x_{i+1} depends on the computed values at several previous points $x_i, x_{i-1}, \ldots$.

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Explicit vs Implicit methods

Additionally, numerical methods can be classified as either:

- **Explicit methods**, which provide a direct formula for computing y_{i+1}, such as:

$$y_{i+1} = F(x_i, y_i)$$

- **Implicit methods**, where y_{i+1} appears on both sides of the formula and must be computed by solving an equation like:

$$y_{i+1} = F(x_i, x_{i+1}, y_i, y_{i+1})$$

To compute y_{i+1} in implicit methods, we often need to solve nonlinear equations numerically. Standard techniques for this include the *Bisection method* and the *Newton-Raphson method*, among others.

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Euler's Explicit Method

Consider the well-posed initial value problem (IVP):

$$y' = f(x, y), \quad y(x_0) = y_0, \quad \text{for } a \leq x \leq b.$$

Discretization of the Interval

We divide the interval $[a, b]$ into N equal subintervals using a uniform grid of **mesh points**. Let

$$h = \frac{b-a}{N}$$

be the *step size*, and define these points as:

$$x_i = a + ih, \quad \text{for } i = 0, 1, \ldots, N-1.$$

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Euler's Explicit Method

Integral Form of the IVP

Since $f(x, y)$ is continuous, the differential equation can be rewritten in integral form over each subinterval $[x_i, x_{i+1}]$ as:

$$y(x_{i+1}) = y(x_i) + \int_{x_i}^{x_{i+1}} f(s, y(s))\, ds.$$

Numerical Approximation of the Integral

We now approximate the integral using the *rectangle rule*:

$$\int_a^b F(x)\, dx \approx (b-a) \cdot F(a).$$

Substituting this approximation into the integral form gives:

$$y(x_{i+1}) \approx y(x_i) + (x_{i+1} - x_i) \cdot f(x_i, y_i) = y(x_i) + h \cdot f(x_i, y_i)$$

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Euler's Method Given the well-posed IVP

$$y' = f(x, y), \quad y(x_0) = y_0, \quad a \le x \le b$$

Let the interval $[a, b]$ be divided into N equal subintervals of **step size** $h = \frac{b-a}{N}$, and let the **mesh points** be $x_i = x_0 + ih, \quad$ for $i = 0, 1, 2, \ldots, N-1$.
Then the Euler Method is a **recursive algorithm** for approximating the solution $y(x)$ at these mesh points using the formula:

$$y_{i+1} = y_i + h \cdot f(x_i, y_i), \quad x_{i+1} = x_i + h. \tag{1}$$

Formula (1) is called the **explicit Euler's method** or **forward Euler's method**.
If, instead of (1), we use

$$y_{i+1} = y_i + h \cdot f(x_{i+1}, y_{i+1}), \quad x_{i+1} = x_i + h, \tag{2}$$

then (2) is called the **implicit Euler's method** or **backward Euler's method**.

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Euler's Method Given the well-posed IVP

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Euler's Explicit Method

Example 1: Consider the well-posed IVP:

$$y' = 5xy; \quad y(0) = 1, \quad 0 \leq x \leq 1$$

Using Euler's explicit method, find an approximate value of y corresponding to $x = 1$.

Solution: for $N = 5$ and $h = \frac{b-a}{N} = \frac{1-0}{5} = 0.2$

i	x_i	y_i	$y_{i+1} = y_i + 0.2 \cdot 5x_iy_i$
0	0.0	1.0000	$y_1 = 1 + 0.2 \cdot 0 = 1.0000$
1	0.2	1.0000	$y_2 = 1 + 0.2 \cdot 1.0 = 1.2000$
2	0.4	1.2000	$y_3 = 1.2 + 0.2 \cdot 2.4 = 1.6800$
3	0.6	1.6800	$y_4 = 1.68 + 0.2 \cdot 5.04 = 2.6880$
4	0.8	2.6880	$y_5 = 2.688 + 0.2 \cdot 10.752 = 4.8384$

The approximate value is $y(1) \approx \boxed{4.8384}$.

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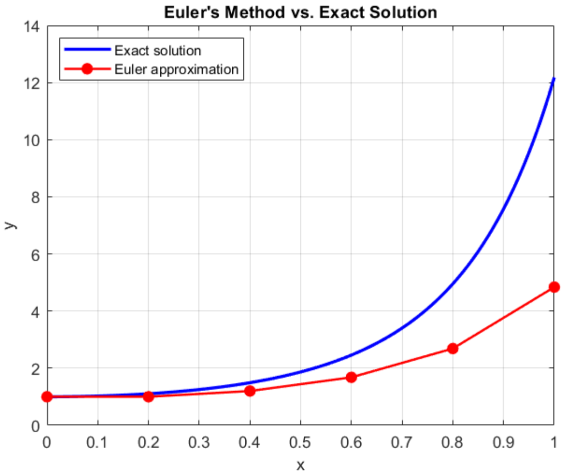
i	x_i	y_i	$y_{i+1} = y_i + 0.2 \cdot 5x_i y_i$
0	0.0	1.0000	$y_1 = 1 + 0.2 \cdot 0 = 1.0000$
1	0.2	1.0000	$y_2 = 1 + 0.2 \cdot 1.0 = 1.2000$
2	0.4	1.2000	$y_3 = 1.2 + 0.2 \cdot 2.4 = 1.6800$
3	0.6	1.6800	$y_4 = 1.68 + 0.2 \cdot 5.04 = 2.6880$
4	0.8	2.6880	$y_5 = 2.688 + 0.2 \cdot 10.752 = 4.8384$

The approximate value is $y(1) \approx \boxed{4.8384}$.

Euler's Explicit Method



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The approximate value is $y(1) \approx \boxed{2.5422}$

Euler's Explicit Method

Example 3: Consider the well-posed IVP:

$$y' = 1 + x\sin(xy); \quad y(0) = 0, \quad 0 \leq x \leq 2$$

Using Euler's explicit method, find an approximate value of y corresponding to $x = 2$.

Solution: for $N = 5$ and $h = \frac{2-0}{5} = 0.4$

We apply Euler's method: $y_{i+1} = y_i + h \cdot f(x_i, y_i)$ with $f(x, y) = 1 + x\sin(xy)$.

i	x_i	y_i	$f(x_i, y_i)$	y_{i+1}
0	0.0	0.0000	$1 + 0 \cdot \sin(0) = 1.0000$	$0.0000 + 0.4 \cdot 1 = 0.4000$
1	0.4	0.4000	$1 + 0.4 \cdot \sin(0.16) \approx 1.0638$	$0.4000 + 0.4 \cdot 1.0638 \approx 0.8255$
2	0.8	0.8255	$1 + 0.8 \cdot \sin(0.6604) \approx 1.4645$	$0.8255 + 0.4 \cdot 1.4645 \approx 1.4113$
3	1.2	1.4113	$1 + 1.2 \cdot \sin(1.6936) \approx 1.9481$	$1.4113 + 0.4 \cdot 1.9481 \approx 2.1906$
4	1.6	2.1906	$1 + 1.6 \cdot \sin(3.5049) \approx 2.4783$	$2.1906 + 0.4 \cdot 2.4783 \approx 3.1829$

The approximate value is $y(2) \approx \boxed{3.1829}$.

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We apply Euler's method: $y_{i+1} = y_i + h \cdot f(x_i, y_i)$ with $f(x, y) = 1 + x\sin(xy)$.

i	x_i	y_i	$f(x_i, y_i)$	y_{i+1}
0	0.0	0.0000	$1 + 0 \cdot \sin(0) = 1.0000$	$0.0000 + 0.4 \cdot 1 = 0.4000$
1	0.4	0.4000	$1 + 0.4 \cdot \sin(0.16) \approx 1.0638$	$0.4000 + 0.4 \cdot 1.0638 \approx 0.8255$
2	0.8	0.8255	$1 + 0.8 \cdot \sin(0.6604) \approx 1.4645$	$0.8255 + 0.4 \cdot 1.4645 \approx 1.4113$
3	1.2	1.4113	$1 + 1.2 \cdot \sin(1.6936) \approx 1.9481$	$1.4113 + 0.4 \cdot 1.9481 \approx 2.1906$
4	1.6	2.1906	$1 + 1.6 \cdot \sin(3.5049) \approx 2.4783$	$2.1906 + 0.4 \cdot 2.4783 \approx 3.1829$

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Using Euler's explicit method, find an approximate value of y corresponding to $x = 2$.

Solution: for $N = 5$ and $h = \frac{2-0}{5} = 0.4$

We apply Euler's method: $y_{i+1} = y_i + h \cdot f(x_i, y_i)$ with $f(x, y) = 1 + x\sin(xy)$.

i	x_i	y_i	$f(x_i, y_i)$	y_{i+1}
0	0.0	0.0000	$1 + 0 \cdot \sin(0) = 1.0000$	$0.0000 + 0.4 \cdot 1 = 0.4000$
1	0.4	0.4000	$1 + 0.4 \cdot \sin(0.16) \approx 1.0638$	$0.4000 + 0.4 \cdot 1.0638 \approx 0.8255$
2	0.8	0.8255	$1 + 0.8 \cdot \sin(0.6604) \approx 1.4645$	$0.8255 + 0.4 \cdot 1.4645 \approx 1.4113$
3	1.2	1.4113	$1 + 1.2 \cdot \sin(1.6936) \approx 1.9481$	$1.4113 + 0.4 \cdot 1.9481 \approx 2.1906$
4	1.6	2.1906	$1 + 1.6 \cdot \sin(3.5049) \approx 2.4783$	$2.1906 + 0.4 \cdot 2.4783 \approx 3.1829$

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Implicit (Backward) Euler's method

Example 4: Consider the well-posed IVP:

$$y' = xy + 1, \quad y(0) = 1$$

Use Euler's method with a step size of $h = 0.25$ to find approximate values of the solution at $x = 1$.

Solution:

- $n = 0$: $y_0 = 1, x_0 = 0, x_1 = x_0 + h = 0 + 0.25 = 0.25$

$$y_1 = y_0 + h \cdot f(x_1, y_1) = 1 + 0.25 \cdot (0.25y_1 + 1)$$

$$y_1 = 1 + 0.0625y_1 + 0.25$$

$$0.9375y_1 = 1.25 \Rightarrow y_1 = \frac{1.25}{0.9375} \approx 1.3333$$

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Solution:

- $n = 0$: $y_0 = 1, x_0 = 0, x_1 = x_0 + h = 0 + 0.25 = 0.25$

$$y_1 = y_0 + h \cdot f(x_1, y_1) = 1 + 0.25 \cdot (0.25y_1 + 1)$$

$$y_1 = 1 + 0.0625y_1 + 0.25$$

$$0.9375y_1 = 1.25 \Rightarrow y_1 = \frac{1.25}{0.9375} \approx 1.3333$$

Implicit (Backward) Euler's method

Solution continued:

- $n = 1$: $x_1 = 0.25,\ y_1 \approx 1.3333$, so

$$x_2 = x_1 + h = 0.25 + 0.25 = 0.5$$

$$y_2 = y_1 + h \cdot f(x_2, y_2) = 1.3333 + 0.25 \cdot (0.5y_2 + 1)$$

$$y_2 = 1.3333 + 0.125y_2 + 0.25$$

$$0.875y_2 = 1.5833 \Rightarrow y_2 = \frac{1.5833}{0.875} \approx 1.8101$$

Implicit (Backward) Euler's method

Solution continued:

- $n = 2$: $x_2 = 0.5,\ y_2 \approx 1.8101$, so

$$x_3 = x_2 + h = 0.5 + 0.25 = 0.75$$

$$y_3 = y_2 + h \cdot f(x_3, y_3) = 1.8101 + 0.25 \cdot (0.75y_3 + 1)$$

$$y_3 = 1.8101 + 0.1875y_3 + 0.25$$

$$0.8125y_3 = 2.0601 \Rightarrow y_3 = \frac{2.0601}{0.8125} \approx 2.5362$$

Implicit (Backward) Euler's method

Solution continued:

- $n = 3$: $x_3 = 0.75,\ y_3 \approx 2.5362$, so

$$x_4 = x_3 + h = 0.75 + 0.25 = 1$$

$$y_4 = y_3 + h \cdot f(x_4, y_4) = 2.5362 + 0.25 \cdot (1.0y_4 + 1)$$

$$y_4 = 2.5362 + 0.25y_4 + 0.25$$

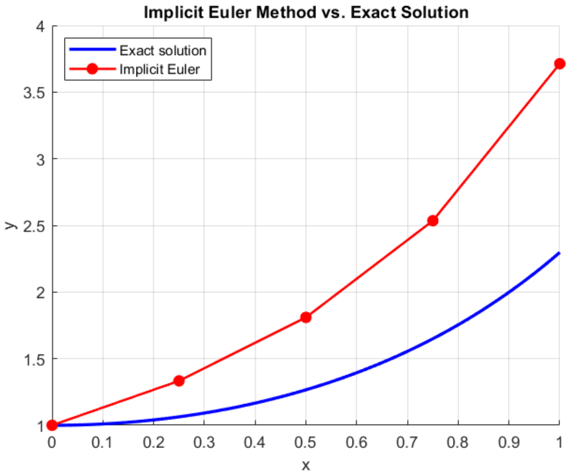
$$0.75y_4 = 2.7862 \Rightarrow y_4 = \frac{2.7862}{0.75} \approx 3.7149$$

Implicit (Backward) Euler's method

Approximate values using Implicit Euler's Method with $h = 0.25$

n	x_n	y_n (approx)
0	0.00	1.0000
1	0.25	1.3333
2	0.50	1.8101
3	0.75	2.5362
4	1.00	3.7149

Implicit (Backward) Euler's method



Geometric Interpretation of Euler's Method: Direction Fields

Consider the well-posed initial value problem:

$$y' = f(x, y), \quad y(x_0) = y_0$$

The derivative y' gives the slope of the tangent to the solution $y(x)$:

This slope is determined by the differential equation. At the point (x_0, y_0), it is

$$y' = \frac{dy}{dx} = f(x_0, y_0)$$

This slope can be represented as a *short line segment* or *arrow* indicating the direction of the solution curve.

Direction Field:

By evaluating $f(x, y)$ at multiple points, we construct a family of such directional arrows - **direction field** - a visualization to the behavior of solutions. Since solutions do not cross and nearby tangents nearly match (Picard-Lindelöf Theorem), the direction field offers a complete **qualitative** picture of the system's dynamics.

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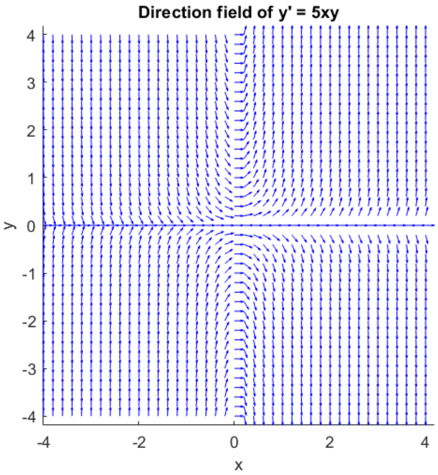

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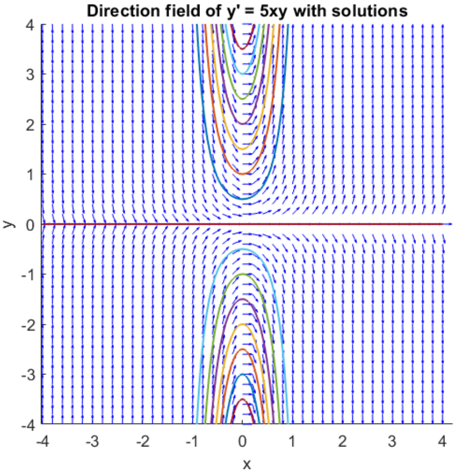

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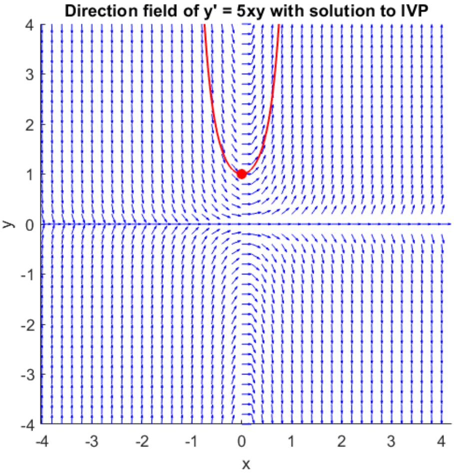

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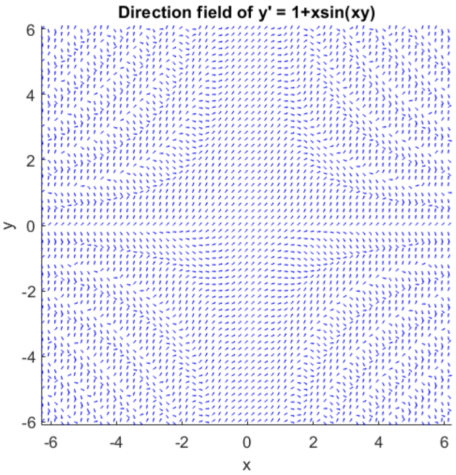

Table of Contents

Derivation

Consider the initial value problem:

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0.$$

To approximate $y(x)$ at $x = x_0 + h$, we apply the Taylor expansion of $y(x)$ about x_0:

$$y(x_0 + h) = y(x_0) + hy'(x_0) + \frac{h^2}{2!}y''(x_0) + \frac{h^3}{3!}y^{(3)}(x_0) + \cdots + \frac{h^p}{p!}y^{(p)}(x_0) + \mathcal{O}(h^{p+1}).$$

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Derivation

Truncating the series after p terms gives the **Taylor method of order** p:

$$y_{n+1} = y_n + hy'(x_n) + \frac{h^2}{2!}y''(x_n) + \cdots + \frac{h^p}{p!}y^{(p)}(x_n),$$

where $y_n \approx y(x_n)$, and $x_{n+1} = x_n + h$.

Each derivative $y^{(k)}(x_n)$ can be recursively computed using the chain rule:

$$\begin{aligned} y' &= f(x,y), \\ y'' &= \frac{d}{dx}f(x,y) = f_x + f_y y', \\ y^{(3)} &= \frac{d}{dx}(f_x + f_y y') = \cdots, \end{aligned}$$

and so on, using repeated applications of the total derivative:

$$\frac{d^k y}{dx^k} = \frac{d^{k-1}}{dx^{k-1}}(f(x,y)).$$

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Euler's method is a first-order Taylor method:

For $p = 1$, we get:

$$y_{n+1} = y_n + hf(x_n, y_n),$$

which is **Euler method**.

Example 1: Solve the well-posed IVP:

$$y' = 5xy, \quad y(0) = 1, 0 \leq x \leq 1$$

by Taylor's series method.

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Taylor series methods

We compute derivatives

$$
\begin{aligned}
y'(x) &= 5xy \quad \Rightarrow \quad y'(0) = 5 \cdot 0 \cdot y(0) = 0, \\
y''(x) &= \frac{d}{dx}(5xy) = 5y + 5xy' \quad \Rightarrow \quad y''(0) = 5 \cdot 1 + 5 \cdot 0 \cdot 0 = 5, \\
y^{(3)}(x) &= \frac{d}{dx}(5y + 5xy') = 5y' + 5y' + 5xy'' = 10y' + 5xy'' \\
&\Rightarrow \quad y^{(3)}(0) = 10 \cdot 0 + 5 \cdot 0 \cdot 5 = 0, \\
y^{(4)}(x) &= \frac{d}{dx}(10y' + 5xy'') = 10y'' + 5y'' + 5xy^{(3)} = 15y'' + 5xy^{(3)} \\
&\Rightarrow \quad y^{(4)}(0) = 15 \cdot 5 + 5 \cdot 0 \cdot 0 = 75.
\end{aligned}
$$

Taylor series methods

Now we apply the Taylor series expansion around $x = 0$:

$$y(x) = y(0) + \frac{x^1}{1!}y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y^{(3)}(0) + \frac{x^4}{4!}y^{(4)}(0) + \cdots$$

Substitute the known values:

$$y(x) \approx 1 + 0 + \frac{5}{2}x^2 + 0 + \frac{75}{24}x^4 + \cdots = 1 + \frac{5}{2}x^2 + \frac{25}{8}x^4 + \cdots$$

This is the Taylor approximation to the solution of the IVP.

Picard iteration

Example 1: Solve the well-posed IVP:

$$y' = 5xy, \quad y(0) = 1, 0 \le x \le 1$$

by Picard's iteration.

Step 1: Rewrite IVP as Integral Equation

$$\begin{aligned} & y' = 5xy, \quad y(0) = 1 \\ \Rightarrow \quad & y(x) = y_0 + \int_0^x f(s, y(s))\, ds \\ & y(x) = 1 + \int_0^x 5sy(s)\, ds \end{aligned}$$

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Picard iteration

Step 2: Define Iteration Sequence

Using method of successive approximations:

$$
\begin{aligned}
y_0(x) &= 1 \\
y_1(x) &= 1 + \int_0^x 5s \cdot y_0(s)\, ds = 1 + \int_0^x 5s \cdot 1\, ds = 1 + \frac{5x^2}{2} \\
y_2(x) &= 1 + \int_0^x 5s \left(1 + \frac{5s^2}{2}\right) ds = 1 + \frac{5x^2}{2} + \frac{25x^4}{8} \\
y_3(x) &= 1 + \int_0^x 5s \left(1 + \frac{5s^2}{2} + \frac{25s^4}{8}\right) ds \\
&= 1 + \frac{5x^2}{2} + \frac{25x^4}{8} + \frac{125x^6}{48} \\
y_4(x) &= 1 + \int_0^x 5s \left(1 + \frac{5s^2}{2} + \frac{25s^4}{8} + \frac{125s^6}{48}\right) ds \\
&= 1 + \frac{5x^2}{2} + \frac{25x^4}{8} + \frac{125x^6}{48} + \frac{625x^8}{384}
\end{aligned}
$$

Taylor vs Picard

Comparison of Taylor Series and Picard Iteration

Given IVP: $y' = 5xy, \quad y(0) = 1$

- **Taylor Series Method:**

$$y(x) = 1 + \frac{5x^2}{2!} + \frac{25x^4}{4!} + \frac{125x^6}{6!} + \cdots$$

- **Picard Iteration:**

$$y_4(x) \approx 1 + \frac{5x^2}{2} + \frac{25x^4}{8} + \frac{125x^6}{48} + \frac{625x^8}{384} + \cdots$$

Comparison of all the three methods

Approximations for the IVP $y' = 5xy,\ y(0) = 1$ at $x = 1$

- **Exact solution:**

$$y(1) = e^{\frac{5}{2}} \approx \boxed{12.1825}$$

- **Euler's method (step size $h = 0.2$):**

$$y(1) \approx \boxed{4.8384}$$

- **Taylor method (4th-degree polynomial):**

$$y(x) \approx 1 + \frac{5x^2}{2!} + \frac{25x^4}{4!} + \frac{125x^6}{6!} + \frac{625x^8}{8!}$$

$$y(1) \approx 1 + \frac{5}{2} + \frac{25}{24} + \frac{125}{720} + \frac{625}{40320} \approx \boxed{11.892}$$

- **Picard iteration (4th step):**

$$y_4(1) = 1 + \frac{5}{2} + \frac{25}{8} + \frac{125}{48} + \frac{625}{384} \approx \boxed{11.9245}$$

Taylor series methods

Example 2: Solve the well-posed IVP:

$$y' = x^2y + 1, \quad y(0) = 1, 0 \leq x \leq 1$$

by Taylor's series method.

We compute derivatives

$$\begin{aligned}
y'(x) &= x^2y + 1, y'(0) = 0^2 \cdot 1 + 1 = 1 \\
y''(x) &= \frac{d}{dx}(x^2y + 1) = 2xy + x^2y' \\
y''(0) &= 2 \cdot 0 \cdot 1 + 0^2 \cdot 1 = 0 \\
y'''(x) &= \frac{d}{dx}(2xy + x^2y') = 2y + 2xy' + 2xy' + x^2y'' \\
&= 2y + 4xy' + x^2y'' \\
y'''(0) &= 2 \cdot 1 + 4 \cdot 0 \cdot 1 + 0^2 \cdot 0 = 2
\end{aligned}$$

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We compute derivatives

$$\begin{aligned}
y'(x) &= x^2y + 1, y'(0) = 0^2 \cdot 1 + 1 = 1 \\
y''(x) &= \frac{d}{dx}(x^2y + 1) = 2xy + x^2y' \\
y''(0) &= 2 \cdot 0 \cdot 1 + 0^2 \cdot 1 = 0 \\
y'''(x) &= \frac{d}{dx}(2xy + x^2y') = 2y + 2xy' + 2xy' + x^2y'' \\
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y'''(0) &= 2 \cdot 1 + 4 \cdot 0 \cdot 1 + 0^2 \cdot 0 = 2
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Taylor series methods

We compute derivatives

$$y^{(4)}(x) = \frac{d}{dx}(2y + 4xy' + x^2y'')$$
$$= 2y' + 4y' + 4xy'' + 2xy'' + x^2y'''$$
$$= 6y' + 6xy'' + x^2y'''$$
$$y^{(4)}(0) = 6 \cdot 1 + 6 \cdot 0 \cdot 0 + 0^2 \cdot 2 = 6$$

Taylor expansion around $x = 0$:

$$y(x) \approx y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 =$$
$$= 1 + x + 0 + \frac{2}{6}x^3 + \frac{6}{24}x^4 = 1 + x + \frac{x^3}{3} + \frac{x^4}{4}$$

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$$y^{(4)}(x) = \frac{d}{dx}(2y + 4xy' + x^2y'')$$
$$= 2y' + 4y' + 4xy'' + 2xy'' + x^2y'''$$
$$= 6y' + 6xy'' + x^2y'''$$
$$y^{(4)}(0) = 6 \cdot 1 + 6 \cdot 0 \cdot 0 + 0^2 \cdot 2 = 6$$

Taylor expansion around $x = 0$:

$$y(x) \approx y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 =$$
$$= 1 + x + 0 + \frac{2}{6}x^3 + \frac{6}{24}x^4 = 1 + x + \frac{x^3}{3} + \frac{x^4}{4}$$

This is the Taylor approximation to the solution of the IVP.

Picard iteration

Example 2: Solve the well-posed IVP:

$$y' = x^2y + 1, \quad y(0) = 1, 0 \leq x \leq 1$$

by Picard's iteration.

Step 1: Rewrite IVP as Integral Equation

$$\begin{aligned} & y' = x^2y + 1, \quad y(0) = 1 \\ \Rightarrow \quad & y(x) = y_0 + \int_0^x f(s, y(s))\, ds \\ & y(x) = 1 + \int_0^x (s^2y(s) + 1)\, ds \end{aligned}$$

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Picard iteration

Step 2: Define Iteration Sequence

Using method of successive approximations:

$$
\begin{aligned}
y_0(x) &= 1\\
y_1(x) &= 1 + \int_0^x (s^2 \cdot y_0(s) + 1)\, ds = 1 + \int_0^x (s^2 + 1)\, ds = 1 + \frac{x^3}{3} + x\\
y_2(x) &= 1 + \int_0^x (s^2 \cdot y_1(s) + 1)\, ds\\
&= 1 + \int_0^x s^2 \left(1 + s + \frac{s^3}{3}\right) + 1\, ds\\
&= 1 + \int_0^x \left(s^2 + s^3 + \frac{s^5}{3} + 1\right)\, ds\\
&= 1 + x + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^6}{18}\\
y_3(x) &= 1 + \int_0^x s^2 \left(1 + s + \frac{s^3}{3} + \frac{s^4}{4} + \frac{s^6}{18}\right) + 1\, ds
\end{aligned}
$$

Taylor vs Picard

Comparison of Taylor Series and Picard Iteration

Given IVP: $y' = x^2 y + 1, \quad y(0) = 1$

- **Taylor Series Method:**

$$y(x) = 1 + x + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

- **Picard Iteration:**

$$y_4(x) \approx 1 + x + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^6}{18} + \frac{x^7}{28} + \frac{x^9}{162} + \frac{x^5}{5} + \frac{x^8}{8} + \frac{x^{10}}{90} + \frac{x^{11}}{308} + \frac{x^{13}}{2106}$$

Comparison of all the three methods

Approximations for the IVP $y' = x^2y + 1,\ y(0) = 1$ at $x = 1$

- **Euler's method (step size $h = 0.1$):**

$$y(1) \approx \boxed{2.542182}$$

- **Taylor method (4th-degree polynomial):**

$$y(1) \approx 1 + 1 + \frac{1^3}{3} + \frac{1^4}{4} = 1 + 1 + \frac{1}{3} + \frac{1}{4} = \frac{31}{12} \approx \boxed{2.5833}$$

- **Picard iteration (4th step):**

$$y_4(1) \approx 1 + 1 + \frac{1}{3} + \frac{1}{4} + \frac{1}{18} + \frac{1}{28} + \frac{1}{162} + \frac{1}{5} + \frac{1}{8} + \frac{1}{90} + \frac{1}{308} + \frac{1}{2106} \approx \boxed{2.5930}$$