Multinomial theorem

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In mathematics, the **multinomial theorem** describes how to expand a power of a sum in terms of powers of the terms in that sum. It is the generalization of the binomial theorem to polynomials.

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Theorem

For any positive integer m and any nonnegative integer n, the multinomial formula tells us how a sum with m terms expands when raised to an arbitrary power n:

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} \prod_{t=1}^{m} x_t^{k_t}\,,$$

where

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1!\, k_2! \cdots k_m!}$$

is a **multinomial coefficient**. The sum is taken over all combinations of nonnegative integer indices k_1 through k_m such that the sum of all k_i is n. That is, for each term in the expansion, the exponents of the x_i must add up to n. Also, as with the binomial theorem, quantities of the form x^0 that appear are taken to equal 1 (even when x equals zero).

In the case $m = 2$, this statement reduces to that of the binomial theorem.

Example

The third power of the trinomial $a + b + c$ is given by

$$(a+b+c)^3 = a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3b^2a + 3b^2c + 3c^2a + 3c^2b + 6abc.$$

This can be computed by hand using the distributive property of multiplication over addition, but it can also be done (perhaps more easily) with the multinomial theorem, which gives us a simple formula for any coefficient we might want. It is possible to "read off" the multinomial coefficients from the terms by using the multinomial coefficient formula. For example:

$$a^2b^0c^1 \text{ has the coefficient } \binom{3}{2,0,1} = \frac{3!}{2!\cdot 0!\cdot 1!} = \frac{6}{2\cdot 1\cdot 1} = 3$$
$$a^1b^1c^1 \text{ has the coefficient } \binom{3}{1,1,1} = \frac{3!}{1!\cdot 1!\cdot 1!} = \frac{6}{1\cdot 1\cdot 1} = 6.$$

Alternate expression

The statement of the theorem can be written concisely using multiindices:

$$(x_1 + \cdots + x_m)^n = \sum_{|\alpha|=n} \binom{n}{\alpha} x^\alpha$$

where $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_m)$ and $x^\alpha = x_1{}^{\alpha_1} x_2{}^{\alpha_2} \cdots x_m{}^{\alpha_m}$.

Proof

This proof of the multinomial theorem uses the binomial theorem and induction on m.

First, for $m = 1$, both sides equal $x_1{}^n$ since there is only one term $k_1 = n$ in the sum. For the induction step, suppose the multinomial theorem holds for m. Then

$$\begin{aligned}&(x_1 + x_2 + \cdots + x_m + x_{m+1})^n = (x_1 + x_2 + \cdots + (x_m + x_{m+1}))^n \\ &= \sum_{k_1+k_2+\cdots+k_{m-1}+K=n} \binom{n}{k_1,k_2,\ldots,k_{m-1},K} x_1^{k_1} x_2^{k_2} \cdots x_{m-1}^{k_{m-1}} (x_m + x_{m+1})^K\end{aligned}$$

by the induction hypothesis. Applying the binomial theorem to the last factor,

$$\begin{aligned}&= \sum_{k_1+k_2+\cdots+k_{m-1}+K=n} \binom{n}{k_1,k_2,\ldots,k_{m-1},K} x_1^{k_1} x_2^{k_2} \cdots x_{m-1}^{k_{m-1}} \sum_{k_m+k_{m+1}=K} \binom{K}{k_m,k_{m+1}} x_m^{k_m} x_{m+1}^{k_{m+1}} \\ &= \sum_{k_1+k_2+\cdots+k_{m-1}+k_m+k_{m+1}=n} \binom{n}{k_1,k_2,\ldots,k_{m-1},k_m,k_{m+1}} x_1^{k_1} x_2^{k_2} \cdots x_{m-1}^{k_{m-1}} x_m^{k_m} x_{m+1}^{k_{m+1}}\end{aligned}$$

which completes the induction. The last step follows because

$$\binom{n}{k_1,k_2,\ldots,k_{m-1},K}\binom{K}{k_m,k_{m+1}} = \binom{n}{k_1,k_2,\ldots,k_{m-1},k_m,k_{m+1}},$$

as can easily be seen by writing the three coefficients using factorials as follows:

$$\frac{n!}{k_1!k_2!\cdots k_{m-1}!K!}\frac{K!}{k_m!k_{m+1}!} = \frac{n!}{k_1!k_2!\cdots k_{m+1}!}.$$

Multinomial coefficients

The numbers

$$\binom{n}{k_1,k_2,\ldots,k_m} = \frac{n!}{k_1!\,k_2!\cdots k_m!},$$

which can also be written as

$$= \binom{k_1}{k_1}\binom{k_1+k_2}{k_2}\cdots\binom{k_1+k_2+\cdots+k_m}{k_m} = \prod_{i=1}^{m}\binom{\sum_{j=1}^{i} k_j}{k_i}$$

are the multinomial coefficients. Just like "n choose k" are the coefficients when a *binomial* is raised to the n^{th} power (e.g., the coefficients are 1,3,3,1 for $(a+b)^3$, where $n = 3$), the multinomial coefficients appear when a *multinomial* is raised to the n^{th} power (e.g., $(a+b+c)^3$).

Sum of all multinomial coefficients

The substitution of $x_i = 1$ for all i into:

$$\sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m} = (x_1+x_2+\cdots+x_m)^n ,$$

gives immediately that

$$\sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} = m^n .$$

Number of multinomial coefficients

The number of terms in a multinomial sum, $\#_{n,m}$, is equal to the number of monomials of degree n on the variables $x_1, \ldots, x_m$:

$$\#_{n,m} = \binom{n+m-1}{m-1} .$$

The count can be performed easily using the method of stars and bars.

Interpretations

Ways to put objects into boxes

The multinomial coefficients have a direct combinatorial interpretation, as the number of ways of depositing n distinct objects into m distinct bins, with k_1 objects in the first bin, k_2 objects in the second bin, and so on.[1]

Number of ways to select according to a distribution

In statistical mechanics and combinatorics if one has a number distribution of labels then the multinomial coefficients naturally arise from the binomial coefficients. Given a number distribution $\{n_i\}$ on a set of N total items, n_i represents the number of items to be given the label i. (In statistical mechanics i is the label of the energy state.)

The number of arrangements is found by

- Choosing n_1 of the total N to be labeled 1. This can be done $\binom{N}{n_1}$ ways.
- From the remaining $N - n_1$ items choose n_2 to label 2. This can be done $\binom{N-n_1}{n_2}$ ways.
- From the remaining $N - n_1 - n_2$ items choose n_3 to label 3. Again, this can be done $\binom{N-n_1-n_2}{n_3}$ ways.

Multiplying the number of choices at each step results in:

$$\binom{N}{n_1}\binom{N-n_1}{n_2}\binom{N-n_1-n_2}{n_3}\cdots = \frac{N!}{(N-n_1)!n_1!}\frac{(N-n_1)!}{(N-n_1-n_2)!n_2!}\frac{(N-n_1-n_2)!}{(N-n_1-n_2-n_3)!n_3!}\cdots.$$

Upon cancellation, we arrive at the formula given in the introduction.

Number of unique permutations of words

The multinomial coefficient is also the number of distinct ways to permute a multiset of *n* elements, and k_i are the multiplicities of each of the distinct elements. For example, the number of distinct permutations of the letters of the word MISSISSIPPI, which has 1 M, 4 Is, 4 Ss, and 2 Ps is

$$\binom{11}{1,4,4,2} = \frac{11!}{1!\,4!\,4!\,2!} = 34650.$$

(This is just like saying that there are 11! ways to permute the letters—the common interpretation of factorial as the number of unique permutations. However, we created duplicate permutations, because some letters are the same, and must divide to correct our answer.)

Generalized Pascal's triangle

One can use the multinomial theorem to generalize Pascal's triangle or Pascal's pyramid to Pascal's simplex. This provides a quick way to generate a lookup table for multinomial coefficients.

The case of $n = 3$ can be easily drawn by hand. The case of $n = 4$ can be drawn with effort as a series of growing pyramids.

See also

- Multinomial distribution
- Stars and bars (combinatorics)

References

1. National Institute of Standards and Technology (May 11, 2010). "NIST Digital Library of Mathematical Functions". Section 26.4. Retrieved August 30, 2010.

External links

- `mutinom.m` function in Specfun (http://octave.sourceforge.net/specfun/) (since 1.1.0) package of Octave-Forge (http://octave.sourceforge.net/index.html) for GNU Octave. SVN version (http://octave.svn.sf.net/viewvc/octave/trunk/octave-forge/main/specfun/inst/multinom.m)
- Hazewinkel, Michiel, ed. (2001), "Multinomial coefficient", *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4