Geometry of Determinants

1) Vectors, Linear Spaces and Affine Geometry

A "vector" is defined by the Oxford English Dictionary to be "a quantity having direction as well as magnitude, especially as determining the position of one point in space relative to another". A "vector space" is "a space consisting of vectors, together with the associative and commutative operation of addition of vectors, and the associative and distributive operation of multiplication of vectors by scalars".

At first sight the concept of vector is intuitively clear. However, as soon as we begin a detailed analysis we are confronted by the "zero vector", which determines the position of one point in space relative to *itself* and is characterized by having neither magnitude nor direction! In any case the concept of "direction" is only meaningful in terms of the underlying geometry of points, in which it is defined.

In order to understand the concept of a vector space, we also need to consider how the magnitude of a vector relates to the multiplication of vectors by scalars (real numbers) and what is meant by addition of vectors. We shall see that axiom systems for vectors, may be applied to more general objects than vectors. We will therefore give a set of axioms for a "linear" space, and we will show that a vector space (as defined by the Oxford English Dictionary) is a linear space.

A **linear structure** consists of a set **S**, which contains **the zero element 0** and on which are defined the **linear operations** of **scalar multiplication**, i.e. $s \mapsto rs$, for every real number r, and in particular **negation**, i.e. $s \mapsto -s = (-1)s$, and the binary operation of **addition**, i.e. $\langle s_1, s_2\rangle \mapsto s_1 + s_2$. $s_1 + s_2$ is the **sum** of s_1 and s_2.

A linear structure **S** is a **linear space** if it satisfies the following axioms:
L1) *For every real number r,* $r\mathbf{0} = \mathbf{0}$ *and* $\mathbf{0} + \mathbf{0} = \mathbf{0}$.
L2) *For each element s of* **S**, $0s = \mathbf{0}$, $1s = s$ *and for any real numbers* r_1 *and* r_2,
$r_1s + r_2s = (r_1 + r_2)s = (r_2 + r_1)s = r_2s + r_1s$ *and* $r_1(r_2s) = (r_1r_2)s = (r_2r_1)s = r_2(r_1s)$.
L3) *The operation "+" is associative on* **S**.
L4) *For each ordered pair* $\langle s_1, s_2\rangle$ *of elements of* **S**,
(i) $s_1 + s_2 = s_2 + s_1$; *i.e. "+" is commutative on* **S**.
(ii) *for every real number r,* $r(s_1 + s_2) = rs_1 + rs_2$.
((L4)(ii) implies L4)(i) (let $r = 2$); L4)(i) implies L4)(ii) (see discussion on page 11)).

A **subspace T** of a linear space **S** is a linear substructure of **S**, i.e. a subset of **S**, which is closed under the operations of **S**. If T_0 is a set of elements of a linear space **S**, then the closure of T_0 under the operations of **S** is a subspace **T** of **S**; indeed, it is the intersection of all subspaces of **S** containing T_0. T_0 is a **spanning set for S** if its closure under the operations of **S** is **S** itself.

THEOREM 1.1
A subset **T** *of a linear space* **S** *is a subspace of* **S** *if and only if for every pair* t_1 *and* t_2 *of elements of* **T** *and real numbers* r_1 *and* r_2*, the element* $(r_1t_1 + r_2t_2)$ *of* **S** *is in* **T**.
PROOF
$\Rightarrow$ **T** is closed under the operations of **S**.
$\Leftarrow$ Setting $t_2 = \mathbf{0}$ shows that **T** is closed under scalar multiplication; setting $r_1 = r_2 = 1$ shows that **T** is closed under addition. ||

A **linear endomorphism** of a subspace **T** of a linear space **S** is a map $f : \mathbf{T} \to \mathbf{T}$, which preserves the linear operations of **T**, which are the linear operations of **S**; i.e. $(rt)f = r(tf)$ and $(t_1 + t_2)f = t_1f + t_2f$ for r in **R** and for t, t_1 and t_2 in **T**. Equivalently, $f : \mathbf{T} \to \mathbf{T}$ is a linear endomorphism if $(r_1t_1 + r_2 t_2)f = r_1(t_1f) + r_2(t_2f)$ for r_1 and r_2 in **R** and for t_1 and t_2 in **T**. A **linear automorphism** is an invertible linear endomorphism.

In particular, by axioms L2) and L4)(ii), scalar multiplication by a real number r is a linear endomorphism of any subspace of **S**, which is a linear automorphism if r is non-zero. More generally, a map (resp. invertible map) from a linear space to a linear space is a **linear homomorphism** (resp. **linear isomorphism**) if it maps the zero element to the zero element, maps scalar multiples by a real number to scalar multiples by the same real number and maps sums to sums.

THEOREM 1.2
The linear structure **O**, *whose only element is* **0**, *is a linear space, which is a subspace of every linear space.*
PROOF
O contains only the element **0,** which must therefore be the image of any (linear) operation on **O**. Hence **O** satisfies L1). Axioms L2), L(3) and L(4) for **O** also follow from the uniqueness of **0** in **O**. **O** is a subspace of any linear space **S** by L1) for **S**. ||

O is **the zero-dimensional** linear space.

THEOREM 1.3
Let e be a non-zero element of a linear structure, let **R***e be the linear substructure whose elements are of the form re, for r a real number and let* **R***e satisfy the axioms* L1) *and* L2). *Then* **R***e is a linear space and is a subspace of any linear space containing e.*
PROOF
We must check that **R***e* satisfies the axioms L3 and L4).
Let $s_1 = r_1e$, $s_2 = r_2e$ and $s_3 = r_3e$. Then:

$$(s_1 + s_2) + s_3 = (r_1e + r_2e) + r_3e = (r_1 + r_2)e + r_3e = ((r_1 + r_2) + r_3)e = (r_1 + (r_2 + r_3))e = r_1e + (r_2 + r_3)e = r_1e + (r_2e + r_3e) = s_1 + (s_2 + s_3).$$

$$s_1 + s_2 = r_1e + r_2e = (r_1 + r_2)e = (r_2 + r_1)e = r_2e + r_1e = s_2 + s_1.$$

$$r(s_1 + s_2) = r(r_1e + r_2e) = r((r_1 + r_2)e) = (r(r_1 + r_2))e = (rr_1 + rr_2)e = (rr_1)e + (rr_2)e = r(r_1e) + r(r_2e) = rs_1 + rs_2.$$

R*e* is closed under the linear operations of any linear space **S** containing the element e by L1) and L2) for **S**. ||

Let e be a non-zero element of a linear structure, let **R***e* be the linear substructure whose elements are of the form re, for r a real number and let **R***e* satisfy the axioms L1) and L2). Then the linear space **R***e* of theorem 1.3 is a **one-dimensional** linear space with **oriented basis** $\langle e \rangle$. If e' is any non-zero element of **R***e*, then $e' = re$, for some non-zero real number r and $\langle e' \rangle$ is an oriented basis for **R***e*$'$ = **R***e*; $\langle e \rangle$ and $\langle e' \rangle$ have the **same orientation** if r is positive; $\langle e \rangle$ and $\langle e' \rangle$ have **opposite orientation** if r is negative. The linear space **R***e* is **orientable** and is **oriented** by its oriented bases. In particular scalar multiplication in **R***e* by positive numbers preserves orientation and scalar multiplication in **R***e* by negative numbers reverses orientation.

The structure **R** of real numbers itself, with its operations of addition and multiplication, is a one-dimensional orientable linear space, where **0** manifests as the real number 0 and **R** may be oriented by its **canonical** oriented basis <1>. If **R***e* is a one-dimensional linear space with oriented basis <*e*>, then the map from the oriented basis <1> of **R** to the oriented basis <*e*> of **R***e*, induces the **canonical** linear isomorphism $r \approx re$ of **R** onto **R***e*. A linear space **L** is one-dimensional with oriented basis <*e*> if and only if *e* corresponds to 1 under a linear isomorphism from **R** to **L**.

Clifford(1878) discusses the two distinct meanings of multiplication. Thus, the significance of "$2 \times 3 = 6$" could be e.g. "A 2 cm by 3 cm rectangle has area 6 cm^2."; alternatively, the significance of "$2 \times 3 = 6$" could be e.g. "Doubling a 3 cm length line-segment yields a 6 cm length line-segment.". *Scalar multiplication* always has Clifford's latter meaning. When **R** is regarded as a linear space it is the space of **scalars**.

THEOREM 1.4
Let **S** *be a linear space.*
If **S** $\neq$ **O**, *then for any real number r, the scalar multiplication* $s \mapsto rs$ *is a linear endomorphism of* **S**, *which is an automorphism unless* $r = 0$ *(in which case* $s \mapsto rs$ *reduces to the* **zero map** $s \mapsto \mathbf{0}$*).*
If **S** = **O**, *then all scalar multiplications and indeed all linear endomorphisms and indeed all maps from* **O** *to* **O** *reduce to the zero map* $\mathbf{0} \mapsto \mathbf{0}$.
If **S** = **R***e, for some non-zero e in* **S**, *then every linear automorphism of* **S** *is a scalar multiplication by some non-zero real number t; such an automorphism is orientation-preserving or orientation-reversing according as t is positive or negative.*
PROOF
We have already noted that scalar multiplications are linear endomorphisms by axioms L2 and L4)(ii)). The inverse of the scalar multiplication $s \mapsto rs$, when $r \neq 0$, is the scalar multiplication $s \mapsto r^{-1}s$.
The only map from **O** to itself is the zero map $\mathbf{0} \mapsto \mathbf{0}$.
If f is a linear automorphism of **S** = **R***e*, where $e \neq \mathbf{0}$, then $ef = te$, for some non-zero real number t. For any $s = re$ in **S**, $sf = (re)f = r(ef) = r(te) = (rt)e = (tr)e = t(re) = ts$, whence f is the scalar multiplication $s \mapsto ts$. The pair of oriented bases <*e*> and <*te*> for **S** have the same or opposite orientation as according as t is positive or negative. ||

The underlying geometry of points for vector spaces as linear spaces is Affine Geometry. This geometry is more general than Euclidean Geometry because it does not consider properties defined in terms of circles and angles (Euclid's axioms III and IIII). The basic objects of Affine Geometry are **points**, **oriented line-segments**, which have **oriented length** and determine **oriented lines**, **oriented triangles** and **parallelograms**, which have **oriented area** and determine **oriented planes**, and **oriented tetrahedra** and **parallelepipeds**, which have **oriented volume** and determine an **orientation** of **Space**. (In modern terminology "lines" are "straight lines" according to Euclid).

We give a set of axioms for Affine Geometry, in terms of vectors, which represent oriented lengths, bivectors, which represent (sums of) oriented areas, and trivectors, which represent (sums of) oriented volumes. The concepts of **set**, **point**, **direction**, **opposite direction**, **oriented length**, **oriented area** and **oriented volume** are left to the intuition and are not defined. Lines, planes and Space itself are specified by the *sets* of the points that lie on them.

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A1) *Let P and P' be distinct points. Then the ordered pair $\langle P, P'\rangle$ determines an oriented line-segment, which has an oriented length. That is to say that it has a direction and a magnitude (i.e. a positive real number) associated with that direction.*

A **non-zero vector** is an oriented length. If P and P' are distinct points, then we write $v = PP'$ if v is the oriented length of the oriented line-segment with **vertices** $\langle P, P'\rangle$. We automatically obtain the vector $(-v) = P'P$, which has the same magnitude (length) as v, but whose direction is opposite to that of v. Also, for any positive real number r, we write rv for the non-zero vector whose direction is that of v and whose magnitude is the magnitude of v multiplied by r. We also write $(-r)v = r(-v) = (-(rv))$; in particular $(-1)v = 1(-v) = -v$. For any point P, we write $\mathbf{0} = PP$, where $\mathbf{0}$ manifests as the **zero vector** and for any real number r and vector v, we write $r\mathbf{0} = \mathbf{0} = 0v$.

A **vector structure** (resp. a **vector space)** is a linear structure (resp. a linear space) whose elements are vectors.

A2) *Vectors (including the zero vector) are translatable. That is to say that if $v = PP'$ for some ordered pair of (not necessarily distinct) points $\langle P, P'\rangle$, then for any point Q, there is a point Q' such that $v = QQ'$. Further, if $w = PQ$, then also $w = P'Q'$.*

We refer to Axiom A2) as "Playfair's axiom". It follows from Playfair's axiom that for any vector v, if $v = PP'$, then there is a transformation of the underlying Affine Geometry which maps any point Q to the point Q', where $QQ' = PP' = v$. Such a transformation is a **translation** and we may interpret the vector v as a representation of this translation. In particular the zero vector $\mathbf{0}$ represents the **identity translation**. Playfair's axiom also tells us that vectors are invariant under translations.

We may use Playfair's axiom to define addition of vectors and to show that such addition is associative and commutative.

Let v_1, v_2 and v_3 be vectors. Then given any point P, there are unique points Q, R and S, which satisfy $v_1 = PQ$, $v_2 = QR$ and $v_3 = RS$. $(v_1 + v_2)$ is well defined as the vector PR. In fact, if $v_1 = P'Q'$ and $v_2 = Q'R'$, then $PP' = QQ' = RR'$, whence $P'R' = PR$. Also $(v_1 + v_2) + v_3 = PR + RS = PS = PQ + QS = v_1 + (v_2 + v_3)$. If vectors are interpreted as translations, then vector addition is interpreted as composition of translations and is automatically associative.

Let $v = PP'$ and $w = PQ$ and let Q' be the point such that $v = QQ'$ and $w = P'Q'$. Then $v + w = PP' + P'Q' = PQ' = PQ + QQ' = w + v$.

A **line, with an associated pair of mutually opposite directions, through a point** P is specified by the set of points, whose elements are P and every point P', such that the oriented line-segment determined by $\langle P, P'\rangle$ has either of the given directions.

A3) *Let P and P' be distinct points. Then there is a unique line containing P and P'. This line is through both of the points P and P' and its associated mutually opposite directions are those of the mutually opposite vectors PP' and $P'P$.*

Distinct lines are **parallel** if they have the same pair of mutually opposite directions. If **L** is a line and Q is a point not on **L**, then there is a unique line **L′** through Q parallel to **L**. (This is the traditional version of *Playfair's axiom of parallels*, which is equivalent to Euclid V in Euclidean Geometry). In fact, if the distinct points P and P' determine the line **L** and $v = PP'$, then by *our* version of Playfair's axiom there is a point Q', such that $v = QQ'$; Q and Q' determine the required line **L′**.

A4) *Let P_0 and P_1 be distinct points determining a line* **L**. *Then the relation "$P < P'$ if the vector PP' has the same direction as the vector P_0P_1" is an ordering of the points of* **L**, *which is isomorphic to the natural ordering of the real numbers. In fact, for any points P and P' of* **L**, *$PP' = rP_0P_1$, for some real number r. P and P' coincide if and only if $r = 0$, when $PP' = 0P_0P_1 = \mathbf{0}$; $P < P'$ if and only r is positive and $P' < P$ if and only if r is negative. Also, for every real number r, and for any point P of* **L**, *there is a point P' of* **L**, *such that $PP' = rP_0P_1$.*

Axiom A4) is called the "continuity axiom" as it actually *defines* the real numbers in terms of the *continuity* of points on a line. (See appendix A for details).

THEOREM 1.5
Let **S** *be a vector structure. Then the vector substructure* **O** *of* **S**, *whose only element is the zero vector* **0**, *is the zero-dimensional vector space. If v is a non-zero vector, then the vector substructure* **R***v* *of* **S**, *consisting of the zero vector and all those vectors whose directions are the same as, or are opposite to, the direction of v, is a one-dimensional orientable vector space, whose orientation is given by the direction of its oriented basis $<v>$.*
PROOF
If **0** is construed to be the zero vector, then the zero-dimensional linear space **O** is a vector space.
If v is a non-zero vector, then there are distinct points P_0 and P_1, such that $v = P_0P_1$. By axiom A4) there is an order isomorphism, given by $r \approx P_r$, from the real numbers **R** to the points on the line **L** determined by P_0 and P_1, where P_r is the point on **L** such that $P_0P_r = rP_0P_1 = rv$. It follows from Playfair's axiom and continuity considerations that the correspondence $r \approx P_r$ induces an isomorphism of the one-dimensional linear space **R** onto the vector structure **R***v*, which is ipso facto a one-dimensional vector space oriented by the direction of its oriented basis $<v>$. (See appendix A for details). ||

Let O, "the Origin", be an arbitrarily chosen fixed point. Then every point V distinct from O may be labelled by the vector $v = OV$. The vector v is the **label** of the point V **with respect to the Origin** O and we say it is **the point** v. The Origin is **the point 0**.

Let O be the Origin and let **L** be a line through O. Then O partitions $\mathbf{L}\backslash\{O\}$ into two **half-lines**, which are **sides of L**. Let O, V and V' be distinct points such that O is the Origin and V and V' are a pair of points on a line **L** through O. Let $v = OV$ and $v' = OV'$. Then by axiom A4), there is a non-zero real number r, such that $v' = rv$. r is positive if V and V' are on the same side of **L** and r is negative if V and V' are on opposite sides.

THEOREM 1.6

Let **S** *be a vector space, whose elements are the labels of points with respect to an Origin O. Let* **O** *be the zero vector space. Let* $\mathbf{0} \neq u = OU$ *and* $\mathbf{0} \neq u' = OU'$ *be labels* in **S** *of points U and U′ which may coincide; let* $\mathbf{L} = \mathbf{R}u$ *and* $\mathbf{L'} = \mathbf{R}u'$ *be the one-dimensional vector subspaces of* **S** *with oriented bases* $<u>$ *and* $<u'>$*, respectively. Then:*

The zero vector **0** *is the label of the Origin O.*

The elements of the one-dimensional vector space $\mathbf{L} = \mathbf{R}u$ *are the labels of the points on the line through the Origin O and the point U; specifically, ru is the label of the point* U_r*, where* $OU_r = rOU$*.*

Either $\mathbf{L} \cap \mathbf{L'} = \mathbf{O}$*, whence* **L** *and* **L′** *represent distinct orientable lines through the Origin O, or for some non-zero real number r,* $u' = ru$*, in which case* **L** *and* **L′** *represent the same orientable line through the origin and have the same or opposite orientation according as r is positive or negative.* ||

If $\mathbf{T}_1$ and $\mathbf{T}_2$ are subspaces of a linear space **S**, then so is their **intersection** $\mathbf{T}_1 \cap \mathbf{T}_2$ and so also is their **sum** $\mathbf{T}_1 + \mathbf{T}_2$, whose elements are sums $t_1 + t_2$, for t_1 in $\mathbf{T}_1$ and t_2 in $\mathbf{T}_2$. If $\mathbf{T}_1 \cap \mathbf{T}_2 = \mathbf{O}$, then the sum $\mathbf{T}_1 + \mathbf{T}_2$ is **direct** and is denoted $\mathbf{T}_1 \oplus \mathbf{T}_2$.

THEOREM 1.7

Let $\mathbf{T}_1$ *and* $\mathbf{T}_2$ *be subspaces of a linear space* **S***, then their intersection* $\mathbf{T}_1 \cap \mathbf{T}_2$ *and their sum* $\mathbf{T}_1 + \mathbf{T}_2$ *are well defined as subspaces of* **S***. If* $\mathbf{T}_1 \cap \mathbf{T}_2 = \mathbf{O}$ *then every element of* $\mathbf{T}_1 \oplus \mathbf{T}_2$ *has a unique decomposition* $t_1 + t_2$*, for* t_1 *in* $\mathbf{T}_1$ *and* t_2 *in* $\mathbf{T}_2$*.*

PROOF

If $\mathbf{T}_1$ and $\mathbf{T}_2$ are both closed under the linear operations of **S**, then so is their intersection.
Let $(t_1 + t_2)$ and $(t'_1 + t'_2)$ be elements of $\mathbf{T}_1 + \mathbf{T}_2$.
Then $(t_1 + t_2) + (t'_1 + t'_2) = (t_1 + t'_1) + (t_2 + t'_2)$ is in $\mathbf{T}_1 + \mathbf{T}_2$, by axioms L3) and L4)(i);
$r(t_1 + t_2) = rt_1 + rt_2$ is in $\mathbf{T}_1 + \mathbf{T}_2$, by axiom L4)(ii).
If $(t_1 + t_2) = (t'_1 + t'_2)$, then $t_1 - t'_1 = -(t_2 - t'_2)$ is in both $\mathbf{T}_1$ and $\mathbf{T}_2$; if $\mathbf{T}_1 \cap \mathbf{T}_2 = \mathbf{O}$, then $t_1 - t'_1 = -(t_2 - t'_2) = \mathbf{0}$, whence $t_1 = t'_1$ and $t_2 = t'_2$. ||

If **T** is a subspace of a linear space **S**, then every element s of **S** determines a **coset** $[s]_\mathbf{T}$ of **T** in **S**, which is the set $\{(s + t) : t \text{ in } \mathbf{T}\}$ and is also written as $s + \mathbf{T}$. In particular, $[\mathbf{0}]_\mathbf{T} = \mathbf{T}$. Every element of **S** belongs to a unique coset of **T** in **S**. Equivalently the cosets of **T** in **S** form a partition of **S**. Scalar multiplication and addition of cosets of **T** in **S** are defined by $r[s]_\mathbf{T} = [rs]_\mathbf{T}$ and $[s]_\mathbf{T} + [s']_\mathbf{T} = [s + s']_\mathbf{T}$ for real numbers r and for s and s' in **S**. The linear structure formed by the cosets of **T** in **S** is the **quotient space S/T**.

Let $\mathbf{S}_0$ be a subspace of a linear space **S**. Then a **projection of S onto $\mathbf{S}_0$** is a linear homomorphism from **S** to $\mathbf{S}_0$, which is invariant on $\mathbf{S}_0$. Let **T** be any subspace of **S**, which is such that $\mathbf{S} = \mathbf{S}_0 \oplus \mathbf{T}$. Then the **projection of S onto $\mathbf{S}_0$ along the cosets of T** is that projection, which maps all the elements of **T** to **0**, specifically $s_0 + t \longmapsto s_0$. The **inclusion map from $\mathbf{S}_0$ to S**, which maps each element to itself, is also a linear homomorphism.

THEOREM 1.8

Let **T** *be a subspace of a linear space* **S** *and let* $\mathbf{S}_0$ *be any subspace of* **S**, *which is such that* $\mathbf{S} = \mathbf{S}_0 \oplus \mathbf{T}$*. Then there is a linear isomorphism from* **S/T** *to* $\mathbf{S}_0$*, which maps each coset of* **T** *in* **S** *to its unique representative element in* $\mathbf{S}_0$*.*

PROOF

Let $s + \mathbf{T}$ be a coset of $\mathbf{T}$ in $\mathbf{S}$. Then, because $\mathbf{S} = \mathbf{S}_0 + \mathbf{T}$, there is s_0 in $\mathbf{S}_0$, which is such that $s = s_0 + t$, for some t in $\mathbf{T}$, and thence $s + \mathbf{T} = s_0 + \mathbf{T}$. Further, s_0 is unique in $\mathbf{S}_0$, because $\mathbf{S}_0 \cap \mathbf{T} = \mathbf{O}$. ||

Let f be a linear homomorphism from a linear space $\mathbf{S}$ to a linear space $\mathbf{S}'$. Then:
The **kernel** of f, Ker(f) is the set $\{s \text{ in } \mathbf{S} : sf = \mathbf{0}\}$.
The **image** of f, Im(f) is the set $\{s' \text{ in } \mathbf{S}' : s' = sf, \text{ for some } s \text{ in } \mathbf{S}\}$.

THEOREM 1.9

Let f be a linear homomorphism from a linear space $\mathbf{S}$ *to a linear space* $\mathbf{S}'$. *Then* Ker(f) *is a subspace of* $\mathbf{S}$ *and* Im(f) *is a subspace of* $\mathbf{S}'$.

PROOF

Let s_1 and s_2 be elements of $\mathbf{S}$ and let r_1 and r_2 be real numbers.
Then $(r_1s_1 + r_2s_2)f = r_1(s_1f) + r_2(s_2f)$.
If $s_1f = s_2f = \mathbf{0}$, then $(r_1s_1 + r_2s_2)f = \mathbf{0}$.
If $s'_1 = s_1f$ and $s'_2 = s_2f$, then $r_1s'_1 + r_2s'_2 = (r_1s_1 + r_2s_2)f$ ||

THEOREM 1.10

Let f be a linear endomorphism of a linear space $\mathbf{S}$ *and let* $\mathbf{S}_0$ *be any subspace of* $\mathbf{S}$, *which is such that* $\mathbf{S} = \mathbf{S}_0 \oplus$ Ker(f). *Then f may be construed as a composition of a projection of* $\mathbf{S}$ *onto* $\mathbf{S}_0$ *along the cosets of* Ker(f), *a linear isomorphism* $f|\mathbf{S}_0$ *from* $\mathbf{S}_0$ *to* Im(f) *and the inclusion map from* Im(f) *to* $\mathbf{S}$ *(which is well defined to be a linear homomorphism).* $f|\mathbf{S}_0$ *is a linear automorphism if and only if* $\mathbf{S}_0 =$ Im(f). *f, itself, is a linear automorphism if and only if* Ker(f) $= \mathbf{O}$ *if and only if* Im(f) $= \mathbf{S}$.

PROOF

Let $s = s_0 + t$, where s_0 is in $\mathbf{S}_0$ and t is in Ker(f). Then $sf = s_0f$.
The projection maps s in $\mathbf{S}$ to s_0 in $\mathbf{S}_0$.
The linear isomorphism $f|\mathbf{S}_0$ maps s_0 in $\mathbf{S}_0$ to s_0f in Im(f) and is a linear automorphism if and only if $\mathbf{S}_0 =$ Im(f).
The inclusion maps s_0f in Im(f) to s_0f in $\mathbf{S}$.
f is a linear automorphism if and only if $\mathbf{S}_0 = \mathbf{S}$. ||

THEOREM 1.11

The linear structure $\mathbf{R}^n$ *of ordered n-tuples of real numbers is a linear space whose operations are given by,* $\mathbf{0} = (0, \ldots, 0)$; $r(x_1, \ldots, x_n) = (rx_1, \ldots, rx_n)$ *for all r in* $\mathbf{R}$ *and* $(x_1, \ldots, x_n) + (x'_1, \ldots, x'_n) = (x_1 + x'_1, \ldots, x_n + x'_n)$. *Every element of* $\mathbf{R}^n$ *may be expressed uniquely as a sum* $(x_1, \ldots, x_n) = x_1\varepsilon_1 + \ldots + x_n\varepsilon_n$, *where for m from* 1 *to n,* ε_m *is the ordered n-tuple whose* m^{th} *term is* 1 *and whose other terms are all zero.* ||

We have already defined the canonical oriented basis of the linear space $\mathbf{R} = \mathbf{R}^1$ to be <1>. The **canonical oriented basis of the linear space** $\mathbf{R}^2$ is $<(1, 0), (0, 1)>$ and the **canonical oriented basis of the linear space** $\mathbf{R}^3$ is $<(1, 0, 0), (0, 1, 0), (0, 0, 1)>$. The **canonical oriented basis of the linear space** $\mathbf{R}^n$ is $<\varepsilon_1, \ldots, \varepsilon_n>$, where, for m from 1 to n, ε_m is the ordered n-tuple whose mth term is 1 and whose other terms are all zero.

2) Plane Geometry

Let e_1 and e_2 be a pair of elements of a linear space **S**. Then e_1 and e_2 are **linearly independent** if there is no one-dimensional subspace of **S** containing both e_1 and e_2; e_1 and e_2 are **linearly dependent**, otherwise. In fact, e_1 and e_2 are linearly independent if and only if the ordered pair of real numbers (0, 0) is the *only* ordered pair of real numbers (x, y) such that $xe_1 + ye_2 = \mathbf{0}$.

Let e_1 and e_2 be a pair of linearly independent elements of a linear space **S**. Then the subspace $\mathbf{T} = \mathbf{R}e_1 \oplus \mathbf{R}e_2$ of **S** is an **orientable two-dimensional** linear space with **oriented basis** $<e_1, e_2>$. **T** is in fact the intersection of all those subspaces of **S**, which contain e_1 and e_2.

The linear space $\mathbf{R}^2$ of ordered pairs of real numbers is an orientable two-dimensional linear space, where **0** manifests as the ordered pair (0, 0) and $\mathbf{R}^2$ may be oriented by its canonical oriented basis <(1, 0), (0, 1)>.

If $\mathbf{T} = \mathbf{R}e_1 \oplus \mathbf{R}e_2$ is a two-dimensional linear space, with oriented basis $<e_1, e_2>$, then the mapping $(1, 0) \mapsto e_1$ and $(0, 1) \mapsto e_2$ from the oriented basis <(1, 0), (0, 1)> of $\mathbf{R}^2$ to the oriented basis $<e_1, e_2>$ of **T** induces a linear isomorphism $(x, y) \approx xe_1 + ye_2$ of $\mathbf{R}^2$ onto **T**. A linear space **T** is two-dimensional with oriented basis $<e_1, e_2>$ if and only if there is a linear isomorphism from $\mathbf{R}^2$ onto **T** for which e_1 is the image of (1, 0) and e_2 is the image of (0, 1).

Let **T** be a two dimensional linear space. Then $\mathbf{T}[2\times1] = \{\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} : t_1 \text{ and } t_2 \text{ in } \mathbf{T}\}$ is the set of ordered pairs of elements of **T** expressed as (2×1) column matrices. This representation allows us to define a left action of **R**(2), the (2×2) real matrices, on **T**[2×1] by $\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} x_1t_1 + y_1t_2 \\ x_2t_1 + y_2t_2 \end{pmatrix}$. We may also define a right action of $\mathrm{End}_{\mathbf{R}}(\mathbf{T})$, the linear endomorphisms of **T**, on **T**[2×1] by $\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} f = \begin{pmatrix} t_1 f \\ t_2 f \end{pmatrix}$. **R**(2) is closed under matrix multiplication and it is straightforward to show that $\mathrm{End}_{\mathbf{R}}(\mathbf{T})$ is closed under composition.

LEMMA 2.1

Let **T** *be a two-dimensional linear space, let M and N be* (2×2) *real matrices and let f and g be linear endomorphisms of* **T**. *Then for all* $\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$ *in* **T**[2×1]*:*

$$(MN)\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = M(N\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}),\ M\left(\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} g\right) = \left(M\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}\right) g \text{ and } \left(\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} f\right) g = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}(fg).$$

Thus, the expressions $MN\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$, $M\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} g$ *and* $\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} fg$ *are unambiguous.*

PROOF

For the first equality let $M = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$ and $N = \begin{pmatrix} x'_1 & y'_1 \\ x'_2 & y'_2 \end{pmatrix}$.

$$(MN)\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}\begin{pmatrix} x'_1 & y'_1 \\ x'_2 & y'_2 \end{pmatrix}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} x_1x'_1 + y_1x'_2 & x_1y'_1 + y_1y'_2 \\ x_2x'_1 + y_2x'_2 & x_2y'_1 + y_2y'_2 \end{pmatrix}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} =$$

$$\begin{pmatrix} (x_1x'_1 + y_1x'_2)t_1 + (x_1y'_1 + y_1y'_2)t_2 \\ (x_2x'_1 + y_2x'_2)t_1 + (x_2y'_1 + y_2y'_2)t_2 \end{pmatrix} \text{ and}$$

$$M(N\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}) = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}\begin{pmatrix} x'_1t_1 + y'_1t_2 \\ x'_2t_1 + y'_2t_2 \end{pmatrix} = \begin{pmatrix} x_1(x'_1t_1 + y'_1t_2) + y_1(x'_2t_1 + y'_2t_2) \\ x_2(x'_1t_1 + y'_1t_2) + y_2(x'_2t_1 + y'_2t_2) \end{pmatrix}$$

For the second equality, let $M = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$.

Then $\left(M\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}\right)g = \begin{pmatrix} x_1t_1 + y_1t_2 \\ x_2t_1 + y_2t_2 \end{pmatrix}g = \begin{pmatrix} (x_1t_1 + y_1t_2)g \\ (x_2t_1 + y_2t_2)g \end{pmatrix}$

$$= \begin{pmatrix} x_1(t_1g) + y_1(t_2g) \\ x_2(t_1g) + y_2(t_2g \end{pmatrix} = M\begin{pmatrix} t_1g \\ t_2g \end{pmatrix} = M\left(\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}g\right).$$

For the third equality $\left(\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}f\right)g = \begin{pmatrix} t_1f \\ t_2f \end{pmatrix}g = \begin{pmatrix} (t_1f)g \\ (t_1f)g \end{pmatrix} = \begin{pmatrix} t_1(fg) \\ t_2(fg) \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}(fg)$.

||

THEOREM 2.2

Let **T** = **R**$e_1 \oplus$ **R**e_2 *be a two-dimensional linear space with oriented basis* $<e_1, e_2>$. *Then the equalities,* $\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}f$ *induce a 1-1-1 correspondence between the elements* $\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$ *of* **T**[2×1], *the matrices* $\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$ *of* **R**(2) *and the linear endomorphisms f of* End$_{\mathbf{R}}$(**T**)*;* $t_1 = x_1e_1 + y_1e_2 = e_1f$ *and* $t_2 = x_2e_1 + y_2e_2 = e_2f$.

Let M and N be matrices of **R**(2) *and let f and g be linear endomorphisms of* End$_{\mathbf{R}}$(**T**), *such that* $M\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}f$ *and* $N\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}g$. *Then* $MN\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}fg$.

PROOF

The 1-1 correspondence $\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \approx \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$ is immediate.

Linear endomorphisms of **T** determine and are determined by the image of the oriented basis $<e_1, e_2>$. If $t_1 = e_1f$ and $t_2 = e_2f$, then $(xe_1 + ye_2)f = xt_1 + yt_2$. This gives the 1-1 correspondence $\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \approx f$.

$$MN\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = M\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}g = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}fg.$$

||

Let **T** = **R**$e_1 \oplus$ **R**e_2 be a two-dimensional linear space with oriented basis $<e_1, e_2>$. Then the (2×2) matrix M **represents** the linear endomorphism f of **T with respect to** the oriented basis $<e_1, e_2>$ of **T** if $M\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}f$.

Recall (or check) that if $M = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$ is a real 2×2 matrix, then the matrix adj(M), the **adjugate** of M is the matrix $\begin{pmatrix} y_2 & -y_1 \\ -x_2 & x_1 \end{pmatrix}$, satisfying:

$$M\,\mathrm{adj}(M) = \mathrm{adj}(M)\,M = \begin{pmatrix} x_1y_2 - x_2y_1 & 0 \\ 0 & x_1y_2 - x_2y_1 \end{pmatrix} = \det(M)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where the real number $\det(M) = x_1y_2 - x_2y_1$ is the **determinant** of the matrix M. M is thus shown to be invertible ($M^{-1} = (1/\det(M))\mathrm{adj}(M)$) if and only if $\det(M) \neq 0$.

THEOREM 2.3

Let $\mathbf{T} = \mathbf{R}e_1 \oplus \mathbf{R}e_2$ *be a two-dimensional linear space with oriented basis* $\langle e_1, e_2\rangle$. *Let* $\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = M\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} f$, *where* $M = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$ *is a real* (2×2) *matrix, f is a linear endomorphism of* $\mathbf{T}$, *and* $\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ *and* $\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$ *are ordered pairs of elements of* $\mathbf{T}$, *expressed as* (2×1) *column matrices. Then the following are equivalent.*

1) $\langle t_1, t_2\rangle$ *is an oriented basis for* $\mathbf{T}$, *whence* $\mathbf{T} = \mathbf{R}t_1 \oplus \mathbf{R}t_2$.
2) t_1 *and* t_2 *are linearly independent.*
3) *The matrix M is invertible.*
4) *The linear endomorphism f is a linear automorphism.*

PROOF

1) ⇒ 2)

If $\langle t_1, t_2\rangle$ is an oriented basis for $\mathbf{T} = \mathbf{R}t_1 \oplus \mathbf{R}t_2$, then $(x, y) \approx xt_1 + yt_2$ is a linear isomorphism of $\mathbf{R}^2$ onto $\mathbf{T}$. Neither t_1 nor t_2 can be zero because $\mathbf{T}$ is two-dimensional. Also $xt_1 + yt_2 = \mathbf{0}$ if and only if $x = y = 0$, whence $\mathbf{R}t_1 \cap \mathbf{R}t_2 = \mathbf{O}$.

2) ⇒ 3)

M is not invertible if and only if $\det(M) = x_1y_2 - x_2y_1 = 0$.
$t_1 = x_1e_1 + y_1e_2$ and $t_2 = x_2e_1 + y_2e_2$, whence $y_2t_1 - y_1t_2 = \det(M)e_1 = \mathbf{0}$ if M is not invertible, but then, if neither t_1 nor t_2 is zero, $\mathbf{R}t_1 \cap \mathbf{R}t_2 \neq \mathbf{O}$.

3) ⇒ 4)

Let N be the inverse of M in $\mathbf{R}(2)$ and g be the linear endomorphism in $\mathrm{End}_{\mathbf{R}}(\mathbf{T})$ such that $N\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} g$. Then $\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} fg = MN\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = NM\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} gf$. Hence f is invertible (with inverse g).

4) ⇒ 1)

The canonical linear isomorphism from $\mathbf{R}^2$ to $\mathbf{T}$ mapping (1, 0) to e_1 and (0, 1) to e_2 is composed with f yielding a canonical linear isomorphism from $\mathbf{R}^2$ to $\mathbf{T}$ mapping (1, 0) to t_1 and (0, 1) to t_2. ||

THEOREM 2.4

Let $\mathbf{T} = \mathbf{R}e_1 \oplus \mathbf{R}e_2$ *be a two-dimensional linear space with oriented basis* $\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ *expressed as a* (2×1) *column matrix with entries from* $\mathbf{T}$. *Let* $M\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} f$, *where M is a real* (2×2) *matrix and f is a linear endomorphism of* $\mathbf{T}$. *Let* $\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$ *be an oriented basis of* $\mathbf{T}$, *expressed as a* (2×1) *column matrix with entries from* $\mathbf{T}$ *and suppose that* $\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = P\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} h$, *where P is an invertible real* (2×2) *matrix and h is a linear automorphism of* $\mathbf{T}$. *Then* $PMP^{-1}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} f$ *and* $M\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} h^{-1}fh$.

PROOF

$PMP^{-1}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = PM\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = P\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} f = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} f$.

$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} h^{-1}fh = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} fh = M\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} h = M\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$. ||

If a matrix M represents a linear endomorphism f with respect to a given oriented basis of a two-dimensional linear space, then any matrix, which represents f with respect to some oriented basis, is a conjugate PMP^{-1} of M and also, any linear endomorphism, which is represented by M with respect to some oriented basis, is a conjugate $h^{-1}fh$ of f, where P and h are an invertible matrix and a linear automorphism (invertible linear endomorphism), respectively.

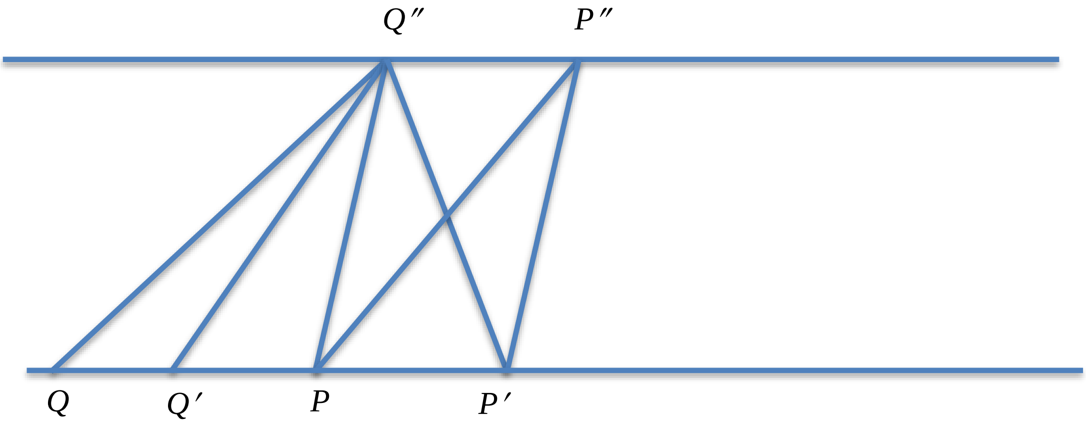
We have shown above that vectors, defined as the oriented lengths of oriented line-segments satisfy all the axioms for linear spaces apart from L4)(ii), which states that for every real number r, $r(v_1 + v_2) = rv_1 + rv_2$, for vectors v_1 and v_2. We only need to establish L4)(ii) for linearly independent pairs of vectors, because if v_1 and v_2 are linearly dependent then they belong to a vector space of dimension at most one, for which L4)(ii) is automatic (Theorem 1.2 page 2).

Let v_1 and v_2 be a pair of (linearly independent) vectors. It follows from the commutativity of vector addition, that $m(v_1 + v_2) = mv_1 + mv_2$ for positive integers m; $n^{-1}(v_1 + v_2) = n^{-1}(nn^{-1}v_1 + nn^{-1}v_2) = n^{-1}(n(n^{-1}v_1 + n^{-1}v_2)) = n^{-1}v_1 + n^{-1}v_2$ for positive integers n; $(v_1 + v_2) + (-v_2) + (-v_1) = \mathbf{0}$, whence $(-1)(v_1 + v_2) = (-1)v_2 + (-1)v_1 = (-1)v_1 + (-1)v_2$ and it is immediate that $0(v_1 + v_2) = \mathbf{0} = 0v_1 + 0v_2$. Hence L4)(ii) holds for rational r and a continuity argument would extend this to all real numbers. However, we prefer an argument based on the Affine Geometry equivalent of Euclid's theory of similar triangles from his sixth book. (See corollary 2.6 p.12)

A5) *Let P, P' and P'' be points, which are not collinear. Then the ordered triple $<P, P', P''>$ forms the vertices of an oriented triangle, which has an oriented area. The orientation of the triangle is determined by the cyclic order of its vertices and an area (positive real number) associated with that orientation. The ordered triple $<P, P'', P'>$ determines the same triangle with the opposite orientation.*

A **non-zero bivector** is an oriented area. If P, P' and P'' are distinct non-collinear points, then we write $b = PP'P''$ if b is the oriented area of the oriented triangle with **vertices** $<P, P', P''>$. We automatically obtain the bivector $PP''P' = (-b)$, which has the same magnitude (area) as b, but whose orientation is opposite to that of b. Also, for any positive real number r, we write rb for the bivector whose orientation is that of b and whose magnitude is the magnitude of b multiplied by r. Also $(-r)b = r(-b) = (-(rb))$ and, in particular, $(-1)b = 1(-b) = -b$. If $<P, P', P''>$ is an ordered triple of collinear points, then we write $\mathbf{0} = PP'P''$, where $\mathbf{0}$ manifests as the **zero bivector** and for any real number r and for any bivector b, we write $r\mathbf{0} = \mathbf{0} = 0b$. It is immediate that if b is a non-zero bivector, then $\mathbf{R}b$, the set of real multiples of b, is a one-dimensional linear space. (If vectors are generalized to higher dimensions, then bivectors become *sums* of oriented areas).

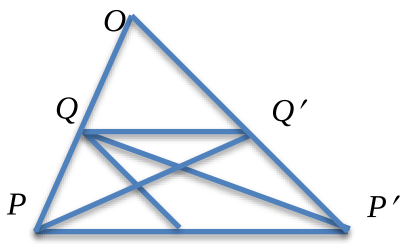
A6) *Let the points P and P' determine a line* $\mathbf{L}$ *and let P'' be a point not on* $\mathbf{L}$. *Let Q and Q' be points on* $\mathbf{L}$ *such that $QQ' = rPP'$ for some real number r and let Q'' be any point on the line through P'' parallel to* $\mathbf{L}$. *Then $QQ'Q'' = rPP'Q'' = rPP'P''$.*



Informally, axiom A6) says that if one edge of an oriented triangle is stretched and/or reversed, while the opposite vertex is fixed, then the oriented area of the oriented triangle is stretched by the same amount and/or reversed; it also says that if one edge of the oriented triangle is fixed and the opposite vertex is moved parallel to the fixed edge, then the oriented area of the oriented triangle is fixed. These informal statements are combined to give the formal statement of the axiom.

THEOREM 2.5

Let O, P and P' be three non-collinear points. Let Q lie on the line determined by O and P and let Q' lie on the line determined by O and P'.
Let the vectors $OQ = rOP$ and $OQ' = r'OP'$.
Then $r = r'$ if and only if the line determined by Q and Q' is parallel to the line determined by P and P'.
Further, if indeed $r = r'$, then $QQ' = rPP'$.
PROOF SKETCH
We illustrate the case $0 < r, r' < 1$



$OQ = rOP$, whence $OQP' = rOPP'$ and $OQ' = r'OP'$, whence $OQ'P = r'OP'P$.
The following are equivalent: $r = r'$; $OQP' + OQ'P = \mathbf{0}$;
$(OQQ' + Q'QP') + (OQ'Q + QQ'P) = \mathbf{0}$; $(Q'QP' + QQ'P) = \mathbf{0}$;
"The line determined by Q and Q' is parallel to the line determined by P and P'".
(The last equivalence also follows from A6).
Finally, we construct the line through Q parallel to OP' to meet PP' etc. ||

COROLLARY 2.6

Let v_1 and v_2 be vectors and let r be a real number. Then $r(v_1 + v_2) = rv_1 + rv_2$; i.e. vectors satisfy axiom L4)(ii).
PROOF
This is already established for linearly dependent vectors so we may assume that there are three non-collinear points O, P and P' such that $OP = (-v_1)$ and $OP' = v_2$.
Let $OQ = rOP = (-rv_1)$ and let $OQ' = rOP' = rv_2$.
Then $rv_1 + rv_2 = QO + OQ' = QQ' = rPP' = r(PO + OP') = r(v_1 + v_2)$. ||

Geometry of Determinants

Recall that a line, with an associated pair of mutually opposite directions, through a point P is specified by the set of points, whose elements are P and every point P', such that the oriented line-segment determined by $<P, P'>$ has either of the given directions. We may recast this definition to say that a **line**, **with an associated pair of mutually opposite orientations through a point** P is specified by the set of points, whose elements are P and every point P', such that the vector PP' is an element of the one-dimensional vector space $\mathbf{R}v$, for some non-zero vector v; the line may be **oriented** by the orientation (direction) of the oriented basis $<v>$ of $\mathbf{R}v$.

A **plane, with an associated pair of mutually opposite orientations, through a point** P is specified by the set of points, whose elements are P and every point Q, such that the vector PQ is an element of a two-dimensional vector space $\mathbf{\Pi}$. $\mathbf{\Pi} = \mathbf{R}v_1 \oplus \mathbf{R}v_2$ for some and hence, by theorem 2.3, any pair of linearly independent vectors v_1 and v_2 of $\mathbf{\Pi}$, which is then **oriented** by its oriented basis $<v_1, v_2>$.

THEOREM 2.7
Any oriented triangle determines a plane and an orientation for that plane.
PROOF
Let $<P, P', P''>$ be the vertices of an oriented triangle. Let $v_1 = PP'$ and $v_2 = PP''$. Then the set of points Q, such that the vector PQ is an element of $\mathbf{R}v_1 \oplus \mathbf{R}v_2$ specifies the required **plane, with oriented basis** $<v_1, v_2>$. ||

Let $\mathbf{\Pi}_P$ be an oriented plane determined by an oriented triangle $<P, P', P''>$ and let $\mathbf{\Pi}_Q$ be a distinct oriented plane determined by an oriented triangle $<Q, Q', Q''>$. Suppose further that the vectors $PQ = P'Q' = P''Q''$ (resp. $PQ = P'Q'' = P''Q'$). Then $\mathbf{\Pi}_P$ and $\mathbf{\Pi}_Q$ are **parallel with the same orientation** (resp. $\mathbf{\Pi}_P$ and $\mathbf{\Pi}_Q$ are **parallel with the opposite orientation**). In either case $\mathbf{\Pi}_Q$ is the image of $\mathbf{\Pi}_P$ under a translation.

If $<P, P', P''>$ and $<Q, Q', Q''>$ determine oriented triangles in the same plane, then loosely speaking they have the same orientation if one may be continuously deformed into the other preserving the cyclic order of the vertices. This corresponds to a *sense of rotation* in Euclidean Geometry. (Also see corollary 2.21 on page 19).

A7) *(Playfair's axiom for planes) Let the ordered triples of points,* $<P, P', P''>$ *and* $<Q, Q', Q''>$ *determine oriented triangles and let the vector* $PQ = P'Q' = P''Q''$ *represent a translation mapping* $<P, P', P''>$ *to* $<Q, Q', Q''>$*. Then the oriented areas* $PP'P''$ *and* $QQ'Q''$ *are equal. Equivalently, bivectors are preserved under translations. In particular, given a plane* $\mathbf{\Pi}$ *and a point* Q*, not in* $\mathbf{\Pi}$*, there is exactly one plane through* Q *parallel to* $\mathbf{\Pi}$*.*

Let $<P, P', P''>$ be an ordered triple of points and let $\mathbf{S}$ be a vector space, containing the vectors $v_1 = PP'$ and $v_2 = PP''$. The ordered triple $<P, P', P''>$ comprises the vertices of an oriented triangle if and only if the points P, P' and P'' are not collinear if and only if the vectors v_1 and v_2 are linearly independent. If indeed v_1 and v_2 are linearly independent, then the two-dimensional vector subspace $\mathbf{\Pi} = \mathbf{R}v_1 \oplus \mathbf{R}v_2$ of $\mathbf{S}$, whose elements are of the form $xv_1 + yv_2$, for real numbers x and y, consists of all those vectors PQ, where Q is in the plane through the vertices of the oriented triangle. We define the **wedge product, exterior product** or **Grassmann product** of v_1 and v_2 to be the bivector $v_1 \wedge v_2 = 2PP'P''$.

The Grassmann product $v_1 \wedge v_2$ of vectors $v_1 = PP'$ and $v_2 = PP''$ in a vector space **S** is the zero bivector **0** if and only if v_1 and v_2 are linearly dependent. Otherwise, v_1 and v_2 determine a unique two-dimensional subspace $\mathbf{R}v_1 \oplus \mathbf{R}v_2$ of **S** and $v_1 \wedge v_2$ may be construed as the oriented area of the parallelogram with adjacent edges PP' and PP'' taken in order. $PP''P' = -PP'P''$, whence the wedge product is antisymmetric; that is to say $v_2 \wedge v_1 = -(v_1 \wedge v_2)$.

Let **S** be a vector space, whose elements are the labels of points with respect to an Origin O. Let v_1 be the point V_1 and v_2 be the point V_2. Then $OV_1V_2 = (1/2)(v_1 \wedge v_2)$. This corresponds to the Euclidean property that the area of a triangle is half of its base multiplied by its height. Indeed $OV_1V_2 = (1/2)(V_1V_2 \wedge PO) = (1/2)(OP \wedge V_1V_2)$ for any point P on the line through V_1 and V_2. The wedge product is a "multiplication" with Clifford's former meaning; it produces a bivector from two vectors. (See second paragraph of page 3).

LEMMA 2.8

Let **S** *be a vector space, whose elements are the labels of points with respect to an Origin O. Let* v_1 *be the point* V_1 *and* v_2 *be the point* V_2*, such that the ordered triple* $\langle O, V_1, V_2\rangle$ *comprises the vertices of an oriented triangle. Let* $v = xv_1 + yv_2$ *be any point V in the two-dimensional vector subspace* $\mathbf{\Pi} = \mathbf{R}v_1 \oplus \mathbf{R}v_2$*. Then:*

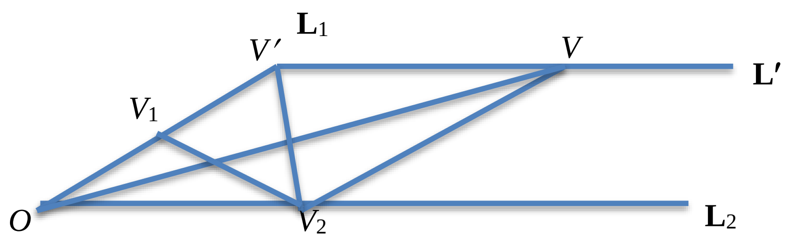
$v \wedge v_2 = x(v_1 \wedge v_2) = (-x)(v_2 \wedge v_1)$ *and* $v_1 \wedge v = y(v_1 \wedge v_2) = (-y)(v_2 \wedge v_1)$.

PROOF SKETCH

We illustrate the case $v \wedge v_2 = x(v_1 \wedge v_2)$, where x is positive:

Let V' be the point labelled by the vector xv_1. Then $OV' = xOV_1$ and V is on the line through V', parallel to the line through O and V_2. In fact, $V'V, = yOV_2$.

$$v \wedge v_2 = 2OVV_2 = 2OV'V_2 = x(2OV_1V_2) = x(v_1 \wedge v_2).$$



If V_1 in the above diagram is moved to the opposite side of O on $\mathbf{L}_1$, then this illustrates the case $v \wedge v_2 = x(v_1 \wedge v_2)$, where x is negative. The case $x = 0$ is $\mathbf{0} = \mathbf{0}$. ||

THEOREM 2.9

Let $\mathbf{\Pi}$ *be a two-dimensional vector space and let* $\mathbf{\Pi}^{(2)}$ *be the set of all bivectors, which are the wedge products of vectors in* $\mathbf{\Pi}$*. Then:*

$v \wedge (xw_1 + yw_2) = x(v \wedge w_1) + y(v \wedge w_2)$ *and* $(xw_1 + yw_2) \wedge v = x(w_1 \wedge v) + y(w_2 \wedge v)$ *for vectors* v*,* w_1 *and* w_2 *of* $\mathbf{\Pi}$ *and scalars x and y. I.e. the wedge product is linear in both arguments.*

PROOF

By anti-symmetry of the wedge product, we need only consider linearity in one of the arguments. We may also consider scalar multiplication and vector addition separately. It is immediate that $v \wedge rw = r(v \wedge w)$ for vectors v and w and scalars r.

Let v, w_1 and w_2 be vectors of $\mathbf{\Pi}$ and let $<u_1, u_2>$ be an oriented basis for $\mathbf{\Pi}$, for which $v = u_1$ and where $w_1 = x_1u_1 + y_1u_2$ and $w_2 = x_2u_1 + y_2u_2$ for scalars x_1, y_1, x_2 and y_2.

Then $v\wedge(w_1 + w_2) = u_1\wedge((x_1 + x_2)u_1 + (y_1 + y_2)u_2) = u_1\wedge(y_1 + y_2)u_2$

$= (y_1 + y_2)(u_1\wedge u_2) = y_1(u_1\wedge u_2) + y_2(u_1\wedge u_2) = (u_1\wedge y_1u_2) + (u_1\wedge y_2u_2)$

$= (u_1\wedge(x_1u_1 + y_1u_2)) + (u_1\wedge(x_2u_1 + y_2u_2)) = (v\wedge w_1) + (v\wedge w_2)$. ||

THEOREM 2.10 (Determinants of (2×2) matrices are signed ratios of oriented areas).
Let $\mathbf{\Pi}$ *be a two-dimensional vector space and let* $\mathbf{\Pi}^{(2)}$ *be the set of all bivectors, which are the wedge products* $v_1\wedge v_2$, *of vectors* v_1 *and* v_2 *in* $\mathbf{\Pi}$. *Let* $<u_1, u_2>$ *be an oriented basis of* $\mathbf{\Pi}$. *Then* $\mathbf{\Pi}^{(2)}$ *may be expressed as a one-dimensional linear space* $\mathbf{R}(u_1\wedge u_2)$. *In fact, let* $<v_1, v_2>$ *be an ordered pair of elements of* $\mathbf{\Pi}$ *and let* M *be the unique* (2×2) *real matrix for which* $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = M\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, *where the oriented basis* $<u_1, u_2>$ *and the ordered pair* $<v_1, v_2>$ *are expressed as* (2×1) *column matrices with entries from* $\mathbf{\Pi}$. *Then* $v_1\wedge v_2 = \det(M)(u_1\wedge u_2)$.

PROOF

Let $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$.

Then $v_1\wedge v_2 = (x_1u_1 + y_1u_2)\wedge(x_2u_1 + y_2u_2) = (x_1y_2 - y_1x_2)(u_1\wedge u_2)$ as required. ||

LEMMA 2.11
Let M *and* N *be* (2×2) *real matrices and* P *be an invertible* (2×2) *real matrix; then:*
1) The (2×2) *real matrix* MN *is invertible if and only if both* M *and* N *are invertible.*
2) $\det(MN) = \det(M)\det(N)$.
3) $\det(P^{-1}) = 1/\det(P)$ *and, in particular,* $\det(PMP^{-1}) = \det(M)$.

PROOF

1) If M and N are both invertible then $(MN)^{-1} = N^{-1}M^{-1}$.
If MN is invertible then $MN(MN)^{-1}$ is the identity matrix, whence $M^{-1} = N(MN)^{-1}$ and $(MN)^{-1}MN$ is also the identity matrix, whence $N^{-1} = (MN)^{-1}M$

2) We need only consider the case M and N both invertible, for otherwise the equality $\det(MN) = \det(M)\det(N)$ reduces to the equality $0 = 0$. Let $\mathbf{\Pi}$ be a two-dimensional vector space and let $<u_1, u_2>$, $<v_1, v_2>$ and $<w_1, w_2>$ be oriented bases of $\mathbf{\Pi}$, such that $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = N\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = M\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, whence $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = MN\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$.

Then $\det(MN)(u_1 \wedge u_2) = w_1 \wedge w_2 = \det(M)(v_1 \wedge v_2) = \det(M)\det(N)(u_1 \wedge u_2)$.

3) $\det(P^{-1})\det(P) = \det(P^{-1}P) = \det(I) = 1$, where I is the identity matrix.
For the particular case, we again need only consider the case M invertible. ||

THEOREM 2.12
Let $\mathbf{\Pi}$ *be a two-dimensional vector space and let* $\mathbf{\Pi}^{(2)}$ *be the set of all bivectors, which are the wedge products* $v_1\wedge v_2$, *of vectors* v_1 *and* v_2 *in* $\mathbf{\Pi}$. *Let* f *be a linear endomorphism of* $\mathbf{\Pi}$. *Then* f *induces a linear endomorphism* $f^{(2)} : v_1\wedge v_2 \longmapsto v_1f\wedge v_2f$ *of the one-dimensional linear space* $\mathbf{\Pi}^{(2)}$. *In fact* $f^{(2)}$ *is scalar multiplication by* $\det(f)$, *where* $\det(f)$ *is well defined to be the determinant of any matrix* M *which represents* f, *i.e. where for some oriented basis* $<u_1, u_2>$ *of* $\mathbf{\Pi}$, $M\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}f = \begin{pmatrix} u_1f \\ u_2f \end{pmatrix}$.

PROOF

If $<u_1, u_2>$ is replaced with any other oriented basis then M is replaced by a conjugate, PMP^{-1} for some invertible P and $\det(PMP^{-1}) = \det(M)$. ||

Geometry of Determinants

Let Π be a two-dimensional vector space and let f be a linear automorphism of Π. Then f is **orientation preserving** (resp. **orientation reversing**) if det(f) is positive (resp. det(f) is negative). The oriented bases $<v_1f, v_2f>$ and $<v_1, v_2>$ have the same or opposite orientation according as the bivectors $v_1f \wedge v_2f$ and $v_1 \wedge v_2$ have the same or opposite orientation according as det(f) is positive or negative. Further, f **preserves oriented area** if det(f) = 1 and **reverses oriented area** if det(f) = -1.

Let Π be a two-dimensional vector space whose elements are the labels of points with respect to an Origin O and let f be a linear automorphism of Π. Then f is **elementary** if f is not the identity, but has an **invariant line**. I.e. Π has a subspace $\mathbf{L} = \mathbf{R}v$ for some non-zero v of Π, such that $wf = w$ for all w in $\mathbf{L}$. Further for any linear automorphism h of Π, f is elementary with invariant line $\mathbf{R}v$ if and only if $h^{-1}fh$ is elementary with invariant line $\mathbf{R}(vh)$.

A square matrix is **elementary** if it is not the identity and is conjugate to a matrix, which differs from the identity only in the first row, the first element of that row being non-zero. In the (2×2) case an elementary matrix is a conjugate of a matrix of the form $\begin{pmatrix} x_1 & y_1 \\ 0 & 1 \end{pmatrix}$, where $x_1 \neq 0$. In particular, $\begin{pmatrix} 1 & 0 \\ x_2 & y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y_2 & x_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is an elementary matrix (or the identity) provided that $y_2 \neq 0$. The determinant of any conjugate of the matrix $\begin{pmatrix} x_1 & y_1 \\ 0 & 1 \end{pmatrix}$ is x_1.

THEOREM 2.13
Let Π be a two-dimensional vector space, whose elements are the labels of points with respect to an Origin O, let f be a linear automorphism of Π and let M be a (2×2) real matrix, which represents f with respect to some oriented basis of Π. Then f is an elementary linear automorphism if and only if M is an elementary matrix.
PROOF
Let f be an elementary linear automorphism with invariant line $\mathbf{R}v$. Let $<v_1, v_2>$ be any oriented basis of Π, satisfying $v = v_2$. Then f is represented by the matrix $\begin{pmatrix} x_1 & y_1 \\ 0 & 1 \end{pmatrix}$ with respect to this oriented basis, where $v_1f = x_1v_1 + y_1v_2$. Any matrix representative of f is conjugate to this matrix. (Theorem 2.4 page 10).
Conversely, let M be a conjugate of the matrix $\begin{pmatrix} x_1 & y_1 \\ 0 & 1 \end{pmatrix}$. Then $\begin{pmatrix} x_1 & y_1 \\ 0 & 1 \end{pmatrix}$ represents f with respect to some oriented basis $<v_1, v_2>$ (Theorem 2.4 page 10), whence f has an invariant line $\mathbf{R}v$, where $v = v_2$. ||

Let $\mathbf{T}$ be any two-dimensional linear space and let f be a linear automorphism of $\mathbf{T}$. Then f is **elementary** if it may be represented by an elementary matrix with respect to some oriented basis of $\mathbf{T}$.

An elementary linear automorphism of a two-dimensional vector space, whose elements are the labels of points with respect to an Origin, is a **shear** if every point, which is not on the invariant line, moves parallel to it; it is an **edge-rescaling** if every point on some line through the origin, other than the invariant line, is rescaled by some non-zero scalar r; a rescaling is a commutative composition, of an **edge-stretch** of scale $|r|$ and either the identity or an **edge-reversal** according as r is either positive or negative.

THEOREM 2.14

Let $\boldsymbol{\Pi}$ *be a two-dimensional vector space, whose elements are the labels of points with respect to an Origin O. Then, with respect to some oriented basis* $<v_1, v_2>$ *of* $\boldsymbol{\Pi}$*, the elementary matrix* $\begin{pmatrix} x_1 & y_1 \\ 0 & 1 \end{pmatrix}$ *is a representation of a shear if* $x_1 = 1$ *and it is a representation of an edge-rescaling if* $y_1 = 0$.

Shears and edge-stretches are orientation-preserving. In particular, shears preserve oriented areas. Edge-reversals are orientation-reversing and, in particular reverse oriented areas.

Every elementary linear automorphism, which is not itself the identity, a shear or an edge-rescaling, may be expressed as a composition of a shear followed by an edge-rescaling or as a composition of an edge-rescaling followed by a shear.

PROOF

Let $x_1 = 1$ and let $w_1 = v_1 + y_1v_2$ and $w_2 = v_2$. Then $(xw_1 + yw_2) – (xv_1 + yv_2) = (xy_1)v_2$
Let $y_1 = 0$ and let $w_1 = x_1v_1$ and $w_2 = v_2$. Then the line $\mathbf{R}v_1$ is fixed or rescaled.
The determinant of a shear is 1. The determinant of an edge-stretch is its scale factor and the determinant of an edge-reversal is -1.

$$\begin{pmatrix} x_1 & y_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & y_1/x_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & y_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad \|$$

Let O be the Origin, let $\mathbf{L}$ be a line through O and let $\boldsymbol{\Pi}$ be a plane containing $\mathbf{L}$. Then $\mathbf{L}$ partitions $\boldsymbol{\Pi}\backslash\mathbf{L}$ into two **half-planes**, which are **sides of** $\boldsymbol{\Pi}$ **determined by** $\mathbf{L}$.

THEOREM 2.15

Let $\boldsymbol{\Pi}$ *be a two-dimensional vector space, whose elements are the labels of points with respect to an Origin O and let* $\mathbf{L}$*, a line through O, be a one-dimensional subspace of* $\boldsymbol{\Pi}$*. Let* v_1 *and* v_2 *be labels of points, which do not lie on* $\mathbf{L}$*. Then* $\mathbf{L}$ *is the invariant line of a unique elementary linear automorphism mapping* v_1 *to* v_2*. Further for every elementary linear automorphism f with invariant line* $\mathbf{L}$ *and for any v in* $\boldsymbol{\Pi}\backslash\mathbf{L}$*, v and vf are on the same side of* $\boldsymbol{\Pi}$ *determined by* $\mathbf{L}$ *if and only if f is orientation-preserving and are on opposite sides of* $\boldsymbol{\Pi}$ *determined by* $\mathbf{L}$ *if and only if f is orientation-reversing.*

PROOF

Let $<u_1, u_2>$ be an oriented basis of $\boldsymbol{\Pi}$, where $u_1 = v_1$ and u_2 is a non-zero element of $\mathbf{L}$. Then $v_2 = x_1u_1 + y_1u_2$ for some scalars $x_1 \neq 0$ and y_1 and the matrix $\begin{pmatrix} x_1 & y_1 \\ 0 & 1 \end{pmatrix}$ represents the unique elementary linear automorphism f, say, mapping v_1 to v_2 and fixing $\mathbf{L}$, with respect to $<u_1, u_2>$. Further every elementary linear automorphism is represented by such a matrix, which fixes the sides of $\boldsymbol{\Pi}$ determined by $\mathbf{L}$ if and only if $x_1 > 0$ and reverses them if and only if $x_1 < 0$. ||

KEY LEMMA 2.16

Let v_1 *and* v_2 *be elements of a vector space* $\mathbf{S}$*, whose elements are the labels of points with respect to an Origin O. Then:*

1) $v_1\wedge v_2 = v_1\wedge(v_1 + v_2) = (-v_2)\wedge(v_1 + v_2) = (-v_2)\wedge v_1$.

2) If v_1 *and* v_2 *are linearly independent, then they form an oriented basis for a two-dimensional subspace* $\boldsymbol{\Pi}$ *of* $\mathbf{S}$*; the linear automorphism of* $\boldsymbol{\Pi}$*, which maps that oriented basis* $<v_1, v_2>$ *to the oriented basis* $<(-v_2), v_1>$ *of* $\boldsymbol{\Pi}$*, preserves oriented area as it may be expressed as the composition of three shears.*

PROOF

1) It follows from theorem 2.9 that the wedge product is both left-distributive and right-distributive over vector addition and that $v_1 \wedge v_1 = v_2 \wedge v_2 = (v_1 + v_2) \wedge (v_1 + v_2) = \mathbf{0}$.

2) $\begin{pmatrix} -v_2 \\ v_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and we must show that that the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the product of three matrices which represent shears.

In fact $\begin{pmatrix} -v_2 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -v_2 \\ v_1 + v_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_1 + v_2 \end{pmatrix} =$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

||

COROLLARY 2.17

Let **Π** *be a two-dimensional vector space, whose elements are the labels of points with respect to an Origin O. Then the linear automorphism* $v \mapsto (-v)$*, which reverses the direction of every vector of* **Π** *preserves oriented area. In fact* $v \mapsto (-v)$*, may be expressed as a composition of six shears, the first of which may be chosen arbitrarily.*

PROOF

The matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right)^2$ represents the linear automorphism $v \mapsto -v$ with respect to any oriented basis of **Π**. Hence $v \mapsto (-v)$ may be expressed as the composition of a sequence of six shears. Let s be a given shear. Then we may choose a point V_1, with vector label v_1 on the invariant line of s and we may choose v_2 to be the vector label of any point on the line, parallel to OV_1, whose labels satisfy $vs - v = v_1$. Then s has the matrix representation $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ with respect to the oriented basis $\langle v_1, v_2 \rangle$.

||

THEOREM 2.18 (Extending the Steinitz exchange lemma for two dimensions)

Let $\mathbf{T} = \mathbf{R}e_1 \oplus \mathbf{R}e_2$ *be a two-dimensional linear space with oriented basis* $\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ *expressed as a* (2×1) *column matrix with entries from* **T**. *Let* T_0 *be a spanning set for* **T**. *Then there is an ordered pair* $\langle t_1, t_2 \rangle$ *of elements of* T_0 *and a pair of matrices* M_1 *and* M_2*, each of which is either the identity matrix or is an elementary matrix, which satisfy* $\begin{pmatrix} t_1 \\ e_2 \end{pmatrix} = M_1 \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ *and* $\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = M_2 \begin{pmatrix} t_1 \\ e_2 \end{pmatrix}$*. Specifically:*

If $M_2 M_1 = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$*, then* $M_1 = \begin{pmatrix} x_1 & y_1 \\ 0 & 1 \end{pmatrix}$ *and* $M_2 = \begin{pmatrix} 1 & 0 \\ x_2/x_1 & (x_1 y_2 - x_2 y_1)/x_1 \end{pmatrix}$.

Let $\Delta_1 = x_1$ *and let* $\Delta_2 = x_1 y_2 - x_2 y_1$. *Then* $\det(M_1) = \Delta_1$ *and* $\det(M_2) = \Delta_2/\Delta_1$.

Further, let T_0 *contain the element* $(-t)$*, whenever it contains the element t. Then* t_1 *and* t_2 *may be chosen so that* Δ_1 *and* Δ_2 *are both positive.*

PROOF

Every element t of T_0 may be expressed as $t = xe_1 + ye_2$ where x and y are real numbers. Since T_0 is a spanning set for **T**, at least one element $t_1 = x_1 e_1 + y_1 e_2$ in **T** is such that x_1 is non-zero (for otherwise, e_1 is not in the span of T_0) and x_1 may be chosen to be positive if the further condition of the theorem holds.

$\begin{pmatrix} t_1 \\ e_2 \end{pmatrix} = M_1 \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$, where $M_1 = \begin{pmatrix} x_1 & y_1 \\ 0 & 1 \end{pmatrix}$ and in particular $\begin{pmatrix} t_1 \\ e_2 \end{pmatrix}$ is an oriented basis of **T**, whence every element t of T_0 may be expressed as $t = x't_1 + y'e_2$ where x' and y' are real numbers. Since T_0 is a spanning set for **T**, at least one element $t_2 = x'_2 t_1 + y'_2 e_2$ in **T** is such that y'_2 is non-zero (for otherwise, e_2 is not in the span of T_0) and y'_2 may be chosen to be positive if the further condition of the theorem holds.

We may also write $t_2 = x_2e_1 + y_2e_2$. Then $\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = M_2\begin{pmatrix} t_1 \\ e_2 \end{pmatrix}$, where $M_2M_1 = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$ and in particular $\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$ is an oriented basis of **T**.

$\det(M_1) = x_1 = \Delta_1$ and $\det(M_2)\det(M_1) = \det(M_2M_1) = x_1y_2 - x_2y_1 = \Delta_2$

$$M_2 = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}\begin{pmatrix} x_1 & y_1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}\begin{pmatrix} 1/x_1 & -y_1/x_1 \\ 0 & 1 \end{pmatrix}.$$
$$= \begin{pmatrix} 1 & 0 \\ x_2/x_1 & (x_1y_2 - x_2y_1)/x_1 \end{pmatrix}.$$
||

LEMMA 2.19

Let **T** = **R***e*$_1$ ⊕ **R***e*$_2$ *be a two-dimensional linear space with oriented basis* $<e_1, e_2>$. *Let f be a linear automorphism of* **T** *and let M be the* (2×2) *real matrix, which represents f with respect to* $<e_1, e_2>$. *Then* $M = M_0M_2M_1$, *where* M_1 *is the identity matrix or is an elementary matrix with positive determinant,* M_2 *is the identity matrix or is an elementary matrix with positive determinant and if* det(*f*) *is positive then* M_0 *is one of the four matrices* $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, *while if* det(*f*) *is negative,* M_0 *is one of the four matrices* $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *or* $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

PROOF

$M\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} e_1f \\ e_2f \end{pmatrix}$ and $\{e_1f, -e_1f, e_2f, -e_2f\}$ is a spanning set for **T**, which contains $-t$, whenever it contains t.

$\begin{pmatrix} t_1 \\ e_2 \end{pmatrix} = M_1\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$, where t_1 is one of the four elements $\pm e_1f$ and $\pm e_2f$, which is chosen so that $\det(M_1)$ is positive. $\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = M_2\begin{pmatrix} t_1 \\ e_2 \end{pmatrix}$, where t_2 is one of the two elements $\pm e_2f$, if $t_1 = \pm e_1f$, or else t_2 is one of the two elements $\pm e_1f$, if $t_1 = \pm e_2f$; in either case t_2 is chosen so that $\det(M_2)$ is positive.

The rows of the matrix M may, or may not, be swapped and either, or neither, or both rows may have all of their entries multiplied by -1 to obtain the matrix M_2M_1 and M_2M_1 may be pre-multiplied by one of the matrices specified in the hypotheses of the lemma to obtain the matrix M. ||

THEOREM 2.20

Let Π *be a two-dimensional vector space, whose elements are the labels of points with respect to an Origin. Then any linear automorphism f of* Π *may be expressed as a composition* $f = g_1g_2g_0$, *where* g_1 *is the identity or is elementary with positive determinant,* g_2 *is the identity or is elementary with positive determinant and* g_0 *is a sequence of shears if* det(*f*) *is positive and is a sequence of shears followed by exactly one edge-reversal if* det(*f*) *is negative.*

PROOF

g_1 and g_2 are found using lemma 2.18; g_0 is found using key lemma 2.16. ||

COROLLARY 2.21

Let Π *be a two-dimensional vector space, whose elements are the labels of points with respect to an Origin. Then a linear automorphism f may be described as a continuous deformation of* Π *if and only if* det(*f*) *is positive.*

PROOF

Shears and edge-stretches are continuous deformations; edge-reversals are not. ||

THEOREM 2.22

Let **Π** *be a two-dimensional vector space, whose elements are the labels of points with respect to an Origin O and let* **L**, *a line through O, be a one-dimensional subspace of* **Π**. *Then the non-zero elements of the quotient space* **Π**/**L** *are the lines, which are parallel to* **L** *in* **Π**. *(The zero element of* **Π**/**L** *manifests as* **L**, *itself).* ||

LEMMA 2.23

Let **Π** *be a two-dimensional vector space, whose elements are the labels of points with respect to an Origin and let f be a non-zero linear endomorphism of* **Π**, *which is not a linear automorphism. Then* Ker(f) *and* Im(f) *are one-dimensional subspaces i.e. lines through the Origin of* **Π**. *If* Ker(f) $\cap$ Im(f) = **O**, *then* **Π** = Ker(f) $\oplus$ Im(f). *Otherwise* Ker(f) $\cap$ Im(f) $\neq$ **O** *and then* Ker(f) = Im(f).

PROOF

Π = Ker(f) $\oplus$ $\mathbf{S_0}$, where $\mathbf{S_0}$ is a subspace of **Π** obtained by extending an oriented basis of Ker(f) to an oriented basis of **Π**. Any such subspace $\mathbf{S_0}$ is linearly isomorphic to Im(f). ||

THEOREM 2.24

Let **Π** *be a two-dimensional vector space, whose elements are the labels of points with respect to an Origin and let f be a non-zero linear endomorphism of* **Π**, *which is not a linear automorphism.*

1) Ker(f) $\cap$ Im(f) = **O** *if and only if f has a matrix representation* $\begin{pmatrix} 0 & 0 \\ 0 & y_2 \end{pmatrix}$ *in* **R**(2) *with respect to an oriented basis* <v_1, v_2> *of* **Π**, *where* v_1 *is an element of* Ker(f), v_2 *is an element of* Im(f) *and* $y_2 \neq 0$. *If this is so, then f may be construed as a composition of a projection of* **Π** *onto* Im(f) *along the cosets of* Ker(f), *a linear automorphism of* Im(f) *and the inclusion map from* Im(f) *to* **Π**; *the linear automorphism of* Im(f) *is multiplication by* y_2.

2) Ker(f) $\cap$ Im(f) $\neq$ **O** *if and only if f has a matrix representation* $\begin{pmatrix} 0 & 0 \\ y_2 & 0 \end{pmatrix}$ *in* **R**(2) *with respect to an oriented basis* <v_1, v_2> *of* **Π**, *where* v_1 *is an element of* Ker(f), v_2 *is any element of* **Π** *such that* v_1 *and* v_2 *are linearly independent and* $y_2 \neq 0$. *If this is indeed so, then f = sg, where s is a sequence of shears mapping* v_1 *to* v_2 *and* ($-v_2$) *to* v_1 *and g is a non-zero linear endomorphism of* **Π**, *for which* Ker(g) $\cap$ Im(g) = **O**.

PROOF

1) $\Leftarrow$ Let f have the matrix representation $\begin{pmatrix} 0 & 0 \\ 0 & y_2 \end{pmatrix}$ with respect to an oriented basis <v_1, v_2> of **Π**. It is immediate that det(f) = 0. Also $v_1f = 0$, whence v_1 is an element of Ker(f) and $v_2f = y_2v_2$, whence $v_2 = (1/y_2)(v_2f)$ is an element of Im(f).

$\Rightarrow$ If Ker(f) $\cap$ Im(f) = **O**, then by lemma 2.23 Ker(f) $\oplus$ Im(f) = **Π**. Let <v_1, v_2> be any oriented basis of **Π**, where v_1 is an element of Ker(f) and v_2 is an element of Im(f). Then $v_1f = 0$ and v_2f is a non-zero element of Im(f), whence it must be of the form y_2v_2 for some real number y_2 and f has the matrix representation $\begin{pmatrix} 0 & 0 \\ 0 & y_2 \end{pmatrix}$ with respect to the oriented basis <v_1, v_2>.

The final part of 1) follows from theorem 1.10 on page 7.

2) $\Leftarrow$ Let f have the matrix representation $\begin{pmatrix} 0 & 0 \\ y_2 & 0 \end{pmatrix}$ with respect to an oriented basis $<v_1, v_2>$ of $\mathbf{\Pi}$. It is immediate that $\det(f) = 0$. Also, $v_1 f = 0$, whence v_1 is an element of Ker(f) and $v_2 f = y_2 v_1$, whence $v_2 f$, which is an element of Im(f), is also an element of Ker(f).

$\Rightarrow$ If Ker(f) $\cap$ Im(f) $\neq$ **O**, then let $<v_1, v_2>$ be any oriented basis of $\mathbf{\Pi}$, where v_1 is an element of Ker(f) $\cap$ Im(f) and v_2 is any element of $\mathbf{\Pi}$ such that v_1 and v_2 are linearly independent. Then $v_1 f = 0$ and $v_2 f$ is a non-zero element of Im(f) = Ker(f) , whence it must be of the form $y_2 v_1$ for some non-zero real number y_2 and f has the matrix representation $\begin{pmatrix} 0 & 0 \\ y_2 & 0 \end{pmatrix}$ with respect to the oriented basis $<v_1, v_2>$.

For the final part of 2) let f have the matrix representation $\begin{pmatrix} 0 & 0 \\ y_2 & 0 \end{pmatrix}$ with respect to the oriented basis $<v_1, v_2>$. Let s be the sequence of three shears, which maps the oriented basis $<v_1, v_2>$ of $\mathbf{\Pi}$ to the oriented basis $<-v_2, v_1>$. Then $f = sg$, where g has the matrix representation $\begin{pmatrix} 0 & 0 \\ 0 & y_2 \end{pmatrix}$ with respect to the oriented basis $<v_2, v_1>$. Indeed:

$$\begin{pmatrix} 0 & 0 \\ y_2 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & y_2 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & y_2 \end{pmatrix}\begin{pmatrix} -v_2 \\ v_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & y_2 \end{pmatrix}\begin{pmatrix} v_2 \\ v_1 \end{pmatrix}.$$

s has the effect of swapping columns in the matrix representation. The minus sign may be ignored because it is applied to the elements of a column of zeros. ||

Appendix A) The Continuity axiom

Let u and u' be vectors which point in the same direction. Then there is a line **L** containing points O, U and U' such that U and U' are on the same side of O. We write $|OP| < |OQ|$ if P lies between O and Q on **L**.

For every positive integer m, we may use Playfair's axiom in one dimension to lay m copies of the directed line segment OU end to end to obtain a point U_m and we may similarly lay n copies of OU' end to end to obtain a point OU'_n for any positive integer n.

Let S be the set of rational numbers $\{m/n : |OU_m| < |OU'_n|\}$. Then S is an initial segment of the positive rationals and has the positive real least upper bound r, say, by the theory of Dedekind cuts. If $m/n < r$, then $|OU_m| < |OU'_n|$; if $m/n > r$, then $|OU_m| > |OU'_n|$. Thus, we define $u' = ru$.

It is straightforward to modify this discussion if u and u' are vectors which point in opposite directions and we obtain $u' = ru$ for some negative real r. Also $\mathbf{0} = 0u$. .