Let $\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$ be an ordered basis of $\mathbf{E}$ and let $\mathbf{v}_1 = |\mathbf{v}_1|\mathbf{u}_1$, $\mathbf{v}_2 = |\mathbf{v}_2|(\cos\theta\mathbf{u}_1 + \sin\theta\mathbf{u}_2)$ and $\mathbf{v}_3 = |\mathbf{v}_3|(\sin\varphi\cos\psi\,\mathbf{u}_1 + \sin\varphi\sin\psi\,\mathbf{u}_2 + \cos\varphi\,\mathbf{u}_3)$, where the ordered orthonormal basis $\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle$ is obtained from $\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$ by the2 Gramm-Schmidt orthonormalization process. $\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$ has the same orientation as $\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle$. $\mathbf{u}_1\mathbf{u}_2 = +\mathbf{u}_3$ if this orientation is positive and $\mathbf{u}_1\mathbf{u}_2 = -\mathbf{u}_3$ if this orientation is negative. In particular, $\mathbf{u}_1\mathbf{u}_2 + \mathbf{u}_2\mathbf{u}_1 = \mathbf{0}$.

$\mathbf{v}_2\mathbf{v}_1 = |\mathbf{v}_1||\mathbf{v}_2|(\cos\theta\mathbf{u}_1^2 + \sin\theta\mathbf{u}_1\mathbf{u}_2) = |\mathbf{v}_1||\mathbf{v}_2|(-\cos\theta + \sin\theta(\mathbf{u}_1\mathbf{u}_2))$;
$\mathbf{v}_2\mathbf{v}_1 = |\mathbf{v}_1||\mathbf{v}_2|(\cos\theta\mathbf{u}_1^2 + \sin\theta\mathbf{u}_2\mathbf{u}_1) = |\mathbf{v}_1||\mathbf{v}_2|(-\cos\theta - \sin\theta(\mathbf{u}_1\mathbf{u}_2))$.
Hence $(\mathbf{v}_1\mathbf{v}_2 + \mathbf{v}_2\mathbf{v}_1)/2$ is the *scalar* $-|\mathbf{v}_1||\mathbf{v}_2|\cos\theta = -\mathbf{v}_1\bullet\mathbf{v}_2$ and
$(\mathbf{v}_1\mathbf{v}_2 - \mathbf{v}_2\mathbf{v}_1)/2$ is the *vector* $|\mathbf{v}_1||\mathbf{v}_2|\sin\theta(\mathbf{u}_1\mathbf{u}_2) = \mathbf{v}_1\times\mathbf{v}_2$.

If $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly dependent non-zero *vectors*, then $\mathbf{w}_1 = |\mathbf{w}_1|\mathbf{u}$ and $\mathbf{w}_1 = \pm|\mathbf{w}_2|\mathbf{u}$, for some unit *vector* $\mathbf{u}$, where the sign is "+" or is "-", according as $\mathbf{w}_1$ and $\mathbf{w}_2$ point in the same direction or in opposite directions and $\mathbf{w}_1\mathbf{w}_2 = \mathbf{w}_2\mathbf{w}_1 = |\mathbf{w}_1||\mathbf{w}_2|\mathbf{u}^2 = -|\mathbf{w}_1||\mathbf{w}_2|\cos 0$, if $\mathbf{w}_1$ and $\mathbf{w}_2$ point in the same direction and $\mathbf{w}_1\mathbf{w}_2 = \mathbf{w}_2\mathbf{w}_1 = -|\mathbf{w}_1||\mathbf{w}_2|\mathbf{u}^2 = -|\mathbf{w}_1||\mathbf{w}_2|\cos\pi$, if $\mathbf{w}_1$ and $\mathbf{w}_2$ point in opposite directions. In both cases $\mathbf{w}_1\mathbf{w}_2 = (\mathbf{w}_1\mathbf{w}_2 + \mathbf{w}_2\mathbf{w}_1)/2 = -\mathbf{w}_1\bullet\mathbf{w}_2$. Further, in both cases $(\mathbf{w}_1\mathbf{w}_2 - \mathbf{w}_2\mathbf{w}_1)/2 = \mathbf{0} = \mathbf{w}_1\times\mathbf{w}_2$.

$$\begin{aligned}\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 = |\mathbf{v}_1||\mathbf{v}_2||\mathbf{v}_3|(&\cos\theta\sin\varphi\cos\psi\,\mathbf{u}_1^3 + \sin\theta\sin\varphi\cos\psi\,(\mathbf{u}_1\mathbf{u}_2\mathbf{u}_1) + \\ &\cos\theta\sin\varphi\sin\psi\,(\mathbf{u}_1^2\mathbf{u}_2) + \sin\theta\sin\varphi\sin\psi\,(\mathbf{u}_1\mathbf{u}_2^2) + \\ &\cos\theta\cos\varphi\,(\mathbf{u}_1^2\mathbf{u}_3) + \sin\theta\cos\varphi\,(\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3))\end{aligned}$$

$\mathbf{u}_1^3 = -\mathbf{u}_1$ and $\mathbf{u}_1\mathbf{u}_2\mathbf{u}_1 = \mathbf{u}_2$ are palindromic and are *vectors*;
$\mathbf{u}_1^2\mathbf{u}_2 = -\mathbf{u}_2 = \mathbf{u}_2\mathbf{u}_1^2$, $\mathbf{u}_1\mathbf{u}_2^2 = -\mathbf{u}_1 = \mathbf{u}_2^2\mathbf{u}_1$ and $\mathbf{u}_1^2\mathbf{u}_2 = -\mathbf{u}_2 = \mathbf{u}_2\mathbf{u}_1^2$ are *vectors*;
$\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3 + \mathbf{u}_3\mathbf{u}_2\mathbf{u}_1 = 0$; in fact, $\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3$ is the positive or negative *scalar* ± 1, according as the orientation of the original ordered basis $\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$ is negative or positive.

$$\begin{aligned}\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 &= |\mathbf{v}_1||\mathbf{v}_2||\mathbf{v}_3|(\sin\theta\cos\varphi\,(\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3) - (\cos\theta\sin\varphi\cos\psi + \sin\theta\sin\varphi\sin\psi)\mathbf{u}_1 \\ &\qquad + (\sin\theta\sin\varphi\cos\psi - \cos\theta\sin\varphi\sin\psi)\mathbf{u}_2 - (\cos\theta\cos\varphi)\mathbf{u}_3) \\ &= |\mathbf{v}_1||\mathbf{v}_2||\mathbf{v}_3|(\sin\theta\cos\varphi(\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3) - \sin\varphi\,(\cos(\psi-\theta)\mathbf{u}_1 + \sin(\psi-\theta)\mathbf{u}_2) - \cos\theta\cos\varphi\,\mathbf{u}_3)\end{aligned}$$

$$\mathbf{v}_3\mathbf{v}_2\mathbf{v}_1 = |\mathbf{v}_1||\mathbf{v}_2||\mathbf{v}_3|(\sin\theta\cos\varphi\,(\mathbf{u}_3\mathbf{u}_2\mathbf{u}_1) - \sin\varphi\,(\cos(\psi-\theta)\mathbf{u}_1 + \sin(\psi-\theta)\mathbf{u}_2) - \cos\theta\cos\varphi\,\mathbf{u}_3)$$

Hence $(\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 - \mathbf{v}_3\mathbf{v}_2\mathbf{v}_1)/2$ is the *scalar* $|\mathbf{v}_1||\mathbf{v}_2|||\mathbf{v}_3|\sin\theta\cos\varphi\,(\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3) = \mathrm{Vol}[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]\,\psi$, where $\mathrm{Vol}[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = |\mathbf{v}_1||\mathbf{v}_2|||\mathbf{v}_3|\sin\theta\cos\varphi$ is the volume of the parallelepiped with adjacent edges $\mathbf{v}_1 = OV_1$, $\mathbf{v}_2 = OV_2$ and $\mathbf{v}_3 = OV_3$ and $\psi = \pm 1$, the sign of ψ being opposite to the sign of the orientation of the ordered basis $\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$.

It is immediate that $(\mathbf{v}_l\mathbf{v}_m\mathbf{v}_n - \mathbf{v}_n\mathbf{v}_m\mathbf{v}_l)/2 = \pm(\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 - \mathbf{v}_3\mathbf{v}_2\mathbf{v}_1)/2$, where the ordered triple $\langle l, m, n \rangle$ is a permutation of the sequence $\langle 1, 2, 3 \rangle$ and where the sign is "+" or is "-", according as the permutation is even or is odd. (In particular, $\mathbf{u}_l\mathbf{u}_m\mathbf{u}_n = \mathbf{u}_1\mathbf{u}_2\mathbf{u}_3$).

$(\mathbf{v}_1\times\mathbf{v}_2)\bullet\mathbf{v}_3 = (|\mathbf{v}_1||\mathbf{v}_2|\sin\theta(\mathbf{u}_1\times\mathbf{u}_2))\bullet|\mathbf{v}_3|\cos\varphi\,\mathbf{u}_3 = |\mathbf{v}_1||\mathbf{v}_2|||\mathbf{v}_3|\sin\theta\cos\varphi\,((\mathbf{u}_1\times\mathbf{u}_2)\bullet\mathbf{u}_3)$; thus $(\mathbf{v}_1\times\mathbf{v}_2)\bullet\mathbf{v}_3 = -(\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 - \mathbf{v}_3\mathbf{v}_2\mathbf{v}_1)/2$ follows from $(\mathbf{u}_1\times\mathbf{u}_2)\bullet\mathbf{u}_3 = -(\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3)$, both being equal either to 1, or to -1, according to the orientation of the ordered basis. It also follows from $(\mathbf{u}_1\times\mathbf{u}_2)\bullet\mathbf{u}_3 = \mathbf{u}_1\bullet(\mathbf{u}_2\times\mathbf{u}_3)$, that $(\mathbf{v}_1\times\mathbf{v}_2)\bullet\mathbf{v}_3 = \mathbf{v}_1\bullet(\mathbf{v}_2\times\mathbf{v}_3)$.

If $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ are linearly dependent *vectors*, then it is straightforward to show that $(\mathbf{w}_1\mathbf{w}_2\mathbf{w}_3 - \mathbf{w}_3\mathbf{w}_2\mathbf{w}_1)/2 = (\mathbf{w}_1\times\mathbf{w}_2)\bullet\mathbf{w}_3 = 0$.

We also give a direct algebraic proof that $(\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 - \mathbf{v}_3\mathbf{v}_2\mathbf{v}_1)/2 = -\mathbf{v}_1\bullet(\mathbf{v}_2\times\mathbf{v}_3)$, using the quaternion identity $(\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 - \mathbf{v}_3\mathbf{v}_2\mathbf{v}_1)/2 = (\mathbf{v}_2\mathbf{v}_3\mathbf{v}_1 - \mathbf{v}_1\mathbf{v}_3\mathbf{v}_2)/2$.

$$
\begin{aligned}
(\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 - \mathbf{v}_3\mathbf{v}_2\mathbf{v}_1)/2 &= (\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 - \mathbf{v}_3\mathbf{v}_2\mathbf{v}_1)/4 + (\mathbf{v}_2\mathbf{v}_3\mathbf{v}_1 - \mathbf{v}_1\mathbf{v}_3\mathbf{v}_2)/4. \\
&= (\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 - \mathbf{v}_1\mathbf{v}_3\mathbf{v}_2)/4 + (\mathbf{v}_2\mathbf{v}_3\mathbf{v}_1 - \mathbf{v}_3\mathbf{v}_2\mathbf{v}_1)/4. \\
&= \mathbf{v}_1(\mathbf{v}_2\mathbf{v}_3 - \mathbf{v}_3\mathbf{v}_2)/4 + (\mathbf{v}_2\mathbf{v}_3 - \mathbf{v}_3\mathbf{v}_2)\mathbf{v}_1/4 \\
&= \mathbf{v}_1(\mathbf{v}_2\times\mathbf{v}_3)/2 + (\mathbf{v}_2\times\mathbf{v}_3)\mathbf{v}_1/2 \\
&= -\mathbf{v}_1\bullet(\mathbf{v}_2\times\mathbf{v}_3).
\end{aligned}
$$

Any vector $\mathbf{v}$ may be expressed as a linear combination of the elements of any basis of $\mathbf{E}$. Specifically, let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis of $\mathbf{E}$. Then:

$$[\mathbf{v}_1\bullet(\mathbf{v}_2\times\mathbf{v}_3)]\mathbf{v} = [\mathbf{v}_2\bullet(\mathbf{v}_3\times\mathbf{v})]\mathbf{v}_1 + [\mathbf{v}_1\bullet(\mathbf{v}\times\mathbf{v}_3)]\mathbf{v}_2 + [\mathbf{v}\bullet(\mathbf{v}_1\times\mathbf{v}_2)]\mathbf{v}_3.$$

In fact:
$$
\begin{aligned}
(\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 - \mathbf{v}_3\mathbf{v}_2\mathbf{v}_1)\mathbf{v} &= \mathbf{v}_1\mathbf{v}_2\mathbf{v}_3\mathbf{v} - \mathbf{v}_3\mathbf{v}_2\mathbf{v}_1\mathbf{v} \\
&= \mathbf{v}_1\mathbf{v}_2\mathbf{v}_3\mathbf{v} - \mathbf{v}_1\mathbf{v}\mathbf{v}_3\mathbf{v}_2 + \mathbf{v}_1\mathbf{v}\mathbf{v}_3\mathbf{v}_2 - \mathbf{v}_3\mathbf{v}\mathbf{v}_1\mathbf{v}_2 + \mathbf{v}_3\mathbf{v}\mathbf{v}_1\mathbf{v}_2 - \mathbf{v}_3\mathbf{v}_2\mathbf{v}_1\mathbf{v} \\
&= \mathbf{v}_1(\mathbf{v}_2\mathbf{v}_3\mathbf{v} - \mathbf{v}\mathbf{v}_3\mathbf{v}_2) + (\mathbf{v}_1\mathbf{v}\mathbf{v}_3 - \mathbf{v}_3\mathbf{v}\mathbf{v}_1)\mathbf{v}_2 + \mathbf{v}_3(\mathbf{v}\mathbf{v}_1\mathbf{v}_2 - \mathbf{v}_2\mathbf{v}_1\mathbf{v}).
\end{aligned}
$$

$(\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 + \mathbf{v}_3\mathbf{v}_2\mathbf{v}_1)/2$ is the *vector* $(-\mathbf{v}_2\bullet\mathbf{v}_3)\mathbf{v}_1 + (\mathbf{v}_1\bullet\mathbf{v}_3)\mathbf{v}_2 + (-\mathbf{v}_1\bullet\mathbf{v}_2)\mathbf{v}_3$. In fact:
$$
\begin{aligned}
(\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 + \mathbf{v}_3\mathbf{v}_2\mathbf{v}_1)/2 &= (\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 + \mathbf{v}_1\mathbf{v}_3\mathbf{v}_2)/2 - (\mathbf{v}_1\mathbf{v}_3\mathbf{v}_2 + \mathbf{v}_3\mathbf{v}_1\mathbf{v}_2)/2 + (\mathbf{v}_3\mathbf{v}_1\mathbf{v}_2 + \mathbf{v}_3\mathbf{v}_2\mathbf{v}_1)/2 \\
&= \mathbf{v}_1(\mathbf{v}_2\mathbf{v}_3 + \mathbf{v}_3\mathbf{v}_2)/2 - ((\mathbf{v}_1\mathbf{v}_3 + \mathbf{v}_3\mathbf{v}_1)/2)\mathbf{v}_2 + \mathbf{v}_3(\mathbf{v}_1\mathbf{v}_2 + \mathbf{v}_2\mathbf{v}_1)/2.
\end{aligned}
$$

We may obtain two vector identities for the expression $(\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 - \mathbf{v}_2\mathbf{v}_3\mathbf{v}_1)/2$.
$$
\begin{aligned}
(\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 - \mathbf{v}_2\mathbf{v}_3\mathbf{v}_1)/2 &= (\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 + \mathbf{v}_2\mathbf{v}_1\mathbf{v}_3)/2 - (\mathbf{v}_2\mathbf{v}_1\mathbf{v}_3 + \mathbf{v}_2\mathbf{v}_3\mathbf{v}_1)/2 \\
&= -\mathbf{v}_3(\mathbf{v}_1\bullet\mathbf{v}_2) + \mathbf{v}_2(\mathbf{v}_1\bullet\mathbf{v}_3).
\end{aligned}
$$
Also, it follows from $(\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 - \mathbf{v}_3\mathbf{v}_2\mathbf{v}_1)/2 = (\mathbf{v}_2\mathbf{v}_3\mathbf{v}_1 - \mathbf{v}_1\mathbf{v}_3\mathbf{v}_2)/2$ that
$$
\begin{aligned}
(\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 - \mathbf{v}_2\mathbf{v}_3\mathbf{v}_1)/2 &= (\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 + \mathbf{v}_3\mathbf{v}_2\mathbf{v}_1)/4 + (\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 - \mathbf{v}_3\mathbf{v}_2\mathbf{v}_1)/4 \\
&\quad - (\mathbf{v}_2\mathbf{v}_3\mathbf{v}_1 + \mathbf{v}_1\mathbf{v}_3\mathbf{v}_2)/4 - (\mathbf{v}_2\mathbf{v}_3\mathbf{v}_1 - \mathbf{v}_1\mathbf{v}_3\mathbf{v}_2)/4 \\
&= (\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 + \mathbf{v}_3\mathbf{v}_2\mathbf{v}_1)/4 - (\mathbf{v}_2\mathbf{v}_3\mathbf{v}_1 + \mathbf{v}_1\mathbf{v}_3\mathbf{v}_2)/4 \\
&= (\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 - \mathbf{v}_1\mathbf{v}_3\mathbf{v}_2)/4 - (\mathbf{v}_2\mathbf{v}_3\mathbf{v}_1 - \mathbf{v}_3\mathbf{v}_2\mathbf{v}_1)/4 \\
&= \mathbf{v}_1(\mathbf{v}_2\times\mathbf{v}_3)/2 - (\mathbf{v}_2\times\mathbf{v}_3)\mathbf{v}_1/2 = \mathbf{v}_1\times(\mathbf{v}_2\times\mathbf{v}_3).
\end{aligned}
$$
In particular, we obtain the Jacobi identity $\mathbf{v}_1\times(\mathbf{v}_2\times\mathbf{v}_3) + \mathbf{v}_2\times(\mathbf{v}_3\times\mathbf{v}_1) + \mathbf{v}_3\times(\mathbf{v}_1\times\mathbf{v}_2) = \mathbf{0}$.
Further, let $<\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4>$ be an ordered quadruple of vectors. Then:

$(\mathbf{v}_1\times\mathbf{v}_2)\bullet(\mathbf{v}_3\times\mathbf{v}_4) = (\mathbf{v}_1\bullet\mathbf{v}_3)(\mathbf{v}_2\bullet\mathbf{v}_4) - (\mathbf{v}_1\bullet\mathbf{v}_4)(\mathbf{v}_2\bullet\mathbf{v}_3)$. In fact:
$$
\begin{aligned}
(\mathbf{v}_1\times\mathbf{v}_2)\bullet(\mathbf{v}_3\times\mathbf{v}_4) &= \mathbf{v}_1\bullet(\mathbf{v}_2\times(\mathbf{v}_3\times\mathbf{v}_4)) \\
&= \mathbf{v}_1\bullet((\mathbf{v}_2\bullet\mathbf{v}_4)\mathbf{v}_3 - (\mathbf{v}_2\bullet\mathbf{v}_3)\mathbf{v}_4) = (\mathbf{v}_1\bullet\mathbf{v}_3)(\mathbf{v}_2\bullet\mathbf{v}_4) - (\mathbf{v}_1\bullet\mathbf{v}_4)(\mathbf{v}_2\bullet\mathbf{v}_3).
\end{aligned}
$$