Shelving Strings

Let N(S) be the free monoid on set S. N(S) consists of formal sums w of the form

$$w = n_1 s_1 + ... + n_k s_k \qquad (1)$$

where $k \geq 0$ and the $n_i > 0$ and $s_i \neq s_{i+1}$ for $1 \neq i < k$ holds. When $k = 0$ the sum is empty and the result is 0, the identity element of monoid N(S). For 0 in N, $0s =^{df} 0$ for any s in S. The operation + in N(S) is simple concatenation. We will write expresssions in bold if we are defining them for the first time.

Notation1: The **length** of an element w of N(S) is written **|w|** and is $n_1 + ... + n_k$ where w is written as in (1) above. k in this representation is called the "**block length**" of w .
For any w in N(S), an **initial segment** of w is a u in N(S) such that

$$w = u + w_1 \text{ for some } w_1 \text{ in } N(S).$$

IS(w) = $\{u \in N(S) \mid w = u + w_1$ for some w_1 in $N(S)\}$. An initial segment of w is called a **proper initial segment** of w if it is distinct from w itself. Hence 0 is a proper initial segment of any non zero element of N(S).

For any w in N(S), a **terminal segment** of w is a v in N(S) such that

$$w = w_0 + v \text{ for some } w_0 \text{ in } N(S).$$

TS(w) = $\{v \in N(S) \mid w = w_0 + v$ for some w_0 in $N(S)\}$. A terminal segment of w is called a **proper terminal segment** of w if it is distinct from w itself. Hence 0 is a proper terminal segment of any non zero element of N(S).
Note that the sets IS(w) and TS(w) include the improper terminal and initial segment of w, namely w itself. 0 is the only word with no proper terminal or initial segments: {0} = IS(0) = TS(0).

We write $\mathbf{u \leq_I v}$ for u an initial segment of v and write $\mathbf{u \leq_T v}$ for u a terminal segment of v.
Accordingly as part of this notational definition we have $u \leq_I v \iff \mathbf{v \geq_I u}$ and $u \leq_T v \iff \mathbf{v \geq_T u}$.
Notice that $\leq_I$ and $\leq_T$ are partial orderings on N(S).
For cardinality of a set T, we write $|T|$.

Section 1: Shelving operations on Strings

Def1: For any u, v in N(S) define
$\mathbf{u \lozenge v} = \{w \mid w = u_0 + z + v_1$ where $u = u_0 + z$ and $v = z + v_1$ for some u_0 , z , v_1 }.
We can call $u \lozenge v$ the set of shelvings of u by v.

Prop1: $|u \Diamond v| = |TS(u) \cap IS(v)|$.

pf: For w in $u \Diamond v$ define $(w)f = z$ where $w = u_0 + z + v_1$ and $u = u_0 + z$, $v = z + v_1$ for some strings v_1, u_0, z. To show f is a well defined suppose we have v_1', u_0', z' with $w = u_0' + z' + v_1'$ and $u = u_0' + z'$, $v = z' + v_1'$. So $u_0 + z + v_1 = u_0' + z' + v_1'$, $u = u_0 + z = u_0' + z'$, $v = z + v_1 = z' + v_1'$. Suppose $|z'| = |z| + d$ for some integer d. Then $|u_0| + |z| = |u_0'| + |z| + d$ and $|z| + |v_1| = |z| + d + |v_1'|$. Hence $|u_0| = |u_0'| + d$ and $|v_1| = d + |v_1'|$. So $|u_0| + |z| + |v_1| = |u_0'| + |z'| + |v_1'|$
$= |u_0| - d + |z| + d + |v_1| - d$ and so d=0.

This shows $z=z'$, $u_0 = u_0'$, and $v_1 = v_1'$. Hence f is well defined, $f: u \Diamond v \to TS(u) \cap IS(v)$.

Now define $g: TS(u) \cap IS(v) \to u \Diamond v$. Let $(z)g = u_0 + z + v_1$ where $u = u_0 + z$, $v = z + v_1$. where z is clearly an element of $TS(u) \cap IS(v)$. Because of this latter we have $u = u_0 + z$, $v = z + v_1$ for some u_0 and v_1. And hence we have $(z)g = u_0 + z + v_1$ an element of $u \Diamond v$. u_0 and v_1 are uniquely determined by z and so (z)g is well defined.

It remains to show that $((w)f)g = w$ and $((z)g)f = z$ where w in $u \Diamond v$ and z is in $TS(u) \cap IS(v)$. First show $((w)f)g = w$: Take w in $u \Diamond v$, arbitrary.
Then $w = u_0 + z + v_1$ and $u = u_0 + z$, $v = z + v_1$ for some strings v_1, u_0, z. We have shown above that z is uniquely determined by w; $z = (w)f$. Now from the definition of g for any z in $TS(u) \cap IS(v)$, $(z)g = u_0 + z + v_1$ where $u = u_0 + z$, $v = z + v_1$, Hence $((w)f)g = (z)g = w$.

It remains to show $((z)g)f = z$ for any z in $TS(u) \cap IS(v)$: Take such a z. Then $u = u_0 + z$, $v = z + v_1$ for unique strings, u_0, v_1. By definition of g, $(z)g = u_0 + z + v_1$ where $u = u_0 + z$, $v = z + v_1$.

$((z)g)f = (u_0 + z + v_1)f = z$ from the definition of f, where $u = u_0 + z$, $v = z + v_1$. Thus f and g are inverse to each other, completing the proof.//

Notation2: Use the notation "**Iuv**" or **I(u,v)** when the brackets and comma are necessary, for $TS(u) \cap IS(v) = Iuv = I(u,v)$ for u and v two elements of N(S). Note Iuv is always non empty as it always contains 0. Further note that in general $Iuv \neq Ivu$, these sets having essentially little to do with each other .

Examples: 1) The simplest case of forming the shelving set , $u \Diamond v = \{w \mid w = u_0 + z + v_1$ where $u = u_0 + z$ and $v = z + v_1$ for some u_0, z, $v_1\}$, is where there are no nontrivial z, namely

$TS(u) \cap IS(v) = \{0\}$. In this case $u \Diamond v = \{u+v\}$.

For two distinct letters a, b, $a \Diamond b = \{a + b\}$.

Also Cecelia ◊ Zeamer = {CeceliaZeamer}

Note that $u \Diamond v \supseteq \{u+v\}$ for all u and v, as 0 is in $TS(u) \cap IS(v)$ for any u and v.

2) $a \Diamond a = \{aa, a\}$. This same pattern is respected when a given word w has no nontrivial terminal segments which are also initial, that is when $TS(w) \cap IS(w) = \{0\}$. For instance

silvie◊silvie = {silviesilvie, silvie}.

3) Now consider the two strings aba and abababa. Then aba ◊ abababa = Initial Segments of v.

We do an analysis:

Initial Segments of v: a, ab, aba, abab, ababa, ababab, abababa

Terminal Segments of u: a, ba, aba

2 Terminal Segments of u being initial segments of v: a, aba

3 Shelvings of u by v: abaabababa, ababababa, abababa

4) Two words that are powers of the same word have lots of shelvings.

aa ◊ aaa = {aaaaa, aaaa, aaa}

aaa ◊ aaaaa = {aaaaaaaa, aaaaaaa, aaaaaa, aaaaa}

In fact it is easy to see that na ◊ ma = {(n+m)a, (n+m-1)a,…,Max(m,n)a} so

|na ◊ ma| = n+m-Max(m,n)+1 and this is a maximum for the cardinality of s ◊ t where

|s| = n and |t| = m.

5) Consider the shelving of abacabadabacabazabacabadabacaba with itself:

abacabadabacabazabacabadabacaba◊abacabadabacabazabacabadabacaba =

{abacabadabacabazabacabadabacabaabacabadabacabazabacabadabacaba,

abacabadabacabazabacabadabacababacabadabacabazabacabadabacaba,

abacabadabacabazabacabadabacabacabadabacabazabacabadabacaba,

abacabadabacabazabacabadabacabadabacabazabacabadabacaba

abacabadabacabazabacabadabacabazabacabadabacaba

abacabadabacabazabacabadabacaba}

This completes the initial examples: Shelving is an operation that is fairly restrictive re its results.

The following definition extends shelving to a binary operation on the subsets of N(S).

Def2: Let U , V be two subsets of N(S) then U ◊ V = ∪{u ◊ v| u in U, v in V}. Note that {u}◊{v} = u ◊ v.

Examples 6) The operation, ◊ , the power set of N(S) can be associative for some triples:

Let U = {aba}, V = {ba}, W = {a}. U ◊ V = {ababa, aba}. V ◊ W = {baa, ba}

(U ◊ V) ◊ W = { ababaa, abaa, ababa, aba}. U ◊ (V ◊ W) = { ababaa, abaa, ababa, aba}.

7) The operation, ◊ , on the power set of N(S) is not associative for some triples:

Let U = {a}, V = {bc}, W = {abx } . U ◊ V = {abc}. V ◊ W = {bcabc}.

$(U \lozenge V) \lozenge W = \{ abcabc, abc\}$. $U \lozenge (V \lozenge W) = \{ abcabc\}$. Hence $(U \lozenge V) \lozenge W \neq U \lozenge (V \lozenge W)$. So our operation on the subsets of N(S) is not associative in general.

8) $0 \lozenge v = v \lozenge 0 = \{v\}$.

Def3: For any z in N(S) define $+_z$, a binary operation on N(S) as follows:

For u,v in N(S) let $u +_z v = u_0 + z + v_1$ where $u = u_0 + z$, $v = z + v_1$, if $z \in TS(u) \cap IS(v)$.
$u + v$ if $z \notin TS(u) \cap IS(v)$. Note that $+_0 = +$.

Example : 9) We must notice that $+_z$ is not necessarily associative: Let $u = a$, $v = b$, $w = a+b+t$, where a and b are distinct in S, $z = a + b$ and t is some element of N(S).

$(u +_z v) +_z w = (a+b) +_z w = w$. But $u +_z (v +_z w) = a +_z (b +_z (a+b+t)) = a+b+w$. Thence

$(u +_z v) +_z w \neq u +_z (v +_z w)$ and associativity fails for this triple.

Notation3: For any $n \geq 0$, Let $N(S)_n = \{w \text{ in } N(S) | \ |w| \geq n\}$.

For a non zero z we say a sum $u +_z v$ is **contracting** if $z \in Iuv$, iff $|u +_z v| < |u + v|$. We say a sum, $u +_z v$ is **flat** if $z \notin Iuv$ iff $|u +_z v| = |u + v|$. For $z = 0$ the sums are all flat. A contracting sum always implies that z in the $+_z$ is non zero.

Prop2: a) For any z in N(S), u is an initial segment and v is a terminal segment of $u +_z v$. Hence $|u| + |v| \geq |u +_z v| \geq Max(|u|,|v|)$.

b) Further, for u, v in N(S) $u \lozenge v = \{u +_z v | \ z \text{ in } N(S)\}$,

$= \{u +_z v | \ z \text{ in } N(S) \text{ and } u +_z v \text{ contracting}\} \cup \{u+v\}$,

$= \{u +_z v | \ z \text{ in } Iuv\}$.

pf: a) From the Def3, if $z \in Iuv$, $u +_z v = u_0 + z + v_1$ where $u = u_0 + z$, $v = z + v_1$ and

so $u +_z v = u + v_1 = u_0 + v$ giving u an initial segment, and v is a terminal segment of $u +_z v$.

If $z \notin Iuv$, $u +_z v = u + v$ making the conclusion of the Prop2 clear. This completes the proof.

b) $\{u +_z v | \ z \text{ in } N(S)\} = \{u +_z v | \ z \text{ in } Iuv\} \cup \{u +_z v | \ z \text{ in } N(S) - Iuv\} = \{u +_z v | \ z \text{ in } N(S) \text{ and } u +_z v \text{ contracting}\} \cup \{u+v\} = \{u +_z v | \ z \text{ in } Iuv\}$.

Prop3: For z,u,v,t in N(S) ,

a) if $|z| \leq |v|$ then $u +_z v$ contracting (flat) iff $u +_z (v + t)$ is contracting(flat) and

b) if $|z| \leq |u|$ then $u +_z v$ contracting (flat) iff $(t + u) +_z v$ is contracting(flat).

pf: Given z,u,v,t in N(S). Assume $|z| \leq |v|$. $u +_z v$ contracting <=> z is in Iuv and z is non zero

<=> z is in Iu(v+t) and z is non zero <=> $u +_z (v + t)$ is contracting. Further, $u +_z v$ flat <=> z is not in Iuv or z = 0 <=> z is not in Iu(v+t) or z = 0 <=> $u +_z (v + t)$ is flat.

Now assume $|z| \le |u|$. Then $u +_z v$ contracting <=> z is in Iuv and z is non zero

<=> z is in I(t+u)v and z is non zero <=> $(t + u) +_z v$ is contracting. Further, $u +_z v$ flat <=> z is not in Iuv or z = 0 <=> z is not in Iu(t+u)v or z = 0 <=> $(t+u) +_z v$ is flat. //

Prop4: u,v,w, y,z in N(S) with $|z|, |y| \le |v|$ => $(u +_y v) +_z w = u +_y (v +_z w)$.

pf: We treat 4 cases determined by whether $u +_y v$ and $v +_z w$ are contracting or flat.

Suppose first that both $u +_y v$ and $v +_z w$ are flat. Then $u +_y v = u + v$ and $v +_z w = v + w$, and

$u +_y (v +_z w) = u +_y (v + w) = u + (v + w)$, since $|y| \le |v|$ makes Prop 3a) apply and this shows that $u +_y (v + w)$ is flat because $u +_y v$ is flat. Now $u + (v + w) = (u + v) + w = (u + v) +_z w$, since $|z| \le |v|$

makes Prop3b) apply and this shows that $(u + v) +_z w$ is flat because $v +_z w$ is flat. Hence

$u +_y (v +_z w) = (u + v) +_z w = (u +_y v) +_z w$, and associativity follows in this case.

Now for the second case let's assume that $u +_y v$ and $v +_z w$ are both contracting.

Then $u = u_0 + y$, $v = y + v_1$, $v = v_0 + z$, $w = z+w_1$ for u_0 , v_1 , v_0 ,w_1 in N(S) perhaps 0. So

$(u +_y v) +_z w = (u_0 + y + v_1) +_z w = (u_0 + v) +_z w = (u_0 + v_0 + z)+_z (z+w_1) = u_0 + v_0 + z+w_1 =$

$u_0 + v + w_1 = u_0 + (v + w_1) = (u_0 + y) +_y (y + v_1 + w_1) = u+_y (v_0 + z + w_1) =$

$u+_y ((v_0 + z) +_z (z + w_1)) = u +_y (v +_z w)$, and associativity holds in this case also.

Thirdly lets assume $u +_y v$ is contracting and $v +_z w$ is flat. Then $v +_z w = v + w$ and

$u = u_0 + y$, $v = y + v_1$. $(u +_y v) +_z w = (u_0 + y + v_1) +_z w = (u_0 + v) +_z w = (u_0 + v) + w$ since

Prop3b implies that $(u_0 + v) +_z w$ is flat because $v +_z w$ is flat and $|z| \le |v|$. Now

$(u_0 + v) + w = u_0 + (v + w) = (u_0 + y) +_y (y + v_1 + w) = u +_y (v + w) = u +_y (v +_z w)$. Hence associativity holds in this case as well.

Finally, assume $u +_y v$ is flat and $v +_z w$ is contracting. Then $u +_y v = u + v$ and

$v = v_0 + z$, $w = z + w_1$. $(u +_y v) +_z w = (u + v) +_z w = (u + v_0 + z) +_z (z + w_1) =$

$u + v_0 + z + w_1 = u + (v + w_1) = u +_y (v + w_1)$ since by Prop3a) $u +_y v$ flat implies that $u +_y (v + w_1)$ is flat. Now $u +_y (v + w_1) = u +_y (v_0 + z + w_1) = u +_y ((v_0 + z) +_z (z + w_1)) = u +_y (v +_z w)$. Hence

$(u +_y v) +_z w = u +_y (v +_z w)$ and associativity. //

Thm1: For z in N(S), $N(S)_{|z|}$ with the empty string, 0, adjoined, is a monoid under binary operation $+_z$.

pf: Take u,v,w in $N(S)_{|z|}$. Since $|z| \leq |v|$ Prop4 applies and we have $(u +_z v) +_z w = u +_z (v +_z w)$. This makes $N(S)_{|z|}$ with binary operation $+_z$, a semigroup. Adjoining the empty string, 0, extends $N(S)_{|z|}$ to a monoid. //

As we shall prove in the next section, Thm1 allows us to write sums of the form,

$u_1 +_z u_2 +_z \ldots +_z u_n$, for any n, without ambiguity.

Section 2: Single Binary Operation Associativity and the Free Magma

Let F = N(S) be the free monoid on set S and let $F_B = N(S \cup \{(\ ,\ )\})$ where the left bracket, "(", and the right bracket, ")", are not in S. Now (+ p + q +) = "(pq)" where the brackets here are single character strings and elements of the basis of F_B and the + is natural concatenation in the free monoid F_B. Define $\mathcal{M} = \cap\{T|\ S \subseteq T \subseteq F_B - \{0\},\ p, q \in T \Rightarrow (pq) \in T\}$.

Clearly $F_B \supseteq \mathcal{M}$ and $\mathcal{M}$ does not contain 0 as all the T intersected to form M do not contain 0. M is non empty because $\{T|\ S \subseteq T \subseteq F_B - \{0\},\ p, q \in T \Rightarrow (pq) \in T\}$ contains $F_B - \{0\}$ and all the T contain S.

So $\mathcal{M} \supseteq S$.

Further, we have a binary operation on $\mathcal{M}$,

$\mathcal{M} \times \mathcal{M} \to \mathcal{M}$, $(p,q) \to (pq)$. $\mathcal{M}$ is in fact the free magma on M where a magma is a set equipped with a not necessarily commutative binary operation.

Notation3: In any free monoid F if $u, v \in F$, then u is a **subword** of v iff $v = u_0 + u + u_1$ for some

$u_0 , u_1 \in F$. u is a **proper subword** of v iff $u_0 + u_1 \neq 0$, i.e. this sum is non-empty. Notice that the relation "subword of" is a partial ordering on F.

We can define a homomorphism, $F_B \to F \subseteq F_B$ induced by the mapping $S \cup \{(\ ,\ )\} \to F$ taking any s in S to itself and taking the symbols, (and), to zero. This mapping is written by underlining its argument: $x \to \underline{x}$. The underlined x is the string x with all its brackets replaced by the empty word. For example,

if r,s,t in S then magna element z = (((rs)(st))(tr)) is such that $\underline{z}$ = rssttr .

Let the number of brackets occurring in an element x of F_B be denoted by (x)br. Let the number of left brackets (i.e. "(")be denoted (x)Lbr and the number of right brackets (i.e. ") ") occurring in x be denoted (x)Rbr.

Prop5: Principle of induction for $\mathcal{M}$: If $S \subseteq T \subseteq \mathcal{M}$ and $p, q \in T \Rightarrow (pq) \in T$ then $T = \mathcal{M}$.

pf: Assume we have a subset T of F_B such that $S \subseteq T \subseteq \mathcal{M}$ and $p, q \in T \Rightarrow (pq) \in T$. It suffices to show that $\mathcal{M} \subseteq T$. But T is such that $S \subseteq T \subseteq F_B - \{0\}$, $p,q \in T \Rightarrow (pq) \in T$ so by definition of $\mathcal{M}$,

$\mathcal{M} \subseteq T$. Hence $\mathcal{M} = T$, completing the proof.//

We can use this principle of induction to prove many things about $\mathcal{M}$.

Prop6: a) u in $\mathcal{M}$ and $u \notin S \Rightarrow u = (pq)$ for some $p, q \in \mathcal{M}$.

b) u in $\mathcal{M}$ $\Rightarrow$ $|u| = 3|\underline{u}| - 2$.

c) $(u)br = 2(|\underline{u}| - 1)$.

d) u in $\mathcal{M}$ $\Rightarrow (u)Lbr = (u)Rbr = |\underline{u}| - 1$.

e) u in $\mathcal{M}$ $\Rightarrow$ If $u = v+w$ for non zero v and w, then $(v)Lbr > (v)Rbr$ and $(w)Lbr < (w)Rbr$.

f) u in $\mathcal{M}$ $\Rightarrow$ If v is a non zero proper terminal or initial segment of u then $v \notin \mathcal{M}$.

g) u in $\mathcal{M}$ and $u \notin S \Rightarrow u = (pq)$ for some unique $p, q \in \mathcal{M}$.

pf: a) Let $T = \{x \text{ in } \mathcal{M} \mid x \in S \text{ or } x = (pq) \text{ for some } p,q \in \mathcal{M}\}$. Then $S \subseteq T \subseteq \mathcal{M}$, Also if $p,q \in T$ then $x = (pq)$ is in $\mathcal{M}$ for some $p,q \in \mathcal{M}$ so (pq) is in T. Hence $T = \mathcal{M}$ by Prop5 and a) follows.

b) Let $T = \{x \text{ in } \mathcal{M} \mid |x| = 3|\underline{x}| - 2\}$ Then $S \subseteq T \subseteq \mathcal{M}$. Further, if $p,q \in T$ then

$|(pq)| = 2 + |p| + |q| = 2 + 3|\underline{p}| - 2 + 3|\underline{q}| - 2 = 3[|\underline{p}| + |\underline{q}|] - 2 = 3\,|\underline{pq}| - 2$. Hence $(pq) \in T$ and $T = \mathcal{M}$ by Prop5. This proves b).

c) Clearly for any u in $\mathcal{M}$, $|u| = (u)br + |\underline{u}| = 3|\underline{u}| - 2$ and so $(u)br = 2(|\underline{u}| - 1)$ proving c).

d) It suffices to prove that $(u)Lbr = (u)Rbr$ in light of c). So let $T = \{x \text{ in } \mathcal{M} \mid (x)Lbr = (x)Rbr\}$. Then $S \subseteq T \subseteq \mathcal{M}$ since elements of S have no bracket occurrences. Now if $p, q \in T$ then (pq) is in $\mathcal{M}$ and $((pq))Lbr = 1 + (p)Lbr + (q)Lbr = (p)Rbr + (q)Rbr + 1 = ((pq))Rbr$. Hence by Prop5 $T = \mathcal{M}$, and the result follows.

e) Let $T = \{u \text{ in } \mathcal{M} \mid$ If $u = v+w$ for non zero v and w, then $(v)Lbr > (v)Rbr$ and $(w)Lbr < (w)Rbr\}$.

Then $S \subseteq T \subseteq \mathcal{M}$ since if $s = v+w$ either v or w is zero and the implication is vacuously true. Now suppose p and q are in T. We show that (pq) is also in T. To do this we must show :

$(pq) = v+w$ for some non zero v and w in $F_B \Rightarrow (v)Lbr > (v)Rbr$ and $(w)Lbr < (w)Rbr$. (*)

Suppose that $(pq) = v+w$ for some non zero v and w in F_B . We have 3 cases:

1) v = "(" $+ p_0$ and $w = p_1 + q +$")" where p_1 is non zero, $p = p_0 + p_1$.

2) v = "("$+p$ and $w = q +$")",

3) v = "("$+p + q_0$ and $w = q_1+$")" for non zero q_0 , $q = q_0 + q_1$.

In 1), if $p_0 = 0$, (v)Lbr = 1 > 0 = (v)Rbr and (w)Lbr = (p)Lbr + (q)Lbr + 0 = (p)Rbr + (q)Rbr < (p)Rbr + (q)Rbr + 1 = (w)Rbr, where we have used part d). So in this case we get (v)Lbr > (v)Rbr and (w)Lbr < (w)Rbr, as required.

Now assume p_0 non zero. (v)Lbr = (p_0)Lbr + 1 > (p_0)Rbr + 1 > (p_0)Rbr = (v)Rbr. Here we have used the fact that p is in T. (w)Lbr = (p_1)Lbr + (q)Lbr = (p_1)Lbr + (q)Rbr < (p_1)Rbr + (q)Rbr < (p_1)Rbr + (q)Rbr +1 = [p_1 + q)]Rbr = (w)Rbr. Again we have (v)Lbr > (v)Rbr and (w)Lbr < (w)Rbr, as required. In case 1), therefore, (*) holds.

In 2), (v)Lbr = 1 + (p)Lbr = 1+ (p)Rbr > (p)Rbr = (v)Rbr. We have used part d and the fact that p is in $\mathcal{M}$. Further, (w)Lbr = (q)Lbr = (q)Rbr < (q)Rbr+1 = (w)Rbr. In case 2), therefore, (*) holds.

In 3) suppose first that $q_1 = 0$. Then (v)Lbr = 1+ (p)Lbr +(q_0)Lbr = 1 + (p)Rbr + (q)Lbr = 1 + (p)Rbr + (q)Rbr > (p)Rbr + (q)Rbr = (v)Rbr, so (v)Lbr > (v)Rbr. Also, (w)Lbr = 0 < 1 = (w)Rbr. So, in the case where $q_1 = 0$, (v)Lbr > (v)Rbr and (w)Lbr < (w)Rbr, as required.

Now suppose q_1 is non zero. Then (v)Lbr = 1+ (p)Lbr +(q_0)Lbr > 1 + (p)Rbr + (q_0)Rbr = 1+[(pq_0]Rbr > [(pq_0]Rbr = (v)Rbr. Also, (w)Lbr = (q_1)Lbr < (q_1)Rbr< 1+(q_1)Rbr = (w)Rbr, since q is in T. Hence in all cases we have (v)Lbr > (v)Rbr and (w)Lbr < (w)Rbr, as required and (*) holds in case 3) as well.

Hence (*) holds for all cases so (pq) is in T. Hence T = $\mathcal{M}$, proving e).

f) Suppose u is in $\mathcal{M}$ and v is a proper initial segment of u. Then either v = 0 in which case v is not in $\mathcal{M}$, or u = v +w where both v and w are non zero. In this case (v)Lbr > (v)Rbr by e) and so $v \notin \mathcal{M}$ by d).

g) Suppose u in $\mathcal{M}$ and $u \notin S$. Then by a) u = (pq) for some $p, q \in \mathcal{M}$. Let u = (vw) with $v,w \in \mathcal{M}$ be another such representation. If |p| > |v| then v is a proper initial segment of p and so by f) $v \notin \mathcal{M}$ and we have a contradiction. If |p| < |v| then p is a proper initial segment of v and so by f) $p \notin \mathcal{M}$ and we have another contradiction. Hence |p| = |v| and so p = v and q = w, and (vw) = (pq). Thus the representation of u in the form (pq) for $p, q \in \mathcal{M}$, is unique.//

Notation 4:

We define a graph, Assoc, with vertices $\mathcal{M}$, and edges consisting of ordered pairs of vertices of form (m,n) where m=n or m = m0 + x + m1 and n = m0 + y + m1 where either x = (a(bc)) and y = ((ab)c) or x = ((ab)c) and y = (a(bc)) , for some a,b,c in $\mathcal{M}$.

Def4: $m \leftrightarrow n$ iff m=n or $m = m_0 + x + m_1$ and $n = m_0 + y + m_1$ where either x = (a(bc)) and y = ((ab)c) or x = ((ab)c) and y = (a(bc)) , for some a,b,c in $\mathcal{M}$.

We write "$m \leftrightarrow n$" to indicate that (m,n) is an edge of Assoc. Notice that the relation, $\leftrightarrow$, on $\mathcal{M}$, is reflexive and symmetric by definition.

We define a relation, Δ, on $\mathcal{M}$ by defining Δ to be the transitive closure of $\leftrightarrow$. This means that for m and n in $\mathcal{M}$, $m \ \Delta \ n$ iff there is a sequence $m_0,\ldots,m_k$ in $\mathcal{M}$ such that $k \geq 0$, $m = m_0$, $n = m_k$, and $m_i \leftrightarrow m_{i+1}$ for $0 \leq i < k$. Otherwise put, $m \ \Delta \ n$ iff there is a path from m to n in Assoc.

Prop7: For any m in $\mathcal{M}$, m is in S or $m \ \Delta \ m'$ for some m' in $\mathcal{M}$, where m' = (sB) for some s in S and B in $\mathcal{M}$.

pf: Let $T = \{m \text{ in } \mathcal{M} \mid m \in S \text{ or } m \ \Delta \ m' \text{ for some } m' = (sB) \text{ for } s \in S \text{ and } B \in \mathcal{M}\}$. Then $S \subseteq T \subseteq \mathcal{M}$. Now suppose p, q are in T. Then $m = (pq) \in \mathcal{M} - S$. If $p \in S$ then m = (sB) where s = p is in S and B = q is in $\mathcal{M}$. Since $m \ \Delta \ m$, m is in T. On the other hand if $p \notin S$ then since p is in T, $p \ \Delta \ p'$ where p' = (sB) for some s in S and B in $\mathcal{M}$. Hence $m = (pq) = ((sB)q) \ \Delta \ (s(Bq))$ which shows that m is in T since $m \ \Delta \ (s(Bq))$ where s is in S and (Bq) is in $\mathcal{M}$. Thus in all cases m is in T. So by Prop5, the principle of induction for $\mathcal{M}$, $T = \mathcal{M}$. This proves Prop7.//

Thm2: For m and n in $\mathcal{M}$, $\underline{m} = \underline{n} \iff m \ \Delta \ n$.

pf: (<=) Referring to Notation4, to establish

$$\underline{m} = \underline{n} \Leftarrow m \ \Delta \ n \qquad [1]$$

it suffices to show that $\underline{m} = \underline{n}$ when $m \leftrightarrow n$. To this end note that if m=n there is nothing to prove. So assume $m \neq n$. Then from Def4 we have $m = m_0 + x + m_1$ and $n = m_0 + y + m_1$ where either

[2] x = (a(bc)) and y = ((ab)c)

or

[3] x = ((ab)c) and y = (a(bc)), for some a,b,c in $\mathcal{M}$.

In the case of [2] $m = m_0 + (a(bc)) + m_1$ and $n = m_0 + ((ab)c) + m_1$. In the case of [3] $m = m_0 + ((ab)c) + m_1$ and $n = m_0 + (a(bc)) + m_1$. In both cases, $\underline{m} = \underline{m_0} + abc + \underline{m_1} = \underline{n}$. This proves (<=).

(=>): We prove for m in $\mathcal{M}$ by induction on $L = |\underline{m}|$ that,

$$n \text{ in } \mathcal{M} \Rightarrow (\underline{m} = \underline{n} \Rightarrow m \ \Delta \ n) \quad [4]$$

Suppose L = 1. Then for any n in $\mathcal{M}$ with $\underline{m} = \underline{n}$, n and m are in S. Hence $m = \underline{m} = \underline{n} = n$ and so $m \ \Delta \ n$ follows. Now suppose we have established [4] in the case that L< N for some integer N >1. Consider m in $\mathcal{M}$ with $L = |\underline{m}| = N$. Take n in $\mathcal{M}$ with $\underline{m} = \underline{n}$. Since $|\underline{m}| = |\underline{n}| = N > 1$, m and n are not in S. Hence by Prop7, $m \ \Delta \ (sB)$ and $n \ \Delta \ (tC)$ for some s.t in S and B,C in $\mathcal{M}$. Since $\underline{m} = \underline{n}$, and we have proved [1], $\underline{m} = s + \underline{B}$ and $\underline{n} = t + \underline{C}$. Hence s = t and $\underline{B} = \underline{C}$. Now $|\underline{B}| < N$ and C is in $\mathcal{M}$, so by induction hypothesis, $B \ \Delta \ C$. So $m \ \Delta \ (sB) \ \Delta \ (sC) \ \Delta \ (tC) \ \Delta \ n$, showing $m \ \Delta \ n$. This completes the induction, completing the proof of =>.//

Alternative proof of general associative law (Stephen Jackson):

S is a set of **letters**

W is a set of words together with a length function on words defined inductively as follows:
Every letter *x* in *S* is a word; length*x* = 1
If *p* and *q* are words then (*pq*) is a word; length(*pq*) = length*p* + length*q*
Nothing else is a word.
Associative Law: If *p*, *q* and *r* are words ((*pq*)*r*) = (*p*(*qr*)).
Every word of length 1 is a letter.
Every word of length 2 has the form (*xy*), where *x* and *y* are letters.
Every word of length 3 has the form ((*xy*)*z*) or (*x*(*yz*)) = ((*xy*)*z*), where *x*, *y* and *z* are letters.
Words of length 1 and 2 are canonical words. Words of length 3 are canonical if they are of the form ((*xy*)*z*).
Let *p* be a canonical word of length *n*. Then (x*p*) is a canonical word of length (*n*+1) for *x* in *S.* Nothing else is a canonical word.
Theorem
Every word is equal to a canonical word if we assume the Associative law above.
Proof by induction on the length of a word. Suppose true for length*w* < *n where n>1.*
Let *w* be a word. If *w* is a letter we are done. If w is not a letter then
w = (*pq*), where *p* and *q* are words. Since length(p) and length(q) <n p and q are equal to canonical words by induction hypothesis. So we can assume p and q are canonical words and we have w = (pq).
If p is a letter then
w = (pq) is a canonical word and we are done. Otherwise *w* = ((xP*)*q), where P is a canonical word and x is a letter.
But ((xP*)*q) = (x(Pq)) by the associative law and (Pq) is equal to a canonical word by the induction hypothesis.
So we are done. QED
Using the Theorem we can take any two words which give the same letter string when we erase all the brackets, and come up with two canonical words to which they are both equal. But since these two canonical words have the same letter string they are equal. Hence the original two words are equal. //

The above theorem establishes the efficacy of the usually stated associative law, ((ab)c) = (a(bc)) , to allow us to write long sums without regard as to how we bracket them. We will use very similar arguments to show that we can also write long sums using the $+_z$ operations without concern for the order of application of the operations, that is, without concern for bracketing. We will further develop this treatment of associativity for multiple operations using the ideas above.

Section 3: Inter Associativity of a Set of Binary Operations

Let Q be a non empty set and let O be a set of binary operations on Q. Let *Q* and *O* be sets of symbols isomorphic to Q and O respectively. As we are pursuing this development within ZFC we have in the background the Axiom of foundation which implies that a binary operation on any set cannot be an element of that set. Hence we have that Q and O are disjoint. We are further able to take *Q* and *O* so that Q, O, *Q*, and *O* are pairwise disjoint. We can further presume that the symbols "*(*" and "*)*" are not in Q, O, *Q*, or *O* . We can then form the free monoid on the disjoint union of the sets *Q, O,* and {*(,)*}. This is $F_{QBO} = N(Q \cup O \cup \{(,)\})$. Note we use the italicized brackets to form expressions in F_{QBO}.

We use italics to connect elements of Q with the corresponding elements in *Q* and elements of O with the corresponding elements in *O. q* in *Q* corresponds to q in Q. We write the set of operations O in terms of an index set, Z, so that each element of O is the operation on Q, $+_z$. $|Z| = |O|$ and is disjoint from all sets O, Q , *Q, O*. Further $+_z = +_y$ for x and y in Z iff y = z. As $O = \{+_z \mid z \in Z\}$, $+_z$ in O corresponds to $+_z$ in *O.*

The simple + without index or italicization is always concatenation or numerical addition depending on context.

Def5: We define the set of expressions that represent arithmetic operations in Q using O, in their guises as *Q* and *O.* This set is a subset of F_{QBO} and is defined as follows:

$$M_{QO} = \cap\{T \mid F_{QBO} \supseteq T \supseteq Q \text{ and } (+_z \in O,\ p,\ q \in T) \Rightarrow (p +_z q) \in T\}.$$

In the expression "$(p +_z q)$" in the above definition, juxtaposition is string concatenation. In this the string "*(*" is concatenated with the string p which is concatenated with the string "$+_z$", which is a single symbol which is concatenated with the string q which is concatenated with the string "*)*".
Let $S_M = \{T \mid F_{QBO} \supseteq T \supseteq Q \text{ and } (+_z \in O,\ p,q \in T) \Rightarrow (p +_z q) \in T\}$. Then $M_{QO} = \cap S_M$ from Def5.
Now note that F_{QBO} is an element of S_M . Hence S_M is non empty and so $F_{QBO} \supseteq M_{QO} \supseteq Q$.

Prop8: Principle of induction for M_{QO} :

$$[Q \subseteq T \subseteq M_{QO} \text{ and } (+_z \in O,\ p,q \in T) \Rightarrow (p +_z q) \in T\ ] \Rightarrow T = M_{QO}.$$

Pf: Suppose a set T satisfies the premise of our implication. $T \subseteq M_{QO}$ follows from this premise. So it suffices to show that $M_{QO} \subseteq T$. But $F_{QBO} \supseteq T \supseteq Q$ and since T satisfies our implication's premise,
$(+_z \in O,\ p,q \in T) \Rightarrow (p +_z q) \in T$. Hence $T \in S_M$. So $M_{QO} = \cap S_M \subseteq T$.
Therefore $T = M_{QO}$, completing the proof.//

Prop9: M_{QO} is in S_M, that is,

a) $Q \subseteq M_{QO}$

b) $(+_z \in O, p,q \in M_{QO}) \Rightarrow (p +_z q) \in M_{QO}$.

Pf: From remarks preceding Prop8, $Q \subseteq M_{QO}$, so a) holds. Now presume $(+_z \in O, p,q \in M_{QO})$. Then $+_z \in O$, p, q $\in M_{QO} = \cap S_M$. Let T be an arbitrary element of S_M. Then p, q $\in$ T. Hence since T is in SM, $(p +_z q) \in$T. Since T is arbitrary in S_M, $(p +_z q) \in \cap S_M = M_{QO}$. Thus $(+_z \in O$, p, q $\in M_{QO}) \Rightarrow (p +_z q) \in M_{QO}$, completing the proof of b).//

We define three useful monoid homomorphisms from $F_{QBO} \rightarrow N$. These are :

Br: $F_{QBO} \rightarrow N(\{(,)\}) \rightarrow N$ where the first homomorphism is the projection from $F_{QBO} \rightarrow N(\{(,)\})$ induced by taking the elements of Q and O to 0 leaving in any string only the symbols in $\{(,)\}$ and the 2nd homomorphism is the length homomorphism $N(\{(,)\}) \rightarrow N$. br is thus the homomorphism from $F_{QBO} \rightarrow N$ which counts the number of brackets in each string in F_{QBO}.

Lbr: $F_{QBO} \rightarrow N(\{(\}) \rightarrow N$ where the first homomorphism is the projection from $F_{QBO} \rightarrow N(\{(\})$ induced by taking the elements of Q , O and $\{)\}$ to 0 leaving in any string only the symbol "(" and the 2nd homomorphism is the length homomorphism $N(\{(\}) \rightarrow N$. Lbr is thus the homomorphism from $F_{QBO} \rightarrow N$ which counts the number of left brackets in each string in F_{QBO}.

Rbr: $F_{QBO} \rightarrow N(\{)\}) \rightarrow N$ where the first homomorphism is the projection from $F_{QBO} \rightarrow N(\{)\})$ induced by taking the elements of Q , O and $\{(\}$ to 0 leaving in any string only the symbols, ")" and the 2nd homomorphism is the length homomorphism $N(\{)\}) \rightarrow N$. Rbr is thus the homomorphism from $F_{QBO} \rightarrow N$ which counts the number of right brackets in each string in F_{QBO}.

Sl: $F_{QBO} \rightarrow N(Q \cup O)$ is the projection homomorphism obtained by taking both brackets in B to 0 and letting Sl be the identity on Q and O.

When applying any of the above as well as other specially defined functions to a member p of F_{QBO}, we will write the result without using brackets as context will be sufficient to avoid ambiguity of meaning. Hence pBr $=^{df}$ (p)Br, and similarly for Lbr, Rbr etc.

Prop10:

a) $p \in M_{QO} \Rightarrow$ pRbr = pLbr = pBr/2, in particular pBr is even.

b) $p \in M_{QO} \Rightarrow p \in Q$ or $p = (r +_z s)$ for some $+_z \in O$ and r,s $\in M_{QO}$.

c) $p \in M_{QO}$, $p = p_0 + p_1$ with p_0, p_1 non zero $\Rightarrow$ p_0Lbr $> p_0$Rbr and p_1Lbr $< p_1$Rbr .

d) No proper terminal or initial segment of an element of M_{QO} is an element of M_{QO}.

e) $p \in M_{QO} \Rightarrow p \in Q$ or $p = (r +_z s)$ for some unique $+_z \in O$ and r,s $\in M_{QO}$.

f) $p \in M_{Q0} \Rightarrow |p| = pBr + |pSl|$

g) $p \in M_{Q0} \Rightarrow pSl = p_0 +_{z1} p_1 +_{z2} p_2 \ldots +_{zn} p_n$ for some $n \geq 0$, p_i in Q and $+_{zi} \in O$.

pf: a) Let $T = \{ p \in M_{Q0} \mid pRbr = pLbr = pBr/2 \}$. Clearly $M_{Q0} \supseteq T \supseteq Q$. By Prop9b),

$p,q \in T, +_z \in O \Rightarrow (p+_z q) \in M_{Q0}$ and $(p+_z q)Rbr = pRbr + qRbr +1$

$$= ½(pBr + qBr +2)$$
$$= ½((p+_z q)Br)$$
$$= pLbr + qLbr +1$$
$$= (p+_z q)Lbr \; .$$

Hence $(p+_z q) \in T$. Therefore $p,q \in T, +_z \in O \Rightarrow (p+_z q) \in T$. Hence by Prop8, the principle of induction for M_{Q0}, $T = M_{Q0}$, completing the proof of a), as for $p \in M_{Q0}$, $pBr/2 = pLbr = pRbr$.

b) Suppose $p \in M_{Q0}$ and $p \notin Q$. Suppose further that $p \neq (r +_z s)$ for some $+_z \in O$ and $r,s \in M_{Q0}$. Let $T = Q \cup \{(r +_z s) \mid +_z \in O \text{ and } r,s \in M_{Q0}\}$. Then by Prop9, $T \subseteq M_{Q0}$. But then T is a subset of M_{Q0} containing Q. Further, $+_z \in O$ and $r,s \in T \Rightarrow +_z \in O$ and $r,s \in M_{Q0} \Rightarrow (r +_z s) \in T$. By Prop8, the principle of induction for M_{Q0}, $T = M_{Q0}$. This contradiction proves $p = (r +_z s)$ for some $+_z \in O$ and $r,s \in M_{Q0}$, and this proves b).

c) We prove the implication in b) by induction on $|p|$. If $|p| = 1$ then the premise of c) is always false so the implication follows vacuously. So assume c) holds for p with $|p| < n$ where n is an integer, $n>1$ and consider a p with $|p| = n$. Since $|p|>1$ we have $p = (r +_z s)$ where $+_z \in O$ and $r,s \in M_{Q0}$. Now $p = p_0 + p_1$ with p_0, p_1 non zero in F_{QBO}. Note r and s are non zero since they are in M_{Q0} and M_{Q0} does not contain 0. Now let us say that p_0 and p_1 divide r if $p_0 = (r_0$, $p_1 = r_1 +_z s)$ where $r = r_0 + r_1$ with both r_0, r_1 non zero. Let us say that p_0 and p_1 divide s if $p_0 = (r +_z s_0$, $p_1 = s_1)$ where $s = s_0 + s_1$ with both s_0, s_1 non zero. If p_0 and p_1 divide r then $p_0Lbr = 1 + r_0Lbr > 1 + r_0Rbr > r_0Rbr = p_0Rbr$ and $p_1Lbr = r_1Lbr + sLbr = r_1Lbr + sRbr < r_1Rbr + sRbr < r_1Rbr + sRbr +1 = p_1Rbr$. So in this case $p_0Lbr > p_0Rbr$ and $p_1Lbr < p_1Rbr$ where we have applied our induction hypothesis to r and used a). If p_0 and p_1 divide s we get the same result applying our induction hypothesis to s and also using a). We may thence assume that p_0 and p_1 divide neither r nor s. We thence have 4 cases:

1) $p_0 = ($ and also $p_1 = r +_z s)$ $p_0Lbr = 1 > 0 = p_0Rbr$ and $p_1Lbr = rLbr + sLbr = rRbr + sRbr < rRbr + sRbr+1 = r +_z s)Rbr = p_1Rbr$.
2) $p_0 = (r$ and also $p_1 = +_z s)$ $p_0Lbr = 1+rLbr = 1+rRbr > rRbr = p_0Rbr$ and $p_1Lbr = sLbr = sRbr < sRbr +1 = +_z s)Rbr = p_1Rbr$.
3) $p_0 = (r +_z$ and also $p_1 = s)$ $p_0Lbr = 1+rLbr = 1+rRbr > rRbr = p_0Rbr$ and $p_1Lbr = sLbr = sRbr < sRbr +1 = s)$ $Rbr = p_1Rbr$.
4) $p_0 = (r +_z s$ and also $p_1 =)$ $p_0Lbr = 1+rLbr +sLbr = 1+rRbr +sRbr > rRbr +(s)Rbr = p_0Rbr$ and $p_1Lbr = 0 < 1 = p_1Rbr$.

In each of these 4 cases and hence in all cases we have p_0Lbr > p_0Rbr and p_1Lbr < p_1Rbr. This proves c).

d) Suppose r is a proper initial segment of some p $\in$ M_{QO} . If r is 0 then r is not in M_{QO} since this set does not contain 0. So we can assume r is non zero. Since it is a proper initial segment there is a non zero s so that p = r+s . By c) rLbr >rRbr and sLbr < sRbr . But if r were in M_{QO}, rLbr = rRbr , a contradiction by a). Hence r is not in M_{QO}.
A similar argument shows that if s is a proper terminal segment of p $\in$ M_{QO} then s is not in M_{QO}. This proves d).

e) We know from b) that p $\in$ M_{QO} => p $\in$ *Q* or p = *(*r $+_z$ s*)* for some $+_z \in$ *O* and r,s $\in$ M_{QO}. To show that this representation is unique we presume we have a p $\in$ M_{QO} with two such representations. So assume p = *(*r $+_z$ s*)* = *(*u $+_y$ v*)* where $+_y \in$ *O* and u,v $\in$ M_{QO} . If r≠u then as r and u are in M_{QO} , u is a proper initial segment of r or r is a proper initial segment of v which contradicts d). So r = u. Similarly s = v since otherwise s would be a proper terminal segment of v or v would be a proper terminal segment of s, contradicting v. Hence p = *(*r $+_z$ s*)* = *(*r $+_y$ s*)* forcing $+_z$ = $+_y$. This proves that the representation of p in form *(*r $+_z$ s*)* for some $+_z \in$ *O* and r, s $\in$ M_{QO} is in fact unique, proving e).

f) This follows since |p| counts all symbol occurrences in p while (p)Br counts all occurrences of brackets in p and |pSl| all occurrences of non bracket symbols in p.

g) Let T = {p$\in$ pSl | pSl = $p_0 +_{z1} p_1 +_{z2} p_2 \ldots +_{zn} p_n$ for some n≥0, p_i in *Q* and $+_{zi} \in$ *O.*}
Then clearly *Q* $\subseteq$ T $\subseteq$ M_{QO} Further suppose p,q are in T and $+_z \in$ *O*. Then *(*p$+_z$q*)* is in M_{QO}. But *(*p$+_z$q*)*Sl = pSl $+_z$qSl and since p and q are in T so is *(*p$+_z$q*)* . By the principle of induction for M_{QO}, M_{QO} = T. Hence for any p in M_{QO} pSl has the required form. This proves g) completing the proof of Prop10. //

We now consider the natural evaluation of expressions in M_{QO}. For each p in M_{QO} there is a uniquely defined element of Q which is the result of the arithmetic process in Q defined by p. To this end we define a function Ev: $M_{QO} \to$ Q which for any p in M_{QO} returns pEv $=^{df}$ (p)Ev , the result of interpreting *Q* and *O* with their corresponding elements and operations in Q. To define Ev we need first to define for any element, p, of M_{QO} the number of bracket placing operations needed to create p from *Q* and *O* . This we call the **level** of an element of M_{QO} and denote it pΛ $=^{df}$ (p)Λ. We define **pΛ = pBr/2.** Note that *q*Λ = 0 for any *q* in *Q.* We can now define a decomposition of M_{QO} in terms of the level function as follows:

For any natural number n ≥ 0, M_{QOn} = {p$\in$ M_{QO} | pΛ = n }.
Note that M_{QO0} = *Q* , and the M_{QOn} are pairwise disjoint. We can accordingly define Ev on M_{QO} by defining it inductively on the M_{QOn} . Clearly M_{QO} = $\cup${ M_{QOn} | n≥0 }.

For p in M_{QO0} = *Q* , p = *q* in *Q,* (p)Ev =(*q*)Ev = q where q in Q corresponds to *q* in *Q.*

Now suppose Ev has been defined on M_{QOn} for n<N where N >0 an integer. Take p in $M_{QON.}$ p is not in *Q,* so by Prop10e), p = *(*x $+_z$ y*)* for unique x and y in M_{QO} and $+_z$ in *O*. Note (x)Λ, (y)Λ < N. As (x)Ev, (y)Ev have already been defined, we can define (p)Ev = ((x)Ev $+_z$ (y)Ev) which is a uniquely determined member of Q. Thus Ev can be considered well defined on M_{QON}.

To prove this lets consider the set D of all Natural numbers n for which Ev is well defined. 0 is in D. Further, if for any N>0, Ev is well defined on M_{QOn} , that is, n is in D, for all n<N, then Ev is well defined on M_{QON} , that is, N is in D. Then by induction on the natural numbers, D is the full set of natural numbers. Hence Ev is well defined on all of M_{QO}.

Def6: The set of operations O on Q are **interassociative** if

$$(a +_y (b +_z c)) = ((a +_y b) +_z c)$$

for any a,b,c in Q and $+_y$, $+_z$ in O.

We aim to prove in what follows that for interassociative operations, two elements in M_{QO} denote the same element of Q if the two expressions are identical when their brackets are removed. Using the above notation this means precisely: If the operations O on Q are interassociative then for p,q in M_{QO} pSl = qSl => (p)Ev = (q)Ev.

***************************END of PAPER*************************************

What's below is notes.

□3 Multi-operation Associativity

Let Q be an arbitrary non empty set and let F_{QBO} = N(Q∪{(,)}∪O) where the "left bracket", (, and the "right bracket",), are not in Q or in O. Further Q and O are disjoint so the sets Q, {(,)}, and O are pairwise disjoint. O is the set of binary operation symbols indexed by a set, Z. So O can be written O = $\{+_z \mid z \in Z\}$. Here each "$+_z$" is to be regarded as a single symbol, accordingly |Z| = |O|.

Define

$$M_{QO} = \cap\{T \mid Q \subseteq T \subseteq F_{QBO} , (p, q \in T \text{ and } +_z \in O) \Rightarrow (p +_z q) \in T\}.$$

Note that the semigroup F_{QBO} – {0} is one of the sets T of which M_{QO} is the intersection. Hence M_{QO} does not contain the empty string, 0. Further we have that $F_{QBO} \supseteq M_{QO} \supseteq Q$. We also have binary operations on M_{QO}, namely $+_z$: $M_{QO} \times M_{QO} \rightarrow M_{QO}$ where (m,n) → (m $+_z$ n) for any z in Z. Note that the string denoted "(m $+_z$ n)" , is (m$+_z$n) = (+ m + $+_z$ + n +) where (, m, $+_z$, n,) are individual symbols and + is natural concatenation in F_{QBO} . Spaces are inserted for clarity of expression only, and the space is not a symbol in the basis of F_{QBO} or in any other free monoid we will consider.

Def4: For m,n in M_{QO} $m \leftrightarrow n$ iff $m = n$ or $m = m_0 + x + m_1$ and $n = m_0 + y + m_1$ where either $x = (a+_w(b+_zc))$ and $y = ((a+_wb)+_zc)$ or $x = ((a+_wb)+_zc)$ and $y = (a+_w(b+_zc))$, for some a,b,c in M_{QO}, and $+_w$, $+_z$ are in O.

We define Δ to be the transitive closure of ↔. Since ↔ is reflexive and symmetric, Δ is an equivalence relation on M_{QO}. We again adapt the underscore bar notation from the previous section and define $x \mapsto \underline{x}$, an endomorphism of F_{QBO} induced by sending "(" and ")" to 0 (the empty string) and leaving the rest of the basis fixed. This function will also be denoted "usb" when the expression it is applied to makes the underscore too cumbersome. For example, we apply $x \mapsto \underline{x}$ to $((p+_zq)+_yr)$ by writing $[((p+_zq)+_yr)]$usb where we use the square brackets to enclose the argument of usb to avoid confusion with the round brackets which are part of this argument.

Hence $[((p+_zq)+_yr)]\text{usb} = \underline{p} +_z \underline{q} +_y \underline{r}$. Note usb can also be applied to subsets of F_{QBO} such as M_{QO}.

Prop8: Principle of induction for M_{QO}:

$$\text{If } Q \subseteq T \subseteq M_{QO} \text{ and } p, q \in T, +_z \in O \Rightarrow (p+_zq) \in T, \text{ then } T = M_{QO} .$$

pf: Assume we have a subset T of F_{QBO} such that $Q \subseteq T \subseteq M_{QO}$ and $p,q \in T$, $+_z \in O \Rightarrow (p+_zq) \in T$. It suffices to show that $M_{QO} \subseteq T$. But T is such that

$Q \subseteq T \subseteq F_{QBO} - \{0\}$ and $p,q \in T$, $+_z \in O \Rightarrow (p+_zq) \in T$ so by definition of M_{QO}, $M_{QO} \subseteq T$. Hence $M_{QO} = T$, completing the proof.//

Let the number of brackets occurring in an element x of F_{QBO} be denoted by **(x)br.** Let the number of left brackets (i.e. "(")be denoted **(x)Lbr** and the number of right brackets (i.e. ")") occurring in x be denoted **(x)Rbr**.

Prop9, below, shows that for any m in M_{QO}, $\underline{m}$ is of the form

$$\underline{m} = s_1+_1s_2+_2 \ldots +_{n-1} s_n$$

where $n \geq 1$, $s_i \in Q$, $+_j \in O$, for $1 \leq i \leq n$ and $1 \leq j \leq n-1$ where this representation is unique.

For $\underline{m} = s_1+_1s_2+_2 \ldots +_{n-1}s_n$ in $\underline{M_{QO}}$ let $n = Ql(\underline{m})$ be the **Q-length** of $\underline{m}$. Q is an arbitrary set hence we use Ql to denote the function that gives us the number of Q symbols occurring in $\underline{m}$.

Prop9: $\underline{M_{QO}} = \{s_1+_1s_2+_2 \ldots +_{n-1}s_n \in F_{QBO} \mid n \geq 1, s_i \in Q, +_j \in O, \text{ for } 1 \leq i \leq n \text{ and } 1 \leq j \leq n-1\}$.

pf: Let $\sum = \{ s_1+_1s_2+_2 \ldots +_{n-1}s_n \in F_{QBO} \mid n \geq 1 , s_i \in Q, +_j \in O, \text{ for } 1 \leq i \leq n \text{ and } 1 \leq j \leq n-1\}$.

Let $T = \{m \in \mathcal{M}_{QO} \mid \underline{m} \in \sum\}$. Since for any s in Q, $\underline{s} = s \in \sum$, $Q \subseteq T \subseteq \mathcal{M}_{QO}$. Now suppose $p, q \in T$, $+_z \in O$. Then $\underline{(p+_zq)} = \underline{p+_zq} = \underline{p}+_z\underline{q}$ which concatenation is clearly in $\sum$. Hence $\underline{(p+_zq)} \in \sum$, so $(p+_zq) \in T$. By Prop8, $T = \mathcal{M}_{QO}$. So $\underline{\mathcal{M}_{QO}} \subseteq \sum$.

Now prove $\sum \subseteq \underline{\mathcal{M}_{QO}}$. Since any element of $\sum$ is of the form $\sigma = s_1+_1s_1+_2 \ldots +_{n-1}s_n$ we proceed by induction on n. If n = 1 then $\sigma = s_1 \in \underline{\mathcal{M}_{QO}}$. Suppose we have shown $\sigma \in \underline{\mathcal{M}_{QO}}$ where $n < k$ for some $k > 1$. Now take a σ where n = k. Then $\sigma = s_1+_1s_2+_2 \ldots +_{n-1}s_n = \underline{m} +_{n-1} s_n$, for some $m \in \mathcal{M}_{QO}$, where we have used the fact that by induction $s_1+_1s_2+_2 \ldots +_{n-1}s_{n-1} \in \underline{\mathcal{M}_{QO}}$.

Now $\sigma = \underline{m} +_{n-1} s_n = \underline{(m +_{n-1} s_n)} \in \underline{\mathcal{M}_{QO}}$ since $(m +_{n-1} s_n) \in \mathcal{M}_{QO}$. This completes the induction showing $\sum \subseteq \underline{\mathcal{M}_{QO}}$.

Hence $\sum = \underline{\mathcal{M}_{QO}}$, completing the proof.//

Prop10: a) u in $\mathcal{M}_{QO}$ and $u \notin Q \Rightarrow u = (p+_zq)$ for some $p, q \in \mathcal{M}_{QO}$, and $+_z \in O$.

b) u in $\mathcal{M}_{QO} \Rightarrow |u| = 4Sl(u) - 3 = 2|\underline{u}| -1$.

c) u in $\mathcal{M}_{QO} \Rightarrow (u)br = 2(Sl(u) -1) = |\underline{u}| - 1 = (|u| - 1)/2$ and |u| is odd.

d) u in $\mathcal{M}_{QO} \Rightarrow (u)Lbr = (u)Rbr = (|\underline{u}| -1)/2 = Sl(u) -1$.

e) u in $\mathcal{M}_{QO} \Rightarrow$ If u = v+w for non zero v and w, then (v)Lbr > (v)Rbr and (w)Lbr < (w)Rbr .

f) u in $\mathcal{M}_{QO} \Rightarrow$ If v is a non zero proper terminal or initial segment of u then $v \notin \mathcal{M}_{QO}$.

g) u in $\mathcal{M}_{QO}$ and $u \notin Q \Rightarrow u = (p+_zq)$ for some unique $p, q \in \mathcal{M}_{QO}, +_z \in O$.

pf: a) Let $T = \{u \in \mathcal{M}_{QO} \mid$ u is in Q or $u = (p+_zq)$ for some $p, q \in \mathcal{M}_{QO}\}$. Then $Q \subseteq T \subseteq \mathcal{M}_{QO}$. Suppose $p, q \in T$, $+_z \in O$. Then p and q are in $\mathcal{M}_{QO}$ so $(p+_zq) \in T$. Hence $T = \mathcal{M}_{QO}$ and a) follows.

b) Let $T = \{u \in \mathcal{M}_{QO} \mid |u| = 4Sl(u) - 3 = 2|\underline{u}| -1\}$. Then $Q \subseteq T \subseteq \mathcal{M}_{QO}$, since if $s \in Q$, $1 = |s| = Sl(s) = |\underline{s}|$ and so $|s| = 4Sl(s) - 3 = 2|\underline{s}| -1$. Now suppose $p, q \in T$, $+_z \in O$. $|(p+_zq)| = 1 + |p| + 1 + |q| + 1 =$

$3 + 4Sl(p) - 3 + 4Sl(q) - 3 = 4(Sl(p) + Sl(q)) - 3 = 4Sl((p+_zq)) - 3$. Further, $|(p+_zq)| =$

$3 + 2|\underline{p}| -1 +2|\underline{q}| -1 = 2(|\underline{p}|+|\underline{q}|)+1 = 2(|\underline{(p+_zq)}|-1) + 1 = 2|\underline{(p+_zq)}|-2 + 1 = 2|\underline{(p+_zq)}|-1$. Hence $(p+_zq) \in T$. Therefore $T = \mathcal{M}_{QO}$ proving b).

c) Let $T = \{u \in \mathcal{M}_{QO} \mid (u)br = 2(Sl(u) -1)\}$. Then $Q \subseteq T \subseteq \mathcal{M}_{QO}$, since if $s \in Q$, $0 = (s)br = 2(Sl(s)-1)$. Take $p, q \in T$, $+_z \in O$. $((p+_zq))br = (p)br + (q)br +2 = 2(Sl(p) -1) +2(Sl(q) -1)+2$ $= 2(Sl(p) + Sl(q)) - 2 = 2Sl(\underline{(p+_zq)}) - 2 = 2(Sl(\underline{(p+_zq)}) - 1)$. Hence $(p+_zq) \in T$. By Prop8, $T = \mathcal{M}_{QO}$.

Hence for u $\in \mathcal{M}_{QO}$ (u)br = 2(Sl(u) -1) . Now to finish the proof of c), notice from b) that $|\underline{u}|$ = 2Sl(u) - 1 and so (u)br = 2Sl(u) – 2 = $|\underline{u}|$ - 1 = (|u| - 1)/2 by b). Note from this last equation that |u| must be odd, so c) is proved.

d) Since (u)br = (u)Lbr + (u)Rbr, to show (u)Lbr = (u)Rbr = ($|\underline{u}|$ -1)/2 = Sl(u) -1, it suffices to show that (u)Lbr = (u)Rbr. To this end let T = {u $\in \mathcal{M}_{QO}$ | (u)Lbr = (u)Rbr }. Then Q $\subseteq$T$\subseteq \mathcal{M}_{QO}$, since if s $\in$ Q , (s)Lbr = 0 = (s)Rbr. Now take p, q $\in$ T, $+_z \in$ O . Then

$$((p+_zq))Lbr = 1 + (p)Lbr + (q)Lbr = 1 + (p)Rbr + (q)Rbr = ((p+_zq))Rbr .$$

By Prop8, T = $\mathcal{M}_{QO}$. This proves d).

e) Let T = {u $\in \mathcal{M}_{QO}$ | u = v+w for non - zero v and w => (v)Lbr > (v)Rbr and (w)Lbr < (w)Rbr }.
Implication (1) is vacuously true for u in Q, so Q $\subseteq$T$\subseteq \mathcal{M}_{QO}$. Now take p, q $\in$ T, $+_z \in$ O and let u = ($p+_zq$). We verify implication (1) for u. Suppose u = v + w for non zero v and w in F_{QBO} . We have the following cases:

1) v = “(“ + p_0 , w = p_1 + “$+_z$” + q + ”)” , where p = p_0 + p_1 and either of the p_i can be 0;
2) v = “(“ + p + “$+_z$” + q_0 , w = q_1 + ”)” , where q = q_0 + q_1 and either of the q_i can be 0.

In 1) if p_i both non zero, (v)Lbr = 1 + (p_0)Lbr > 1 + (p_0)Rbr > (v)Rbr, so (v)Lbr > (v)Rbr and also (w)Lbr < (w)Rbr by d) as (v)Lbr + (w)Lbr = (v)Rbr + (w)Rbr. [2]
We have used the fact that p is in T to get (p_0)Lbr > (p_0)Rbr. If p_0 = 0 then v = “(“ and so (v)Lbr = 1 > 0 = (v)Rbr and so also (w)Lbr < (w)Rbr. If p_1 = 0 then v = “(“ + p, so (v)Lbr = 1+ (p)Lbr = 1+ (p)Rbr > (p)Rbr = (v)Rbr and so also (w)Lbr < (w)Rbr. Thus in case 1) implication [1] holds for u.
In 2) if both q_i non zero, (w)Rbr = (q_1)Rbr + 1 > (q_1)Lbr + 1 > (q_1)Lbr = (w)Lbr. Hence also (v)Lbr > (v)Rbr by [2] . If q_1 = 0 then w = “)“ and so (w)Rbr = 1 > 0 = (w)Lbr and so also (v)Lbr > (v)Rbr. If q_0 = 0 then w = q + ”)”, so (w)Rbr = (q)Rbr+1 = (q)Lbr +1 > (q)Lbr = (w)Lbr. Thus in case 2) implication [1] holds for u.

Therefore in all cases implication [1] holds for u so u is in T. Therefore by Prop8, T = $\mathcal{M}_{QO}$. This proves e).

f) Suppose u $\in \mathcal{M}_{QO}$ and suppose v is a non zero proper initial segment of u. Then u = v + w where both v and w are non zero. By e), (v)Lbr > (v)Rbr. Therefore v$\notin \mathcal{M}_{QO}$ by d). Similarly suppose v is a proper terminal segment of u. Then u = w + v where both v and w are non zero. By e) (v)Lbr < (v)Rbr, and so by d) v$\notin \mathcal{M}_{QO}$. This proves f).

g) Suppose u in $\mathcal{M}_{QO}$ and u $\notin$ Q. Then by a) u = (p $+_z$ q) for some p, q in $\mathcal{M}_{QO}$ and $+_z \in$ O. Suppose we have another such representation of u = (r $+_w$ s) . Note we have r, s in $\mathcal{M}_{QO}$ and $+_w \in$ O. Since p,q,r,s are in $\mathcal{M}_{QO}$ none of these strings are 0 (i.e. empty) . Hence if p $\neq$ r then either p is a proper initial segment of r or r is a proper initial segment of p. In the first case p cannot be in $\mathcal{M}_{QO}$ and in the second r cannot be in M_{QO} . This contradiction forces p = r and so $+_w = +_z$ and s = q. This shows the uniqueness of our representation of u in the form $(p +_z q)$, proving g). This completes the proof of Prop10.//

Prop11: For any m in M_{QO} , $m \in Q$ or $m\ \Delta\ m'$ for some $m' \in M_{QO}$, where $m' = (s+_zB)$ for some s in Q and B in M_{QO} and $+_z \in O$.

pf: Let $T = \{m \in M_{QO} \mid m \in Q$ or $m\ \Delta\ m'$ for some $m' = (s+_zB)$ for $s \in Q$ and $B \in M_{QO}$, $+_z \in O\}$. Then $Q \subseteq T \subseteq M_{QO}$. Now suppose p, q are in T and $+_z \in O$. Then $m = (p+_zq) \in M_{QO} - Q$. If $p \in Q$ then $m = (s+_zB)$ where $s = p \in Q$, $B = q \in M_{QO}$ and $+_z \in O$. Since $m\ \Delta\ m$, $m \in T$.

So we may assume that $p \notin Q$. Then since $p \in T$, $p\ \Delta\ p'$ where $p' = (s+_yB)$ for some s in Q and B in M_{QO} and $+_y \in O$. Hence $m = (p+_zq)\ \Delta\ ((s+_yB)+_zq)\ \Delta\ (s+_y(B+_zq))$ which shows that $m \in T$ since $m\ \Delta\ (s+_y(B+_zq))$ where $s \in Q$ and $(B+_zq)$ is in M_{QO}. Thus in all cases m is in T. So by Prop8, the principle of induction for M_{QO}, $T = M_{QO}$. This proves Prop11.//

Thm3: For m and n in M_{QO}, $\underline{m} = \underline{n} \Leftrightarrow m\ \Delta\ n$.

pf: (<=) Referring to Def4, to establish

$$\underline{m} = \underline{n} \Leftarrow m\ \Delta\ n \qquad [1]$$

it suffices to show that $\underline{m} = \underline{n}$ when $m \leftrightarrow n$. To this end note that if m=n there is nothing to prove. So assume $m \neq n$. Then from Def4 we have $m = m_0 + x + m_1$ and $n = m_0 + y + m_1$ where either

[2] $x = (a+_y(b+_zc))$ and $y = ((a+_yb)+_z c)$

or

[3] $x = ((a+_yb)+_zc)$ and $y = (a+_y(b+_zc))$, for some a,b,c in M_{QO}.

In the case of [2] $m = m_0 + (a+_y(b+_zc)) + m_1$ and $n = m_0 + ((a+_yb)+_z c) + m_1$. In the case of [3] $m = m_0 + ((a+_yb)+_zc) + m_1$ and $n = m_0 + (a+_y(b+_zc)) + m_1$. In both cases, $\underline{m} = \underline{n}$. This proves (<=).

(=>): We prove for m in M_{QO} by induction on $L = |\underline{m}|$ that,

$$n \text{ in } M_{QO} \Rightarrow (\underline{m} = \underline{n} \Rightarrow m\ \Delta\ n) \qquad [4]$$

Suppose L = 1. Then for any n in M_{QO} with $\underline{m} = \underline{n}$, n and m are in Q. Hence $m = \underline{m} = \underline{n} = n$ and so $m\ \Delta\ n$ follows. Now suppose we have established [4] in the case that L< N for some integer N >1. Consider m in M_{QO} with $L = |\underline{m}| = N$. Take n in M_{QO} with $\underline{m} = \underline{n}$. Since $|\underline{m}| = |\underline{n}| = N > 1$, m and n are not in Q. Hence by Prop11, $m\ \Delta\ (s+_yB)$ and $n\ \Delta\ (t+_zC)$ for some s.t in Q and $B,C \in M_{QO}$, $+_y$, $+_z \in O$. Since $\underline{m} = \underline{n}$, $\underline{m} = s +_y \underline{B}$ and $\underline{n} = t +_z \underline{C}$. Hence s = t, $+_y = +_z$, and $\underline{B} = \underline{C}$. Now $|\underline{B}| < N$ and C is in M_{QO}, so by induction hypothesis, [4] applies and so $B\ \Delta\ C$. So

$m \,\Delta\, (s+_y B) \,\Delta\, (s+_y C) \,\Delta\, (t +_z C) \,\Delta\, n$, showing $m \,\Delta\, n$. This completes the induction, establishing [4] for any m. Hence we have $m, n \in M_{QO} \Rightarrow (\underline{m} = \underline{n} \Rightarrow m \,\Delta\, n)$ completing the proof of (=>) and hence the proof of Thm3. //

Theorem 3 implies that given a set of binary operations O, on a set Q, any two bracketings of a sum $s_0+_1 s_1+_2 \ldots +_n s_n$ are equal iff the operations in O satisfy the associative law for a set of operations, namely,

$$(a+_y (b +_z c)) = ((a +_y b) +_z c) \qquad (2)$$

for any a,b,c in Q and $+_y$, $+_z$ in O. We say that such a set of binary operations on a given set is **interassociative**.

Theorem 3 can be restated to say:

Thm3: If a set of binary operations O on a set Q is interassociative then any two bracketings of a sum $s_0+_1 s_1+_2 \ldots +_n s_n$, are equal, where $+_i \in O$, and $s_j \in Q$.

□ 4 Associative Shelving Sums in S

Recall F = N(S) and for any z in F, $+_z$ is a binary operation on F which is not necessarily associative. Here we study sums made in F from successive use of the $+_z$ operations, namely elements of the set $O_F = \{+_z \mid z \in F\}$. To this end we form a free monoid which consists of a set of strings that contains all bracketed O_F sums from F. These O_F sums will be a subset of this set of strings and will allow us to prove what we need to prove about the O_F sums that can be formed in F. For the sake of precision and to avoid the occurrence of ambiguous expressions, we will always use + for string concatenation no matter what free monoid we are in and we will form a copy of F, this being a new set $<F> = \{<f> \mid f \in F\}$'. This is the set of strings obtained by concatenating "<" and ">" on the front and back of each string in F where we will assume that the symbols "<" and ">" are not elements of $S \cup \{(,)\} \cup O$.

For notational convenience we will, in this section let $O = O_F = \{+_z \mid z \in F\}$. This set of strings we denote as EF, and it is the free monoid on the union of sets F, {(,)}, and O, namely,

$$EF = N(<F> \cup \{(,)\} \cup O) = F_{FBO} \;,$$

where we take these three sets, <F>, {(,)}, and O to be pairwise disjoint. We use EF, short for "Expressions from F" to denote F_{FBO}. We then apply the theory developed in the last section. Q from the last section is replaced by F. B = {(,)} as before. O = is the set of symbols, $O_F = \{+_z \mid z \in F\}$.

We find the possible expressions involving elements of F, B, and the set of operation symbols $O = O_F = \{+_z \mid z \in F\}$.

This set, which we call M_{FO} , is defined as follows:

$$M_{FO} = \cap\{T \mid F \subseteq T \subseteq EF \text{ and } (p, q \in T , z \in F \Rightarrow (p+_z q) \in T)\},$$
$$= \cap\{T \mid F \subseteq T \subseteq EF \text{ and } (p, q \in T , \square \in O \Rightarrow (p \,\square\, q) \in T)\} .$$

Note that the $(p+_zq)$ here is a string of symbols in EF where p and q are such strings and "(", "$+_z$", and ")" are elements of the basis of EF. So $(p+_zq)$ is formed through concatenation in EF. M_{FO} does not contain the empty word though it does contain the string "0" which is the string consisting of the 0 of F.

M_{FO} contains all the sums that can be formed from elements of F with no assumptions of associativity.

M_{FO} is the free multi - operation magma on F where the operations range through $O = O_F$. M_{FO} is in fact an instance of M_{QO} treated above. For this reason the propositions proved for M_{QO} also hold for M_{FO}. Propositions 8, 9 and Prop10 thus have the following propositions 12, 13, and 14 about M_{FO} as corollaries or rather instantiations.

Prop12: Principle of induction for M_{FO} :

$$\text{If } F \subseteq T \subseteq M_{FO} \text{ and } p, q \in T, +_z \in O_F \Rightarrow (p+_zq) \in T, \quad \text{then } T = M_{FO} .$$

Notation6: For $\underline{m} = s_1+_1s_2+_2 \ldots +_{n-1}s_n$ in $\underline{M_{FO}}$ let $n = Sl(\underline{m})$ be the F-<u>length</u> of $\underline{m}$. Recall that here the s_i are elements of F and can be the 0 of F.

Let the number of brackets occurring in an element x of EF be denoted by (x)br. Let the number of left brackets occurring in x (i.e. "(") be denoted (x)Lbr and the number of right brackets (i.e. ")") occurring in x be denoted (x)Rbr.

Prop13: $\underline{M_{FO}} = \{s_1+_1s_2+_2 \ldots +_{n-1}s_n \in EF \mid n\geq 1, s_i \in F, +_j \in O_F, \text{ for } 1\leq i\leq n \text{ and } 1\leq j\leq n-1\}$.

Prop14: a) u in M_{FO} and $u \notin F \Rightarrow u = (p+_zq)$ for some $p, q \in M_{FO}$, and $+_z \in O_F$.

b) u in M_{FO} $\Rightarrow$ $|u| = 4Sl(u) - 3 = 2|\underline{u}| -1$.

c) u in M_{FO} $\Rightarrow$ $(u)br = 2(Sl(u) -1) = |\underline{u}| - 1 = (|u| - 1)/2$ and $|u|$ is odd.

d) u in M_{FO} $\Rightarrow$ $(u)Lbr = (u)Rbr = (|\underline{u}| -1)/2 = Sl(u) -1$.

e) u in M_{FO} $\Rightarrow$ If $u = v+w$ for non-zero v and w, then $(v)Lbr > (v)Rbr$ and $(w)Lbr < (w)Rbr$.

f) u in M_{FO} $\Rightarrow$ If v is a non-zero proper terminal or initial segment of u then $v \notin M_{FO}$.

g) u in $\mathcal{M}_{FO}$ and $u \notin F \Rightarrow u = (p+_z q)$ for some unique $p, q \in \mathcal{M}_{FO}$, and $+_z \in O_F$.

pf: Follows from Prop 10 above. //

We can now define an evaluation function from $\mathcal{M}_{FO}$ into F, which gives the value in F for an expression u in $\mathcal{M}_{FO}$ obtained when the operation symbols $O = O_F$ are used as actual operations in F. Recalling that $O = O_F = \{+_z \mid z \in F\}$, we let each occurrence of an element of O in an expression u in $\mathcal{M}_{FO}$ become the actual binary operation on F that it corresponds to and then compute the resulting element of F that u actually denotes. This element will be denoted (u)ev, read the "evaluation of u" . We first construct the evaluation function ev and then prove that if p,q in $\mathcal{M}_{FO}$ and $\underline{p} = \underline{q} = s_1 +_{z_1} s_2 +_{z_2} \ldots +_{z_{n-1}} s_n$, $n \geq 1$, $s_i \in F$, $+_{z_j} \in O_F$, for $1 \leq i \leq n$ and where for $1 \leq j \leq n-1$, z_j is in $I(s_j, s_{j+1})$, then $(p)ev = (q)ev = s_1 +_{z_1} s_2 +_{z_2} \ldots +_{z_{n-1}} s_n$ where the last computation's result does not depend on the way the sum is bracketed. We call a sum in F of the above form a **tractable sum**. Note a binary sum, $s_1 +_{z_1} s_2$, is tractable exactly when z_1 is in Is_1s_2 .

We now construct our evaluation function, ev. ev is a subset of $\mathcal{M}_{FO} \times F$ containing $\Delta F = \{(f,f) | f \in F\}$. In fact we define it as follows:

$$ev = \cap\{\Delta F \subseteq T \subseteq \mathcal{M}_{FO} \times F \mid (p,f), (q,g) \in T, z \in F \Rightarrow ((p+_z q), f+_z g) \in T\}, \quad (1)$$
$$= \cap \Upsilon .$$

$\mathcal{M}_{FO} \times F$ is one of the T whose intersection is taken in (1). Also note if $(p,f), (q,g) \in ev$, $z \in F$ then for any T in Υ, (p,f), (q,g) are in T and so also $((p+_z q), f+_z g) \in T$. So $((p+_z q), f+_z g) \in \cap \Upsilon = ev$. So we have that ev itself is an element of Υ, in fact its smallest element.
Hence

$$(p,f), (q,g) \in ev , z \in F \Rightarrow ((p+_z q), f+_z g) \in ev . \quad (2)$$

We show that ev is a well defined function ev: $\mathcal{M}_{FO} \to F$.

We must show that for each element $p \in \mathcal{M}_{FO}$ there is a unique f such that $(p,f) \in ev$. Let D be the domain of ev, that is, $D = \{d \in \mathcal{M}_{FO} \mid$ there is an f in F such that $(d,f) \in ev\}$. Since $\Delta F \subseteq ev$, $F \subseteq D \subseteq \mathcal{M}_{FO}$. From (2) we have that if
$d,d' \in D$ and $z \in F \Rightarrow (d +_z d') \in D$. By Prop12, the principle of induction for $\mathcal{M}_{FO}$, we have that $D = \mathcal{M}_{FO}$. So ev is defined on all of $\mathcal{M}_{FO}$.

Now to show that ev is a function, let $W = \{d \in \mathcal{M}_{FO} \mid$ there is a unique f in F such that $(d,f) \in ev\}$. For any f in F, (f,f) is an element of ev. If (f,g) is in ev with $f \neq g$ consider $K = \Delta F \cup (\mathcal{M}_{FO} - F) \times F$. Now

$\Delta F \subseteq K \subseteq \mathcal{M}_{FO} \times F$ and

$(p,f), (q,g) \in K$, $z \in F \Rightarrow ((p +_z q), f +_z g) \in K$. Hence K contains ev. Note that every element of K which is not in ΔF has a bracket occurring in its first component. Therefore (f,g) is not in K and hence not in ev. Thus $F \subseteq W \subseteq \mathcal{M}_{FO}$.

Now suppose w and w' are elements of W. Then there are unique f and f' in F such that (w,f) and (w',f') are in ev. Then

for any z in F $((w +_z w'), f +_z f')$ is in ev by (2). Suppose $((w +_z w'), g)$ is also in ev $g \neq f +_z f'$. Then let $L = ev - \{((w +_z w'), g)\}$.

L contains ΔF. Also note that (p,h) (p',h') in L, z' in F => $((p +_{z'} p'), h +_{z'} h')$ is in L unless $((p +_{z'} p'), h +_{z'} h') = ((w +_z w'), g)$. But if this happens p = w, p' = w', z = z' by Prop14 g) . So $((w +_z w'), h +_z h') = ((w +_z w'), g)$. Now (w,h) and (w',h') are in ev as are (w,f) and (w',f'). But by the definition of W we must have h = f and h' = f'. Hence $g = h +_z h' = f +_z f'$. This contradicts our supposition that a $((w +_z w'), g)$ is in ev with $g \neq f +_z f'$. Thus $(w +_z w')$ is in W. Thus w,w' in W and z in F => $(w +_z w')$ in W and we have $F \subseteq W \subseteq \mathcal{M}_{FO}$

By Prop12 $W = \mathcal{M}_{FO}$. This proves $ev: \mathcal{M}_{FO} \to F$ is a well defined function. We take (2) above and single it out as a separate proposition because of it allows us to compute particular values of ev.

Prop15: For p , q in $\mathcal{M}_{FO}$ and z in F, $((p +_z q))ev = (p)ev +_z (q)ev$.

Pf: Take p , q in $\mathcal{M}_{FO}$ and z in F. Since we have proven above that $ev: \mathcal{M}_{FO} \to F$ is a function, (p)ev and (q)ev exist unique elements of F such that $(p,(p)ev), (q,(q)ev) \in ev$. From (2) we have, $(p)ev +_z (q)ev = ((p +_z q))ev$, since (2) says

$(p,f), (q,g) \in ev$, $z \in F \Rightarrow ((p +_z q), f +_z g) \in ev$. //

Thm4: If p in $\mathcal{M}_{FO}$ and $\underline{p} = s_1 +_{z_1} s_2 +_{z_2} \ldots +_{z_{n-1}} s_n$ where $z_k \in I(s_k\ s_{k+1})$ then if q is in $\mathcal{M}_{FO}$ with $\underline{q} = \underline{p}$, then (p)ev = (q)ev.

Pf:

Shelving Closures

Def5: Let U be a subset of F = N(S) and let

$\underline{U} = \cap\{T \mid U \subseteq T \subseteq F, (p, q \in T) \Rightarrow (p \Diamond q) \subseteq T\}$. We call $\underline{U}$ the shelving closure of U.

If $U = \underline{U}$ we say U is **closed under shelving**.

We have a principle of induction for the shelving closure of an arbitrary subset U of the free monoid $F = N(S)$:

Prop: For U a subset of $F = N(S)$,

$$\text{If } U \subseteq T \subseteq \underline{U} \text{ and } (z \in F,\ p, q \in T \Rightarrow p+_z q \in T) \text{ then } T = \underline{U}.$$

pf: So take a subset T of F with $U \subseteq T \subseteq \underline{U}$ and ($z \in F$, $p, q \in T \Rightarrow p+_z q \in T$). We show that T must be all of $\underline{U}$. For this it suffices to show that $T \supseteq \underline{U}$ since we already have $T \subseteq \underline{U}$. Now by Def5

$$\underline{U} = \cap\{T \mid U \subseteq T \subseteq F,\ (p, q \in T) \Rightarrow (p \lozenge q) \subseteq T\}.$$

So all we need to do is show that for our T, $U \subseteq T \subseteq F$ and $(p, q \in T) \Rightarrow (p \lozenge q) \subseteq T$.

$U \subseteq T \subseteq F$ is obvious since $T \subseteq \underline{U}$ and $\underline{U} \subseteq F$. So it only remains to show that if arbitrary $p, q \in T$ then $(p \lozenge q) \subseteq T$. But $(p \lozenge q) = \{p +_z q \mid z \text{ in } N(S)\}$ and this is contained in T since by assumption $p, q \in T \Rightarrow p +_z q \in T$ for any z in F. Hence $(p, q \in T) \Rightarrow (p \lozenge q) \subseteq T$ and this completes the proof of Prop13.//

We collect some useful facts about the shelving closure of a set of strings in the following proposition.

Prop: For $U \subseteq F$, non empty,

a) $\underline{U} = \cap\{T \mid U \subseteq T \subseteq F,\ (p, q \in T, z \in F) \Rightarrow (p +_z q) \in T\}$.

b) $U \subseteq \underline{U} \subseteq F$.

c) $(p, q \in \underline{U}, z \in F) \Rightarrow (p +_z q) \in \underline{U}$.

d) $0 \in \underline{U} \Leftrightarrow 0 \in U$ and $\underline{\{0\}} = \{0\}$.

e) $w \in \underline{U}$ => There exist u, v in U such $u \leq_I w$ and $v \leq_T w$.

pf: a) Suffices to show that

$$\{T \mid U \subseteq T \subseteq F,\ (p, q \in T) \Rightarrow (p \lozenge q) \subseteq T\} = \{T \mid U \subseteq T \subseteq F,\ (p, q \in T, z \in F) \Rightarrow (p+_z q) \in T\}. \quad [1]$$

Let LHS, RHS be the left hand side, right hand side of [1]. Suppose T is in the RHS. Then $U \subseteq T \subseteq F$, $(p, q \in T, z \in F) \Rightarrow (p +_z q) \in T$. But this means that $p, q \in T \Rightarrow p \lozenge q \subseteq T$ by Prop2 b). Hence $T \in$ LHS and so RHS $\subseteq$ LHS. Now suppose T is in LHS. Then for $p, q \in T$, $z \in F$, $p +_z q \in \{p +_z q \mid z \text{ in } F\} = p \lozenge q \subseteq T$ by Prop2b). Hence $T \in$ RHS. Therefore LHS $\subseteq$ RHS. Hence LHS = RHS proving a).

b) Follows from $F \in \{T \mid U \subseteq T \subseteq F,\ (p, q \in T, z \in F) \Rightarrow (p +_z q) \in T\}$ and part a).

c) Suppose $(p, q \in \underline{U}, z \in F)$. Then by a) if T is such that $U \subseteq T \subseteq F$ and $(x, y \in T, v \in F) \Rightarrow (x +_v y) \in T$, then p, q is in T. Thus $p +_z q$ is in T. Hence we have $p +_z q \in \underline{U}$. This proves b).

d) $0 \notin \underline{U}$ implies we have a T such that $U \subseteq T \subseteq F$, $(p, q \in T, z \in F) \Rightarrow (p +_z q) \in T$ and $0 \notin T$. Hence $0 \notin U$. So $0 \notin \underline{U} \Rightarrow 0 \notin U$. Now suppose $0 \notin U$. Let $F' = F-\{0\}$. Then $U \subseteq F' \subseteq F$, and $(p, q \in F'\ z \in F) \Rightarrow (p+_z q) \in F'$. From the definition of $\underline{U}$, $\underline{U} \subseteq F'$. Hence $0 \notin \underline{U}$. This proves $0 \notin U \Rightarrow 0 \notin \underline{U}$. Therefore we have $0 \notin \underline{U} \Leftrightarrow 0 \notin U$, hence $0 \in \underline{U} \Leftrightarrow 0 \in U$. This proves the first statement of d). To show $\underline{\{0\}} = \{0\}$ we need only show that $\underline{\{0\}} \subseteq \{0\}$. But notice that $\{0\} \subseteq \{0\} \subseteq F$, $(p, q \in \{0\}, z \in F) \Rightarrow (p +_z q) \in \{0\})$. So $\underline{\{0\}} \subseteq \{0\}$. This completes the proof of d).

e) Using Prop12, let $T = \{w \in \underline{U} \mid$ There exist u, v in U such $u \leq_I w$ and $v \leq_T w\}$. Since for any u, including of course u in U, $u \leq_I u$ and $u \leq_T u$, $T \supseteq U$. Hence $U \subseteq T \subseteq \underline{U}$. To show $T = \underline{U}$, it remains only to show that for $z \in F$, $p, q \in T \Rightarrow p +_z q \in T$. So take $z \in F$ and $p, q \in T$. Then by Prop2a) $p \leq_I p +_z q$ and $q \leq_T p +_z q$. But p and q are in T so we have u, v in U such that $u \leq_I p$ and $v \leq_T q$. Note here we have used the defining property of T once for p and again for q.

So $u \leq_I p +_z q$ and $v \leq_T p +_z q$. Hence $p +_z q \in T$. Therefore $z \in F$, $p, q \in T \Rightarrow p +_z q \in T$ and so $T = \underline{U}$. This completes the proof of e).

This completes the proof of Prop14.//

Notation6: $U,V \subseteq F$,and * a binary operation on F then let $U*V = \{u*v \mid u \in U, v \in V\}$.

of O is a set of binary operations on F let $U_O V = \cup\{U*V \mid * \in O\}$.

Let $U \subseteq F = N(S)$ and let $O = \{+_z \mid z \in F\}$ let $U_0 = U$. Assuming U_n defined, let $U_{n+1} = \cup \{U_n +_Z U \mid z \in F\} = U_n\ _O\ U$.

Prop: Let $U \subseteq F = N(S)$ and let $O = \{+_z \mid z \in F\}$ with U_n defined as above,

a) $U_n \subseteq \underline{U}$ for $n \geq 0$.

b) $U_n \subseteq U_{n+1}$ for $n \geq 0$.

c) $\underline{U} = \cup \{ U_n \}_{n\geq 0}$.

pf: a) By Prop14b) $U = U_0 \subseteq \underline{U}$

Using induction on n, assume $U_n \subseteq \underline{U}$ Then $U_{n+1} = \cup \{U_n +_Z U \mid z \in F\}$. So it suffices to show that for $z \in F$ $U_n +_Z U \subseteq \underline{U}$ Take x in U_n and y in U. Then x and y are in $\underline{U}$. Then by Prop13c) $x +_z y$ is in $\underline{U}$. Hence $U_n +_Z U \subseteq \underline{U}$ and $U_{n+1} \subseteq \underline{U}$. This completes the induction proving a).

b) $U_{n+1} = \cup \{U_n +_Z U \mid z \in F\}$. Take w in U_n . By Prop14e) and Prop15a), we have u and v in U such that $u \leq_I w$ and $v \leq_T w$. Hence $w +_v v = w$ so w is in U_{n+1} . Hence $U_n \subseteq U_{n+1}$.

c) Let $V = \cup \{ U_n \}_{n\geq 0}$. By b) $U \subseteq V \subseteq F$. By a) $V \subseteq \underline{U}$. By Prop14, to show $V = \underline{U}$ it suffices to show that $p,q \in V \Rightarrow (p+_z q) \in V$ for any z in F.