Geometric Properties of the Dot and Cross Products

Let O be a point in Space, the "Origin". Then we may choose a unit of length and a primitive direction by choosing a point U such that the distance from O to U is one unit of length. We may define the **unit** vector $u = OU$ to be a unit magnitude associated with the direction from O to U. $(-u) = (-1)u$ is a unit magnitude in the opposite direction. More generally, for any real number r, the vector ru has magnitude $|r|$ and direction that of u or $(-u)$ according as r is positive or negative. $0u = 0$, the zero vector.

The points O and U determine a line with its two senses of direction, but also determine a plane through O, perpendicular to OU, which has two senses of rotation about the axis OU, specifically anti-clockwise and clockwise as observed by looking from U towards O.

A vector $u' = OU'$ is a unit vector if and only if U' is a point on the unit sphere, i.e., the sphere of radius one unit. Unit vectors u and u' are orthogonal if and only if $\angle UOU'$ is a right angle.

More generally, the dot product of a vector with a zero vector is the zero scalar 0.
Let $0 \neq v = OV$ and $0 \neq v' = OV'$.

If O, V and V' are collinear and V and V' are on the same side of O, then $v \bullet v' = |v||v'|$, while if V and V' are on opposite sides of O, then $v \bullet v' = -|v||v'|$.
Otherwise, $v \bullet v' = |v||v'| \cos \theta$, where $\theta = \angle VOV'$, $0 < \theta < \pi$.

It follows from Euclid I, proposition 4, that the dot product is invariant under orthogonal transformations and in fact invariance of the dot product characterises orthogonal transformations among the linear automorphisms of space.

$v \bullet v' = |v||v'| \cos \theta = [|v + v'|^2 - |v|^2 - |v'|^2]/2 = [|v|^2 + | v'|^2 - |v - v'|^2]/2$ follows from the Law of Cosines. Hence the vanishing of the dot product characterises orthogonality of directions.

If v and v' are not collinear, then they determine a parallelogram with adjacent edges OV and OV', which in turn determine a plane area and a sense of rotation of that plane. Such a plane area and sense of rotation is a bivector. The magnitude of the bivector is the area of the parallelogram i.e., $|v||v'| \sin \theta$ (positive because $0 < \theta < \pi$). We represent the sense of rotation by a unit vector perpendicular to the plane.

The (right-handed) cross product of v and v' is $v \times v' = (|v||v'| \sin \theta)n$, where n is the unit vector, perpendicular to the plane of v and v', such that if the thumb and the forefinger of the right hand point in the directions of v and v', respectively, then the vector n points to the side of the hand faced by the palm. The cross product is a vector which represents the bivector. Its length is the area of the bivector and its direction is the perpendicular to its plane determined by its sense of rotation.

We see that the corresponding left-handed cross product is given by is $v' \times v = -(v \times v')$. It is conventional that $v \times v'$ is chosen to be the right cross product and $v' \times v$ to be the left cross product. A completely symmetric theory of vector algebra is obtained by choosing them to be the other way round.

Geometric Properties of the Dot and Cross Products

Let u be a given unit vector OU and let v_1 and v_2 be any pair of vectors.
Then $u\bullet(v_1+v_2) = u\bullet v_1 + u\bullet v_2$ and $u\times(v_1+v_2) = u\times v_1 + u\times v_2$
$u\bullet(v_1+v_2)$ is the signed magnitude of the projection of (v_1+v_2) in the direction of u, which is the sum of the signed magnitudes of the projections of v_1 and v_2 in the direction of u, which is the sum $u\bullet v_1 + u\bullet v_2$.

The map $v \mapsto (v - (u\bullet v)u)$ is the projection of v onto the plane perpendicular to u.
The map $(v - (u\bullet v)u) \mapsto u\times(v - (u\bullet v)u) = u\times v$, rotates that projection through a right angle, in an anti-clockwise sense when looking along the axis *UO.*
Both of these maps are linear hence so is their composition $v \mapsto u\times v$.

More generally, the dot and cross products are linear in both arguments.

Let $<u_1, u_2, u_3>$ be an ordered orthonormal basis of space.
Then $u_1\bullet u_1 = u_2\bullet u_2 = u_3\bullet u_3 = 1$ and $u_2\bullet u_3 = u_3\bullet u_1 = 0$
Also $u_1\times u_1 = u_2\times u_2 = u_3\times u_3 = 1$
$u_2\times u_3 = u_1$ if and only if $u_3\times u_1 = u_2$ if and only if $u_1\times u_2 = u_3$ if and only if $<u_1, u_2, u_3>$ is right-handed.
$u_2\times u_3 = -u_1$ if and only if $u_3\times u_1 = -u_2$ if and only if $u_1\times u_2 = -u_3$ if and only if $<u_1, u_2, u_3>$ is left-handed.

Let $v_1 = x_1u_1 + y_1u_2 + z_1u_3$, $v_2 = x_2u_1 + y_2u_2 + z_2u_3$ and $v_3 = x_3u_1 + y_3u_2 + z_3u_3$.
Then $|v_1|^2 = x_1^2 + y_1^2 + z_1^2$ and $v_1\bullet v_2 = x_1x_2 + y_1y_2 + z_1z_2$
$v_1\times v_2 = \pm[(y_1z_2 - z_1y_2)u_1 + (z_1x_2 - x_1z_2)u_2 + (x_1y_2 - y_1x_2)u_3]$, where the positive (resp. negative) sign corresponds to right-handedness (resp. left-handedness).
Assuming that $<u_1, u_2, u_3>$ is right-handed:
$v_1\bullet(v_2\times v_3) = x_1(y_2z_3 - z_2y_3) + y_1(z_2x_3 - x_2z_3) + z_1(x_2y_3 - y_2x_3) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$ and is the signed volume of the parallelepiped with edges v_1, v_2 and v_3. The sign is positive or negative according as the handedness of the ordered basis $<v_1, v_2, v_3>$ is the same as or is opposite to the handedness of the ordered orthonormal basis $<u_1, u_2, u_3>$.