Coordinate-free Quaternions

1) Vectors and Linear Spaces

Our model of the physical Space, which underlies Classical Physics, is that described by the notions and theorems of Euclidean Geometry. A **vector** is the magnitude and direction of an oriented line-segment. If the line-segment joins points P and Q, then we write $v = PQ$ for the vector whose magnitude is the length of the line-segment and whose direction is that of an arrow pointing from P towards Q. We immediately obtain $(-v) = QP$, for the vector which has the same magnitude as that of v, but whose direction is opposite to that of v. For any point P, we write $0 = PP$, for the **zero vector** "0", which has zero magnitude and undefined direction.

We use the term **scalar** as a synonym for real number. The **zero scalar** "0" is distinguished from the zero vector "0" by context. The structure consisting of the set of scalars together with the constants 0 and 1 and the operations of addition and multiplication is denoted **R**.

If v is a non-zero vector and r is a positive scalar, then rv is the vector whose magnitude is that of v multiplied by the magnitude $|r| = r$ of r and whose direction is that of v; if r is a negative scalar, then rv is the vector whose magnitude is that of v multiplied by the magnitude $|r| = -r$ of r and whose direction is opposite to that of v; if $r = 0$, then rv is the zero vector. If v is the zero vector, then so is rv for any scalar r. Further, for any scalar r, $r(-v) = (-r)v = (-(rv))$; in particular $(-1)v = (-v)$. The map $v \mapsto rv$ is **scalar multiplication**.

Vectors (including the zero vector) are translatable. I.e. if $v = PP'$ for some ordered pair of (not necessarily distinct) points $<P, P'>$, then for any point Q, there exists a point Q' such that $v = QQ'$ and, similarly, for any point R', there exists a point R such that $v = RR'$. If $v = PP' = QQ'$ and $w = PQ$, then also $w = P'Q'$ because translations preserve relative displacements.

Let v_1 and v_2 be a pair of vectors. Then there are points P, Q and R such that $v_1 = PQ$ and $v_2 = QR$. We write $(v_1 + v_2) = PR$. This is called the "Triangle Law of Addition" (even though P, Q and R may be collinear). The expression $(v_1 + v_2)$ is well defined, for if $v_1 = P'Q'$ and $v_2 = Q'R'$, then $PP' = QQ'$ and $QQ' = RR'$, whence it follows from $PP' = RR'$, that $PR = P'R'$. Let $v_1 = PQ$ and $v_2 = PS$. Then $(v_1 + v_2) = PR$, where $PS = QR$, whence also $PQ = SR$. This is called the "Parallelogram Law of Addition" (even though P, Q and S may be collinear). The map $<v_1, v_2> \mapsto (v_1 + v_2)$ is **vector addition.** We write $(v_1 - v_2)$ for $(v_1 + (-v_2))$

A **linear structure** consists of a set **S**, which contains **the zero element** 0 and on which are defined the **linear operations** of **scalar multiplication**, $s \mapsto rs$, for every scalar r, **negation**, $s \mapsto (-s) = (-1)s$ and **addition** $<s_1, s_2> \mapsto (s_1 + s_2)$. $(s_1 - s_2)$ is $(s_1 + (-s_2))$.

A linear structure **S** is a **linear space** if it satisfies the following axioms:

L1) *For every scalar r,* $r0 = 0$ *and* $(0 + 0) = 0$.

L2) *For each element s of* **S**, $0s = 0$, $1s = s$ *and for any scalars* r_1 *and* r_2,
$(r_1s + r_2s) = (r_1 + r_2)s = (r_2 + r_1)s = (r_2s + r_1s)$ *and* $r_1(r_2s) = (r_1r_2)s = (r_2r_1)s = r_2(r_1s)$.

L3) *For each ordered triple* $<s_1, s_2, s_3>$ *of elements of* **S**, $((s_1 + s_2) + s_3) = (s_1 + (s_2 + s_3))$*;*
i.e. addition is associative on **S**.

L4) *For each ordered pair* $<s_1, s_2>$ *of elements of* **S** *and every scalar r,*
(i) $(s_1 + s_2) = (s_2 + s_1)$*; i.e. addition is commutative on* **S** *and*
(ii) $r(s_1 + s_2) = (rs_1 + rs_2)$*; i.e. scalar multiplication is distributive over addition on* **S**.

THEOREM 1.1
1) The vectors of Euclidean Geometry form a linear space under scalar multiplication and vector addition.
2) The scalars form a linear space under real-number multiplication and addition
PROOF
1), L1) and L2) are straightforward.
For L3) we apply the "triangle law"; let $v_1 = P_0P_1$, $v_2 = P_1P_2$ and $v_3 = P_2P_3$:
$((v_1 + v_2) + v_3) = (P_0P_2 + P_2P_3) = P_0P_3 = (P_0P_1 + P_1P_3) = (v_1 + (v_2 + v_3))$.
For L4)(i) we apply the "parallelogram law"; let $v_2 = P_1P_2 = P_0P_4$; then $v_1 = P_0P_1 = P_4P_2$:
$(v_1 + v_2) = (P_0P_1 + P_1P_2) = P_0P_2 = (P_0P_4 + P_4P_2) = (v_2 + v_1)$
If v_1 and v_2 have the same or opposite direction, then L4(ii) follows from L2). Otherwise L4(ii) follows from the theory of similar triangles (Euclid's Elements book 6).
2) follows from the so-called "laws of Arithmetic". ||

We denote the **linear space of vectors** of Euclidean Geometry by "**E**". We may represent the points of physical Space by elements of **E**. Specifically, we choose an arbitrary, but fixed, point O, the **Origin**, and then represent each point P, by the vector OP. We identify vectors with the points they represent, or not, as convenient. For example, the zero vector 0, regarded as a point of physical Space, is the Origin.

A **subspace** of a linear space **S** is a linear substructure of **S**, i.e. a subset that is closed under the operations of **S**. **S**, itself, and the **zero space O**, whose only element is the zero element, are **improper** subspaces; all other subspaces are **proper** subspaces. The proper subspaces of the linear space **E** of points of physical Space are the lines through the Origin and the planes through the Origin.

THEOREM 1.2
A subset **T** *of a linear space* **S** *is a subspace of* **S** *if and only if for every pair t_1 and t_2 of elements of* **T** *and scalars r_1 and r_2, the element $(r_1t_1 + r_2t_2)$ of* **S** *is in* **T**.
PROOF
$\Rightarrow$ **T** is closed under the operation of **S**.
$\Leftarrow$ Setting $t_2 = 0$ shows that **T** is closed under scalar multiplication; setting $r_1 = r_2 = 1$ shows that **T** is closed under addition. ||

If $\mathbf{T}_1$ and $\mathbf{T}_2$ are subspaces of a linear space **S**, then so are their **intersection** $\mathbf{T}_1 \cap \mathbf{T}_2$ and their **sum** $\mathbf{T}_1 + \mathbf{T}_2$, whose elements are sums $(t_1 + t_2)$, t_1 in $\mathbf{T}_1$ and t_2 in $\mathbf{T}_2$. If $\mathbf{T}_1 \cap \mathbf{T}_2 = \mathbf{O}$, then the sum $\mathbf{T}_1 + \mathbf{T}_2$ is **direct** and is denoted $\mathbf{T}_1 \oplus \mathbf{T}_2$.

THEOREM 1.3
Let $\mathbf{T}_1$ *and* $\mathbf{T}_2$ *be subspaces of a linear space* **S**. *Then their intersection* $\mathbf{T}_1 \cap \mathbf{T}_2$ *and their sum* $\mathbf{T}_1 + \mathbf{T}_2$ *are well defined as subspaces of* **S**. *If* $\mathbf{T}_1 \cap \mathbf{T}_2 = \mathbf{O}$, *then every element of* $\mathbf{T}_1 \oplus \mathbf{T}_2$ *has a unique decomposition* $(t_1 + t_2)$, t_1 *in* $\mathbf{T}_1$ *and* t_2 *in* $\mathbf{T}_2$.
PROOF
$\mathbf{T}_1$ and $\mathbf{T}_2$ are both closed under the operations of **S**, whence so is their intersection.
Let $(t_1 + t_2)$ and $(t'_1 + t'_2)$ be elements of $\mathbf{T}_1 + \mathbf{T}_2$ and let r be a scalar.
Then $r(t_1 + t_2) = (rt_1 + rt_2)$ is in $\mathbf{T}_1 + \mathbf{T}_2$ by axiom L4)(ii) and
$((t_1 + t_2) + (t'_1 + t'_2)) = ((t_1 + t'_1) + (t_2 + t'_2))$ is in $\mathbf{T}_1 + \mathbf{T}_2$ by axioms L3) and L4)(i).
If $(t_1 + t_2) = (t'_1 + t'_2)$, then $(t_1 - t'_1) = (-(t_2 - t'_2))$ is in $\mathbf{T}_1 \cap \mathbf{T}_2$.
if $\mathbf{T}_1 \cap \mathbf{T}_2 = \mathbf{O}$, then $(t_1 - t'_1) = (t_2 - t'_2) = 0$, whence $t_1 = t'_1$ and $t_2 = t'_2$. ||

If T_0 is a set of elements of a linear space **S**, then a **linear combination** of the elements of T_0 is an element $(r_1t_1 + \ldots + r_kt_k)$ of **S**, where $t_1, \ldots, t_k$ are elements of T_0 and $r_1, \ldots, r_k$ are scalars. (Internal brackets are removed unambiguously from the compound sum $(r_1t_1 + \ldots + r_kt_k)$ by applications of associativity, i.e. by using axiom L3)). T_0 is **linearly independent** if no element t of T_0 is a linear combination of the elements of $T_0\backslash\{t\}$; equivalently, T_0 is **linearly independent** if $(r_1t_1 + \ldots + r_kt_k) = 0$ implies that $r_1 = \ldots = r_k = 0$ for any linear combination $(r_1t_1 + \ldots + r_kt_k)$ of the elements of T_0; otherwise, T_0 is **linearly dependent**. The empty set $\varnothing$ is vacuously linearly independent.

The set of all linear combinations of elements of a subset T_0 of a linear space **S** is a subspace **T** of **S**. Indeed, **T** is the intersection of all the subspaces of **S** containing T_0. T_0 is a **spanning set** for **T**. A subspace **T** of a linear space **S** is **finite-dimensional** if **T** has a finite spanning set.

LEMMA 1.4
Let T_0 be a finite spanning set for a finite-dimensional subspace **T** *of a linear space* **S**. *Then either T_0 is linearly independent or T_0 has a proper subset which is a spanning set for* **T**.
PROOF
If T_0 is not linearly independent, then it contains an element t, which is a linear combination l_t, say, of the elements of $T_0\backslash\{t\}$. t may be replaced in any linear combination of elements of T_0 by l_t, whence the proper subset $T_0\backslash\{t\}$ of T_0 is a spanning set for **T**. ||

A spanning set for a subspace **T** of a linear space **S** is a **basis** of **T** if it is linearly independent. The empty set $\varnothing$ is defined to be a basis for the zero subspace **O** of **S**.

THEOREM 1.5
Let **T** *be a finite-dimensional subspace of a linear space* **S**. *Then* **T** *has a basis.*
PROOF
T has a finite spanning set T_0 by definition. If T_0 is not a basis of **T** then T_0 has a proper subset, which is a basis of **T** by iteratively applying lemma 1.4 to remove elements of T_0, one at a time, while retaining the spanning condition. ||

Let **T** be a finite-dimensional subspace of a linear space **S**. Then an **ordered basis** of **T** consists of the elements of a basis of **T** arranged in a sequence.

THEOREM 1.6 (Steinitz "Exchange" Lemma)
Let **T** *be a finite-dimensional subspace of a linear space* **S**, *let $<s_1, \ldots, s_n>$ be an ordered basis of* **T** *and let T_0 be a spanning set for* **T**. *Then there is a sequence $<t_1, \ldots, t_n>$ of elements of T_0 and a sequence $B_0, \ldots, B_n$ of ordered bases of* **T** *such that $B_0 = <s_1, \ldots, s_n>$, $B_n = <t_1, \ldots, t_n>$ and for $0 < i < n$, $B_i = <t_1, \ldots, t_i, s_{i+1}, \ldots, s_n>$. That is to say that the elements of a basis may be replaced, one at a time, by the elements of a spanning set.*
PROOF
By induction:
We are given that B_0 is an ordered basis of **T**.
Suppose that $B_k = <t_1, \ldots, t_k, s_{k+1}, \ldots, s_n>$ for some $k < n$, is an ordered basis of **T**.
Then any element of $T_0\backslash\{t_1, \ldots, t_k\}$ has an expression $(r_1t_1 + \ldots + r_kt_k) + (r_{k+1}s_{k+1} + \ldots + r_ns_n)$ for some scalars $r_1, \ldots, r_n$.
(The term $(r_1t_1 + \ldots + r_kt_k)$ is replaced by 0 when $k = 0$).

Further, for at least one such element, $r_{k+1} \neq 0$, for otherwise s_{k+1} could not be expressed as a linear combination of elements of T_0. Let t_{k+1} be chosen to be such an element.
Then $s_{k+1} = (1/r_{k+1})(t_{k+1} - (r_1t_1 + \ldots + r_kt_k) - (r_{k+2}s_{k+2} + \ldots + r_ns_n))$.
(The term $(r_{k+2}s_{k+2} + \ldots + r_ns_n)$ is replaced by 0 when $k = (n - 1)$).

Any linear combination of elements of B_k may be replaced by a linear combination of elements of B_{k+1} by substituting $(1/r_{k+1})(t_{k+1} - (r_1t_1 + \ldots + r_kt_k) - (r_{k+2}s_{k+2} + \ldots + r_ns_n))$ for s_{k+1}, wherever s_{k+1} occurs, whence B_{k+1} spans **T** because B_k spans **T**.

Assume that B_{k+1} is linearly dependent. Then there is a linear combination of elements of B_{k+1}, $(x_1t_1 + \ldots + x_kt_k) + x_{k+1}t_{k+1} + (x_{k+2}s_{k+2} + \ldots + x_ns_n) = 0$, where the x_i for i from 1 to n are not all zero. In particular, $x_{k+1} \neq 0$, because $B_{k+1}\backslash\{t_{k+1}\} = B_k\backslash\{s_{k+1}\}$ is linearly independent. Hence t_{k+1} is a linear combination of elements of $B_{k+1}\backslash\{t_{k+1}\} = B_k\backslash\{s_{k+1}\}$, whence so is s_{k+1}, because $(t_{k+1} - r_{k+1}s_{k+1})$ is also a linear combination of elements of $B_k\backslash\{s_{k+1}\}$. But this contradicts the induction hypothesis that B_k is a basis of **T** and is, in particular, linearly independent. Hence B_{k+1} is linearly independent. ||

COROLLARY 1.7
Let **T** *be a finite-dimensional subspace of a linear space* **S**. *Then every basis of* **T** *has the same number of elements.*
PROOF
Let B and B' be bases of **T**.
Every element of the basis B may be replaced by an element of B' because B' is a spanning set for **T**, whence B' contains at least as many elements as B.
Every element of the basis B' may be replaced by an element of B because B is a spanning set for **T**, whence B contains at least as many elements as B'. ||

Let **T** be a finite-dimensional subspace of a linear space **S**. Then the **dimension** of **T**, dim(**T**) is the number of elements in a basis of **T**. All the linear spaces discussed in this paper are finite-dimensional and from now on we define the term **linear space** to mean finite-dimensional linear space.

The zero linear space **O** whose basis is $\varnothing$ has dimension zero. The linear space **E** of the points of physical Space has dimension three. Planes through the Origin have dimension two and lines through the Origin have dimension one. (Notwithstanding the original geometric significance of the word "dimension", it just means here the number of elements in a basis of a linear space – no more and no less). The linear space **R** of scalars has dimension one; **R** has a **canonical ordered basis** <1>. We may identify the "linear space of scalars" with the "scalars as the real numbers", or not, as convenient. Linear spaces of dimension n exist for all positive integers n:

THEOREM 1.8
The linear structure $\mathbf{R}^n$ *of ordered n-tuples of scalars is an n-dimensional linear space whose operations are given by:*
$0 = (0, \ldots, 0)$,
$r(x_1, \ldots, x_n) = (rx_1, \ldots, rx_n)$ *for all scalars r and n-tuples* $(x_1, \ldots, x_n)$,
$((x_1, \ldots, x_n) + (x'_1, \ldots, x'_n)) = ((x_1 + x'_1), \ldots, (x_n + x'_n))$ *for all n-tuples* $(x_1, \ldots, x_n)$ *and* $(x'_1, \ldots, x'_n)$.
$\mathbf{R}^n$ *has an ordered basis* $<\varepsilon_1, \ldots, \varepsilon_n>$, *where for k from* 1 *to n*, ε_k *is the ordered n-tuple, whose* k^{th} *term is* 1 *and whose other terms are all zero.* ||

The **canonical ordered basis** of the linear space $\mathbf{R}^n$ is $<\varepsilon_1, \ldots, \varepsilon_n>$, where for k from 1 to n, ε_k is the ordered n-tuple, whose k^{th} term is 1 and whose other terms are all zero. Thus, $\mathbf{R} = \mathbf{R}^1$ has canonical ordered basis <1>, $\mathbf{R}^2$ has canonical ordered basis <(1, 0), (0, 1)>, $\mathbf{R}^3$ has canonical ordered basis <(1, 0, 0), (0, 1, 0), (0, 0, 1)> etc.

Let $\mathbf{S}$ and $\mathbf{S}'$ be linear spaces. Then a map $f : \mathbf{S} \to \mathbf{S}'$ is a **linear homomorphism** if it respects the linear operations i.e. $(rs)f = r(sf)$ and $(s_1 + s_2)f = s_1f + s_2f$ for r in $\mathbf{R}$ and for s, s_1 and s_2 in $\mathbf{S}$. Equivalently, f is a linear homomorphism if $(r_1s_1 + r_2s_2)f = r_1(s_1f) + r_2(s_2f)$ for r_1 and r_2 in $\mathbf{R}$ and for s_1 and s_2 in $\mathbf{S}$. We say that f maps the sequence $<s_1, \ldots, s_n>$ of elements of $\mathbf{S}$ to the sequence $<s'_1, \ldots, s'_n>$ of elements of $\mathbf{S}'$, if $s'_k = s_kf$ for k from 1 to n. A linear homomorphism is completely determined by its action on a basis. Specifically, let $<s_1, \ldots, s_n>$ be an ordered basis of $\mathbf{S}$ and let its image under f be $<s'_1, \ldots, s'_n>$. Then any element of $\mathbf{S}$ may be expressed in the form $s = (r_1s_1 + \ldots + r_ns_n)$ for some scalars $r_1, \ldots, r_n$ and then $sf = (r_1s'_1 + \ldots + r_ns'_n)$. The linear homomorphism f is a **linear isomorphism** if it is invertible as a map, i.e. if it is 1-1 and onto. The identity map is a linear isomorphism.

THEOREM 1.9
Let $\mathbf{S}$, $\mathbf{S}'$ *and* $\mathbf{S}''$ *be linear spaces and let* $f : \mathbf{S} \to \mathbf{S}'$ *and* $g : \mathbf{S}' \to \mathbf{S}''$ *be linear homomorphisms. Then:*
1)The composition $fg : \mathbf{S} \to \mathbf{S}''$, $s \mapsto (sf)g$ *is a linear homomorphism.*
2) If $f : \mathbf{S} \to \mathbf{S}'$ *is a linear isomorphism, then its inverse* $f^{-1}: \mathbf{S}' \to \mathbf{S}$ *is a linear isomorphism.*
3) The following conditions are equivalent:

(i) f *is a linear isomorphism.*
(ii) $\dim(\mathbf{S}) = \dim(\mathbf{S}')$ *and* f *maps every ordered basis of* $\mathbf{S}$ *to an ordered basis of* $\mathbf{S}'$.
(iii) $\dim(\mathbf{S}) = \dim(\mathbf{S}')$ *and* f *maps some ordered basis of* $\mathbf{S}$ *to an ordered basis of* $\mathbf{S}'$.

PROOF
1) $(rs)(fg) = ((rs)f)g$ — definition of fg
$= (r(sf))g$ — f is a linear homomorphism
$= r((sf)g)$ — g is a linear homomorphism
$= r(s(fg))$ — definition of fg

$(s_1 + s_2)(fg) = ((s_1 + s_2)f)g$ — definition of fg
$= (s_1f + s_2f)g$ — f is a linear homomorphism
$= ((s_1f)g + (s_2f)g)$ — g is a linear homomorphism
$= (s_1(fg)+ s_2(fg))$ — definition of fg

2) It suffices to prove that f^{-1} is a linear homomorphism.
Let s', s'_1 and s'_2 be elements of $\mathbf{S}'$, such that $s' = sf$, $s'_1 = s_1f$ and $s'_2 = s_2f$ for some elements s, s_1 and s_2 of $\mathbf{S}$ and let r be a scalar.
Then $(rs')f^{-1} = (r(sf))f^{-1} = ((rs)f)f^{-1}$ — f is a linear homomorphism
$= (rs)(ff^{-1})$
$= rs = r(s'f^{-1})$

$(s'_1 + s'_2)f^{-1} = (s_1f + s_2f)f^{-1} = ((s_1 + s_2)f)f^{-1}$ — f is a linear homomorphism
$= (s_1 + s_2)(ff^{-1})$
$= (s_1 + s_2) = (s'_1f^{-1} + s'_2f^{-1})$

3) (i) $\Rightarrow$ (ii) Let $<s_1, \ldots, s_n>$ be an ordered basis of **S** and let $<s'_1, \ldots, s'_n>$ be its image under f. If $\{s'_1, \ldots, s'_n\}$ is linearly dependent, then f is not 1-1 because some non-zero element of **S** is mapped by f to the zero element of **S′** along with the zero element of **S**. If $\{s'_1, \ldots, s'_n\}$ does not span **S′**, then f is not onto.
(ii) $\Rightarrow$ (iii) Since **S** has an ordered basis the general case implies the particular case.
(iii) $\Rightarrow$ (i) Let $<s_1, \ldots, s_n>$ be an ordered basis of **S** and let its image under f, $<s'_1, \ldots, s'_n>$, be an ordered basis of **S′**. Then f is invertible and f^{-1} is in fact the linear homomorphism which maps $<s'_1, \ldots, s'_n>$ to $<s_1, \ldots, s_n>$. ||

Let **S** be an n-dimensional linear space and let $<s_1, \ldots, s_n>$ be an ordered basis of **S**. Then a **coordinate map** is a linear isomorphism from **S** to $\mathbf{R}^n$, which maps s_i to ε_i for i from 1 to n, where $<\varepsilon_1, \ldots, \varepsilon_n>$ is the canonical ordered basis of $\mathbf{R}^n$. The image of an element s in **S**, where $s = r_1s_1 + \ldots + r_ns_n$ is the n-tuple comprising the **coordinates** $(r_1, \ldots, r_n)$ of s. Thus, elements of **S** may be labelled by their coordinates. However, the labelling depends on choosing a *fixed* ordered basis of **S**. A property of linear spaces is **coordinate-free** if it does not depend on any specific coordinate map.

2) Linear Endomorphisms and Algebras

Let **S** be a linear space. Then a **linear endomorphism** of **S** is a linear homomorphism from **S** to itself. A **linear automorphism** of **S** is an invertible linear endomorphism, equivalently a **linear automorphism** of **S** is a linear isomorphism from **S** to itself. We write **gl**(**S**) for the set of all linear endomorphisms of **S** and we write GL(**S**) for the set of all linear automorphisms of **S**. (This terminology is borrowed from Lie theory).

THEOREM 2.1
*The set GL(**S**) of all linear automorphisms of a linear space **S** is a group under the operation of composition of functions.*
PROOF
The identity map is a linear automorphism. By theorem 1.9 1) and 2), GL(**S**) is closed under composition of maps and taking inverses. Hence GL(**S**) is a subgroup of the group of all invertible maps from the set **S** to itself. ||

LEMMA 2.2
*If **S** is an n-dimensional linear space, then **gl**(**S**) may be given the structure of an n^2-dimensional linear space. The zero element is the zero map $0 : s \mapsto 0$; if f is an element of **gl**(**S**) and r is a scalar, then rf is the map $rf : s \mapsto r(sf)$ and if f_1 and f_2 are elements of **gl**(**S**), then $(f_1 + f_2)$ is the map $(f_1 + f_2) : s \mapsto (sf_1 + sf_2)$.*
PROOF
We must show (i) that $0, f \mapsto rf$ and $<f_1, f_2> \mapsto (f_1 + f_2)$ are well defined as the zero element, the linear operation of scalar multiplication and the linear operation of addition, respectively, on **gl**(**S**), (ii) that the linear structure **gl**(**S**) satisfies the axioms for a linear space and (iii) that **gl**(**S**) has a basis with n^2 elements.
(i) It is immediate that the zero map is in **gl**(**S**).
Let s_1 and s_2 be elements of **S** and let r_1 and r_2 be scalars.

Then $(r_1s_1 + r_2s_2)(rf)$	$= r((r_1s_1 + r_2s_2)f)$	definition of rf
	$= r(r_1(s_1f) + r_2(s_2f))$	f is a linear endomorphism
	$= r(r_1(s_1f)) + r(r_2(s_2f))$	L4)(ii) for **S**
	$= r_1(r(s_1f)) + r_2(r(s_2f))$	L2) for **S**
	$= r_1(s_1(rf)) + r_2(s_2(rf))$	definition of rf. Hence rf is in **gl**(**S**).

$(r_1s_1 + r_2s_2)(f_1 + f_2) = ((r_1s_1 + r_2s_2)f_1 + (r_1s_1 + r_2s_2)f_2$ definition of $(f_1 + f_2)$
$= ((r_1(s_1f_1) + r_2(s_2f_1)) + (r_1(s_1f_2) + r_2(s_2f_2)))$ f_1 and f_2 are linear endomorphisms
$= (r_1(s_1f_1) + r_1(s_1f_2) + r_2(s_2f_1) + r_2(s_2f_2))$ L3) and L4)(i) for **S**
$= (r_1(s_1f_1 + s_1f_2) + r_2(s_2f_1 + s_2f_2))$ L4)(ii) for **S**
$= (r_1(s_1(f_1 + f_2)) + r_2(s_2(f_1 + f_2)))$ definition of $(f_1 + f_2)$
Hence $(f_1 + f_2)$ is in **gl**(**S**).

(ii) The properties of the zero element (axiomL1)) and the associativity and commutativity of addition (axioms L3) and L4)(i)) are immediate.
Let f be a linear endomorphism and let r_1 and r_2 be scalars. Then for all s in **S**:
$s((r_1 + r_2)f) = (r_1 + r_2)(sf) = r_1(sf) + r_2(sf) = s(r_1f) + s(r_2f) = s(r_1f + r_2f)$ and
$s(r_1(r_2f)) = (r_1s)(r_2f) = (r_2(r_1s))f = ((r_2r_1)s)f = s((r_2r_1)f) = s((r_1r_2)f)$ etc. Hence axiom L2) holds.
Let f_1 and f_2 be linear endomorphisms and let r be a scalar. Then for all s in **S**:
$s(r(f_1 + f_2)) = (rs)(f_1 + f_2) = (rs)f_1 + (rs)f_2 = s(rf_1) + s(rf_2) = s(rf_1 + rf_2)$ Hence axiom L4)(ii) holds.

(iii) Let $<s_1, \ldots, s_n>$ be a basis of **S** and for i from 1 to n and for j from 1 to n let f_{ij} be the function which maps s_k to 0 if $k \neq i$ and maps rs_i to rs_j for any scalar r. It is immediate that the f_{ij} are linear endomorphisms of **S** for all i and j.
Any linear endomorphism f maps each basis element s_i to an element $x_{i1}s_1 + \ldots + x_{in}s_n$, where $x_{i1}, \ldots, x_{in}$ are scalars. $f = \sum_{i=1}^{n}(\sum_{j=1}^{n} x_{ij}f_{ij})$, whence the f_{ij} span **gl**(**S**).
Let $f = \sum_{i=1}^{n}(\sum_{j=1}^{n} x_{ij}f_{ij})$. Then for any s_i in **S**, $s_if_{kj} = 0$ for all j, if $k \neq i$.
Hence for all i from 1 to n, $s_if = s_i(\sum_{j=1}^{n} x_{ij}f_{ij}) = \sum_{j=1}^{n} x_{ij}(s_if_{ij}) = \sum_{j=1}^{n} x_{ij}s_j$.
If $f = 0$, then for all i, $s_if = 0$, whence for all i and for all j, $x_{ij} = 0$, whence the f_{ij} are linearly independent.
The set $\{f_{ij} : i$ from 1 to n and j from 1 to $n\}$ is a basis of **gl**(**S**) containing n^2 elements. ||

LEMMA 2.3
Let **S** *be a linear space. For every element g of* **gl**(**S**) *the* **left multiplication** $L[g] = f \mapsto gf$ *and the* **right multiplication** $R[g] = f \mapsto fg$ *are both linear endomorphisms of* **gl**(**S**) *i.e. they are both elements of* **gl**(**gl**(**S**)).
PROOF
Let f, f_1 and f_2 be elements of **gl**(**S**) and let r be a scalar; let s be an element of **S**.
Then $s((rf)L[g])$ $= s(g(rf))$ definition of $L[g]$
$= (sg)(rf)$ composition of functions
$= r((sg)f)$ definition of rf
$= r(s(gf))$ composition of functions
$= s(r(gf))$ definition of $r(gf)$
$= s(r(fL[g]))$ definition of $L[g]$
Hence $(rf)L[g] = r(fL[g])$ and
$s((f_1 + f_2)L[g])$ $= s(g(f_1 + f_2))$ definition of $L[g]$
$= (sg)(f_1 + f_2)$ composition of functions
$= ((sg)f_1 + (sg)f_2)$ definition of $(f_1 + f_2)$
$= (s(gf_1) + s(gf_2))$ composition of functions
$= s(gf_1 + gf_2)$ definition of $(gf_1 + gf_2)$
$= s(f_1L[g] + f_2L[g])$ definition of $L[g]$
Hence $(f_1 + f_2)L[g] = (f_1L[g] + f_2L[g])$.

$s((rf)R[g])$	$= s((rf)g)$	definition of $R[g]$
	$= (s(rf))g$	composition of functions
	$= (r(sf))g$	definition of rf
	$= ((rs)f)g$	f is a linear endomorphism
	$= (rs)(fg)$	composition of functions
	$= s(r(fg))$	fg is a linear endomorphism
	$= s(r(fR[g]))$	definition of $R[g]$

Hence $(rf)R[g] = r(fR[g])$ and

$s((f_1 + f_2)R[g])$	$= s((f_1 + f_2)g)$	definition of $R[g]$
	$= (s(f_1 + f_2))g$	composition of functions
	$= (sf_1 + sf_2)g$	definition of $(f_1 + f_2)$
	$= (sf_1)g + (sf_2)g$	g is a linear endomorphism
	$= s(f_1g) + s(f_2g)$	composition of functions
	$= s(f_1g + f_2g)$	definition of $(f_1g + f_2g)$
	$= s(f_1R[g] + f_2R[g])$	definition of $R[g]$

Hence $(f_1 + f_2)R[g] = (f_1R[g] + f_2R[g])$. ||

LEMMA 2.4

Let **S** *be a linear space.* **gl**(**S**) *is a monoid under composition, i.e. composition in* **gl**(**S**) *is associative and the identity linear endomorphism is an identity for* **gl**(**S**) *under composition. Further* **gl**(**S**) *contains all the inverses of its invertible elements.*

PROOF

This is a corollary of theorem 1.9. ||

LEMMA 2.5

Let **S** *be a linear space. The set* $\mathbf{R}_m = \{r_m = s \mapsto rs : r \text{ in } \mathbf{R}\}$ *of scalar multiplications is a linear subspace of* **gl**(**S**) *and for all r in* **R**, $L[r_m] = R[r_m]$.

PROOF

By axioms L2) and L4)(ii) for **S**, scalar multiplications are linear endomorphism of **S**.
Let r and r_0 be scalars. Then for all s in **S**

$s(rr_{0m})$	$=$	$r(sr_{0m})$	definition of rr_{0m}
	$=$	$r(r_0s)$	definition of r_{0m}
	$=$	$(rr_0)s$	axiom L2) for **S**
	$=$	$s((rr_0)_m)$	definition of $(rr_0)_m$

Hence $\mathbf{R}_m$ is closed under scalar multiplication in **gl**(**S**).
Let r_1 and r_2 be scalars. Then for all s in **S**

$s(r_{1m} + r_{2m})$	$=$	$(sr_{1m} + sr_{2m})$	definition of $(r_{1m} + r_{2m})$
	$=$	$(r_1s + r_2s)$	definitions of r_{1m} and r_{2m}
	$=$	$(r_1 + r_2)s$	axiom L2) for **S**
	$=$	$s((r_1 + r_2)_m)$,	definition of $(r_1 + r_2)_m$

Hence $\mathbf{R}_m$ is closed under addition in **gl**(**S**).
Let f be an element of **gl**(**S**) and let r be a scalar. Then for all s in **S**

$s(r_mf)$	$= (sr_m)f$	composition of functions
	$= (rs)f$	definition of r_m
	$= r(sf)$	f is a linear homomorphism
	$= (sf)r_m$	definition of r_m
	$= s(fr_m)$.	composition of functions

Hence $L[r_m] = R[r_m]$ in **gl**(**gl**(**S**)). ||

A linear space **A** is an **algebra** if it admits a product, which is a map $\langle a, b\rangle \mapsto ab$, from ordered pairs of elements of **A** to **A**, with the following properties:
(i) The left multiplications $L[a] : x \mapsto ax$ are elements of $\mathbf{gl}(\mathbf{A})$ for all a in **A** and
(ii) The right multiplications $R[b] : x \mapsto xb$ are elements of $\mathbf{gl}(\mathbf{A})$ for all b in **A**.
An element c of an algebra **A** is **central** if $L[c] = R[c]$, i.e. $cx = xc$ for all x in **A**.
An algebra **A** is **associative** if its product is associative, i.e. $(ab)c = a(bc)$ for all a, b, and c in **A**.
An algebra **A** is **unital** if it contains a **product identity**, i.e. an element $1_{\mathbf{A}}$, which satisfies $1_{\mathbf{A}}a = a1_{\mathbf{A}} = a$, for all a in **A**.
An algebra **A** determines its **opposite** algebra $\mathbf{A}^{\mathrm{opp}}$ with the same underlying linear space, but with the reverse product $\langle a, b\rangle \mapsto ba$.
An algebra **A** is **commutative** if its product is commutative, i.e. $ab = ba$ for all a, b in **A**. Equivalently, **A** is **commutative** if $\mathbf{A} = \mathbf{A}^{\mathrm{opp}}$.
A subspace of an algebra **A**, which is closed under the product on **A**, is an algebra and is a **subalgebra** of **A**. A subalgebra of **A** is **central** if its elements are all central. The **centre** of **A** is the largest central subalgebra of **A**.

THEOREM 2.6
Let **S** *be a linear space. Then the linear space* $\mathbf{gl}(\mathbf{S})$ *of linear endomorphisms of* **S** *is an algebra, whose product is composition. The algebra* $\mathbf{gl}(\mathbf{S})$ *is associative and unital. The centre of* $\mathbf{gl}(\mathbf{S})$ *is the set* $\mathbf{R}_m = \{r_m = s \mapsto rs : r \text{ in } \mathbf{R}\}$ *of scalar multiplications.*
PROOF
By lemma 2.2, $\mathbf{gl}(\mathbf{S})$ is well-defined as a linear space. By lemma 2.3, $\mathbf{gl}(\mathbf{S})$ is an algebra and by lemma 2.4, $\mathbf{gl}(\mathbf{S})$ is associative and unital.
We note that 0_m is the zero linear endomorphism and that 1_m is the identity linear automorphism of **S**. By lemma 2.5, $\mathbf{R}_m$ is a linear subspace of $\mathbf{gl}(\mathbf{S})$, whose elements are all central. Let r_{1m} and r_{2m} be elements of $\mathbf{R}_m$. Then for all s in **S**,

$$s(r_{1m}r_{2m}) = (sr_{1m})r_{2m} = r_2(sr_{1m}) = r_2(r_1s) = (r_2r_1)s = s(r_2r_1)_m.$$

Hence $\mathbf{R}_m$ is closed under composition and is a subalgebra of $\mathbf{gl}(\mathbf{S})$.
To conclude the proof, we must show that given any linear endomorphism f of **S**, which is not in $\mathbf{R}_m$, there must be a linear endomorphism g of **S**, such that $fg \neq gf$.
If f is not in $\mathbf{R}_m$, there must be a pair of linearly independent elements s_1 and s_2 of **S**, such that $s_1f = rs_1 + s_2$. Hence the dimension of **S** is necessarily greater than one. We may extend the ordered pair $\langle s_1, s_2\rangle$ to an ordered basis of **S**. We may define g by $s_1g = 0$, $s_2g = s_1$ and g is arbitrary on the rest of the ordered basis. Then $s_1(fg) = s_1 \neq 0 = s_1(gf)$. ||

COROLLARY 2.7
If **S** *is a one-dimensional linear space then* $\mathbf{gl}(\mathbf{S})$ *is the commutative algebra* $\mathbf{R}_m$ *of scalar multiplications. If* **S** *has dimension greater than one, then* $\mathbf{gl}(\mathbf{S})$ *is not commutative.* ||

An element a of an algebra **A** is a left annihilator if $L[a] = 0$, i.e. if $ax = 0$ for all x in **A**.
An element b of an algebra **A** is a right annihilator if $R[a] = 0$, i.e. if $xa = 0$ for all x in **A**.

LEMMA 2.8
Let **A** *be an algebra, let* $L[\mathbf{A}] = \{L[a] : a \text{ in } \mathbf{A}\}$ *and* $R[\mathbf{A}] = \{R[b] : b \text{ in } \mathbf{A}\}$ *be its set of left multiplications and its set of right multiplications respectively. Then the map* $a \mapsto L[a]$ *is onto and it is 1-1 if and only if* **A** *contains no left annihilators and the map* $b \mapsto R[b]$ *is onto and it is 1-1 if and only if* **A** *contains no right annihilators.* ||

THEOREM 2.9

Let $\mathbf{A}$ *be an algebra, let* $L[\mathbf{A}] = \{L[a] : a \text{ in } \mathbf{A}\}$ *and* $R[\mathbf{A}] = \{R[b] : b \text{ in } \mathbf{A}\}$ *be its set of left multiplications and its set of right multiplications respectively. Let* $\mathbf{K}$ *be the set of all central elements of* $\mathbf{A}$ *and let* $K[\mathbf{A}] = \{f \text{ in } \mathbf{gl}(\mathbf{A}) : f = L[c] = R[c]$, where c is a central element of $\mathbf{A}\}$

Then:

1) $L[\mathbf{A}]$ *is a subspace of* $\mathbf{gl}(\mathbf{A})$ *and the map* $a \mapsto L[a]$ *is a linear homomorphism from* $\mathbf{A}$ *onto* $L[\mathbf{A}]$*, which is a linear isomorphism if and only if* $\mathbf{A}$ *contains no non-zero left annihilators.*

2) $R[\mathbf{A}]$ *is a subspace of* $\mathbf{gl}(\mathbf{A})$ *and the map* $b \mapsto R[b]$ *is a linear homomorphism from* $\mathbf{A}$ *onto* $R[\mathbf{A}]$*, which is a linear isomorphism if and only if* $\mathbf{A}$ *contains no non-zero right annihilators.*

3) $K[\mathbf{A}]$ *is a subspace of* $L[\mathbf{A}] \cap R[\mathbf{A}]$ *and the map* $c \mapsto L[c] = R[c]$ *is a linear homomorphism from* $\mathbf{K}$ *onto* $K[\mathbf{A}]$*, which is a linear isomorphism if and only if* $\mathbf{A}$ *contains neither non-zero left annihilators nor non-zero right annihilators. In particular,* $\mathbf{K}$ *is a linear subspace of* $\mathbf{A}$*.*

4) If $\mathbf{A}$ *is unital, then* $\mathbf{A}$ *contains neither non-zero left annihilators nor no non-zero right annihilators and* $K[\mathbf{A}] = L[\mathbf{A}] \cap R[\mathbf{A}]$*.*

5) If $\mathbf{A}$ *is unital, then* $\mathbf{R_A} = \{r_\mathrm{A} = r1_\mathrm{A} = 1_\mathrm{A}r_m : r \text{ in } \mathbf{R}\}$ *is a central unital subalgebra of* $\mathbf{A}$*.*

PROOF

1) Let $L[a]$, $L[a_1]$and $L[a_2]$ be elements of $L[\mathbf{A}]$ and let r be a scalar. Then for all x in $\mathbf{A}$:

$x(rL[a])$	$= r(xL[a])$	definition of $rL[a]$
	$= r(aR[x])$	
	$= (ra)R[x]$	$R[x]$ is a linear endomorphism
	$= xL[ra])$	
$x(L[a_1] + L[a_2])$	$= (xL[a_1] + xL[a_2])$	definition of $(L[a_1] + L[a_2])$
	$= (a_1R[x] + a_2R[x])$	
	$= (a_1 + a_2)R[x]$	$R[x]$ is a linear endomorphism
	$= xL[(a_1 + a_2)]$	

Hence $L[\mathbf{A}]$ is a subspace of $\mathbf{gl}(\mathbf{A})$ and the map $a \mapsto L[a]$ preserves the linear operations. $a \mapsto L[a]$ is onto and if $a_1 \neq a_2$ then $L[a_1] \neq L[a_2]$ or else $(a_1 - a_2)$ is a left annihilator.

2) Let $R[b]$, $R[b_1]$and $R[b_2]$ be elements of $R[\mathbf{A}]$ and let r be a scalar. Then for all x in $\mathbf{A}$:

$x(rR[b])$	$= r(xR[b])$	definition of $rR[b]$
	$= r(bL[x])$	
	$= (rb)L[x]$	$L[x]$ is a linear endomorphism
	$= xR[rb]$	
$x(R[b_1] + R[b_2])$	$= (xR[b_1] + xR[b_2])$	definition of $(R\,[b_1] + R[b_2])$
	$= (b_1L[x] + b_2L[x])$	
	$= (b_1 + b_2)L[x]$	$L[x]$ is a linear endomorphism
	$= xR[(b_1 + b_2)]$	

Hence $R[\mathbf{A}]$ is a subspace of $\mathbf{gl}(\mathbf{A})$ and the map $b \mapsto R[b]$ preserves the linear operations. $b \mapsto R[b]$ is onto and if $b_1 \neq b_2$ then $R[b_1] \neq R[b_2]$ or else $(b_1 - b_2)$ is a right annihilator.||

3) Let c_1 and c_2 be elements of $\mathbf{K}$ and let r_1 and r_2 be scalars. Then:

$$L[(r_1c_1 + r_2c_2)] = (r_1L[c_1] + r_2L[c_2]) = (r_1R[c_1] + r_2R[c_2]) = R[(r_1c_1 + r_2c_2)]$$

4) If $\mathbf{A}$ is unital and $a \neq 0$, then $a1_\mathbf{A} = a \neq 0$ and if $b \neq 0$, then $1_\mathbf{A}b = b \neq 0$.

If $L[a] = R[b]$, then $a = a1_\mathbf{A} = 1_\mathbf{A}L[a] = 1_\mathbf{A}R[b] = b$.

5) Let $r_{0\mathbf{A}}$, $r_{1\mathbf{A}}$ and $r_{2\mathbf{A}}$ be elements of $\mathbf{R_A}$, let r be a scalar and let x be an element of $\mathbf{A}$. Then:

$r(r_{0\mathbf{A}}) = r(r_01_\mathbf{A}) = (rr_0)1_\mathbf{A} = (rr_0)_\mathbf{A}$;

$(r_{1\mathbf{A}} + r_{2\mathbf{A}}) = (r_11_\mathbf{A} + r_21_\mathbf{A}) = (r_1 + r_2)1_\mathbf{A} = (r_1 + r_2)_\mathbf{A}$;

$1_\mathbf{A}$ is in $\mathbf{R_A}$;

$r_{1\mathbf{A}}r_{2\mathbf{A}} = (r_11_\mathbf{A})r_{2\mathbf{A}} = r_1(1_\mathbf{A}r_{2\mathbf{A}}) = r_1(r_{2\mathbf{A}}) = (r_1r_2)_\mathbf{A}$.

Hence $\mathbf{R_A}$ is a unital subalgebra of $\mathbf{A}$. To show that $\mathbf{R_A}$ is central:

$xr_\mathrm{A} = x(r1_\mathbf{A}) = (r1_\mathbf{A})L[x] = r(1_\mathbf{A}L[x]) = rx = r(1_\mathbf{A}R[x]) = (r1_\mathbf{A})R[x] = (r1_\mathbf{A})x = r_\mathbf{A}x$ ||

Let **A** be an algebra; then, in particular, **A** is a linear space. If **A′** is an algebra then a linear homomorphism f from **A** to **A′** is an **algebra homomorphism** if f preserves the product on **A**; that is $(af)(bf) = (ab)f$ for all a and b in **A**. f is an **algebra anti-homomorphism** if it reverses the product on **A**; that is $(af)(bf) = (ba)f$ for all a and b in **A**. Equivalently, f is an **algebra anti-homomorphism** if it is an algebra homomorphism from **A** to $\mathbf{A}'^{\text{opp}}$. The algebra homomorphism (resp. anti-homomorphism) f is an **algebra isomorphism** (resp. **anti-isomorphism**) if it is invertible as a map, i.e. if it is 1-1 and onto. There are analogous definitions for an **algebra endomorphism** (resp. **anti-endomorphism**) and an **algebra automorphism** (resp. **anti-automorphism**) when **A′** is replaced by **A**.

THEOREM 2.10
Let **A** *be an associative algebra. Then the subspaces* $L[\mathbf{A}]$ *and* $R[\mathbf{A}]$ *of left multiplications and right multiplications, respectively are subalgebras of* $\mathbf{gl}(\mathbf{A})$*. The linear homomorphism* $a \mapsto L[a]$ *from* **A** *onto* $L[\mathbf{A}]$ *is an algebra anti-homomorphism and the linear homomorphism* $b \mapsto R[b]$ *from* **A** *onto* $R[\mathbf{A}]$ *is an algebra homomorphism. Every element of* $L[\mathbf{A}]$ *commutes with every element of* $R[\mathbf{A}]$*. The centre of* **A** *consists of all the central elements of* **A**.
PROOF
Let $L[a]$, $L[a_1]$and $L[a_2]$ be elements of $L[\mathbf{A}]$ and let $R[b]$, $R[b_1]$and $R[b_2]$ be elements of $R[\mathbf{A}]$. Then for all x in **A**:

$$xL[a_1a_2] = (a_1a_2)x = a_1(a_2x) = a_1(xL[a_2]) = (xL[a_2])L[a_1] = x(L[a_2]L[a_1])$$
$$xR[b_1b_2] = x(b_1b_2) = (xb_1)b_2 = (xR[b_1])b_2 = (xR[b_1])R[b_2] = x(R[b_1]R[b_2])$$
$$x(L[a]R[b]) = (xL[a])R[b] = (ax)b = a(xb) = (xR[b])L[a] = x(R[b]L[a])$$

Every element of the centre of **A** must be central.
We have already shown (previous theorem) that the set of all central elements **K** of **A** is a linear subspace of **A.** It remains to show that **K** is closed under the algebra product.
Let c_1 and c_2 be central elements of **A** and let x be an element of **A.**
Then $xL[c_1c_2] = x(L[c_2]L[c_1]) = x(R[c_2]L[c_1]) = x(L[c_1]R[c_2]) = x(R[c_1]R[c_2]) = xR[c_1c_2]$ ||

THEOREM 2.11
Let **A** *be an associative and unital algebra.* $\mathbf{A}^{\text{opp}}$ *is an associative and unital algebra with the same underlying linear space, whence, in particular* $\mathbf{gl}(\mathbf{A}) = \mathbf{gl}(\mathbf{A}^{\text{opp}})$*. The subspaces* $L[\mathbf{A}]$ *and* $R[\mathbf{A}]$ *of left multiplications and right multiplications, respectively, are subalgebras of the algebra* $\mathbf{gl}(\mathbf{A})$*, which are algebra anti-isomorphic and algebra isomorphic, respectively, to* **A***.* $L[\mathbf{A}^{\text{opp}}] = R[\mathbf{A}]$ *and* $R[\mathbf{A}^{\text{opp}}] = L[\mathbf{A}]$*. If* **A** *is commutative, then* $L[\mathbf{A}] = R[\mathbf{A}]$*.* $L[\mathbf{A}]$ *and* $R[\mathbf{A}]$ *are mutually commuting, i.e. every element of* $L[\mathbf{A}]$ *commutes with every element of* $R[\mathbf{A}]$*. The centre of* **A** *is algebra isomorphic to the subalgebra* $L[\mathbf{A}]\cap R[\mathbf{A}]$ *of* $\mathbf{gl}(\mathbf{A})$*. The centre of* **A** *has a subalgebra* $\mathbf{R}_\mathbf{A}$ *that is algebra isomorphic to the real numbers.* ||

THEOREM 2.12
Let **A** *be an associative and unital algebra. Then the map* $R[a] \mapsto L[a]$ *is an algebra anti-isomorphism from* $R[\mathbf{A}]$ *to* $L[\mathbf{A}]$*. This map induces an algebra anti-automorphism on* **A**.
PROOF
$\mathbf{A} = 1_\mathbf{A}R[\mathbf{A}] = \{1_\mathbf{A}R[a] : a \text{ in A}\}$ and $\mathbf{A}^{\text{opp}} = 1_\mathbf{A}L[\mathbf{A}] = \{1_\mathbf{A}L[a] : a \text{ in A}\}$ ||

An algebra **A** is a **division algebra** if for all pairs of non-zero elements a and b, the equations $ax = b$ and $ya = b$ have unique solutions x and y. If **A** is a unital algebra, then an element c of **A** is **invertible** if **A** contains an element c^{-1}, satisfying $cc^{-1} = c^{-1}c = 1_\mathbf{A}$

THEOREM 2.13
Let **A** *be an associative and unital algebra. Then* **A** *is a division algebra if and only if every non-zero element of* **A** *is invertible.*
PROOF
$\Rightarrow$ Let c be a non-zero element of **A**. Then there exists a unique z in **A** such that $cz = 1_{\mathbf{A}}$, whence also $c = 1_{\mathbf{A}}c = (cz)c = c(zc)$ and $zc = 1_{\mathbf{A}}$. Hence c is invertible with $c^{-1} = z$.
$\Leftarrow$ Existence of solutions: $a(a^{-1}b) = (aa^{-1})b = 1_{\mathbf{A}}b = b$ and $(ba^{-1})a = b(a^{-1}a) = b1_{\mathbf{A}} = b$.
Uniqueness of x:
If $ax_1 = b$ and $ax_2 = b$, then $ax_1 - ax_2 = 0$; $a(x_1 - x_2) = 0$; $a^{-1}(a(x_1 - x_2) = 0$;
$(a^{-1}a)(x_1 - x_2) = 0$; $1_{\mathbf{A}}(x_1 - x_2) = 0$; $(x_1 - x_2) = 0$; $x_1 = x_2$.
Uniqueness of y is proved similarly. ||

THEOREM 2.14
Let **D** *be a linear space and let F and G be subsets of* **gl**(**D**) *such that:*
1) F contains the zero endomorphism 0 *and is transitive on the non-zero elements of* **D***; i.e. for all non-zero d in* **D** *and for all d′ in* **D***, there is an element f of F with df = d′.*
2) G contains the zero endomorphism 0 *and is transitive on the non-zero elements of* **D***; i.e. for all non-zero d in* **D** *and for all d′ in* **D***, there is an element g of G with dg = d′.*
3) F and G are mutually commuting; i.e. for all f in F and all g in G, fg = gf.
Then:
(i) Every element of **gl**(**D**) *which commutes with every element of F is in G and every element of* **gl**(**D**) *which commutes with every element of G is in F.*
(ii) F and G are both uniquely transitive on the non-zero elements of **D***; i.e. for all non-zero d in* **D** *and for all d′ in* **D***, there is exactly one element f of F with df = d′ and exactly one element g of G with dg = d′*
(iii) F and G are subalgebras of **gl**(**D**) *which are associative and unital division algebras.*
(iv) Every non-zero element u of **D** *induces an algebra anti-isomorphism from F to G: f* $\mapsto$ *g if uf = ug.*
(v) Every non-zero element u of **D** *determines an associative and unital division algebra* $\mathbf{D}_{uF}$*, with underlying linear space* **D***, product identity* $1_{\mathbf{D}} = u$*, right multiplications F and left multiplications G and u determines an associative and unital division algebra* $\mathbf{D}_{uG}$*, with underlying linear space* **D***, product identity* $1_{\mathbf{D}} = u$*, right multiplications G and left multiplications F.* $\mathbf{D}_{uF}$ *and* $\mathbf{D}_{uG}$ *are opposite algebras, which coincide as a commutative algebra if and only if F = G.*
PROOF
(i) Let h be an element of **gl**(**D**) which commutes with every element of F and let d_0 be a non-zero element of **D**. Then, by transitivity of G, for some g in G, $d_0g = d_0h$.
We show that $g = h$.
Let d be any non-zero element of **D**. Then, by transitivity of F, for some f in F, $d_0 = df$.
Then $dh = (d_0f)h = d_0(fh) = d_0(hf) = (d_0h)f = (d_0g)f = d_0(gf) = d_0(fg) = (d_0f)g = dg$.
The proof that every element of **gl**(**D**) which commutes with every element of G is in F is symmetric.
(ii) The proof of (i) also shows that the transitivity of F (resp. of G) is unique.
(iii) Let f, f_1 and f_2 be elements of F, let r be a scalar and let g be any element of G.
Then for all d in **D**:
$d((rf)g) = (d(rf))g = (r(df))g = ((rd)f)g = (rd)(fg) = (rd)(gf) = (r(dg))f = (dg)(rf) = d(g(rf))$
$d((f_1 + f_2)g) = (d(f_1 + f_2))g = (df_1 + df_2)g = (df_1)g + (df_2)g = d(f_1g) + d(f_2g)$
$= d(gf_1) + d(gf_2) = ((dg)f_1 + (dg)f_2) = (dg)(f_1 + f_2) = d(g(f_1 + f_2))$
$(f_1f_2)g = f_1(f_2g)) = f_1(gf_2) = (f_1g)f_2 = (gf_1)f_2 = g(f_1f_2)$.

Hence F is an algebra and is associative because it is a subalgebra of the associative algebra $\mathbf{gl}(\mathbf{D})$; its unit is the identity linear homomorphism $1_{\mathbf{gl}(\mathbf{D})}$; every non-zero element of F has an inverse by transitivity (If $df = d'$, then f^{-1} is the unique element of F such that $d'f^{-1} = d$). Hence F is an associative and unital division algebra. The proof that G is an associative and unital division algebra is symmetric.
(iv) Let f, f_1 and f_2 be elements of F and let g, g_1 and g_2 be elements of G such that $uf = ug$, $uf_1 = ug_1$ and $uf_2 = ug_2$; let r be a scalar. Then:

$$u(rf) = r(uf) = r(ug) = u(rg),\ u(f_1 + f_2) = (uf_1 + uf_2) = (ug_1 + ug_2) = u(g_1 + g_2) \text{ and}$$
$$u(f_1f_2) = (uf_1)f_2 = (ug_1)f_2 = u(g_1f_2) = u(f_2g_1) = (uf_2)g_1 = (ug_2)g_1 = u(g_2g_1)$$

(v) $\mathbf{D}_{uF}$ and $\mathbf{D}_{uG}$ and their respective linear operations and products are defined as follows:
$\mathbf{D}_{uF} = \{uf : f \text{ in } F\}$, $r(uf) = u(rf)$, $(uf_1 + uf_2) = u(f_1 + f_2)$ and $(uf_1)(uf_2) = u(f_1f_2)$
$\mathbf{D}_{uG} = \{ug : g \text{ in } G\}$, $r(ug) = u(rg)$, $(ug_1 + ug_2) = u(g_1 + g_2)$ and $(ug_1)(ug_2) = u(g_1g_2)$ ||

COROLLARY 2.14 (The "Real Line")
Let $\mathbf{L}$ *be a line through the Origin. Then every non-zero vector* u *of* $\mathbf{L}$ *determines an associative and unital division algebra* $\mathbf{L}_{u\mathbf{gl}(\mathbf{L})}$, *algebra isomorphic to the real numbers, with underlying linear space* $\mathbf{L}$ *and product identity* $1_{\mathbf{L}} = u$, *whose endomorphism algebra* $\mathbf{gl}(\mathbf{L})$ *is the algebra of scalar multiplications* $\mathbf{R}_m$. *If* $r_{1\mathbf{L}} = r_1u$ *and* $r_{2\mathbf{L}} = r_2u$, *then* $(r_1r_2)_{\mathbf{L}} = (r_1r_2)u$.
PROOF
$\mathbf{L}$ is one-dimensional, whence $\mathbf{gl}(\mathbf{L}) = \mathbf{R}_m$. $\mathbf{R}_m$ is commutative, contains the zero linear endomorphism and is transitive on the non-zero vectors of $\mathbf{L}$. ||

3) Linear Endomorphisms and Square Matrices
We have analogous properties in one, two and three dimensions and give proofs in the two-dimensional case. We begin with the trivial case of one dimension.

Recall that a (1×1) matrix is a square array $[x_1]$ of real numbers. The (1×1) matrices form a one-dimensional linear space with zero element $[0]$, scalar multiplication $r[x_1] = [rx_1]$ and addition $[x_1] + [x'_1] = [x_1 + x'_1]$. They also have a product $[x_1][x'_1] = [x_1x'_1]$. . We denote the linear space of (1×1) matrices by $\mathbf{R}(1)$. $\mathbf{R}(1)$ has a basis $\{E_{11}\}$, where $E_{11} = [1]$ is the **identity matrix**, which is a product identity for $\mathbf{R}(1)$.

Let $\mathbf{L}$ be a line through the Origin. Then $\mathbf{L}(1\times1) = \{[v_1]: v_1 \text{ in } \mathbf{L}\}$. $\mathbf{L}(1\times1)$ is the one-dimensional linear space of (1×1) column matrices with an entry from $\mathbf{L}$. The elements of $\mathbf{L}(1\times1)$ are **oriented segments** of $\mathbf{L}$. $\mathbf{L}(1\times1)$ has zero element $[0]$, scalar multiplication $r[v_1] = [rv_1]$ and addition $[v_1] + [v'_1] = [v_1 + v'_1]$. If $\langle v_1\rangle$ is an (ordered) basis of $\mathbf{L}$, then $\{[v_1]\}$ is a basis of $\mathbf{L}(1\times1)$. An oriented segment $[v_1]$ of $\mathbf{L}$ is **degenerate** if $v_1 = 0$. Otherwise $[v_1]$ is **non-degenerate**. $[v_1]$ is non-degenerate if and only if $\langle v_1\rangle$ is an (ordered) basis of $\mathbf{\Pi}$.

There is a left action of $\mathbf{R}(1)$ on $\mathbf{L}(1\times1)$ and there is a right action of $\mathbf{gl}(\mathbf{L}) = \mathbf{R}_m$ on $\mathbf{L}(1\times1)$: $[x_1][v_1] = [xv_1]$ and $[v_1]r_m = [v_1r_m]$.

LEMMA 3.1A
Let M *and* N *be* (1×1) *matrices and let* f *and* g *be linear endomorphisms of some line* $\mathbf{L}$ *through the Origin. Then for any oriented segment* $[v_1]$ *of* $\mathbf{L}$: *1)* $(MN)[v_1] = M(N[v_1])$, *2)* $M([v_1]g) = (M[v_1])g$ *and 3)* $([v_1]f)g = [v_1](fg)$. *I.e. the expressions* $MN[v_1]$, $M[v_1]g$ *and* $[v_1]fg$ *are all unambiguous.* ||

THEOREM 3.2A
Let **L** *be a line through the Origin, and let* <v_1> *be an (ordered) basis of* **L**. *Then the equalities,* $[w_1] = [x_1][v_1] = [v_1]f$, *induce a 1-1-1 correspondence between the oriented segments* $[w_1]$ *of* $\mathbf{L}(1\times1)$, *the matrices* $[x_1]$ *of* $\mathbf{R}(1)$ *and the linear endomorphisms f of* $\mathbf{gl}(\mathbf{L})$*;* $w_1 = x_1v_1 = v_1f$. *Further, let M and N be matrices of* $\mathbf{R}(1)$ *and let f and g be linear endomorphisms of* $\mathbf{gl}(\mathbf{L})$, *such that* $M[v_1] = [v_1]f$ *and* $N[v_1] = [v_1]g$. *Then:*

$(rM)[v_1] = [v_1](rf)$, $(M + N)[v_1] = [v_1](f + g)$ and $MN[v_1] = [v_1]fg$. ||

COROLLARY 3.3A
The linear space $\mathbf{R}(1)$ *of* (1×1) *matrices is an algebra under the matrix product. In fact, if* **L** *is a line through the Origin, then, trivially, for any (ordered) basis* <v_1> *of* **L**, *the correspondence* $M \approx f$ *if* $M[v_1] = [v_1]f$ *is an algebra isomorphism from* $\mathbf{R}(1)$ *to* $\mathbf{gl}(\mathbf{L}) = \mathbf{R}_m$. *M is invertible if and only if f is invertible if and only if* $M[v_1] = [v_1]f$ *is a non-degenerate oriented segment.* ||

Recall that a (2×2) matrix is a square array $\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$ of real numbers. The (2×2) matrices form a four-dimensional linear space with zero element $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, scalar multiplication $r\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} rx_1 & ry_1 \\ rx_2 & ry_2 \end{bmatrix}$ and addition $\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} + \begin{bmatrix} x'_1 & y'_1 \\ x'_2 & y'_2 \end{bmatrix} = \begin{bmatrix} x_1 + x'_1 & y_1 + y'_1 \\ x_2 + x'_2 & y_2 + y'_2 \end{bmatrix}$. They also have a product $\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}\begin{bmatrix} x'_1 & y'_1 \\ x'_2 & y'_2 \end{bmatrix} = \begin{bmatrix} x_1x'_1 + y_1x'_2 & x_1y'_1 + y_1y'_2 \\ x_2x'_1 + y_2x'_2 & x_2y'_1 + y_2y'_2 \end{bmatrix}$. We denote the linear space of (2×2) matrices by $\mathbf{R}(2)$. $\mathbf{R}(2)$ has a basis $\{E_{11}, E_{12}, E_{21}, E_{22}\}$, where E_{ij} is the matrix with entry 1 in the intersection of the i^{th} row and the j^{th} column and entry 0 elsewhere. The **identity matrix** is $E_{11} + E_{22}$, which is a product identity for $\mathbf{R}(2)$.

Let $\mathbf{\Pi}$ be a plane through the Origin. Then $\mathbf{\Pi}(2\times1) = \{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$: v_1 and v_2 in $\mathbf{\Pi}\}$. $\mathbf{\Pi}(2\times1)$ is the four-dimensional linear space of (2×1) column matrices with entries from $\mathbf{\Pi}$. The elements of $\mathbf{\Pi}(2\times1)$ are **oriented parallelograms** of $\mathbf{\Pi}$. $\mathbf{\Pi}(2\times1)$ has zero element $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, scalar multiplication $r\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} rv_1 \\ rv_2 \end{bmatrix}$ and addition $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} v'_1 \\ v'_2 \end{bmatrix} = \begin{bmatrix} v_1 + v'_1 \\ v_2 + v'_2 \end{bmatrix}$. If <$v_1$, v_2> is a basis of $\mathbf{\Pi}$, then $\{\begin{bmatrix} v_1 \\ 0 \end{bmatrix}, \begin{bmatrix} v_2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ v_1 \end{bmatrix}, \begin{bmatrix} 0 \\ v_2 \end{bmatrix}\}$ is a basis of $\mathbf{\Pi}(2\times1)$. An oriented parallelogram $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ of $\mathbf{\Pi}$ is **degenerate** if v_1 and v_2 are linearly dependent. Otherwise $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is **non-degenerate**. $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is non-degenerate if and only if <v_1, v_2> is an ordered basis of $\mathbf{\Pi}$.

There is a left action of $\mathbf{R}(2)$ on $\mathbf{\Pi}(2\times1)$ and there is a right action of $\mathbf{gl}(\mathbf{\Pi})$ on $\mathbf{\Pi}(2\times1)$: $\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} x_1v_1 + y_1v_2 \\ x_2v_1 + y_2v_2 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} f = \begin{bmatrix} v_1f \\ v_2f \end{bmatrix}$.

LEMMA 3.1B

Let M and N be (2×2) *matrices and let f and g be linear endomorphisms of some plane* Π *through the Origin. Then for any oriented parallelogram* $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ *of* Π*: 1)* $(MN)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = M(N\begin{bmatrix} v_1 \\ v_2 \end{bmatrix})$*, 2)* $M(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}g) = (M\begin{bmatrix} v_1 \\ v_2 \end{bmatrix})g$ *and 3)* $(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}f)g = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}(fg)$*. I.e. the expressions* $MN\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$*,* $M\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}g$ *and* $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}fg$ *are all unambiguous.*

PROOF

Let $M = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$ and $N = \begin{bmatrix} x'_1 & y'_1 \\ x'_2 & y'_2 \end{bmatrix}$. Then:

1) $(MN)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} =$

$$\begin{bmatrix} x_1x'_1 + y_1x'_2 & x_1y'_1 + y_1y'_2 \\ x_2x'_1 + y_2x'_2 & x_2y'_1 + y_2y'_2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} (x_1x'_1 + y_1x'_2)v_1 + (x_1y'_1 + y_1y'_2)v_2 \\ (x_2x'_1 + y_2x'_2)v_1 + (x_2y'_1 + y_2y'_2)v_2 \end{bmatrix} \text{ and}$$

$$M(N\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}) = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}\begin{bmatrix} x'_1v_1 + y'_1v_2 \\ x'_2v_1 + y'_2v_2 \end{bmatrix} = \begin{bmatrix} x_1(x'_1v_1 + y'_1v_2) + y_1(x'_2v_1 + y'_2v_2) \\ x_2(x'_1v_1 + y'_1v_2) + y_2(x'_2v_1 + y'_2v_2) \end{bmatrix}.$$

2) $M(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}g) = M\begin{bmatrix} v_1g \\ v_2g \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}\begin{bmatrix} v_1g \\ v_2g \end{bmatrix} = \begin{bmatrix} x_1(v_1g) + y_1(v_2g) \\ x_2(v_1g) + y_2(v_2g) \end{bmatrix}$

$$= \begin{bmatrix} (x_1v_1 + y_1v_2)g \\ (x_2v_1 + y_2v_2)g \end{bmatrix} = \begin{bmatrix} x_1v_1 + y_1v_2 \\ x_2v_1 + y_2v_2 \end{bmatrix}g = (M\begin{bmatrix} v_1 \\ v_2 \end{bmatrix})g.$$

3) $(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}f)g = \begin{bmatrix} v_1f \\ v_2f \end{bmatrix}g = \begin{bmatrix} (v_1f)g \\ (v_2f)g \end{bmatrix} = \begin{bmatrix} v_1(fg) \\ v_2(fg) \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}(fg)$. ||

THEOREM 3.2B

Let Π *be a plane through the Origin, and let* $<v_1, v_2>$ *be an ordered basis of* Π*. Then the equalities,* $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}f$*, induce a 1-1-1 correspondence between the oriented parallelograms* $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ *of* $\Pi(2\times1)$*, the matrices* $\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$ *of* $\mathbf{R}(2)$ *and the linear endomorphisms f of* $\mathbf{gl}(\Pi)$*;* $w_1 = x_1v_1 + y_1v_2 = v_1f$ *and* $w_2 = x_2v_1 + y_2v_2 = v_2f$*.*

Further, let M and N be matrices of $\mathbf{R}(2)$ *and let f and g be linear endomorphisms of* $\mathbf{gl}(\Pi)$*, such that* $M\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}f$ *and* $N\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}g$*. Then:*

$$(rM)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}(rf), (M+N)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}(f+g) \text{ and } MN\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}fg.$$

PROOF

The 1-1 correspondence $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \approx \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$ is immediate.

Linear endomorphisms of Π determine and are determined by the image of the ordered basis $<v_1, v_2>$. If $w_1 = v_1f$ and $w_2 = v_2f$, then $(xv_1 + yv_2)f = xw_1 + yw_2$. This gives the 1-1 correspondence $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \approx f$.

$$(rM)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = M\left(r\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = M\begin{bmatrix} rv_1 \\ rv_2 \end{bmatrix} = M\begin{bmatrix} v_1r_m \\ v_2r_m \end{bmatrix} = M\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}r_m = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}fr_m = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}(rf)$$

$$(M+N)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = M\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + N\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}f + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}g = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}(f+g)$$

$$MN\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = M\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}g = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}fg.$$ ||

COROLLARY 3.3B

The linear space $\mathbf{R}(2)$ *of* (2×2) *matrices is an algebra under the matrix product. In fact, if* Π *is a plane through the Origin, then for any ordered basis* $<v_1, v_2>$ *of* Π*, the correspondence* $M \approx f$ *if* $M\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}v_1\\v_2\end{bmatrix}f$ *is an algebra isomorphism from* $\mathbf{R}(2)$ *to* $\mathbf{gl}(\Pi)$*. M is invertible if and only if f is invertible if and only if* $M\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}v_1\\v_2\end{bmatrix}f$ *is a non-degenerate oriented parallelogram.*

PROOF

The algebra properties of linearity of the product, associativity and possession of a unit are induced by the correspondence.

Let $N \approx g$. Then $MN \approx fg$ and $NM \approx gf$.

$MN = NM$ is the identity matrix if and only if $fg = gf$ is the identity isomorphism. ||

Let Π be a plane through the Origin, and let $\begin{bmatrix}v_1\\v_2\end{bmatrix}$ be a non-degenerate oriented parallelogram of Π. Then the (2×2) matrix M **represents** the linear endomorphism f of Π **with respect to** $\begin{bmatrix}v_1\\v_2\end{bmatrix}$ if $M\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}v_1\\v_2\end{bmatrix}f$.

THEOREM 3.4B

Let Π *be a plane through the Origin, and let* $\begin{bmatrix}v_1\\v_2\end{bmatrix}$ *be a non-degenerate oriented parallelogram of* Π*.. Let* $M\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}v_1\\v_2\end{bmatrix}f$*, where M is a* (2×2) *matrix and f is a linear endomorphism of* Π*. Let* $\begin{bmatrix}v'_1\\v'_2\end{bmatrix}$ *be a non-degenerate oriented parallelogram of* Π *and let* $\begin{bmatrix}v'_1\\v'_2\end{bmatrix} = P\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}v_1\\v_2\end{bmatrix}h$*, where P is an invertible* (2×2) *matrix and h is a linear automorphism of* Π*. Then* $PMP^{-1}\begin{bmatrix}v'_1\\v'_2\end{bmatrix} = \begin{bmatrix}v'_1\\v'_2\end{bmatrix}f$ *and* $M\begin{bmatrix}v'_1\\v'_2\end{bmatrix} = \begin{bmatrix}v'_1\\v'_2\end{bmatrix}h^{-1}fh$*.*

PROOF

$PMP^{-1}\begin{bmatrix}v'_1\\v'_2\end{bmatrix} = PM\begin{bmatrix}v_1\\v_2\end{bmatrix} = P\begin{bmatrix}v_1\\v_2\end{bmatrix}f = \begin{bmatrix}v'_1\\v'_2\end{bmatrix}f$.

$\begin{bmatrix}v'_1\\v'_2\end{bmatrix}h^{-1}fh = \begin{bmatrix}v_1\\v_2\end{bmatrix}fh = M\begin{bmatrix}v_1\\v_2\end{bmatrix}h = M\begin{bmatrix}v'_1\\v'_2\end{bmatrix}$. ||

COROLLARY 3.5B

If a (2×2) *matrix M represents a linear endomorphism f with respect to a given non-degenerate oriented parallelogram of a plane* Π *through the Origin, then any matrix, which represents f with respect to some non-degenerate oriented parallelogram of* Π*, is a conjugate* PMP^{-1} *of M and also, any linear endomorphism of* Π*, which is represented by M with respect to some non-degenerate oriented parallelogram, is a conjugate* $h^{-1}fh$ *of f, where P and h are an invertible matrix and a linear automorphism, respectively.* ||

Recall that a (3×3) matrix is a square array $\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$ of real numbers. The (3×3) matrices form a nine-dimensional linear space with zero element $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, with scalar multiplication $r\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} = \begin{bmatrix} rx_1 & ry_1 & rz_1 \\ rx_2 & ry_2 & rz_2 \\ rx_3 & ry_3 & rz_3 \end{bmatrix}$ and with addition

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} + \begin{bmatrix} x'_1 & y'_1 & z'_1 \\ x'_2 & y'_2 & z'_2 \\ x'_3 & y'_3 & z'_3 \end{bmatrix} = \begin{bmatrix} x_1 + x'_1 & y_1 + y'_1 & z_1 + z'_1 \\ x_2 + x'_2 & y_2 + y'_2 & z_2 + z'_2 \\ x_3 + x'_3 & y_3 + y'_3 & z_3 + z'_3 \end{bmatrix}.$$

They also have a product $\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}\begin{bmatrix} x'_1 & y'_1 & z'_1 \\ x'_2 & y'_2 & z'_2 \\ x'_3 & y'_3 & z'_3 \end{bmatrix} =$

$$\begin{bmatrix} x_1x'_1 + y_1x'_2 + z_1x'_3 & x_1y'_1 + y_1y'_2 + z_1y'_3 & x_1z'_1 + y_1z'_2 + z_1z'_3 \\ x_2x'_1 + y_2x'_2 + z_2x'_3 & x_2y'_1 + y_2y'_2 + z_2y'_3 & x_2z'_1 + y_2z'_2 + z_2z'_3 \\ x_3x'_1 + y_3x'_2 + z_3x'_3 & x_2y'_1 + y_2y'_2 + z_2y'_3 & x_3z'_1 + y_3z'_2 + z_3z'_3 \end{bmatrix}.$$

We denote the linear space of (3×3) matrices by **R**(3).
R(3) has a basis $\{E_{ij} : i = 1 \text{ to } 3, j = 1 \text{ to } 3\}$, where E_{ij} is the matrix with entry 1 in the intersection of the i^{th} row and the j^{th} column and entry 0 elsewhere. The **identity matrix** is $E_{11} + E_{22} + E_{33}$, which is a product identity for **R**(3).

Let **E** be the linear space of vectors. Then **E**(3×1) = $\{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$: v_1, v_2 and v_3 in **E**$\}$. **E**(3×1) is the nine-dimensional linear space of (3×1) column matrices with entries from **E**. The elements of **E**(3×1) are **oriented parallelepipeds** of **E**. **E**(3×1) has zero element $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, scalar multiplication $r\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} rv_1 \\ rv_2 \\ rv_3 \end{bmatrix}$ and addition $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{bmatrix} = \begin{bmatrix} v_1 + v'_1 \\ v_2 + v'_2 \\ v_3 + v'_3 \end{bmatrix}$. If $\langle v_1, v_2, v_3\rangle$ is a basis of **E**, then $\{\begin{bmatrix} v_1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} v_2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} v_3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ v_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ v_2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ v_3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ v_1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ v_2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix}\}$ is a basis of **E**(3×1).
An oriented parallelepiped $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ of **E** is **degenerate** if v_1, v_2 and v_3 are linearly dependent. Otherwise $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is **non-degenerate.** $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is non-degenerate if and only if $\langle v_1, v_2, v_3\rangle$ is an ordered basis of **E**.

There is a left action of **R**(3) on **E**(3×1) and there is a right action of **gl**(**E**) on **E**(3×1):
$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} x_1v_1 + y_1v_2 + z_1v_3 \\ x_2v_1 + y_2v_2 + z_2v_3 \\ x_3v_1 + y_3v_2 + z_3v_3 \end{bmatrix} \text{ and } \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} f = \begin{bmatrix} v_1f \\ v_2f \\ v_3f \end{bmatrix}.$$

LEMMA 3.1C

Let M and N be (3×3) matrices and let f and g be linear endomorphisms of the linear space of vectors **E**. *Then for any oriented parallelepiped* $\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$ *of* **E**: *1)* $(MN)\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = M(N\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix})$,

2) $M(\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}g) = (M\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix})g$ *and 3)* $(\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}f)g = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}(fg)$.

I.e. the expressions $MN\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$, $M\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}g$ *and* $\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}fg$ *are all unambiguous.* ||

THEOREM 3.2C

Let **E** *be the linear space of vectors and let* $<v_1, v_2, v_3>$ *be an ordered basis of* **E**. *Then the equalities,* $\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix} = \begin{bmatrix}x_1 & y_1 & z_1\\x_2 & y_2 & z_2\\x_3 & y_3 & z_3\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}f$, *induce a 1-1-1 correspondence between the oriented parallelepipeds* $\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}$ *of* $\mathbf{E}(3\times1)$, *the matrices* $\begin{bmatrix}x_1 & y_1 & z_1\\x_2 & y_2 & z_2\\x_3 & y_3 & z_3\end{bmatrix}$ *of* $\mathbf{R}(3)$ *and the linear endomorphisms f of* $\mathbf{gl}(\mathbf{E})$; $w_1 = x_1v_1 + y_1v_2 + z_1v_3 = v_1f$, $w_2 = x_2v_1 + y_2v_2 + z_2v_3 = v_2f$ *and* $w_3 = x_3v_1 + y_3v_2 + z_3v_3 = v_3f$.

Further, let M and N be matrices of $\mathbf{R}(3)$ *and let f and g be linear endomorphisms of* $\mathbf{gl}(\mathbf{E})$, *such that* $M\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}f$ *and* $N\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}g$. *Then:*

$(rM)\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}(rf), (M+N)\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}(f+g)$ *and* $MN\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}fg$. ||

COROLLARY 3.3C

The linear space $\mathbf{R}(3)$ *of* (3×3) *matrices is an algebra under the matrix product. In fact, if* **E** *is the linear space of vectors, then for any ordered basis* $<v_1, v_2, v_3>$ *of* **E**, *the correspondence* $M \approx f$ *if* $M\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}f$ *is an algebra isomorphism from* $\mathbf{R}(3)$ *to* $\mathbf{gl}(\mathbf{E})$. *M is invertible if and only if f is invertible if and only if* $M\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}f$ *is a non-degenerate oriented parallelepiped.* ||

Let **E** be the linear space of vectors and let $\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$ be a non-degenerate oriented parallelepiped of **E**. Then the (3×3) matrix M **represents** the linear endomorphism f of **E** **with respect to** $\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$ if $M\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}f$.

THEOREM 3.4C

Let **E** *be the linear space of vectors, and let* $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ *be a non-degenerate oriented parallelepiped of* **E**.. *Let* $M\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} f$, *where M is a* (3×3) *matrix and f is a linear endomorphism of* **E**. *Let* $\begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{bmatrix}$ *a non-degenerate oriented parallelogram of* **Π** *and let* $\begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{bmatrix} = P\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} h$, *where P is an invertible* (3×3) *matrix and h is a linear automorphism of* **Π**. *Then* $PMP^{-1}\begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{bmatrix} = \begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{bmatrix} f$ *and* $M\begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{bmatrix} = \begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{bmatrix} h^{-1}fh$. ||

COROLLARY 3.5C

If a (3×3) *matrix M represents a linear endomorphism f with respect to a given non-degenerate oriented parallelepiped of the linear space* **E** *of vectors, then any matrix, which represents f with respect to some non-degenerate oriented parallelepiped of* **E**, *is a conjugate* PMP^{-1} *of M and also, any linear endomorphism of* **E**, *which is represented by M with respect to some non-degenerate oriented parallelepiped, is a conjugate* $h^{-1}fh$ *of f, where P and h are an invertible matrix and a linear automorphism, respectively.* ||

4) Orientation, Multivectors and Determinants

As with the previous section we begin with the trivial case of one dimension. Let **L** be a line through the origin and let <*v*> be an (ordered) basis of **L**. Then every oriented segment of **L** may be expressed as $[w] = [r][v] = [v]r_m$, for some real number *r*. *v* is the oriented length of the oriented segment [*v*] and *w* is the oriented length of the oriented segment [*w*] and $w = rv$. *r* is the **determinant** det([*r*]) of the (1×1) matrix [*r*] and is also the **determinant** $\det(r_m)$ of the linear automorphism r_m of **gl**(**L**). If $r = 0$, then [*w*] is degenerate (in fact $[w] = [0]$, equivalently $w = 0$); if $r > 0$, [*w*] and [*v*] have the **same orientation**, equivalently, *w* and *v* have the same orientation and r_m is **orientation preserving**; if $r < 0$, [*w*] and [*v*] have **opposite orientation**, equivalently, *w* and *v* have opposite orientation and r_m is **orientation reversing**.

Let **Π** be a plane through the origin and let $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be an oriented parallelogram of **Π**. Then the **bivector** $B = v_1 \wedge v_2$ of **Π** is the **oriented area** of $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. *B* has a magnitude, which is the **area** of the geometric parallelogram with adjacent edges drawn from the origin to the points represented by the vectors v_1 and v_2 and if *B* is non-zero, then it has an **orientation** relative to other non-zero bivectors as described below. *B* is the **Grassmann product** of v_1 and v_2 (also called the **wedge product** or the **exterior product**). The set of bivectors of **Π** is denoted $\Pi^{(2)}$. Bivectors are signed magnitudes, whence they may be multiplied by scalars and may be added to satisfy axioms L1) and L2) for linear spaces:

L1) *For every scalar r,* $r0 = 0$ *and* $(0 + 0) = 0$, *where* 0 *manifests as the* ***zero bivector***

L2) *For each element B of* $\Pi^{(2)}$, $0B = 0$, $1B = B$ *and for any scalars* r_1 *and* r_2,

$(r_1B + r_2B) = (r_1 + r_2)B = (r_2 + r_1)B = (r_2B + r_1B)$ *and* $r_1(r_2B) = (r_1r_2)B = (r_2r_1)B = r_2(r_1B)$.

LEMMA 4.1B
Let Π *be a plane through the origin and let* $B = v_1 \wedge v_2$ *be a bivector of* Π*. Then*
1) $B = 0$ *if and only if* v_1 *and* v_2 *are linearly dependent;*
2) For any scalars x *and* y*,* $v_1 \wedge (xv_1 + v_2) = (v_1 + yv_2) \wedge v_2 = B$*.*
3) For any scalar r*,* $(rv_1) \wedge v_2 = v_1 \wedge (rv_2) = rB$ *.*
PROOF
1) is immediate.
2) The geometric parallelogram with edges determined by v_1 and $(xv_1 + v_2)$ is obtained from the geometric parallelogram with edges determined by v_1 and v_2 by a shear parallel to v_1; the geometric parallelogram with edges determined by $(v_1 + yv_2)$ and v_2 is obtained from the geometric parallelogram with edges determined by v_1 and v_2 by a shear parallel to v_2.
N.B. $\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ xv_1 + v_2 \end{bmatrix}$ and $\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 + yv_2 \\ v_2 \end{bmatrix}$
3) If $r = 0$, then $rb = 0$; otherwise, if $r > 0$, then one of the edges of the geometric parallelogram is rescaled by a factor $|r| = r$ and if $r < 0$, then one of the edges of the geometric parallelogram is rescaled by a factor $|r| = -r$ and also reversed.
N.B. $\begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} rv_1 \\ v_2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ rv_2 \end{bmatrix}$ ||

THEOREM 4.2B
Let Π *be a plane through the Origin. Then:*
1) The Grassmann product, $\langle v_1, v_2 \rangle \mapsto v_1 \wedge v_2$ *from ordered pairs of vectors of* Π *to bivectors of* Π *is antisymmetric and linear in both arguments.*
I.e. if v_1*,* v'_1*,* v_2 *and* v'_2 *are vectors of* Π *and* r *is a scalar then:*
$-(v_2 \wedge v_1)$*. (In particular,* $v_1 \wedge v_1 = 0$*).*
$v_1 \wedge (v_2 + v'_2) = v_1 \wedge v_2 + v_1 \wedge v'_2$ *and* $v_1 \wedge (rv_2) = r(v_1 \wedge v_2)$*,*
$(v_1 + v'_1) \wedge v_2 = v_1 \wedge v_2 + v'_1 \wedge v_2$ *and* $(rv_1) \wedge v_2 = r(v_1 \wedge v_2)$*.*
2) If $w_1 = x_1v_1 + y_1v_2$ *and* $w_2 = x_2v_1 + y_2v_2$ *for scalars* x_1*,* y_1*,* x_2 *and* y_2*, then:*
$$(w_1 \wedge w_2) = (x_1y_2 - x_2y_1)(v_1 \wedge v_2).$$
3) The set $\Pi^{(2)}$ *of bivectors of* Π *is a one-dimensional linear space.*
PROOF
1) $v_1 \wedge v_2 = v_1 \wedge (v_1 + v_2) = (v_1 - (v_1 + v_2)) \wedge (v_1 + v_2) = (-v_2) \wedge (v_1 + v_2) = (-v_2) \wedge v_1 = -(v_2 \wedge v_1)$.
Let v be any vector of Π such that v_1 and v are linearly independent. Then $\{v_1, v\}$ is a basis of Π and there are scalars x, y, x' and y' such that $v_2 = xv_1 + yv$ and $v'_2 = x'v_1 + y'v$.
$v_1 \wedge (v_2 + v'_2) = v_1 \wedge ((xv_1 + yv) + (x'v_1 + y'v)) = v_1 \wedge ((x + x')v_1 + (y + y')v)$
$= v_1 \wedge (y + y)'v = v_1 \wedge (yv) + v_1 \wedge (y'v) = v_1 \wedge (xv_1 + yv) + v_1 \wedge (x'v_1 + y'v) = v_1 \wedge v_2 + v_1 \wedge v'_2$.
The rest of 1) follows from anti-symmetry and lemma 4.1, 3).
2) $(w_1 \wedge w_2) = (x_1v_1 + y_1v_2) \wedge (x_2v_1 + y_2v_2)$
$= (x_1x_2)(v_1 \wedge v_1) + (x_1y_2)(v_1 \wedge v_2) + (y_1x_2)(v_2 \wedge v_1) + (y_1y_2)(v_2 \wedge v_2)$
$= (x_1y_2)(v_1 \wedge v_2) + (y_1x_2)(v_2 \wedge v_1) = (x_1y_2)(v_1 \wedge v_2) - (y_1x_2)(v_1 \wedge v_2)$
$= (x_1y_2 - x_2y_1)(v_1 \wedge v_2)$.
3) If B is any non-zero bivector of Π, then every bivector of Π may be expressed as rB, for some scalar r. It follows that $\Pi^{(2)}$ is the one-dimensional linear space $\mathbf{R}B$, with (ordered) basis $\langle B \rangle$. ||

Let $M = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$ be a (2×2) matrix. Then the **determinant** of M, denoted $\det(M)$ or $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$ is the scalar $(x_1y_2 - x_2y_1)$. The **transpose** of M, $M^T = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$ is the matrix obtained from M by interchanging its rows and columns. (The map $M \mapsto M^T$ is in fact an involutory (i.e. $(M^T)^T = M$) algebra anti-automorphism of $\mathbf{R}(2)$). $\det(M^T) = \det(M)$.

The **adjugate** of M, $\operatorname{adj}(M) = \begin{bmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{bmatrix}$, where:

$X_1 = y_2$ is the **cofactor** of x_1 in the expansion of $\det(M) = (x_1y_2 - x_2y_1)$.
$X_2 = -y_1$ is the cofactor of x_2 in the expansion of $\det(M) = (x_1y_2 - x_2y_1)$.
$Y_1 = -x_2$ is the cofactor of y_1 in the expansion of $\det(M) = (x_1y_2 - x_2y_1)$.
$Y_2 = x_1$ is the cofactor of y_2 in the expansion of $\det(M) = (x_1y_2 - x_2y_1)$.

Let $\Delta = \det(M)$. Then we may "expand Δ by rows" $\Delta = x_1X_1 + y_1Y_1 = x_2X_2 + y_2Y_2$ and we also have "alien cofactor row expansions" $0 = x_1X_2 + y_1Y_2 = x_2X_1 + y_2Y_1$.

It follows that $\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \begin{bmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{bmatrix}^T = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}$.

We may also "expand Δ by columns" $\Delta = x_1X_1 + x_2X_2 = y_1Y_1 + y_2Y_2$ and we also have "alien cofactor column expansions" $0 = x_1X_2 + y_1Y_2 = x_2X_1 + y_2Y_1$.

It follows that $\begin{bmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{bmatrix}^T \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}$.

(In some texts the adjugate of M is defined to be the transpose of our adjugate).

THEOREM 4.3B

Let M be a (2×2) *matrix and let* $\Delta = \det(M)$. *Then either* $\Delta = 0$ *and M is not invertible or M is invertible with inverse* $M^{-1} = (1/\Delta)(\operatorname{adj}(M))^T$.

PROOF

If M is invertible with inverse N, then for any plane Π, through the Origin, there are non-degenerate oriented parallelograms $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ with $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = M\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = N\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$. It follows that $w_1 \wedge w_2 = \det(M)(v_1 \wedge v_2) = \det(M)\det(N)(w_1 \wedge w_2)$, whence $\det(M) \neq 0$. ||

LEMMA 4.4B

Let M and N be (2×2) *matrices and let P be an invertible* (2×2) *matrix; then:*

1) The (2×2) *matrix MN is invertible if and only if both M and N are invertible.*

2) $\det(MN) = \det(M)\det(N)$.

3) $\det(P^{-1}) = 1/\det(P)$ *and, in particular,* $\det(PMP^{-1}) = \det(M)$.

PROOF

1) If M and N are both invertible then $(MN)^{-1} = N^{-1}M^{-1}$.
If MN is invertible then $MN(MN)^{-1} = (MN)^{-1}MN$ is the identity matrix.
$M^{-1} = N(MN)^{-1}$ is the inverse of M and $N^{-1} = (MN)^{-1}M$ is the inverse of N.

2) We need only consider the case M and N both invertible, for otherwise the equality $\det(MN) = \det(M)\det(N)$ reduces to the equality $0 = 0$. Let Π be a plane through the Origin and let $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $\begin{bmatrix} v'_1 \\ v'_2 \end{bmatrix}$ and $\begin{bmatrix} v''_1 \\ v''_2 \end{bmatrix}$ be oriented parallelograms of Π, such that $\begin{bmatrix} v'_1 \\ v'_2 \end{bmatrix} = N\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\begin{bmatrix} v''_1 \\ v''_2 \end{bmatrix} = M\begin{bmatrix} v'_1 \\ v'_2 \end{bmatrix}$, whence $\begin{bmatrix} v''_1 \\ v''_2 \end{bmatrix} = MN\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

Then $\det(MN)\,(v_1 \wedge v_2) = v''_1 \wedge w''_2 = \det(M)(v'_1 \wedge v'_2) = \det(M)\det(N)(v_1 \wedge v_2)$.

3) $\det(P^{-1})\det(P) = \det(P^{-1}P) = \det(I) = 1$, where I is the identity matrix.

For the particular case, we again need only consider the case M invertible. ||

THEOREM 4.5B
Let Π *be a plane through the Origin and let* $\Pi^{(2)}$ *be the set of all bivectors, which are the Grassmann products* $v_1 \wedge v_2$*, of vectors* v_1 *and* v_2 *in* Π*. Let* f *be a linear endomorphism of* Π*. Then* f *induces a linear endomorphism* $f^{(2)} : v_1 \wedge v_2 \longmapsto v_1 f \wedge v_2 f$ *of the one-dimensional linear space* $\Pi^{(2)}$ *of bivectors of* Π*. In fact* $f^{(2)}$ *is scalar multiplication by* $\det(f)$*, where* $\det(f)$ *is well defined to be the determinant of any matrix M which represents f, i.e. where for some oriented parallelogram* $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ *of* Π*,* $M\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} f = \begin{bmatrix} v_1 f \\ v_2 f \end{bmatrix}$*.*

PROOF
If $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is replaced with any other oriented parallelogram then, by Corollary 3.5B, M is replaced by a conjugate, PMP^{-1} for some invertible P and $\det(PMP^{-1}) = \det(M)$. ||

Let Π be a plane through the Origin. Let $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be a non-degenerate parallelogram of Π and let $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = M\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} f$ be a parallelogram of Π, where M is a (2×2) matrix and f is a linear endomorphism of Π.
Then $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ is degenerate, M is not invertible and f is not a linear automorphism, or
$\det(M) = \det(f)$ is positive, $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ is non-degenerate, $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ have the **same orientation**, equivalently $w_1 \wedge w_2$ and $v_1 \wedge v_2$ have the same orientation, and f is an **orientation-preserving** linear automorphism, or
$\det(M) = \det(f)$ is negative, $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ is non-degenerate, $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ have **opposite orientation**, equivalently $w_1 \wedge w_2$ and $v_1 \wedge v_2$ have opposite orientation, and f is an **orientation-reversing** linear automorphism.

Let **E** be the linear space of vectors and let $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ be an oriented parallelepiped of **E**. Then the **trivector** $\tau = v_1 \wedge v_2 \wedge v_3$ of **E** is the **oriented volume** of $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. τ has a magnitude, which is the **volume** of the geometric parallelepiped with adjacent edges drawn from the origin to the points represented by the vectors v_1, v_2 and v_3 and if τ is non-zero, then it has an **orientation** relative to other non-zero bivectors as described below. τ is the **Grassmann product** of v_1, v_2 and v_3 (also called the **wedge product** or the **exterior product**). The set of trivectors of **E** is denoted $\mathbf{E}^{(3)}$. Trivectors are signed magnitudes, whence they may be multiplied by scalars and may be added to satisfy axioms L1) and L2) for linear spaces:
L1) *For every scalar r,* $r0 = 0$ *and* $(0 + 0) = 0$*, where* 0 *manifests as the* ***zero trivector***
L2) *For each element* τ *of* $\mathbf{E}^{(3)}$*,* $0\tau = 0$*,* $1\tau = \tau$ *and for any scalars* r_1 *and* r_2*,*
$(r_1\tau + r_2\tau) = (r_1 + r_2)\tau = (r_2 + r_1)\tau = (r_2\tau + r_1\tau)$ *and* $r_1(r_2\tau) = (r_1r_2)\tau = (r_2r_1)\tau = r_2(r_1\tau)$*.*

LEMMA 4.1C
Let **E** *be the linear space of vectors and let* $\tau = v_1 \wedge v_2 \wedge v_3$ *be a trivector of* **E***. Then*
1) $\tau = 0$ *if and only if* v_1*,* v_2 *and* v_3 *are linearly dependent;*
2) For any scalars x′, x″, y, y″, z and z′, $(v_1 + yv_2 + zv_3) \wedge v_2 \wedge v_3 = v_1 \wedge (x'v_1 + v_2 + z'v_3) \wedge v_3$
$= v_1 \wedge v_2 \wedge (x''v_1 + y''v_2 + v_3) = \tau$*.*
3) For any scalar r, $(rv_1) \wedge v_2 = v_1 \wedge (rv_2) = r\tau$*.*

PROOF
1) is immediate.
2) Analogous to Lemma 4.1C 2)
N.B. $\begin{bmatrix}1 & y & z\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}v_1 + yv_2 + zv_3\\v_2\\v_3\end{bmatrix}$ etc.
3) Analogous to Lemma 4.1C 3)
N.B. $\begin{bmatrix}r & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}rv_1\\v_2\\v_3\end{bmatrix}$ etc. ||

THEOREM 4.2C
Let **E** *be the linear space of vectors. Then:*
1) The Grassmann product, $<v_1, v_2, v_3> \mapsto v_1\wedge v_2\wedge v_3$ *from ordered triples of vectors of* **E** *to trivectors of* **E** *is antisymmetric and linear in all three arguments.*

I.e. if v_1, v'_1, v_2, v'_2, v_3 *and* v'_3 *are vectors of* **Π** *and r is a scalar then:*
$v_1\wedge v_2\wedge v_3 = v_2\wedge v_3\wedge v_1 = v_3\wedge v_1\wedge v_2 = -(v_3\wedge v_2\wedge v_1) = -(v_2\wedge v_1\wedge v_3) = -(v_1\wedge v_3\wedge v_2)$
$(v_1 + v'_1)\wedge v_2\wedge v_3 = v_1\wedge v_2\wedge v_3 + v'_1\wedge v_2\wedge v_3$ *and* $(rv_1)\wedge v_2\wedge v_3 = r(v_1\wedge v_2\wedge v_3)$,
$v_1\wedge(v_2 + v'_2)\wedge v_3 = v_1\wedge v_2\wedge v_3 + v_1\wedge v'_2\wedge v_3$ *and* $v_1\wedge(rv_2)\wedge v_3 = r(v_1\wedge v_2\wedge v_3)$,
$v_1\wedge v_2\wedge(v_3 + v'_3) = v_1\wedge v_2\wedge v_3 + v_1\wedge v_2\wedge v'_3$ *and* $v_1\wedge v_2\wedge(rv_3) = r(v_1\wedge v_2\wedge v_3)$.

2) If $w_1 = x_1v_1 + y_1v_2 + z_1v_3$, $w_2 = x_2v_1 + y_2v_2 + z_2v_3$ *and* $w_3 = x_3v_1 + y_3v_2 + z_3v_3$ *for scalars* x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3 *and* z_3*, then:*

$$(w_1\wedge w_2\wedge w_3) = (x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2 - x_3y_2z_1 - x_2y_1z_3 - x_1y_3z_2)(v_1\wedge v_2\wedge v_3).$$

3) The set $\mathbf{E}^{(3)}$ *of trivectors of* **E** *is a one-dimensional linear space.*
PROOF
1) Let $<l, m, n>$ be any permutation of the sequence <1, 2, 3>
$v_l\wedge v_m\wedge v_n = v_l\wedge(v_l + v_m)\wedge v_n = (v_l - (v_l + v_m))\wedge(v_l + v_m)\wedge v_n$
$= (-v_m)\wedge(v_1 + v_2)\wedge v_n = (-v_m)\wedge v_l\wedge v_n = -(v_m\wedge v_l)\wedge v_n$ etc.
Let v be any vector of **E** such that v and v_2 and v_3 are linearly independent.
Then $\{v, v_2, v_3\}$ is a basis of **E** and there are scalars x, y, x', y', z and z' such that

$$v_1 = xv + yv_2 + zv_3 \text{ and } v'_1 = x'v + y'v_2 + z'v_3.$$

$(v_1 + v'_1)\wedge v_2\wedge v_3 = xv + yv_2 + zv_3) + (x'v + y'v_2 + z'v_3))\wedge v_2\wedge v_3$
$= ((x + x')v + (y + y')v_2 + (z + z')v_3))\wedge v_2\wedge v_3$
$= ((x + x')v)\wedge v_2\wedge v_3 = (xv)\wedge v_2\wedge v_3 + (x'v)\wedge v_2\wedge v_3$ etc. as for Theorem 4.2B
The rest of 1) follows from anti-symmetry and lemma 4.2B, 3).
2) $(w_1\wedge w_2\wedge w_3) = (x_1v_1 + y_1v_2 + z_1v_3)\wedge(x_2v_1 + y_2v_2 + z_2v_3)\wedge(x_2v_1 + y_2v_2 + z_2v_3)$
$= (x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2 - x_3y_2z_1 - x_2y_1z_3 - x_1y_3z_2)(v_1\wedge v_2\wedge v_3)$ by anti-symmetry.
3) If τ is any non-zero trivector of **E**, then every trivector of **E** may be expressed as $r\tau$, for some scalar r. It follows that $\mathbf{E}^{(3)}$ is the one-dimensional linear space $\mathbf{R}\tau$, with (ordered) basis $<\tau>$. ||

Let $M = \begin{bmatrix}x_1 & y_1 & z_1\\x_2 & y_2 & z_2\\x_3 & y_3 & z_3\end{bmatrix}$ be a (3×3) matrix. Then the **determinant** of M, denoted det(M) or $\begin{vmatrix}x_1 & y_1 & z_1\\x_2 & y_2 & z_2\\x_3 & y_3 & z_3\end{vmatrix}$ is the scalar $(x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2 - x_3y_2z_1 - x_2y_1z_3 - x_1y_3z_2)$. The **transpose** of M, $M^T = \begin{bmatrix}x_1 & x_2 & x_3\\y_1 & y_2 & y_3\\z_1 & z_2 & z_3\end{bmatrix}$ is obtained from M by interchanging its rows and columns.

(The map $M \mapsto M^T$ is in fact an involutory (i.e. $(M^T)^T = M$) algebra anti-automorphism of $\mathbf{R}(3)$). $\det(M^T) = \det(M)$. The **adjugate** of M, $\mathrm{adj}(M) = \begin{bmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{bmatrix}$, where:

$X_1 = (y_2z_3 - y_3z_2)$ is the **cofactor** of x_1 in the expansion of $\det(M)$
$X_2 = (y_3z_1 - y_1z_3)$ is the cofactor of x_2 in the expansion of $\det(M)$
$X_3 = (y_1z_2 - y_2z_1)$ is the cofactor of x_3 in the expansion of $\det(M)$
$Y_1 = (z_2x_3 - z_3x_2)$ is the cofactor of y_1 in the expansion of $\det(M)$.
$Y_2 = (z_3x_1 - z_1x_3)$ is the cofactor of y_2 in the expansion of $\det(M)$
$Y_3 = (z_1x_2 - z_2x_1)$ is the cofactor of x_2 in the expansion of $\det(M)$
$Z_1 = (x_2y_3 - x_3y_2)$ is the cofactor of x_2 in the expansion of $\det(M)$
$Z_2 = (x_3y_1 - x_1y_3)$ is the cofactor of x_2 in the expansion of $\det(M)$
$Z_3 = (x_1y_2 - x_2y_1)$ is the cofactor of z_3 in the expansion of $\det(M)$

Let $\Delta = \det(M)$. Then we may "expand Δ by rows"
$\Delta = x_1X_1 + y_1Y_1 + z_1Z_1 = x_2X_2 + y_2Y_2 + z_2Z_2 = x_3X_3 + y_3Y_3 + z_3Z_3$
and we also have "alien cofactor row expansions" $0 = x_lX_m + y_lY_m + z_lZ_m$, where l and m are different selections from the sequence <1, 2, 3>.

It follows that $\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \begin{bmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{bmatrix}^T = \begin{bmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix}$.

We may also "expand Δ by columns" and we also have "alien cofactor column expansions". We leave the details to the reader as they are analogous to the two-dimensional case. (We reiterate that in some texts the adjugate of M is defined to be the transpose of our adjugate).

THEOREM 4.3C
Let M be a (3×3) matrix and let $\Delta = \det(M)$. Then either $\Delta = 0$ and M is not invertible or M is invertible with inverse $M^{-1} = (1/\Delta)(\mathrm{adj}(M))^T$.
PROOF
Analogous to the proof of Theorem 4.3B ||

LEMMA 4.4C
Let M and N be (3×3) matrices and let P be an invertible (3×3) matrix; then:
1) The (3×3) matrix MN is invertible if and only if both M and N are invertible.
2) $\det(MN) = \det(M)\det(N)$.
3) $\det(P^{-1}) = 1/\det(P)$ *and, in particular,* $\det(PMP^{-1}) = \det(M)$.
PROOF
Analogous to Lemma 4.4B. ||

THEOREM 4.5C
Let $\mathbf{E}$ be the linear space of vectors and let $\mathbf{E}^{(3)}$ be the set of all trivectors, which are the Grassmann products $v_1\wedge v_2\wedge v_3$, of vectors v_1, v_2 and v_3 in $\mathbf{E}$. Let f be a linear endomorphism of $\mathbf{E}$. Then f induces a linear endomorphism $f^{(3)} : v_1\wedge v_2\wedge v_3 \longmapsto v_1f\wedge v_2f\wedge v_3f$ of the one-dimensional linear space $\mathbf{E}^{(3)}$ of trivectors of $\mathbf{E}$. In fact $f^{(3)}$ is scalar multiplication by $\det(f)$, *where* $\det(f)$ *is well defined to be the determinant of any matrix M which represents f, i.e. where for some oriented parallelepiped $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ of $\mathbf{E}$, $M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} f = \begin{bmatrix} v_1f \\ v_2f \\ v_3f \end{bmatrix}$.*

PROOF

If $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is replaced with any other oriented parallelepiped then, by Corollary 3.5C, M is replaced by a conjugate, PMP^{-1} for some invertible P and $\det(PMP^{-1}) = \det(M)$. ||

Let **E** be the linear space of vectors. Let $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ be a non-degenerate parallelepiped of **E** and let $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} f$ be a parallelepiped of **E**, where M is a (3×3) matrix and f is a linear endomorphism of **E**.

Then $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ is degenerate, M is not invertible and f is not a linear automorphism, or

$\det(M) = \det(f)$ is positive, $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ is non-degenerate, $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ have the **same orientation**, equivalently $w_1 \wedge w_2 \wedge w_3$ and $v_1 \wedge v_2 \wedge v_3$ have the same orientation, and f is an **orientation-preserving** linear automorphism, or

$\det(M) = \det(f)$ is negative, $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ is non-degenerate, $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ have **opposite orientation**, equivalently $w_1 \wedge w_2 \wedge w_3$ and $v_1 \wedge v_2 \wedge v_3$ have opposite orientation, and f is an **orientation-reversing** linear automorphism.

Let **E** be the space of all vectors and let v and v' be a pair of linearly independent elements of **E**. Let V and V' be the points of physical Space, such that $v = OV$ and $v' = OV'$, where O is the Origin. If B is the non-zero bivector $v \wedge v'$, then $B = 2\Delta OVV'$, where for any points P, Q, and R of physical space $\Delta PQR = (1/2) PQ \wedge PR$ is the **oriented area** of the triangle with vertices P, Q and R taken in alphabetical order. (In particular if $B = v \wedge v'$, then $v' \wedge v = (-B)$). The plane through the Origin $\mathbf{\Pi} = \mathbf{R}v \oplus \mathbf{R}v'$, that is determined by the triangle with vertices O, V and V', is the **plane of** B, for which $\mathbf{\Pi}^{(2)} = \mathbf{R}B$. We define $\mathbf{E}^{(2)}$ to be a linear structure consisting of the set of all bivectors $\{B = v \wedge v' : v \text{ and } v' \text{ in } \mathbf{E}\}$. We show, immediately below, how to define the linear operations on $\mathbf{E}^{(2)}$, noting that, in particular, for any non-zero bivector B, the linear substructure $\mathbf{R}B$ of $\mathbf{E}^{(2)}$ is a one-dimensional linear subspace.

$\mathbf{E}^{(2)}$ contains the zero bivector; $\mathbf{0} = v \wedge v$, say, for any v in **E**. Scalar multiplication of bivectors in $\mathbf{E}^{(2)}$ is defined by identifying rB in $\mathbf{E}^{(2)}$ with rB in $\mathbf{R}B$ for all scalars r and bivectors B. Let $\mathbf{\Pi}$ be a plane through the origin and let B_1 and B_2 be a pair of elements of $\mathbf{\Pi}^{(2)}$. Then $(B_1 + B_2)$ in $\mathbf{E}^{(2)}$ is identified with $(B_1 + B_2)$ in the one dimensional linear space $\mathbf{\Pi}^{(2)}$. Otherwise, let $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$ be distinct planes of bivectors B_1 and B_2, respectively. Then $\mathbf{\Pi}_1 \cap \mathbf{\Pi}_2 \neq \mathbf{O}$, for otherwise the subspace $\mathbf{\Pi}_1 + \mathbf{\Pi}_2$ of the three-dimensional linear space **E** would be a direct sum of a pair of two-dimensional linear spaces and thus be four-dimensional. Hence, $\mathbf{\Pi}_1 \cap \mathbf{\Pi}_2 = \mathbf{R}v$ for some non-zero vector v and there are vectors w_1 in $\mathbf{\Pi}_1$ and w_2 in $\mathbf{\Pi}_2$, for which $B_1 = v \wedge w_1$ and $B_2 = v \wedge w_2$; the bivector sum $B_1 + B_2$ in $\mathbf{E}^{(2)}$ is *defined* to be $v \wedge (w_1 + w_2)$.

LEMMA 4.6

Let $\mathbf{E}$ *be the linear space of vectors and let* $\mathbf{E}^{(2)}$ *be the linear structure consisting of the set of all bivectors* $\{v\wedge v' : v \text{ and } v' \text{ in } \mathbf{E}\}$. *Then bivector addition in* $\mathbf{E}^{(2)}$ *is well defined, commutative and satisfies* $r(B_1 + B_2) = rB_1 + rB_2$, *for bivectors* B_1 *and* B_2.

PROOF SKETCH

If v is replaced by rv, then w_1 and w_2 are replaced by $r^{-1}w_1$ and $r^{-1}w_2$, respectively.
$v\wedge w = v\wedge w'$ if and only if $w - w'$ is an element of $\mathbf{R}v$. The bivector properties described follow from the corresponding properties of vectors. ||

LEMMA 4.7

Let Π_1, Π_2 *and* Π_3 *be the planes of bivectors* B_1, B_2 *and* B_3, *respectively.*
If $\Pi_1\cap\Pi_2\cap\Pi_3 \neq \mathbf{O}$, *then there is a non-zero vector* v *in* $\Pi_1\cap\Pi_2\cap\Pi_3$ *and vectors* w_1 *in* Π_1, w_2 *in* Π_2 *and* w_3 *in* Π_3, *for which* $B_1 = v\wedge w_1$, $B_2 = v\wedge w_2$ *and* $b_3 = B\wedge w_3$.
In this case $(B_1 + B_2) + B_3 = B_1 + (B_2 + B_3) = v\wedge(w_1 + w_2 + w_3)$
Otherwise, $\Pi_1\cap\Pi_2\cap\Pi_3 = \mathbf{O}$ *and there exists an ordered basis* $<v_1, v_2, v_3>$ *of* $\mathbf{E}$, *such that either* $B_1 = v_2\wedge v_3$, $B_2 = v_3\wedge v_1$ *and* $B_3 = v_1\wedge v_2$ *and* $(B_1 + B_2) + B_3 = B_1 + (B_2 + B_3) = 2\Delta V_1V_2V_3$ *or* $B_1 = v_3\wedge v_2$, $B_2 = v_1\wedge v_3$ *and* $B_3 = v_2\wedge v_1$ *and* $(B_1 + B_2) + B_3 = B_1 + (B_2 + B_3) = -2\Delta V_1V_2V_3$, *where* $v_1 = OV_1$, $v_2 = OV_2$, $v_3 = OV_3$, V_1, V_2 *and* V_3, *being points of physical space such that no three of* O, V_1, V_2 *and* V_3 *are collinear.*

PROOF

This is immediate if $\Pi_1\cap\Pi_2\cap\Pi_3 \neq \mathbf{O}$. Otherwise, $\Pi_1\cap\Pi_2\cap\Pi_3 = \mathbf{O}$, but all three of $\Pi_2\cap\Pi_3$, $\Pi_3\cap\Pi_1$ and $\Pi_1\cap\Pi_3$ contain non-zero elements.
$\Pi_1\cap\Pi_2\cap\Pi_3 = \mathbf{O}$, but $\Pi_2\cap\Pi_3 \neq \mathbf{O}$, hence there is a non-zero vector w_1 in $\Pi_2\cap\Pi_3$, but not in Π_1 and similarly there are non-zero vectors w_2 in $\Pi_3\cap\Pi_1$, but not in Π_2 and w_3 in $\Pi_1\cap\Pi_2$, but not in Π_3. w_1, w_2 and w_3 are necessarily linearly independent because none of them can be expressed as a linear combination of the other two.
w_2 and w_3 are in Π_1, whence $B_1 = x(w_2\wedge w_3) = -x(w_3\wedge w_2)$ for some non-zero scalar x and similarly $B_2 = y(w_3\wedge w_1) = -y(w_1\wedge w_3)$ and $B_3 = z(w_2\wedge w_3) = -z(w_3\wedge w_2)$ for some non-zero scalars y and z.
Let $t = \sqrt{|xyz|}$ and let $v_1 = (t/x)w_1$, $v_2 = (t/y)w_2$, and $v_3 = (t/z)w_3$. Then, if the product xyz is positive, $B_1 = v_2\wedge v_3$, $B_2 = v_3\wedge v_1$ and $B_3 = v_1\wedge v_2$ and if the product xyz is negative, $B_1 = v_3\wedge v_2$, $B_2 = v_1\wedge v_3$ and $B_3 = v_2\wedge v_1$.
We consider the case t is positive; the case t is negative is analogous:

$$
\begin{aligned}
(v_2\wedge v_3 + v_3\wedge v_1) + v_1\wedge v_2 &= (v_2\wedge v_3 - v_1\wedge v_3) + v_1\wedge v_2 \\
&= (v_2 - v_1)\wedge v_3 + (v_1 - v_2)\wedge v_2 \\
&= (v_2 - v_1)\wedge(v_3 - v_2) \\
&= (v_2 - v_1)\wedge(v_3 - v_2 + v_2 - v_1) \\
&= (v_2 - v_1)\wedge(v_3 - v_1) = V_1V_2\wedge V_1V_3 = 2\Delta V_1V_2V_3 \\
v_2\wedge v_3 + (v_3\wedge v_1 + v_1\wedge v_2) &= v_2\wedge v_3 - (v_1\wedge v_3 - v_1\wedge v_2) \\
&= v_2 \wedge(v_3 - v_2) - v_1\wedge(v_3 - v_2) \\
&= (v_2 - v_1)\wedge(v_3 - v_2) \\
&= (v_2 - v_1)\wedge(v_3 - v_2 + v_2 - v_1) \\
&= (v_2 - v_1)\wedge(v_3 - v_1) = V_1V_2\wedge V_1V_3 = 2\Delta V_1V_2V_3.
\end{aligned}
$$
||

COROLLARY 4.8

Let O *be the origin and let the points* V_1, V_2 *and* V_3 *form the vertices of the base triangle of a tetrahedron with apex* O. *Then* $\Delta V_1V_2V_3 = \Delta V_0V_2V_3 + \Delta V_0V_3V_1 + \Delta V_0V_1V_2$ ||

THEOREM 4.9

Let **E** *be the linear space of vectors and let* $\mathbf{E}^{(2)}$ *be the linear structure consisting of the set of all bivectors* $\{v \wedge v' : v$ and v in $\mathbf{E}\}$. *Then:*

1) $\mathbf{E}^{(2)}$ *is a three-dimensional linear space. Indeed if* $<v_1, v_2, v_3>$ *is an ordered basis of* **E**, *then* $<v_2 \wedge v_3, v_3 \wedge v_1, v_1 \wedge v_2>$ *is an ordered basis of* $\mathbf{E}^{(2)}$.

2) If $<B_1, B_2, B_3>$ *is an ordered basis of* $\mathbf{E}^{(2)}$, *then there is an ordered basis* $<v_1, v_2, v_3>$ *of* **E** *such that either* $B_1 = v_2 \wedge v_3$, $B_2 = v_3 \wedge v_1$ *and* $B_3 = v_1 \wedge v_2$ *or* $B_1 = v_3 \wedge v_2$, $B_2 = v_1 \wedge v_3$ *and* $B_3 = v_2 \wedge v_1$.

3)The map $<v, v'> \mapsto v \wedge v'$ *from ordered pairs of vectors in* **E** *to their Grassmann products in* $\mathbf{E}^{(2)}$ *is anti-symmetric and linear in both arguments.*

PROOF

1) Axioms L1) and L2) for $\mathbf{E}^{(2)}$ follow from axioms L1) and L2) for the one-dimensional linear space $\mathbf{\Pi}^{(2)}$, where $\mathbf{\Pi}$ is any plane through the origin. Axioms L4)(i) and L4(ii) follow from Lemma 4.6 and axiom L3) (associativity of bivector addition) follows from Lemma 4.7.

Let $x_1v_1 + y_1v_2 + z_1v_3$ and $x_2v_1 + y_2v_2 + z_2v_3$ be any pair of vectors of **E**, where $x_1, y_1, z_1, x_2,$ y_2 and z_2 are scalars. Then:

$$(x_1v_1 + y_1v_2 + z_1v_3) \wedge (x_2v_1 + y_2v_2 + z_2v_3)$$
$$= (y_1z_2 - z_1y_2)(v_2 \wedge v_3) + (z_1x_2 - x_1z_2)(v_3 \wedge v_1) + (x_1y_2 - y_1x_2)(v_1 \wedge v_2).$$

Hence the set $\{v_2 \wedge v_3, v_3 \wedge v_1, v_1 \wedge v_2\}$ spans $\mathbf{E}^{(2)}$.

Let $x(v_2 \wedge v_3) + y(v_3 \wedge v_1) + z(v_1 \wedge v_2) = 0$, where x, y and z are scalars. Then x, y and z cannot all be non-zero for otherwise, we could use the construction of lemma 4.7 to find a triangle, whose vertices W_1, W_2 and W_3 are distinct from the Origin and lie on $\mathbf{R}v_1$, $\mathbf{R}v_2$ and $\mathbf{R}v_3$, respectively, such that $x(v_2 \wedge v_3) + y(v_3 \wedge v_1) + z(v_1 \wedge v_2) = \pm 2\Delta W_1W_2W_3 \neq 0$.

Suppose, without loss of generality that $x = 0$. Then $y(v_3 \wedge v_1) + z(v_1 \wedge v_2) = 0$.

It follows that $(yv_3 - zv_2) \wedge v_1 = 0$ so that $(yv_3 - zv_2)$ and v_1 are linearly dependent.

But this is only possible if $yv_3 - zv_2 = 0$, which in turn is only possible if $y = z = 0$.

Hence the set $\{v_2 \wedge v_3, v_3 \wedge v_1, v_1 \wedge v_2\}$ is linearly independent in $\mathbf{E}^{(2)}$.

2) If $<B_1, B_2, B_3>$ is an ordered basis of $\mathbf{E}^{(2)}$ and $\mathbf{\Pi}_1$, $\mathbf{\Pi}_2$ and $\mathbf{\Pi}_3$ are the planes of B_1, B_2 and B_3, respectively, then $\mathbf{\Pi}_1 \cap \mathbf{\Pi}_2 \cap \mathbf{\Pi}_3 = \mathbf{O}$. The required ordered basis of **E**, $<v_1, v_2, v_3>$, is determined by the construction in the proof of Lemma 4.7.

3) The map $<v, v'> \mapsto v \wedge v'$ is antisymmetric because its restriction to any plane subspace of **E** is antisymmetric; i.e. if $<v, v'>$ determines a plane $\mathbf{\Pi}$, then $v \wedge v = -(v' \wedge v)$ in $\mathbf{\Pi}^{(2)}$.

Linearity in the map's second argument follows from (and indeed motivates) the definition of bivector addition and then linearity in its first argument follows from anti-symmetry. ||

Let **E** be the linear space of vectors and let $<v_1, v_2, v_3>$ be an ordered basis of **E**. Then the ordered bases $<v_2 \wedge v_3, v_3 \wedge v_1, v_1 \wedge v_2>$ and $<v_3 \wedge v_2, v_1 \wedge v_3, v_2 \wedge v_1>$ of $\mathbf{E}^{(2)}$ are, respectively, **cocyclic** and **contracyclic**.

Let $f : \mathbf{E} \to \mathbf{E}$ be any function from the linear space of vectors **E** to itself. Then f **induces** the function $f^{(2)} : \mathbf{E}^{(2)} \to \mathbf{E}^{(2)}$, $v \wedge v' \mapsto vf \wedge v'f$ from the **linear space of bivectors** to itself.

LEMMA 4.10

If **E** *is the linear space of vectors, then for any pair of functions* $f : \mathbf{E} \to \mathbf{E}$ *and* $g : \mathbf{E} \to \mathbf{E}$, $(fg)^{(2)} = f^{(2)}g^{(2)}$. *In particular, the identity on* **E** *induces the identity on* $\mathbf{E}^{(2)}$ *and if* f *and* g *are inverse to each other, then* $f^{(2)}$ *and* $g^{(2)}$ *are inverse to each other.* ||

THEOREM 4.11

Let **E** *be the linear space of vectors and let f be a linear endomorphism of* **E**. *Then $f^{(2)}$ is a linear endomorphism of* $\mathbf{E}^{(2)}$; *f is invertible if and only if $f^{(2)}$ is invertible.*

PROOF

Let B be a bivector and r be a scalar. $B = v \wedge v'$ for some vectors v and v'.

$(rB)f^{(2)} = (r(v\wedge v'))f^{(2)} = (rv\wedge v')f^{(2)} = (rv)f\wedge v'f = (r(vf))\wedge v'f = r(vf\wedge v'f) = r(Bf^{(2)})$

Let B_1 and B_2 be bivectors.

For some vector v and vectors v'_1 and v'_2, $B_1 = v\wedge v'_1$ and $B_2 = v\wedge v'_2$.

$$(B_1 + B_2)f^{(2)} = (v\wedge v'_1 + v\wedge v'_2)f^{(2)} = (v\wedge(v'_1 + v'_2))f^{(2)}$$
$$= vf\wedge(v'_1f + v'_2f) = vf\wedge v'_1f + vf\wedge v'_2f) = B_1f + B_2f$$

If $fg = gf$ is the identity on **E**, then $f^{(2)}g^{(2)} = g^{(2)}f^{(2)}$ is the identity on $\mathbf{E}^{(2)}$. The inverse of any linear automorphism is a linear automorphism (Theorem 1.9 2))

If f is not invertible there must a pair of linearly independent vectors, v and v' for which vf and $v'f$ are linearly dependent; $v\wedge v' \neq 0$, but $(v\wedge v')f^{(2)} = vf\wedge v'f = 0$. ||

Let $\mathbf{E}^{(2)}$ be the linear space of bivectors. Then $\mathbf{E}^{(2)}(3\times1) = \{\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$: B_1, B_2 and B_3 in $\mathbf{E}^{(2)}\}$.

$\mathbf{E}^{(2)}(3\times1)$ is the nine-dimensional linear space of (3×1) column matrices with entries from $\mathbf{E}^{(2)}$. The elements of $\mathbf{E}^{(2)}(3\times1)$ are **oriented face-parallelepipeds** of **E**. $\mathbf{E}^{(2)}(3\times1)$ has zero element $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, scalar multiplication $r\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} rB_1 \\ rB_2 \\ rB_3 \end{bmatrix}$ and addition $\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} + \begin{bmatrix} B'_1 \\ B'_2 \\ B'_3 \end{bmatrix} = \begin{bmatrix} B_1 + B'_1 \\ B_2 + B'_2 \\ B_3 + B'_3 \end{bmatrix}$.

If $\{v_1, v_2, v_3\}$ is a basis of $\mathbf{E}^{(2)}$, then $\{\begin{bmatrix} B_1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} B_2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} B_3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ B_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ B_2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ B_3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ B_1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ B_2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ B_3 \end{bmatrix}\}$ is a basis of $\mathbf{E}^{(2)}(3\times1)$. An oriented face-parallelepiped $\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$ of **E** is **degenerate** if B_1, B_2 and B_3 are linearly dependent. Otherwise $\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$ is **non-degenerate**. $\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$ is non-degenerate if and only if $<B_1, B_2, B_3>$ is an ordered basis of $\mathbf{E}^{(2)}$. (Face-parallelepiped is a non-standard notation).

There is a left action of $\mathbf{R}(3)$ on $\mathbf{E}^{(2)}(3\times1)$ and there is a right action of $\mathbf{gl}(\mathbf{E}^{(2)})$ on $\mathbf{E}^{(2)}(3\times1)$:

$$\begin{bmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} X_1B_1 + YB_2 + Z_1B_3 \\ X_2B_1 + Y_2B_2 + Z_2B_3 \\ X_3B_1 + Y_3B_2 + Z_3B_3 \end{bmatrix} \text{ and } \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}F = \begin{bmatrix} B_1F \\ B_2F \\ B_3F \end{bmatrix}.$$

Let **E** be the linear space of bivectors and let $\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$ be a non-degenerate oriented face-parallelepiped of **E**. Then the (3×3) matrix $\begin{bmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{bmatrix}$ **represents** the linear endomorphism F of **E** **with respect to** $\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$ if $\begin{bmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}F$.

THEOREM 4.12
Let **E** *be the linear space of vectors and let f be a linear endomorphism of* **E**. *Then for any ordered basis* $<v_1, v_2, v_3>$ *of* **E**, $f^{(2)}$ *maps the ordered basis* $<v_2\wedge v_3, v_3\wedge v_1, v_1\wedge v_2>$ *of* $\mathbf{E}^{(2)}$ *to the ordered triple* $<v_2f\wedge v_3f, v_3f\wedge v_1f, v_1f\wedge v_2f>$ *of* $\mathbf{E}^{(2)}$ *which is also an ordered basis of* $\mathbf{E}^{(2}$ *if and only if f is a linear automorphism.*
If $g = -f$, *then* $g^{(2)} = f^{(2)}$.
If f is represented by a matrix $\pm M$ *with respect to the non-degenerate parallelepiped* $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ *of* **E**, *then* $f^{(2)}$ *is represented by* $M^{(2)}$ *with respect to the non-degenerate face-parallelepiped* $\begin{bmatrix} v_2\wedge v_3 \\ v_3\wedge v_1 \\ v_1\wedge v_2 \end{bmatrix}$ *of* $\mathbf{E}^{(2)}$, *(and also with respect to the non-degenerate face-parallelepiped* $\begin{bmatrix} v_3\wedge v_2 \\ v_1\wedge v_3 \\ v_2\wedge v_1 \end{bmatrix}$ *of* $\mathbf{E}^{(2)}$*), where* $M^{(2)}$ *is the adjugate matrix of* $\pm M$.
PROOF
Recall that if $M = \pm\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$ then $M^{(2)} = \begin{bmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{bmatrix}$, where $X_i = (y_jz_k - z_jy_k)$, $Y_i = (z_jx_k - x_jz_k)$ and $Z_i = (x_jy_k - y_jx_k)$ and the sequence $<i, j, k>$ is an even permutation of the sequence $<1, 2, 3>$.
$(x_jv_1 + y_jv_2 + z_jv_3)\wedge(x_kv_1 + y_kv_2 + z_kv_3) = X_i(v_2\wedge v_3) + Y_i(v_3\wedge v_1) + Z_i(v_1\wedge v_2)$ etc.
$v(-f)\wedge v'(-f) = (-vf)\wedge(-v'f) = (-1)(-1)(vf\wedge v'f) = vf\wedge v'f$ ||

LEMMA 4.13
Let M be a (3×3) *matrix such that* $\det(M) = \Delta$ *and let* $M^{(2)}$ *be the adjugate of M. Then* $\det(M^{(2)}) = \Delta^2$.
PROOF
Let $M = \pm\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$ and $M^{(2)} = \begin{bmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{bmatrix}$.
Then $M((M^{(2)})^T) = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}\begin{bmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{bmatrix}^T = \begin{bmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix}$.
Hence $\det(M((M^{(2)})^T)) = \det(M)\det((M^{(2)})^T) = \Delta^3$.
Hence $\Delta\det((M^{(2)})^T) = \Delta^3$.
Hence $\det(M^{(2)}) = \det((M^{(2)})^T) = \Delta^2$. ||

THEOREM 4.14
Let **E** *be the linear space of vectors and let f be a linear endomorphism of* **E**. *Let* $f^{(2)}$ *be the linear endomorphism of* $\mathbf{E}^{(2)}$ *induced by f. Then* $\det(f^{(2)}) = (\det(f))^2 > 0$. ||

THEOREM 4.15
Let **E** *be the linear space of vectors and let F be a linear automorphism of* $\mathbf{E}^{(2)}$. *Then for some linear automorphism f of* **E** *either* $F = f^{(2)}$ *or* $F = -(f^{(2)})$. *If* $F = f^{(2)}$, *then F maps cocyclic ordered bases of* $\mathbf{E}^{(2)}$ *to cocyclic ordered bases and maps contracyclic ordered bases of* $\mathbf{E}^{(2)}$ *to contracyclic ordered bases. If* $F = -(f^{(2)})$, *then F maps cocyclic ordered bases of* $\mathbf{E}^{(2)}$ *to contracyclic ordered bases and maps contracyclic ordered bases of* $\mathbf{E}^{(2)}$ *to cocyclic ordered bases.*

PROOF

Let $<v_1, v_2, v_3>$ be an ordered basis of **E** and let F map the cocyclic ordered basis $<v_2\wedge v_3, v_3\wedge v_1, v_1\wedge v_2>$ of $\mathbf{E}^{(2)}$ to a cocyclic ordered basis $<w_2\wedge w_3, w_3\wedge w_1, w_1\wedge w_2>$ of $\mathbf{E}^{(2)}$ or to a contracyclic oriented basis $<w_3\wedge w_2, w_1\wedge w_3, w_2\wedge w_1>$ of $\mathbf{E}^{(2)}$; in either case $<w_1, w_2, w_3>$ is an ordered basis of **E**, whence there is a linear automorphism f of **E**, which maps the ordered basis $<v_1, v_2, v_3>$ to the ordered basis $<w_1, w_2, w_3>$.

Let $v\wedge v' = (xv_1 + yv_2 + zv_3)\wedge(x'v_1 + y'v_2 + z'v_3)$

$$= (yz' - zy')(v_2\wedge v_3) + (zx' - xz')(v_3\wedge v_1) + (xy' - yx')(v_1\wedge v_2)$$

be an arbitrary element of $\mathbf{E}^{(2)}$.

Then in the former case:

$$(v\wedge v')F = (yz' - zy')(w_2\wedge w_3) + (zx' - xz')(w_3\wedge w_1) + (xy' - yx')(w_1\wedge w_2)$$

$$= (xw_1 + yw_2 + zw_3)\wedge(x'w_1 + y'w_2 + z'w_3) = vf\wedge v'f = (v\wedge v')f^{(2)};$$

in the latter case $(v\wedge v')F = v'f\wedge vf = (v\wedge v')(-f^{(2)})$.

The remainder of the theorem is proved by considering the action of F on the contracyclic oriented basis $<v_3\wedge v_2, v_1\wedge v_3, v_2\wedge v_1>$ of $\mathbf{E}^{(2)}$. ||

COROLLARY 4.16

Let **E** *be the space of all vectors. Then two ordered bases of* $\mathbf{E}^{(2)}$ *have the same orientation if they are both cocyclic or if they are both contracyclic, whilst they have opposite orientation if one is cocyclic and the other is contracyclic.*

PROOF

For any linear automorphism f of **E**, $f^{(2)}$ is orientation-preserving and $(-f^{(2)})$ is orientation-reversing on $\mathbf{E}^{(2)}$ by theorem 4.14. ||

Let **E** be the space of all vectors and $\mathbf{E}^{(2)}$ be the linear space of bivectors. Let $<b_1, b_2, b_3>$ be an ordered basis of $\mathbf{E}^{(2)}$. Then $<b_1, b_2, b_3>$ is well defined to have **cocyclic orientation** if and only if $<b_1, b_2, b_3>$ is cocyclic and to have **contracyclic orientation** if and only if $<b_1, b_2, b_3>$ is contracyclic.

THEOREM 4.17

Let **E** *be the linear space of vectors and let* $F = f^{(2)}$ *be an orientation-preserving linear automorphism of* $\mathbf{E}^{(2)}$*. Let* $M^{(2)} = \begin{bmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{bmatrix}$ *represent* F *with respect to some non-degenerate face-parallelepiped* $\begin{bmatrix} v_2\wedge v_3 \\ v_3\wedge v_1 \\ v_1\wedge v_2 \end{bmatrix}$ *of* $\mathbf{E}^{(2)}$*. (equivalently,* $M^{(2)}$ *represents* F *with respect to the non-degenerate face-parallelepiped* $\begin{bmatrix} v_3\wedge v_2 \\ v_1\wedge v_3 \\ v_2\wedge v_1 \end{bmatrix}$ *of* $\mathbf{E}^{(2)}$*). Let* $0 < \det(F) = \Delta^2$.

Then $M = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$ *represents either* f *or* $(-f)$ *with respect to the non-degenerate parallelepiped* $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$*, and is orientation-preserving or orientation-reversing according as* Δ *is positive or negative;* $x_i = (Y_jZ_k - Z_jY_k)/\Delta$*,* $y_i = (Z_jX_k - X_jZ_k)/\Delta$ *and* $z_i = (X_jY_k - Y_jX_k)/\Delta$*, where the sequence* $<i, j, k>$ *is an even permutation of the sequence* $<1, 2, 3>$*.*

PROOF
For any (3×3) matrix M with determinant Δ:
$\mathrm{adj}(\mathrm{adj}(M))(\mathrm{adj}(M)^T) = \Delta^2 I$, where I is the (3×3) identity matrix, because $\det(\mathrm{Adj}(M)) = \Delta^2$.
$M(\mathrm{adj}(M)^T) = \Delta I$, whence $\mathrm{adj}(\mathrm{adj}(M)) = \Delta M = (-\Delta)(-M) = \mathrm{adj}(\mathrm{adj}(-M))$. ||

5) The Grassmann Algebra

We extend the definition of the Grassman product as follows. Let **E** be the linear space of vectors, let v be a vector and let $B = w \wedge w'$ be a bivector, where w and w' are vectors. Then we produce a trivector: $v \wedge B = v \wedge w \wedge w' = w \wedge w' \wedge v = B \wedge v$.

Let **E** be the linear space of vectors. Then **E** determines the linear space Mult(**E**) of **multivectors**, where the **zero multivector** is of all **grades** and otherwise:

a **multivector of grade zero** is a scalar;
a **multivector of grade one** is a vector;
a **multivector of grade two** is a bivector;
a **multivector of grade three** is a trivector;

Mult(**E**) = $\mathbf{E}^{(0)} \oplus \mathbf{E}^{(1)} \oplus \mathbf{E}^{(2)} \oplus \mathbf{E}^{(3)}$, where $\mathbf{E}^{(0)} = \mathbf{R1}$ is a copy $\{r1 : r \text{ in } \mathbf{R}\}$ of the scalars **R** considered as a *one*-dimensional linear space, $\mathbf{E}^{(1)} = \mathbf{E}$ is the *three*-dimensional linear space of vectors, $\mathbf{E}^{(2)}$ is the *three*-dimensional linear space of bivectors and $\mathbf{E}^{(3)}$ is the *one*-dimensional **linear space of trivectors**. It follows that Mult(**E**) is an *eight*-dimensional linear space. ($\mathbf{E}^{(0}$ is also called the **linear space of** 0-**vectors**).

Let **E** be the linear space of vectors and let f be a linear endomorphism of **E**. Then we define $f^{(0)}$ to be the identity on $\mathbf{E}^{(0)}$, $f^{(1)}$ to be f, itself, $f^{(2)}$ to be the linear endomorphism that f induces on the bivectors and $f^{(3)}$ to be the linear endomorphism (specifically scalar multiplication by $\det(f)$), that f **induces** on the trivectors.

LEMMA 5.1
Let **E** *be the space of all vectors and let f be a linear endomorphism of* **E**. *Then f induces a linear endomorphism* f_1 *of* Mult(**E**)*, specifically if r is scalar, v is a vector, B is a bivector and* τ *is a trivector* $(r\mathbf{1} + v + B + \tau)f_1 = r\mathbf{1}f^{(0)} + vf^{(1)} + Bf^{(2)} + \tau f^{(3)}$. *If f is a linear automorphism then* f_1 *is a linear automorphism.* ||

We define the **Grassmann product** of elements of Mult(**E**):

$$(r\mathbf{1} + v + B + \tau)\wedge(r'\mathbf{1} + v' + B' + \tau')$$
$$= (rr')\mathbf{1} + r(v' + B' + \tau') + r'(v + B + \tau) + v\wedge v' + b\wedge v' + v\wedge b'.$$

We note that taking the Grassmann product of any element m of Mult(**E**) with a scalar $r\mathbf{1}$, on either side, is equivalent to scalar multiplication by r. $(r\mathbf{1})\wedge m = m\wedge(r\mathbf{1}) = rm$.

THEOREM 5.2
Let **E** *be the space of all vectors. Then the linear space* Mult(**E**) *of multivectors is a unital and associative algebra under the Grassmann product.*
PROOF SKETCH
The Grassmann product is linear in both arguments because by definition it is a sum of terms, which are linear in both arguments.

This is not quite trivial for $B\wedge v'$ or $v\wedge B'$. We show that $(B_1 + B_2)\wedge v' = B_1\wedge v' + B_2\wedge v'$.
Let v be any vector in the intersection of the planes of the bivectors B_1 and B_2.
Then $B_1 = v\wedge w_1$ and $B_2 = v\wedge w_2$ for some vectors w_1 and w_2.

$$(B_1 + B_2)\wedge v' = (v\wedge w_1 + v\wedge w_2)\wedge v' = v\wedge(w_1 + w_2)\wedge v'$$
$$= v\wedge w_1\wedge v' + v\wedge w_2\wedge v' = B_1\wedge v' + B_2\wedge v'.$$

We have implicitly used associativity, which follows by noting that any three vectors associate under the Grassmann product to form the same trivector. **1** is the identity. ||

Let **E** be the space of all vectors. Then the **Grassmann Algebra of E**, $Gr(\mathbf{E})$ is the linear space of multivectors under the Grassmann product.

An algebra **A** is **graded** if every non-zero element of **A** is a direct sum of linear subspaces $\mathbf{A}^{(0)} \oplus \mathbf{A}^{(1)} \oplus \ldots \oplus \ldots$, where the non-zero elements of a subspace $\mathbf{A}^{(k)}$ are the elements of grade k and if $a^{(m)}$ and $a^{(n)}$ are elements of $\mathbf{A}^{(m)}$ and $\mathbf{A}^{(n)}$, respectively, then the products $a^{(m)}a^{(n)}$ and $a^{(n)}a^{(m)}$ are elements of $\mathbf{A}^{(m+n)}$. The product has an **anti-commutative structure** if $a^{(m)}a^{(n)} = (-)^{mn}(a^{(n)}a^{(m)})$.

THEOREM 5.3
Let **E** *be the linear space of vectors. Then the Grassmann algebra of* **E** *is graded with no non-zero elements of grade greater than three. The Grassmann product has an anti-commutative structure.* ||

LEMMA 5.4
Let **E** *be the linear space of vectors and let f be a linear endomorphism of* **E**. *Let $e^{(m)}$ and $e^{(n)}$ be elements of $\mathbf{E}^{(m)}$ and $\mathbf{E}^{(n)}$, respectively. Then $(e^{(m)}f^{(m)})\wedge(e^{(n)}f^{(n)}) = (e^{(m)}\wedge e^{(n)})f^{(m+n)}$, where $f^{(m+n)}$ is the zero map if $(m + n) > 3$.* ||

THEOREM 5.5
Let **E** *be the linear space of vectors and let f be a linear endomorphism of* **E**. *Then f induces an algebra endomorphism f_1 of $Gr(\mathbf{E})$; specifically if r is scalar, v is a vector, B is a bivector and τ is a trivector then $(r\mathbf{1} + v + B + \tau)f_1 = r\mathbf{1}f^{(0)} + vf^{(1)} + Bf^{(2)} + \tau f^{(3)}$. If f is a linear automorphism, then f_1 is an algebra automorphism.*
PROOF
f_1 is a linear endomorphism by lemma 5.1. Using lemma 5.4:
$((r\mathbf{1} + v + B + \tau)\wedge(r'\mathbf{1} + v' + B' + \tau'))f_1 =$
$((rr')\mathbf{1} + r(v' + B' + \tau') + r'(v + B + \tau) + v\wedge v' + b\wedge v' + v\wedge b')f_1 =$
$(rr')\mathbf{1}f^{(0)} + r(v'f^{(1)} + B'f^{(2)} + \tau'f^{(3)}) + r'(vf^{(1)} + Bf^{(2)} + \tau f^{(3)}) + v\wedge v'f^{(2)} + b\wedge v'f^{(3)} + v\wedge b'f^{(3)} =$
$(rr')\mathbf{1}f^{(0)} + r(v'f^{(1)} + B'f^{(2)} + \tau'f^{(3)}) + r'(vf^{(1)} + Bf^{(2)} + \tau f^{(3)})$
$+ vf^{(1)}\wedge v'f^{(1)} + Bf^{(2)}\wedge v'f^{(1)} + vf^{(1)}\wedge B'f^{(2)} =$
$((r\mathbf{1}f^{(0)} + vf^{(1)} + Bf^{(2)} + \tau f^{(3)})\wedge(r'\mathbf{1}f^{(0)} + v'f^{(1)} + B'f^{(2)} + \tau'f^{(3)})) =$
$((r\mathbf{1} + v + B + \tau)f_1)\wedge((r'\mathbf{1} + v' + B' + \tau')f_1)$. ||

THEOREM 5.6
Let **E** *be the linear space of vectors and let $Gr(\mathbf{E})$ be the Grassmann algebra of* **E**. *Then $\mathbf{E}^{(0)} = \mathbf{R1}$ is a one-dimensional central subalgebra of $Gr(\mathbf{E})$, $\mathbf{E}^{(0)} \oplus \mathbf{E}^{(2)} = \mathbf{R1} \oplus \mathbf{E}^{(2)}$ is a four-dimensional subalgebra of $Gr(\mathbf{E})$ and $\mathbf{E}^{(0)} \oplus \mathbf{E}^{(3)} = \mathbf{R1} \oplus \mathbf{E}^{(3)}$ is a two-dimensional subalgebra of $Gr(\mathbf{E})$.* ||

The scalars $\mathbf{E}^{(0)} = \mathbf{R1}$ may, if desired, be identified with the real numbers $\mathbf{R}$, which then become a central subalgebra of all the subalgebras of $Gr(\mathbf{E})$. Taking Grassmann products with $r\mathbf{1}$ i.e. $(r\mathbf{1})\wedge m = m\wedge(r\mathbf{1})$, is identified with scalar multiplication by r, i.e. rm in Mult($\mathbf{E}$).

Let $\mathbf{S}$ be a linear space. Then μ is a **linear involution** of $\mathbf{S}$ if it is a linear automorphism whose square is the identity, i.e. $s(\mu)^2 = (s\mu)\mu = s$. If μ is a linear involution of $\mathbf{S}$, it determines a direct sum decomposition $\mathbf{S} = \mathbf{S}\mu^+ \oplus \mathbf{S}\mu^-$, into **characteristic subspaces**, where for s in $\mathbf{S}$, $s = ½(s + s\mu) + ½(s - s\mu)$ in $\mathbf{S}\mu^+ \oplus \mathbf{S}\mu^-$. $\mathbf{S}\mu^+$, the **preserved characteristic subspace** of μ, consists of those elements of $\mathbf{S}$ for which $s\mu = s$ and $\mathbf{S}\mu^-$, the **reversed characteristic subspace** of μ, consists of those elements of $\mathbf{S}$ for which $s\mu = (-s)$. A linear involution μ is **non-trivial** if neither $\mathbf{S}\mu^+$ nor $\mathbf{S}\mu^-$ is $\mathbf{O}$. Equivalently μ is non-trivial if μ is not scalar multiplication by 1 or by (-1).

THEOREM 5.7
Let $\mathbf{E}$ *be the linear space of vectors,* Mult($\mathbf{E}$) *the linear space of Multivectors of* $\mathbf{E}$ *and* $Gr(\mathbf{E})$ *the Grassmann algebra of* $\mathbf{E}$*. Then*
1) Mult($\mathbf{E}$)*admits a linear involution* $m \mapsto m^R$ *for which the preserved characteristic subspace is* $\mathbf{E}^{(0)} \oplus \mathbf{E}^{(1)}$*. It is an algebra anti-automorphism of* $Gr(\mathbf{E})$.
2) Mult($\mathbf{E}$) *admits a linear involution* $m \mapsto (-)^{\mathrm{grade}(m)}m$ *for which the preserved characteristic subspace is* $\mathbf{E}^{(0)} \oplus \mathbf{E}^{(2)}$*. It is an algebra automorphism of* $Gr(\mathbf{E})$ *and its preserved characteristic subspace is a subalgebra of* $Gr(\mathbf{E})$.
3) Mult($\mathbf{E}$) *admits a linear involution* $m \mapsto (-)^{\mathrm{grade}(m)}m^R$*, which is the composition, in either order, of the linear involutions* $m \mapsto m^R$ *and* $m \mapsto (-)^{\mathrm{grade}(m)}m$ *and for which the preserved characteristic subspace is* $\mathbf{E}^{(0)} \oplus \mathbf{E}^{(3)}$*. It is an algebra anti-automorphism of* $Gr(\mathbf{E})$ *and its preserved characteristic subspace is a subalgebra of* $Gr(\mathbf{E})$.
PROOF
1) $r\mathbf{1}^R = r\mathbf{1}$, $v^R = v$, $(v\wedge v')^R = (v'\wedge v) = -(v\wedge v')$ and $(v\wedge v'\wedge v'')^R = (v''\wedge v'\wedge v) = -(v\wedge v'\wedge v'')$.
Thus $m \mapsto m^R$, which reverses the Grassmann product, maps the algebra $Gr(\mathbf{E})$ to its opposite algebra. $(e^{(m)}\wedge e^{(n)})^R = (e^{(n)})^R\wedge(e^{(m)})^R$, for $e^{(m)}$ of grade m and $e^{(n)}$ of grade n.
2) $m \mapsto (-)^{\mathrm{grade}(m)}m$ is the algebra automorphism of $Gr(\mathbf{E})$ induced by scalar multiplication of vectors by (-1).
3) That $m \mapsto (-)^{\mathrm{grade}(m)}m^R$ is as described is immediate. ||

Let $<v_1, v_2, v_3>$ be an ordered basis of the space of all vectors $\mathbf{E}$. $<v_1, v_2, v_3>$ has either **right-handed orientation** or **left-handed orientation**, where the handedness is determined by placing the palm of the hand at the origin and letting the thumb point in the direction of v_1, the index finger point in the direction of v_2 and the middle finger point in the direction of v_3. Let $B_i = v_j\wedge v_k$ and let $\tau = v_j\wedge v_j\wedge v_k$ for all of the three even permutations $<i, j, k>$ of $<1, 2, 3>$. Then $<\mathbf{1}, v_1, v_2, v_3, B_1, B_2, B_3, \tau>$ is the **cocyclic ordered basis** of Mult($\mathbf{E}$) and $<\mathbf{1}, v_1, v_2, v_3, (-B_1), (-B_2), (-B_3), (-\tau)>$ is the **contracyclic ordered basis** of Mult($\mathbf{E}$) **induced** by the ordered basis $<v_1, v_2, v_3>$ and the Grassmann product.

THEOREM 5.8
Let $m \mapsto m^R$*,* $m \mapsto (-)^{\mathrm{grade}(m)}m$ *and* $m \mapsto (-)^{\mathrm{grade}(m)}m^R$ *be the linear involutions of* Mult($\mathbf{E}$) *described in the immediately preceding theorem. Then*
1) $m \mapsto m^R$ *preserves the orientation of* $\mathbf{E}$ *and reverses that of* $\mathbf{E}^{(2)}$ *and* $\mathbf{E}^{(3)}$*.*
2) $m \mapsto (-)^{\mathrm{grade}(m)}m$ *preserves the orientation of* $\mathbf{E}^{(2)}$ *and reverses that of* $\mathbf{E}$ *and* $\mathbf{E}^{(3)}$*.*
3) $m \mapsto (-)^{\mathrm{grade}(m)}m^R$ *preserves the orientation of* $\mathbf{E}^{(3)}$ *and reverses that of* $\mathbf{E}$ *and* $\mathbf{E}^{(2)}$*.* ||