Argand's Proof of the Fundamental Theorem of Algebra

Argand, in his paper of 1806 defines complex numbers as directed lines in a plane and uses this notion to prove the Fundamental Theorem of Algebra.

THEOREM (The Fundamental Theorem of Algebra)
Let $f(z)$ be a non-constant polynomial in z with complex coefficients. Then the equation $f(z) = 0$ has a complex root.

Argand shows that if $f(z)$ is a non-constant polynomial in z with complex coefficients and ζ in **C** satisfies $|f(\zeta)| > 0$, then for some ζ' in **C**, $|f(\zeta')| < |f(\zeta)|$. This proves the fundamental theorem on the assumption that $|f(z)|$ attains its greatest lower bound for some value of z, which Argand assumes.

LEMMA
Let $f(z)$ be a non-constant polynomial in z with complex coefficients and let μ be the greatest lower bound of the set $\{|f(z)| : z \text{ in } \mathbf{C}\}$. Then for some z_0 in **C**, *$|f(z_0)| = \mu$.*
PROOF
Suppose not.
Then there is a sequence $<z_n>$ of elements of **C**, for which $\mu < |f(z_n)| < \mu + 1/n$.
The elements z_n may be selected from the compact disc $\{|z| \leq M : z \text{ in } \mathbf{C}\}$, where M is chosen large enough so that $|f(z)| > \mu + 1$, if $|z| > M$. (Note that for large enough M, $|f(z)| = O(|z|^{\deg f})$ for $|z| > M$).
The sequence $<z_n>$ has a convergent subsequence $<z_i>$ whose limit we may take as z_0.
f is continuous, whence the sequence $<f(z_i)>$ converges to $f(z_0)$.
Finally $|f(z_0)| = \lim |f(z_i)| = \mu$, for contradiction. ||

LEMMA (Argand 1806 and 1814)
Let $f(z)$ be a non-constant polynomial in z with complex coefficients and let ζ in **C** *satisfy $|f(\zeta)| > 0$. Then for some ζ' in* **C**, *$|f(\zeta')| < |f(\zeta)|$.*
PROOF
Let ω be a unit magnitude complex number and let t be a small real number.
Then $f(\zeta + \omega t) = f(\zeta) + A_m\omega^m t^m + o(t^m)$, where A_m is the first non-vanishing coefficient in the expansion of $f(\zeta + \omega t) - f(\zeta)$, (which exists because $f(z)$ is non-constant).
We may always choose the direction of ω so that the direction of $A_m\omega^m$ is opposite to the direction of $f(\zeta)$. (This is Argand's trick).
Let $\zeta' = \zeta + \omega t$, where t is small enough so that terms in t^{m+1} may be neglected and $|A_m\omega^m t^m| < |f(\zeta)|$. Then $|f(\zeta')| = |f(\zeta) + A_m\omega^m t^m| = |f(\zeta)| - |A_m\omega^m t^m| < |f(\zeta)|$. ||