Complex Analysis

Summary

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Summary of the Lecture Complex Analysis
in the Autumn Semester 2020 given by Afonso Bandeira

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Abstract

This is a summary of the topics dealt with in the lecture *Complex Analysis* in the autumn semester 2020 given by Afonso Bandeira at ETH Zürich. This script is based on the lecture notes and the book *Complex Analysis* by Lars Ahlfors.[1] The list of topics is not exhaustive. Many things were left out and only the topics which the author regarded as important are mentioned. This summary should neither be considered as a replacement of the lecture nor as a sufficient preparation for the exam. This summary should only be a reminder to which you can resort in case you quickly want to look something up. No liability is accepted in the event of failure to pass the examination.

If you stumble over mistakes, be it linguistic or thematic, or if you have suggestions what to add or how to improve this script, do not hesitate to contact me at szekerb@student.ethz.ch.

Many Thanks

Balázs Szekér

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[1][Ahlfors] - L. Ahlfors: "Complex analysis. An introduction to the theory of analytic functions of one complex variable." International Series in Pure and Applied Mathematics. McGraw-Hill Book Co. Link to an online PDF: `http://people.math.gatech.edu/~mccuan/courses/6321/lars-ahlfors-complex-analysis-third-edition-mcgraw-hill-science_engineering_math-1979.pdf`

Contents

1   Complex Functions

1.1   Basics

Theorem. $(\cos(\phi) + i\sin(\phi))^n = \cos(n\phi) + i\sin(n\phi)$

Theorem (Riemann Sphere)**.** *Represent $\mathbb{C}$ on the unit Sphere $S^2 \subseteq \mathbb{R}^3$ and vice versa. Representation $S^2 \to \mathbb{C}$:*

$$z = \frac{x_1 + ix_2}{1 - x_3}$$

Representation $\mathbb{C} \to S^2$:

$$x_1 = \frac{z - \overline{z}}{i\left(1 + |z|^2\right)}$$

$$x_2 = \frac{z + \overline{x}}{1 + |z|^2}$$

$$x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$$

Definition (Limit)**.** *The function $f(x)$ is said to have a **limit** A as x tends to a*

$$\lim_{x \to a} f(x) = A$$

if and only if the following is true: For every $\epsilon > 0$ there exists a number $\delta > 0$ with the property that for $x \neq a$: $|x - a| < \delta \implies |f(x) - A| < \epsilon$.

Theorem (Limes Rules)**.** *The following are true:*

$$\lim_{x \to a} \overline{f}(x) = \overline{A}$$
$$\lim_{x \to a} Re(f(x)) = Re(A)$$
$$\lim_{x \to a} Im(f(x)) = Im(A)$$

We say f is continuous at a if

$$\lim_{x \to a} f(x) = f(a)$$

1.2   Analytic Functions

Definition (Derivative)**.** *The **derivative** of a function f is defined as*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Theorem. *Let $f : \mathbb{C} \to \mathbb{R}$ be a function. If we now want to calculate the derivative, of f at a point z_0, then we can approach z_0 from different sides, in particular purely imaginary and purely real. That means:*

$$f'(z_0) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(z_0 + h) - f(z_0)}{h} \in \mathbb{R}$$

$$f'(z_0) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(z_0 + h) - f(z_0)}{ih} \in \mathbb{C}$$

In order for this to be true, $f'(z_0)$ has to be zero. Thus, a real function of a complex variable either has derivative zero, or else the the derivative does not exist.

Theorem. *If we write $z(t) = x(t) + iy(t)$ we find $z'(t) = x'(t) + iy'(t)$.*

Definition (Analytic/Holomorphic). *A complex function f is called **analytic** or **Holomorphic** if f possesses a derivative wherever the function is defined.*

Definition (Cauchy-Riemann Equations). *Let f be an analytic function. If we write $f(z) = f(x,y) = u(x) + iv(y)$ we obtain the **Cauchy-Riemann Equations**.*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Derivation:

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$f'(z) = \lim_{k \to 0} \frac{f(z+ik) - f(z)}{ik} = -i\frac{\partial f}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Lemma. *All analytic functions fulfil the Cauchy-Riemann Equations.*

Theorem. *Derivatives of analytic functions are again analytic.*

Definition (Harmonic). *A function f that fulfils the Laplace equation $\Delta f = 0$ is called **harmonic**.*

Theorem. *All analytic functions are harmonic. In particular for $f = u + iv$, both u and v are harmonic.*

Definition (Harmonic Conjugate). *If two harmonic functions u and v satisfy the Cauchy-Riemann equations, then v is the **harmonic conjugate** of u.*

Theorem. *If $u(x,y)$ and $v(x,y)$ have continuous first-order partial derivatives which satisfy the Cauchy-Riemann differential equations, then $f(z) = u(z) + iv(z)$ is analytic with continuous derivative $f'(z)$, and conversely.*

Theorem (Luca's Theorem). *If all zeros of a polynomial $P(z)$ lie in a half plane, then all zeros of the derivative $P'(z)$ lie in the same half plane.*

Theorem. *Let $P(z)$ be a polynomial of degree n and let $Q(z)$ be a polynomial of degree m. We define $R(z) = \frac{P(z)}{Q(z)}$. The following statements are true.*

 (i) *If $m > n$, then $R(z)$ has a zero of order $m - n$ at ∞.*

 (ii) *If $m < n$, then $R(z)$ has a pole of order $n - m$ at ∞.*

 (iii) *If $n = m$, then $R(\infty) = \frac{a_n}{b_n}$.*

1.3   Sequences and Series

Definition. *A sequence $\{a_n\}_1^\infty$ has the **limit** A if to every $\epsilon > 0$ there exists an N such that for $n \geq N$: $|a_n - A| < \epsilon$. A sequence with a finite limit is called **convergent** and if it is not convergent it is **divergent**.*

Definition (Cauchy Sequence). *A sequence $\{a_n\}_1^\infty$ is called a **Cauchy sequence** if it satisfies the following:*

$$\forall \epsilon > 0 \ \exists N \ : \ \forall n, m > N \ : \ |a_n - a_m| < \epsilon$$

Definition (Convergence). *We can define two different types of **convergence**.*

 (i) ***pointwise:*** $\forall \epsilon > 0 \ \forall x \ \exists N : \forall n > N : |f_n(x) - f(x)| < \epsilon$

 (ii) ***uniformal:*** $\forall \epsilon > 0 \ \exists N : \forall x \ \forall n > N : |f_n(x) - f(x)| < \epsilon$

Proposition. *If f_n are continuous and $f_n \to f$ converges uniformally than f if continuous.*

Theorem (Weierstrass M-Test). *Suppose that $(f_n)_n$ is a sequence of functions defined on a set A and that there is a sequence of non negative numbers $(M_n)_n$ satisfying*

$$\forall n \geq 1, \forall x \in A : \ |f_n(x)| \leq M_n \quad and \quad \sum_{n=1}^\infty M_n < \infty$$

Then the series $\sum_{n=1}^\infty f_n(x)$ converges absolutely and uniformaly on A.

Definition (Radius of Convergence)**.** *The **radius of convergence** of a series $\{a_n\}_1^\infty$ is defined as*

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

Theorem (Abel)**.** *For every power series there exists a number R, $0 \le R \le \infty$, called the radius of convergence, with the following properties:*

  (i) *The series converges absolutely for every z with $|z| < R$. If $0 \le \rho < R$ the convergence is uniform for $|z| \le \rho$.*

  (ii) *If $|z| > R$ the terms of the series are unbounded, and the series is consequently divergent.*

  (iii) *In $|z| < R$ the sum of the series is an analytic function. The derivative can be obtained by termwise differentiation, and the derived series has the same radius of convergence.*

1.4 Exponential and Trigonometric Functions

Definition (Complex Logarithm)**.** *For a complex number w, we define the **complex logarithm** as*

$$\log(w) = \log(|w|) + i \arg(w)$$

*As we can see, $\log$ is **multivalued**, since $e^{ix} = e^{ix+i2\pi k}$ $\forall k \in \mathbb{Z}$. Thus*

$$\log(e^{ix}) = \log\left(\left|e^{ix}\right|\right) + i(x + 2\pi k) = \log(1) + i(x + 2\pi k) = i\,(x + 2\pi k) \quad \forall k \in \mathbb{Z}$$

Definition (Inverse Sine and Cosine)**.** *The **inverses** of $\sin$ and $\cos$ are given by*

$$\arccos(w) = -i \log\left(w \pm \sqrt{w^2 - 1}\right) = \pm i \log\left(w + \sqrt{w^2 - 1}\right)$$
$$\arcsin(w) = \frac{\pi}{2} - \arccos(w)$$

Definition (Branch Cut)**.** *A **branch cut** is a part of a set where a multivalues function is not defined. At this point, two branches are separated.*

Example (Square Root)**.** *Let f be $f(z) = \sqrt{z}$. Say $z = re^{i\theta}$. Since $e^{i\theta} = e^{i(\theta+2\pi)}$, we have two legitimate solutions.*

$$\sqrt{z} = \begin{cases} \sqrt{re^{i\theta}} = \sqrt{r}e^{i\theta/2} \\ \sqrt{re^{i(\theta+2\pi)}} = \sqrt{r}e^{i\theta/2 + i\pi} = -\sqrt{r}e^{i\theta/2} \end{cases}$$

*To make $g(w) = w^2$ injective, we have to restrict $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. With this restriction we now can compute $f(z)$ uniquely. We call such a restriction of the domain a **branch**. Note that there are only two branches of the square root, namely $(0, 2\pi)$ and $(2\pi, 4\pi)$. All other branches deliver the same result as one of these two. $(0, 2\pi)$ maps to $(0, \pi)$ and $(2\pi, 4\pi)$ maps to $(\pi, 2\pi)$. All other branches are equivalent to these branches because they are the same modulo 2π.*

Example (Logarithm)**.** *The logarithm is also a multivalued function. Analogously to the squareroot we can define branch cuts anywhere between 0 and 2π. The specialty about the logarithm is, that we have infinitely many branches.*

2   Analytic Functions as Mappings

2.1   Point Set Topology

Definition (Metric Space)**.** *A set S is a **metric space** if there is defined, for every pair $x, y \in S$, a nonnegative real number $d(x, y)$ in such a way that the following conditions are fulfilled:*

1. *$d(x, y) = 0$ if and only if $x = y$*

2. *$d(y, x) = d(x, y)$*

3. *$d(x, z) \leq d(x, y) + d(y, z)$*

Definition (Neighborhood)**.** *A set $Y \subseteq X$ is called a **neighborhood** of $y \in X$ if it contains a ball $B(y, \delta)$.*

Definition (Open Set)**.** *A set is **open** if it is a neighborhood of each of its elements.*

Theorem. *The following stetements are true.*

(i) *The intersection of a finite number of open sets is open.*

(ii) *The union of any collection of open sets is open.*

(iii) *The union of a finite number of closed sets is closed.*

(iv) *The intersection of any collection of closed sets is closed.*

Definition (Interior)**.** *The **interior** $Int(X)$ of a set X is the largest open set contained in X.*

Definition (Closure)**.** *The **closure** X^- or $\overline{X}$ of a set X is the smalles closed set which contains X. A point belongs to the closure of X if and only if all its neighborhoods intersect X.*

Definition (Boundry)**.** *The **boundry** ∂X of X is the closure minus the interior. A point belongs to the boundry of X if and only if all its neighborhoods intersect both X and X^C.*

Definition (Connectedness)**.** *A subset of a metric space is **connected** if it cannot be represented as the union of two disjoint relatively open sets none of which is empty. (A set S if connected if and only if $\nexists\, A, B$ open and non-empty such that $A \cup B = S$ and $A \cap B = \varnothing$.)*

Theorem. *A open set is connected if it cannot be decomposed into two open sets, and a closed set is connected if it cannot be decomposed into two closed sets.*

Theorem. *The nonempty connected subsets of $\mathbb{R}$ are the intervals.*

Theorem. *Any closed and bounded nonempty set of real numbers has a minimum and a maximum.*

Theorem. *A nonempty open set in the plane is connected if and only if any two of its points can be joined by a polygon which lies in the set.*

Definition (Region)**.** *A nonempty connected open set is called **region**.*

Definition (Completeness)**.** *A metric space is said to be **complete** if every Cauchy sequence is convergent.*

Definition (Compactness)**.** *A set X is **compact** if and only if every open covering of X contains a finite subcovering. Equivalent: S is compact if and only if for any open covering of S there exists a finite subcovering O_x open, such that $S \subseteq \bigcup_{x \in X} O_x$.*

Definition (Totally Bounded)**.** *A set X is **totally bounded** if, $\forall \epsilon > 0$, X can be covered by finitely many balls of radius ϵ.*

Theorem. *A set is compact if and only if it is complete and totally bounded.*

Theorem. *A subset of $\mathbb{R}$ or $\mathbb{C}$ is compact if and only if it is closed and bounded.*

Theorem. *A metric space is compact if and only if every infinite sequence has a limit point.*

Definition (Continuity)**.** *A function is **continuous** if and only if the inverse image of every open set is open. Equivalent: A function is continuous if and only if the inverse image of every closed set is closed.*

Theorem. *Under a continuous mapping the image of every compact set is compact, and consequently closed.*

Theorem. *A continuous real-valued function on a compact set has a maximum and a minimum.*

Theorem. *Under a continuous mapping the image of any connected set is connected.*

Definition (Uniform Continuity). *A function is called **uniformally continuous** if*

$$\forall \epsilon > 0 \ \exists \delta > 0 : \forall x_1, x_2 : \ d(x_1, x_2) < \delta \Rightarrow d(f(x_1), f(x_2)) < \epsilon$$

Theorem. *On a compact set every continuous function is uniformally continuous.*

2.2 Conformality

Definition (Arc). *An **arc** is a continuous function γ that maps $t \in [\alpha, \beta] \subseteq \mathbb{R}$ to a plane (or generally on a subset of $\mathbb{R}^n$). For an arc on $\mathbb{C}$ we can define γ as $z(t) = x(t) + iy(t)$ with $x(t), y(t)$ continuous functions.*

Definition (Differentiable Arc). *An arc is said to be **differentiable** if $z'(t)$ exists and is continuous. If $z'(t) \neq 0$, the arc is said to be **regular**. An arc is **piecewise differentiable** or piecewise regular if the same conditions hold exept for a finite number of values t. At these t points $z(t)$ shall still be continuous with left and right derivatives which are equal to the left and right limits of $z'(t)$.*

Definition (Jordan Arc). *An arc is **simple**, or a **Jordan arc**, if $z(t_1) = z(t_2)$ only for $t_1 = t_2$.*

Definition (Closed Curve). *An arc is a **closed arc** if the endpoints coincide: $z(\alpha) = z(\beta)$.*

Definition (Opposite Arc). *The **opposite arc** of $z(t)$ for $\alpha \leq t \leq \beta$ is the arc $z = z(-t)$ for $-\beta \leq t \leq \alpha$. Opposite arcs are sometimes donated by γ and $-\gamma$, sometimes γ and γ^{-1}, depending on the connection.*

Definition (Analytic Function). *A complex-valued function $f(z)$, defined on an open set Ω, is said to be **analytic** or **holomorphic** in Ω if it has a derivative at each point of Ω.*

Theorem. *An analytic function $f : \Omega \to \mathbb{C}$ in a region Ω with $f'(z) = 0 \ \forall z \in \Omega$, then f is constant in Ω. The same is true if either the real part, the imaginary part, the modulus, or the argument is constant.*

Definition (Image of Arcs). *Suppose that an arc γ with the equation $z = z(t)$ for $\alpha \leq t \leq \beta$ is contained in a region Ω, and let $f(z)$ be defined and continuous in Ω. Then the equation $w = w(t) = f(z(t))$ defines an arc γ' in the w-plane which may be called the **image of** γ.*

Theorem. *If $z'(t)$ exists, we find that $w'(t)$ also exists and is determined by $w'(t) = f'(z(t))z'(t)$.*

Theorem. *Let $f : \Omega \to \mathbb{C}$ be analytic and $z_0 \in \Omega$ such that $f'(z_0) \neq 0$ and $z'(t_0) \neq 0$. Then f takes tangent arcs at z_0 into tangent arcs at $f(z_0)$. The function f takes two arcs at a relativ angle θ at t_0 to arcs with the same relative error.*

Proof. Let z_1 and z_2 be two arcs that cross z_0 at t_0. Then the following is true.

$$z_1(t_0) = z_2(t_0)$$
$$f(z_1(t_0)) = w_1(t_0)$$
$$f(z_2(t_0)) = w_2(t_0)$$
$$\arg\left(w_2'(t_0)\right) - \arg(w_1'(t_0)) = \arg(f'(z_2(t_0))z_2'(t_0)) - \arg(f'(z_1(t_0))z_1'(t_0))$$
$$= \arg(f'(z_2(t_0))) + \arg(z_2'(t_0)) - \arg(f'(z_1(t_0))) - \arg(z_1'(t_0))$$
$$= \arg(z_2'(t_0)) - \arg(z_1'(t_0))$$

$\square$

Theorem. *If f is analytic and $f'(z_0) \neq 0$, then f is conformal at z_0.*

3   Complex Integration

3.1   Fundamental Theorems

Theorem. *If $f(z) = u(t) + iv(t)$ is a continuous function, defined on an interval (a,b), then we set*

$$\int_a^b f(t) \ dt = \int_a^b u(t) \ dt + i \int_a^b v(t) \ dt$$

Theorem. *Let γ be a piecewise differentiable arc with the equation $z = z(t)$ with $a \le t \le b$. If the function $f(z)$ is defined and continuous on γ, then $f(z(t))$ is also continuous and we can set*

$$\int_\gamma f(z) \ dz = \int_a^b f(z(t))z'(t) \ dt$$

Theorem. *Let $-\gamma$ be the opposite arc of γ. That means, $-\gamma(z) = \gamma(-z)$. However, for $-\gamma$, t is in the interval $-b \le t \le -a$. Thus:*

$$\int_{-\gamma} f(z) \ dz = - \int_\gamma f(z) \ dz$$

Theorem. *If an arc γ consists of many subarcs $\gamma_1, \ldots, \gamma_n$, then*

$$\int_\gamma f \ dz = \int_{\gamma_1 + \cdots + \gamma_n} f \ dz = \int_{\gamma_1} f \ dz + \cdots + \int_{\gamma_n} f \ dz$$

Theorem.

$$\int_\gamma f \ \overline{dz} = \overline{\int_\gamma \overline{f} \ dz}$$

Theorem. *If $f = u + iv$, then we can write*

$$\int_\gamma f \ dz = \int_\gamma (u \ dx - v \ dy) + i \int_\gamma (u \ dy + v \ dx)$$

Theorem. *If two arcs γ_1 and γ_2 have the same initial points and the same end point, we require*

$$\int_{\gamma_1} p \ dx + q \ dy = \int_{\gamma_2} p \ dx + q \ dy$$

Theorem. *The integral over any closed curve vanishes.*

Theorem. *The line integral $\int_\gamma p \ dx + q \ dy$, defined in Ω, depends only on the end points of γ if and only if there exists a function $U(x,y)$ in Ω with the partial derivatives $\frac{\partial U}{\partial x} = p$, $\frac{\partial U}{\partial y} = q$.*

Definition (Exact Differential). *An expression $p \ dx + q \ dy$ which can be written as $dU = \frac{\partial U}{\partial x} \ dx + \frac{\partial U}{\partial y} \ dy$ is called an **exact differential**.*

Theorem. *An integral depends only on the end points if and only if the integrand is an exact differential.*

Theorem. *A function f is an exact differential if and only if there exists a function F in Ω such that*

$$\frac{\partial F(z)}{\partial x} = f(z) \qquad \frac{\partial F(z)}{\partial y} = if(z)$$

If this is true, F fulfills the Cauchy-Riemann equations.

Theorem. *The integral $\int_\gamma f \ dz$, with continuous f, depends only on the end points of γ if and only if f is the derivative of an analytic function in Ω.*

Theorem. *If the function $f(z)$ is analytic on R, then*

$$\int_{\partial R} f(z) \ dz = 0$$

Theorem. *Let $f(z)$ be analytic on the set R' obtained from a rectangle R by omitting a finite number of interior points ξ_j. If it is true that $\lim\limits_{z \to \xi_j} (z - \xi_j) f(z) = 0 \ \forall j$, then*

$$\int_{\partial R} f(z) \ dz = 0$$

Theorem (Cauchy's Theorem on a disk)**.** *If $f(z)$ is analytic in an open disk $\Delta = \{z \in \mathbb{C} : |z - z_0| < r\}$, then*

$$\int_{\gamma} f(z) \ dz = 0$$

for every closed curve γ in Δ.

3.2 Cauchy's Integral Formula

Theorem. *If a piecewise differentiable closed curve γ does not pass through the point a, then the following is true.*

$$\int_{\gamma} \frac{dz}{z - a} = 2\pi i$$

Equivalent: $a \in \mathbb{C}$ and $C = \{z \in \mathbb{C} : |z - a| < r\}$

$$\int_{C} \frac{dz}{z - a} = 2\pi i$$

Definition (Winding Number)**.** *Let γ be a closed piecewise differentiable curve and $a \notin \gamma$. Then we define the so called **winding number** $\eta(\gamma, a)$ as*

$$\eta(\gamma, a) := \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - a} \ dz$$

This number is always an integer.

Proposition. *If γ lies inside of a circle, then $\eta(\gamma, a) = 0$ for all points a outside of the same circle.*

Theorem. *Let γ be a closed curve in $\mathbb{C}$. If $a, b \in \mathbb{C}$ are in the same region determined by γ, then $\eta(\gamma, a) = \eta(\gamma, b)$.*

Theorem (Cauchy's Integral Formula)**.** *Let $f : \Delta \to \mathbb{C}$ be analytic in an open disk $\Delta \subset \mathbb{C}$ and let γ be a closed curve in Δ. For any point $a \notin \gamma$ the following is true.*

$$\eta(\gamma, a) f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - a} \ dz$$

Theorem (Cauchy's Integral Formula)**.** *Let f be analytic on an open disk Δ and let γ be a closed curve in Δ. If $z \in \gamma$ and $\eta(\gamma, z) = 1$ the following is true.*

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} \ d\xi$$

Corollary. *Let f, Δ and γ be as above. Assume $\eta(\gamma, a) = 1$. If $a \neq b$, then we can write:*

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - a)(z - b)} \ dz = \frac{1}{2\pi i (a - b)} \left(\int_{\gamma} \frac{f(z)}{z - a} \ dz - \int_{\gamma} \frac{f(z)}{z - b} \ dz \right) = \frac{1}{a - b} \left(\eta(\gamma, a) f(a) - \eta(\gamma, b) f(b) \right)$$

Since $\frac{1}{(z-a)(z-b)} = \frac{1}{a-b} \left(\frac{1}{z-a} - \frac{1}{z-b} \right)$. If however $a = b$, then:

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - a)^2} \ dz = \eta(\gamma, a) f'(a)$$

Theorem. *With the above conditions the following is true.*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\xi)}{(\xi - z)^{n+1}} \ d\xi$$

Lemma. *Suppose that $\phi(\xi)$ is continuous on the arc γ. Then the function*

$$F_n(z) = \int_\gamma \frac{\varphi(\xi)}{(\xi - z)^n}\ d\xi$$

is analytic in each of the regions determined by γ, and its derivative is $F_n'(z) = n F_{n+1}(z)$.

Theorem. *If $f : \Omega \to \mathbb{C}$ is analytic and C is a circle in Ω for all z in the inside of C it is true that*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - z}\ d\xi \quad and \quad f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^{n+1}}\ d\xi$$

Theorem. *If $f : \Omega \to \mathbb{C}$ is analytic in $\Omega \subseteq \mathbb{C}$ a region, then f has n derivatives.*

Theorem (Morera's Theorem)**.** *Let $f : \Omega \to \mathbb{C}$ be continuous in a region Ω. If for all closed paths γ it is true that $\int_\gamma f(z)\ dz = 0$, then f is analytic in Ω.*

Theorem. *Let f be analytic in a region containing $C = \{z : |z - a| < r\}$ and the inside of C. Further, let $\max_{z \in C} |f(z)| \le M$. Then*

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^n}\ d\xi \le \frac{n!}{2\pi} M \frac{1}{r^n} \int_C |d\xi| \le \frac{n!}{2\pi} M \frac{1}{r^n} 2\pi \le \frac{n! M}{r^n}$$

If f is analytic in $B(a, r')$ and $|f(z)| < M\ \forall z \in C$ where $C = \{z : |z - a| < r\}$ with $r < r'$, then $\left| f^{(n)}(z) \right| \le M \frac{n!}{r^n}$.

Theorem (Liouville's Theorem)**.** *If $f : \mathbb{C} \to \mathbb{C}$ is analytic in all $\mathbb{C}$ and bounded, then f must be constant.*

3.3 Local Properties of Analytic Functions

Theorem. *Cauchy's integral formula still holds if f has singularities $\{a_j\}_{j \in J}$. The function f is analytic in $\Delta \setminus \{a_j\}_{j \in J}$ provided that $\lim_{z \to a_j} (z - a_j) f(z) = 0\ \forall j \in J$ as long as $\{a_j\}_{j \in J} \cap \gamma = \varnothing$ and $\{a_j\}_{j \in J} \cap \{z\} = \varnothing$.*

Definition (Removable Singularity)**.** *Suppose that $f(z)$ is analytic in the region Ω' obtained by omitting a point a from the region Ω. A necessary and sufficient condition that there exists an analytic function in Ω which coincided with $f(z)$ in Ω' is that $\lim_{z \to a} (z - a) f(z - a) = 0$. The extended function is uniquely determined by*

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - a}\ d\xi$$

*where C is a circle with a in the inside of C and C and the inside of C being in Ω. We call this a **removable singularity**.*

Theorem. *If $f : \Omega \to \mathbb{C}$ is analytic in a region Ω containing a, then for any $n > 0$ integer*

$$f(z) = f(a) + \frac{(z - a)}{1!} f'(a) + \cdots + \frac{(z - a)^{n-1}}{(n - 1)!} f^{(n-1)}(a) + (z - a)^n f_n(a)$$

where $f_n(z)$ is analytic in Ω and is given by

$$f_n(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - a)^n (\xi - z)}\ d\xi$$

Theorem. *If $f : \Omega \to \mathbb{C}$ is analytic and $\exists a \in \Omega$ such that $f(a) = 0$ and $f^{(n)}(a) = 0$, then $f = 0$ in all of Ω.*

Definition (Order of a Zero)**.** *Take $f : \Omega \to \mathbb{C}$ analytic and not identically zero. Let a be a zero of f. Then, $\exists h$ such that $f^{(h)}(a) \ne 0$. Then, a is a so called **zero of order** h.*

Theorem. *If $f(a)$ and all derivatives $f^{(n)}(a)$ vanish, we can write*

$$f(z) = f_n(z)(z - a)^n$$

with $f_n(a) \ne 0$. Then $\exists \delta > 0$ such that $f_n(z) \ne 0\ \forall z : |a - z| < \delta$. Thus, $f(z)$ is non zero in a punctured neighborhood of a.

Definition (Isolated Zeros)**.** *The zeros of an analytic function which does not vanish identically are **isolated**.*

Definition (Isolated Singularity)**.** *We consider a function $f(z)$ which is analytic in a neighborhood of a, exept perhaps at a itself. In other words, $f(z)$ shall be analytic in a region $0 < |z - a| < \delta$. The point a is called an **isolated singularity** of $f(z)$.*

Definition (Pole)**.** *If $\lim_{z \to a} f(z) = \infty$, the point a is said to be a **pole** of $f(z)$, and we set $f(a) = \infty$.*

Definition (Order of a Pole)**.** *There exists a $\delta' \le \delta$ such that $f(z) \neq 0$ for $0 < |z - a| < \delta'$. In this region the function $g(z) = \frac{1}{f(z)}$ is defined and analytic. But the singularity of $g(z)$ at a is removable, and $g(z)$ has an analytic extension with $g(a) = 0$. Since $g(z)$ does not vanish identically, the zero at a has a finite order, and we can write $g(z) = (z - a)^h g_h(z)$ with $g_h(a) \neq 0$. The number h is the **order of the pole**, and $f(z)$ has the representation $f(z) = (z - a)^{-h} f_h(z)$ where $f_h(z) = \frac{1}{g_h(z)}$ is analytic and different from zero in a neighborhood of a.*

Definition (Meromorphic)**.** *A function $f(z)$ which is analytic in a region Ω, exept for poles, is said to be **meromorphic** in Ω. More precisely, to every $a \in \Omega$ there shall exist a neighborhood $|z - a| < \delta$, contained in Ω, such that either $f(z)$ is analytic in the whole neighborhood, of else $f(z)$ is analytic for $0 < |z - a| < \delta$, and the isolated singularity is a pole.*

Theorem. *If $f(z)$ is identically zero in a subregion of Ω, then it is identically zero in Ω, and the same is true if $f(z)$ vanishes on an arc which does not reduce to a point.*

Theorem. *Define the following two conditions. Let $\alpha \in \mathbb{R}$.*

$$(1) \qquad \lim_{z \to a} |z - a|^\alpha \, |f(z)| = 0$$

$$(2) \qquad \lim_{z \to a} |z - a|^\alpha \, |f(z)| = \infty$$

If (1) holds for a certain α, then it holds for all larger α, and hence for some integer m. Then $(z - a)^m f(z)$ has a removable singularity and vanishes for $z = a$. Either $f(z)$ is identically zero, in which case (1) holds for all α, or $(z - a)^m f(z)$ has a zero of finite order k. In the latter case it follows at once that (1) holds for all $\alpha > h = m - k$, while (2) holds for all $\alpha < h$.
Assume that (2) holds for some α; then it holds for all smaller α, and hence for some integer n. The function $(z - a)^n f(z)$ has a pole of finite order l, and setting $h = n + l$ we find again that (1) holds for $\alpha > h$ and (2) for $\alpha < h$.
There are three possibilities here.

  (i) *Condition (1) holds for all α and $f(z)$ vanishes identically. This case is not very interesting.*

 (ii) *There exists an integer h such that (1) holds for $\alpha > h$ and (2) for $\alpha < h$. In this case we call h the **algebraic order** of $f(z)$ at a. It is positive in the case of a pole, negative in case of a zero and zero if $f(z)$ is analytic but not 0 at a.*

(iii) *Neither (1) nor (2) holds for any α. In this case the point a is called an **essential isolated singularity**. In the neighborhood of an essential singularity $f(z)$ is at the same time unbounded and comes arbitrarily close to zero.*

Theorem (Weierstrass)**.** *An analytic function comes arbitrarily close to any complex value in every neighborhood of an essential singularity. Equivalent is: If a is an essential singularity of f, $\forall \epsilon > 0$ (sufficiently small) $\{f(z) : z \in B(a, \epsilon)\}$ is dense in $\mathbb{C}$.*

Overview Singularities

$$\lim_{z \to a} f(z) \ \text{ exists?} \begin{cases} \text{Yes:} \quad \lim_{z \to a} \frac{1}{f(z)} \ \text{ exists?} \begin{cases} \text{Yes: Removable} \\ \text{No: Zero} \end{cases} \\[2em] \text{No:} \quad \lim_{z \to a} \frac{1}{f(z)} \ \text{ exists?} \begin{cases} \text{Yes: Pole} \\ \text{No: Essential} \end{cases} \end{cases}$$

Removable Singularity Show $\lim_{z \to a} (z - a) f(z) = 0$ and find analytic extension of f in a.

Zero Find $n \geq 1$, $n \in \mathbb{N}$ such that $f(z) = (z-a)^n f_n(z)$ with $f_n(a) \neq 0$ and f_n analytic in a. We want n to satisfy

$$\lim_{z \to a}(z-a)^{-k}f(z) = 0 \ \forall k < n$$

$$\lim_{z \to a}(z-a)^{-k}f(z) = \infty \ \forall k > n$$

$$\lim_{z \to a}(z-a)^{-k}f(z) = f_n(a) \neq 0 \ \text{ if } n = k$$

n can be found inductively by deriving f. Meaning $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0$ but $f^{(n)}(a) \neq 0$.

Poles Show $\lim_{z \to a}|f(z)| = \infty$. f has a pole of order $n \Leftrightarrow \frac{1}{f}$ has a zero of order n. We do the same as for the zeros for $\frac{1}{f}$. We want to find $n \geq 1$, $n \in \mathbb{N}$ such that $f(z) = (z-a)^{-n}f_n(z)$, $f_n(a) < \infty$ and f_n analytic in a and

$$\lim_{z \to a}f(z)(z-a)^k = 0 \ \forall k > n$$

$$\lim_{z \to a}f(z)(z-a)^k = \infty \ \forall k < n$$

Essential Singularity Show that a is not removable and not a pole. Weierstrass Theorem: Show that $f(D(a,r)\backslash\{a\}) = \mathbb{C}$.

Classification For $n \in \mathbb{Z}$ let $\lim_{z \to a}(z-a)^n f(z) \in \mathbb{C}\backslash\{0\}$.

$$n \geq 1 \Rightarrow a \text{ is a pole of order } n$$

$$n \leq -1 \Rightarrow a \text{ is a zero of order } -n$$

$$n = 0 \Rightarrow a \text{ is a removable singularity}$$

$$n \text{ does not exist } \Rightarrow a \text{ is an essential singularity}$$

Find the type of singularity by its expansion. First note, if f has a pole at a of order n, $f(z) = (z-a)^{-n}f_n(z)$, where f_n is analytic in a

$$f_n(z) = \sum_{k=0}^{\infty} \frac{f_n^{(k)}(a)}{k!}(z-a)^k$$

$$f(z) = \frac{f_n(a)}{(z-a)^n} + \frac{f_n'(a)}{(z-a)^{n-1}} + \cdots + \frac{f_n^{(n-1)}(a)}{(n-1)!(z-a)} + \sum_{k=1}^{\infty}\frac{f_n^{(k)}(a)}{k!}(z-a)^{k-1}$$

We call $S(z) = \sum_{k<0}a_k(z-a)^k$ the **essential / principle part** of f. If $S(z)$ is a finite sum $S(z) = \sum_{k=-m}^{-1}a_k(z-a)^k$, then a is a pole of order m. If $\lim_{z \to a}(z-a)f(z) = 0 \Rightarrow \lim_{z \to a}f(z) \in \mathbb{C}\backslash\{0\} \Rightarrow f$ admits a Taylor expantion. If $S \equiv 0$, a is removable. If S is an infinite sum, a is essential.

Theorem. *Let $f(z)$ be an analytic functin which is not identcaly zero in an open disk Δ. Let γ be a closed curve in Δ such that $f(z) \neq 0$ on γ. Suppose f has only finitely many zeros $z_1, z_2, \ldots, z_n$ in Δ. In this case, we can write $f(z) = (z-z_1)(z-z_2)\ldots(z-z_n)g(z)$ where g is analytic and $\neq 0$ in Δ. The **logarithmic derivative** is given as*

$$\frac{f'(z)}{f(z)} = \frac{1}{z-z_1} + \frac{1}{z-z_2} + \cdots + \frac{1}{z-z_n} + \frac{g'(z)}{g(z)}$$

Since $g(z) \neq 0$ in Δ, Cauchy's theorem yields

$$\int_\gamma \frac{g'(z)}{g(z)} \ dz = 0$$

So we find:

$$\eta(\gamma, z_1) + \eta(\gamma, z_2) + \cdots + \eta(\gamma, z_n) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} \ dz$$

Theorem. *Let z_j be the zeros of a function $f(z)$ which is analytic in a disk Δ and does not vanish identically, each zero being counted as many times as its order indicates. For every closed curve γ in Δ which does not pass through a zero*

$$\sum_j \eta(\gamma, z_j) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz$$

where the sum has only a finite number of terms $\neq 0$. Let $w = f(z)$ be the function that maps γ onto a closed curve Γ in the ω-plane, and we find

$$\int_\Gamma \frac{dw}{w} = \int_\gamma \frac{f'(z)}{f(z)}\, dz$$

Hence, we arrive at $\eta(\Gamma, 0) = \sum_j \eta(\gamma, z_j)$.

Theorem. *Take the same conditions as above. Let each $\eta(\gamma, z_j)$ be either 0 or 1. Then, we can derive a formula for the total number of zeros enclosed by γ. This is the case, if γ is a circle. Donate the names $z_j(a)$ to the roots of the equation $f(z) = a$ for some arbitrary complex value. We then obtain*

$$\sum_j \eta(\gamma, z_j(a)) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - a}\, dz$$

Hence,

$$\eta(\Gamma, a) = \sum_j \eta(\gamma, z_j(a))$$

Theorem. *If a and b are in the same region determined by Γ, we know that $\eta(\Gamma, a) = \eta(\Gamma, b)$, and hence we have also $\sum_j \eta(\gamma, z_j(a)) = \sum_j \eta(\gamma, z_j(b))$. If γ is a circle, it follows that $f(z)$ takes the value a and b equally many times inside of γ.*

Theorem. *Suppose that $f(z)$ is analytic at z_0, $f(z_0) = w_0$, and that $f(z) - w_0$ has a zero of order n at z_0. $\forall \epsilon > 0$ small enough, $\exists \delta > 0$ such that for all a with $|a - w_0| < \delta$ the equation $f(z) - a$ has exactly n roots in the dist $|z - z_0| < \epsilon$.*

Corollary. *A nonconstant analytic function maps open sets onto open sets.*

Theorem (Maximum Principle)**.** *If $f(z)$ is analytic and nonconstant in a region Ω, then its absolute value $|f(z)|$ has no maximum in Ω.*

Theorem. *If $f(z)$ is defined and continuous on a closed bounded set E and analytic on the interior of E, then the maximum of $|f(z)|$ on E is attained on the boundary of E.*

3.4 The General Form of Cauchy's Theorem

Definition (Simply Connected)**.** *A region Ω is **simply connected** if and only if its complements with respect to the extended plane is connected. (Equivalent: If Ω has no holes.)*

Theorem. *A region Ω is simply connected if and only if $\eta(\gamma, a) = 0$ for all cycles γ in Ω and all points $a \notin \Omega$.*

Definition (Homology)**.** *A cycle γ in an open set Ω is said to be **homologous to zero** with respect to Ω if $\eta(\gamma, a) = 0$ $\forall a \in \Omega^C$. In symbols we write $\gamma \sim 0$ (mod Ω). We say that $\gamma_1 \sim \gamma_2$ if and only if $\gamma_1 - \gamma_2 \sim 0$ mod Ω.*

Theorem (General Cauchy's Theorem)**.** *If $f(z)$ is analytic in Ω, then*

$$\int_\gamma f(z)\, dz = 0$$

for every cycle γ which is homologous to zero in Ω.

Corollary. *If $f(z)$ is analytic in a simply connected region Ω, then*

$$\int_\gamma f(z)\, dz = 0$$

holds for all cycles in Ω.

Corollary. *If $f(z)$ is analytic and $\neq 0$ in a simply connected region Ω, then it is possible to define single-valued analytic branches of $\log(f(z))$ and $\sqrt[n]{f(z)}$ in Ω.*

Definition (Multiply Connected). *A region which is not simply connected is called **multiply connected**. More precisely, Ω is said to have the **finite connectivity** n if the complement of Ω has exactly n components and **infinite connectivity** if the complement has infinitely many complements. (Colloquially speaking: A region of connectivity n arises by punching n holes in the Riemann sphere.)*

Definition (Modules of Periodicity). *Let $f : \Omega \to \mathbb{C}$ be analytic and Ω be multiply connected. Then we can write*

$$\int_\gamma f \; dz = \int_{c_1\gamma_1+\cdots+c_{n-1}\gamma_{n-1}} f \; dz = c_1 \int_{\gamma_1} f \; dz + \cdots + c_{n-1} \int_{\gamma_{n-1}} f \; dz$$

The numbers

$$P_i = \int_{\gamma_i} f \; dz$$

*depend only on the function and not on γ. They are called **modules of periodicity** or **period of** f.*

3.5   The Calculus of Residues

Theorem (Cauchy's Integral Formula). *If $f(z)$ is analytic in a region Ω, then*

$$\eta(\gamma, a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a} \; dz$$

for every cycle γ which is homologous to zero in Ω.

Definition (Residue). *The **residue** of $f(z)$ at an isolated singularity a is the unique complex number R which makes $f(z) - \frac{R}{z-a}$ the derivative of a single valued analytic function in an annulus $0 < |z - a| < \delta$. Often R is written as $R = Res_{z=a} f(z)$ or $R_j = Res_{z=a_j} f(z)$.*

Definition (Residue). *Equivalent to the above definition is: Let $f : D(z_0, r) \setminus \{z_0\} \to \mathbb{C}$ be holomorphic with the Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$. Then the residue of f at z_0 is $Res_{z_0}(f) := a_{-1}$.*

Theorem. *If $f(z) - \frac{R}{z-a}$ is the derivative of a single valued analytic function, it itself is analytic and thus due to Cauchy's Theorem its integral evaluates to zero for every cyclic γ which is homologous to zero. Thus we obtain*

$$\int_\gamma f(z) - \frac{R}{z - a} \; dz = 0$$

$$\Leftrightarrow \int_\gamma f(z) \; dz = \int_\gamma \frac{R}{z - a} \; dz = 2\pi i R$$

Therefore:

$$Res_{z=a} f(z) = \frac{1}{2\pi i} \int_\gamma f(z) \; dz$$

Theorem. *Let $f(z)$ be analytic exept for isolated singularities a_j in a region Ω. Then*

$$\frac{1}{2\pi i} \int_\gamma f(z) \; dz = \sum_j \eta(\gamma, a_j) Res_{z=a_j} f(z)$$

for any cycle γ which is homologous to zero in Ω and does not pass through any of the points a_j. Often only the case $\eta(\gamma, a_j) = 1$ or $\eta(\gamma, a_j) = 0$ is considered. Then the theorem simplyfies to

$$\frac{1}{2\pi i} \int_\gamma f(z) \; dz = \sum_j Res_{z=a_j} f(z)$$

where the sum is extended over all singularities enclosed by γ.

Theorem. *Let's say that f has a pole at $z = a$ of order h. Then: $(z-a)^h f(z)$ is analytic in a neighborhood of a and so*

$$f(z) = B_h (z-a)^{-h} + B_{h-1}(z-a)^{-(h-1)} + \cdots + B_1(z-a)^{-1} + \varphi(z)$$

with $\varphi(z)$ analytic in a neighborhood of a. Then B_1 is the residue of f at a because $f(z) - \frac{B_1}{z-a}$ is the derivative of a function in an annulus $0 < |z-a| < \delta$.

Definition (Bound). *A cycle γ is said to **bound** the region Ω if and only if $\eta(\gamma, a)$ is defined and equal to 1 for all points $a \in \Omega$ and either undefined or equal to zero for all points $a \notin \Omega$.*

Theorem. *If γ bounds Ω and $f(z)$ is analytic on the set $\Omega + \gamma$, then*

$$\int_\gamma f(z)\ dz = 0$$

and

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z}\ d\xi$$

for all $z \in \Omega$. If $f(z)$ is analytic in $\Omega + \gamma$ exept for isolated singularities in Ω, then

$$\frac{1}{2\pi i}\int_\gamma f(z)\ dz = \sum_j Res_{z=a_j} f(z)$$

where the sum ranges over the singularities $a_j \in \Omega$.

Theorem (Argument Principle). *If $f(z)$ is meromorphic in Ω with the zeros a_j and the poles b_k, then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\ dz = \sum_j \eta(\gamma, a_j) - \sum_k \eta(\gamma, b_k)$$

for every cycle γ which is homologous to zero in Ω and does not pass through any of the zeros or poles.

Theorem (Rouché). *Let γ be homologous to zero in Ω and such that $\eta(\gamma, z)$ is either 0 or 1 for any point z not on γ. Suppose that $f(z)$ and $g(z)$ are analytic in Ω and satisfy the inequality $|f(z) - g(z)| < |f(z)|$ on γ. Then $f(z)$ and $g(z)$ have the same number of zeros enclosed by γ.*

Theorem (Rouché). *Let γ be a null-homologous cycle and Jordan bounding the region D. Let $f, g : U \to \mathbb{C}$ be holomorphic in U such that in $D \cup \gamma = \overline{D} \subseteq U$:*

$$\forall z \in \gamma: \ |f(z) - g(z)| < |f(z)| + |g(z)|$$

Then, f and g have the same numbers of zeros in D (with multiplicity).

Theorem. *If $g : \Omega \to \mathbb{C}$ analytic, then $g(z)\frac{f'(z)}{f(z)}$ has residue $g(a)h$ if a is a zero of order h of f and $-g(a)h$ if a is a pole of order h of f.*

Theorem. *Let $f(z_0) = w_0$, $f'(z_0) \neq 0$. Then there exists $\delta > 0$ such that for $|w - w_0| < \delta$, there exists only one solution to $f(z) = w$ in $|z - z_0| < \epsilon$ for an $\epsilon > 0$.*

Tricks

 a) f has a removable singularity at $z_0 \Rightarrow R = 0$

 b) f has a simple pole at $z_0 \Rightarrow R = \lim_{z \to z_0} (z - z_0) f(z)$

 c) f has a pole of order $k > 1$ at $z_0 \Rightarrow R = \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left((z-z_0)^k f(z)\right)_{|_{z=z_0}}$

 d) f has a simple pole at z_0 and g analytic in a neighborhood of $z_0 \Rightarrow R(f \cdot g) = \mathrm{Res}_{z_0}(f \cdot g) = g(z_0) \cdot \mathrm{Res}_{z_0}(f)$

Remarks

- For essential singularities, compute the whole Laurent Series.

- $\text{Res}(f, z_0) = 0 \not\Rightarrow z_0$ is a removable singularity.

Theorem (Existence of $\log(f)$). *Let D be a simply connected domain and let f be holomorphic on D such that $f(z) \neq 0 \ \forall z \in D$. Then, there exists a holomorphic function g on D such that $f(z) = e^{g(z)}$. (I.e. $\log$ is holomorphic.) Moreover, if $z_0 \in D$ and $f(z_0) = e^{w_0}$, one can pick a function g such that $g(z_0) = w_0$.*

Theorem. *Let $f : D \to \mathbb{C}$ be holomorphic in the region D and $f(z) \neq 0 \ \forall z \in D$. Then, there exists a branch of $\log(f)$ in D such that*

$$\int_\gamma \frac{f'(z)}{f(z)} \ dz = 0$$

for all piecewise regular closed paths $\gamma \in \mathbb{C}$. This theorem holds in particular for simply connected regions.

Theorem. *Let P and Q be polynomials such that $\frac{P(x)}{Q(x)}$ has no poles in $\mathbb{R}$ and $\deg\left(\frac{P(x)}{Q(x)}\right) \leq -2$. Then:*

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \ dx = 2\pi i \sum_{\substack{a \\ Im(a)>0}} res_a\left(\frac{P(z)}{Q(z)}\right)$$

Theorem. *Let P and Q be polynomials and say $\frac{P(x)}{Q(x)}$ has no poles on $(0, \infty)$ and at most a simple pole at 0. Say $\deg \frac{P}{Q} \leq -2$ and let $0 < \alpha < 1$. Then:*

$$\int_0^{\infty} x^\alpha \frac{P(x)}{Q(x)} \ dx = \frac{2\pi i}{1 - e^{2\pi i \alpha}} \sum_{b \neq 0} Res\left(z^\alpha \frac{P(z)}{Q(z)}, b\right)$$

4   Series and Product Developments

4.1   Power Series Expansion

Theorem (Weierstrass)**.** *Suppose that $f_n(z)$ is analytic in the region Ω_n, and that the sequence $\{f_n(z)\}$ converges to a limit function $f(z)$ in a region Ω, uniformaly on every compact subset of Ω. Then $f(z)$ is analytic in Ω. Moreover, $f_n'(z)$ converges uniformaly to $f'(z)$ on every compact subset of Ω.*

Corollary. *Let $\sum_{n=1}^\infty f_n$ be a series of analytic functions in $U \subseteq \mathbb{C}$ a region. Assume $\sum_{n=1}^\infty f_n \to f : U \to \mathbb{C}$ uniformaly on every compact subset of U. (Which means that $S_m(z) = f_1(z) + \cdots + f_m(z)$ converges uniformaly locally to f.) Then, f is analytic in U and $\forall k \in \mathbb{N}$: $\sum_{n=0}^\infty f_n^{(k)}(z) = f^k(z)$.*

Theorem. *If the functions $f_n(z)$ are analytic and $\neq 0$ in a region Ω, and if $f_n(z)$ converges to $f(z)$, uniformaly on every compact subset of Ω, then $f(z)$ is either identically zero or never equal to zero in Ω.*

Theorem (Taylor)**.** *If $f(z)$ is analytic in the region Ω, containing z_0, then the representation*

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \dots$$

is valid in the largest open disk of center z_0 contained in Ω.

Definition (Laurent Series)**.** *A **Laurent Series** is given by*

$$\sum_{n=-\infty}^\infty a_n (z - z_0)^n$$

for a $z_0 \in \mathbb{C}$.

Definition (Radius of Convergence)**.** *The **radius of convergence** is given by*

$$\rho_+ := \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$$
$$\rho_- := \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

Assume $\rho_+ > \rho_-$. Then the Laurent series $\sum_{n\in\mathbb{Z}} a_n(z - z_0)^n$ converges (locally) uniformaly and absolutely in $A = \{z \in \mathbb{C} : \rho_- |z - z_0| < \rho_+\}$

Theorem. *Let A be as before. Let $f : A \to \mathbb{C}$ be analytic. Then f can be expanded to a Laurent series in A and the series is (locally) uniformaly and absolutely convergent. The expansion is unique.*

$$f(z) = \sum_{n\in\mathbb{Z}} a_n(z - z_0)^n$$

Theorem. *If $f : \Omega \backslash \{z_0\} \to \mathbb{C}$ is analytic and has an isolated singularity at z_0, then*

(i) z_0 *is removable $\Leftrightarrow \lim_{z\to z_0}(z - z_0)f(z) = 0 \Leftrightarrow \lim_{z\to z_0}\sum_{k=-\infty}^\infty a_k(z - z_0)^{k+1} = 0 \Leftrightarrow \forall k \le -1 : a_k = 0 \Leftrightarrow f(z) = \sum_{k=0}^\infty a_k(z - z_0)^k$ and $S(z) = \sum_{k=-\infty}^0 a_k(z - z_0)^k \equiv 0$ is the so called **essential part**.*

(ii) z_0 *is a pole of order $n \Leftrightarrow \lim_{z\to z_0}(z - z_0)^n f(z) \in \mathbb{C}\backslash\{0\} \Leftrightarrow \lim_{z\to z_0}\sum_{k=-\infty}^\infty a_k(z - z_0)^{n+k} \in \mathbb{C}\backslash\{0\} \Leftrightarrow \forall k < -n : a_k = 0 \Leftrightarrow f(z) = \sum_{k=-n}^\infty a_k(z - z_0)^k$. The essential part consists of only finitely many terms.*

(iii) z_0 *is an essential singularity $\Leftrightarrow z_0$ is not removable and not a pole $\Leftrightarrow f(z) = \sum_{k=-\infty}^\infty a_k(z - z_0)^k$ where $a_k \neq 0 \; \forall k < 0$.*

Theorem. *Let f be analytic in $\{z : R_1 < |z| < R_2\}$ can be written as $f(z) = \sum_{n=0}^\infty A_n(z - z_0)^n$*

5   Useful Tools

5.1   Infinite Product

Definition (Infinite Product)**.** *The infinite product is given as*

$$\prod_{n=0}^{\infty} P_n = P_1 \cdots \cdots P_n \cdot \dots$$

Theorem. *We say that $\prod_{n=0}^{\infty} P_n$ converges if*

(i) $P_n = \prod_{k=1}^{n} P_k$, $P_n \to P \neq 0$

(ii) *There are finitely many zero factors.*

Theorem. *With the necessary condition $a_n \to 0$ we can write $\prod_{n=0}^{\infty} P_n$ as*

$$\prod_{n=1}^{\infty} (1 + a_n)$$

Theorem. *If no factor is zero, we can understand $\prod_{n=0}^{\infty} P_n$ as follows.*

$$S_M = \sum_{n=1}^{M} \log(1 + a_n)$$

$$P_M = \prod_{n=1}^{M} (1 + a_n)$$

$$P_M = \exp(S_M)$$

If S_M converges, so does P_M. In particular:

Theorem. *The finite product $\prod_{n=1}^{\infty} (1 + a_n)$ with $1 + a_n \neq 0$ converges if and only if $\sum_n \log(1 + a_n)$ converges.*

Theorem. *$\sum_n \log(1 + a_n)$ converges absolutely if and only if $\sum_n a_n$ converges absolutely.*