0 Basics

$$\partial_x \delta(x-y) = -\partial_y \delta(x-y) \quad , \quad \varepsilon_{kij}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$$
$$(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$$
$$Ra \times Rb = R(a \times b) \quad , \quad \nabla \times \nabla \psi = 0$$
$$\nabla \times (\nabla \times A) = \nabla (\nabla \cdot A) - \Delta A$$
$$\nabla (A \cdot B) = (A \cdot \nabla)B + (B \cdot \nabla)A + A \times (\nabla \times B) + B \times (\nabla \times A)$$

with a, b, c, d vectors, A, B vector fields, $R \in SO(3)$ and ψ a scalar potential.

0.1 Spherical Coordinates

$$\vec{e}_r \times \vec{e}_\theta = \vec{e}_\varphi \quad , \quad \vec{e}_\theta \times \vec{e}_\varphi = \vec{e}_r \quad , \quad \vec{e}_\varphi \times \vec{e}_r = \vec{e}_\theta$$

For $\vec{V} = (V_r, V_\theta, V_\varphi)$:

$$\vec{\nabla} \cdot \vec{V} = \frac{1}{r^2}\frac{\partial (r^2 V_r)}{\partial r} + \frac{1}{r \sin(\theta)}\frac{\partial (\sin(\theta) V_\theta)}{\partial \theta} + \frac{1}{r\sin(\theta)}\frac{\partial V_\varphi}{\partial \varphi}$$
$$\vec{\nabla} \times \vec{V} = \frac{1}{r\sin(\theta)}\left[\frac{\partial(\sin(\theta)V_\varphi)}{\partial\theta} - \frac{\partial V_\theta}{\partial\varphi}\right]\vec{e}_r + \frac{1}{r}\left[\frac{1}{\sin(\theta)}\frac{\partial V_r}{\partial\varphi} - \frac{\partial(rV_\varphi)}{\partial r}\right]\vec{e}_\theta + \frac{1}{r}\left[\frac{\partial(rV_\theta)}{\partial r} - \frac{\partial V_r}{\partial\theta}\right]\vec{e}_\varphi$$

0.2 Cylinder Coordinates

$$\vec{e}_r \times \vec{e}_\phi = \vec{e}_z \quad , \quad \vec{e}_\phi \times \vec{e}_z = \vec{e}_r \quad , \quad \vec{e}_z \times \vec{e}_r = \vec{e}_\phi$$

For $\vec{V} = (V_r, V_\phi, V_z)$:

$$\vec{\nabla} \cdot \vec{V} = \frac{1}{r}\frac{\partial(rV_r)}{\partial r} + \frac{1}{r}\frac{\partial V_\phi}{\partial\phi} + \frac{\partial V_z}{\partial z}$$
$$\vec{\nabla} \times \vec{V} = \left[\frac{1}{r}\frac{\partial V_z}{\partial\phi} - \frac{\partial V_\phi}{\partial z}\right]\vec{e}_r + \left[\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right]\vec{e}_\phi + \frac{1}{r}\left[\frac{\partial(rV_\phi)}{\partial r} - \frac{\partial V_r}{\partial\phi}\right]\vec{e}_z$$

1 Electromagnetic Force

Lorentz Force $\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$

Maxwell Equations

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \;,\; \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \;,\; \vec{\nabla} \cdot \vec{B} = 0 \;,\; \vec{\nabla} \times \vec{B} = \frac{\vec{j}}{c^2\varepsilon_0} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

2 Electrostatics

Fixed charge distribution and static electric field.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad , \quad \vec{\nabla} \times \vec{E} = 0$$

2.1 Coulomb's Law

$$\vec{F} = \frac{q}{4\pi\varepsilon_0}\frac{q_1}{|\vec{x}-\vec{y}_1|^3}(\vec{x}-\vec{y}_1) \quad , \quad \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{|\vec{x}-\vec{y}_1|^3}(\vec{x}-\vec{y}_1)$$
$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{|\vec{x}-\vec{y}_i|^3}(\vec{x}-\vec{y}_i) \quad , \quad \vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|^3}(\vec{x}-\vec{y})\; d^3\vec{y}$$

2.1.1 Dirac Delta

$$\int_{-\infty}^{\infty} f(x)\delta(x-y)\; dx = f(y) \;,\; \delta(\vec{x}-\vec{y}) = \delta^{(D)}(\vec{x}-\vec{y}) = \prod_{i=1}^{D}\delta(x_i - y_i)$$

Charge density of a single charge q at a position $\vec{y}$: $\rho(\vec{x}) = q\delta(\vec{x}-\vec{y})$.

Charge distribution of many point-like charges q_i at positions $\vec{y}_i$: $\rho(\vec{x}) = \sum_i q_i \delta(\vec{x}-\vec{y}_i)$

2.2 Gauss' law from Coulomb's law

$$\vec{\nabla}_{\vec{x}}\frac{1}{|\vec{x}-\vec{y}|} = -\frac{(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3} \;\Rightarrow\; \vec{E} = -\vec{\nabla}\Phi(\vec{x})$$
$$\Rightarrow\; \Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}\; d^3\vec{y}$$
$$\Rightarrow\; \vec{\nabla} \cdot \vec{E} = -\vec{\nabla}^2\Phi = -\Delta\Phi = \frac{\rho}{\varepsilon_0}$$

Further: $\nabla^2_{\vec{x}}\frac{1}{|\vec{x}-\vec{y}|} = -4\pi\delta(\vec{x}-\vec{y})$

2.2.1 Integral form of Gauss' law

$$\int_{S(V)} \vec{E} \cdot d\vec{S} = \int_{V(S)} \vec{\nabla} \cdot \vec{E}\; d^3\vec{x} = \frac{1}{\varepsilon_0}\int_{V(S)} \rho(\vec{x})\; d^3\vec{x}$$

2.3 Scalar potential

$$\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \vec{\nabla}\Phi = 0$$
$$\int_S \vec{\nabla} \times \vec{E} \cdot d\vec{S} = \oint_{\partial S} \vec{E} \cdot d\vec{l} = 0$$

Field exerts a force $\vec{F} = q\vec{E} = -q\vec{\nabla}\Phi$. Work needed to transport a test charge from a position $\vec{x}_A$ to a position $\vec{x}_B$:

$$W_{A\to B} = q\int_{\vec{x}_A}^{\vec{x}_B} \vec{\nabla}\Phi \cdot d\vec{l} = q\left(\Phi(\vec{x}_B) - \Phi(\vec{x}_A)\right)$$

Work done is independent of the chosen path.

2.4 Potential energy of a charge distribution

Potential energy of a system is the energy required to bring the charges to their positions from infinity. Discrete case:

$$W = \frac{1}{4\pi\varepsilon_0}\sum_{i=2}^{N}\sum_{j<i}\frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} = \frac{1}{8\pi\varepsilon_0}\sum_{i\neq j}\frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}$$

Continous case:

$$W = \frac{1}{8\pi\varepsilon_0}\int \frac{\rho(\vec{x})\rho(\vec{y})}{|\vec{x}-\vec{y}|}\; d^3\vec{x}d^2\vec{y} = \frac{1}{2}\int \rho(\vec{x})\Phi(\vec{x})\; d^3\vec{x}$$
$$= -\frac{\varepsilon_0}{2}\int \Phi(\vec{x})\vec{\nabla}^2\Phi(\vec{x})\; d^3\vec{x} = \frac{\varepsilon_0}{2}\int \left|\vec{E}\right|^2\; d^3\vec{x}$$

Energy density of the electric field $w = \frac{\varepsilon_0}{2}\left|\vec{E}\right|^2$

2.4.1 Self-energy

Infinities in the total energy occur if we consider discrete distributions with a density $\rho(\vec{x}) = \sum_i q_i\delta(\vec{x}-\vec{x}_i)$

2.5 Charged conductors

Electrons move freely within their mass. $\vec{E}_{\text{inside}} = 0 \Rightarrow \Phi$ const. Flux of electric field through Gauss surface:

$$\text{Flux} = \int \vec{E} \cdot d\vec{S} = E\Delta S$$

Charge enclosed:

$$\text{Charge} = \sigma(\vec{x})\Delta S$$

with $\sigma(\vec{x})$ the charge surface density. Electric field on the surface:

$$E = \frac{\sigma(\vec{x})}{\varepsilon_0}$$

Energy density on the surface of the conductor:

$$w = \frac{\varepsilon_0}{2}\left|\vec{E}\right|^2 = \frac{\sigma^2(\vec{x})}{2\varepsilon_0}$$

Small deformation $\Delta x \Delta S$ leads to a difference in electrostatic energy:

$$\Delta W = \int_{V_{\text{out}}} w\ d^3\vec{x} - \int_{V_{\text{out}}-\Delta x\Delta S} w\ d^3\vec{x} = -\Delta S\Delta x\frac{\sigma^2}{2\varepsilon_0}$$

Force needed to undo deformation: $\Delta W = F\Delta x$. Pressure on the surface:

$$\frac{|F|}{\Delta S} = \frac{|\Delta W|}{\Delta x\Delta S} = \frac{\sigma^2}{2\varepsilon_0}$$

3 Boundary condition problems in electrostatics

3.1 Dirichlet and Neumann boundary conditions

If we know the potential on the boundary: Dirichlet B.C
If we know $\vec{E} = -\vec{\nabla}\phi$ on the boundary: Neumann B.C.

3.2 Green's functions

$$\vec{\nabla}^2_{\vec{x}} G(\vec{x},\vec{y}) = -4\pi\delta(\vec{x}-\vec{y}) \quad , \quad G(\vec{x},\vec{y}) = \frac{1}{|\vec{x}-\vec{y}|} + F(\vec{x},\vec{y})$$

such that $\vec{\nabla}^2_{\vec{x}} F(\vec{x},\vec{y}) = 0$

$$\begin{aligned}\Phi(\vec{y}) =& \frac{1}{4\pi\varepsilon_0}\int_V d^3\vec{x}\ G(\vec{x},\vec{y})\rho(\vec{x}) \\ &- \frac{1}{4\pi}\int_{S(V)} d\vec{S}\cdot\left[\Phi(\vec{x})\vec{\nabla}_{\vec{x}}G(\vec{x},\vec{y}) - G(\vec{x},\vec{y})\vec{\nabla}\Phi(\vec{x})\right]\end{aligned}$$

3.2.1 Dirichlet boundary conditions

We search for a Green's function $G_D(\vec{x},\vec{y})$ which vanishes on the surface $S(V)$. Then we have:

$$\Phi(\vec{y}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3\vec{x}\ G_D(\vec{x},\vec{y})\rho(\vec{x}) - \frac{1}{4\pi}\int_{S(V)} d\vec{S}\cdot\Phi(\vec{x})\vec{\nabla}_{\vec{x}}G_D(\vec{x},\vec{y})$$

3.2.2 Neumann boundary conditions

$$\begin{aligned}\Phi(\vec{y}) =& \frac{1}{4\pi\varepsilon_0}\int_V d^3\vec{x}\ G_N(\vec{x},\vec{y})\rho(\vec{x}) \\ &+ \frac{1}{4\pi}\int_{S(V)} d\vec{S}\cdot G_N(\vec{x},\vec{y})\vec{\nabla}_{\vec{x}}\Phi(\vec{x}) + \langle\Phi\rangle_{S(V)}\end{aligned}$$

With $\langle\Phi\rangle_{S(V)} = \frac{\int_S(V) d\vec{S}\cdot\Phi\hat{n}}{S}$ an unimportant physical constant.

3.3 Explicit solutions

3.3.1 Conductor filling half of space

Left side: free space, right side: conductor. Let x_1 point to the right, x_2 to the top and x_3 out of the plane. $\forall\vec{y} = (y_1,y_2,y_3) \in$ LHS, we define $\vec{y}^* = (-y_1,y_2,y_3) \in$ RHS and

$$F(\vec{x},\vec{y}) = -\frac{1}{\vec{x}-\vec{y}} \quad \Rightarrow \quad G(\vec{x},\vec{y}) = \frac{1}{|\vec{x}-\vec{y}|} - \frac{1}{|\vec{x}-\vec{y}^*|}$$

This G fulfills $G = 0$ on the boundary $(y_1 = 0)$ and $\vec{\nabla}^2 F = 4\pi\delta(\vec{x}-\vec{y}^*) = 0$ because $\vec{x}$ and $\vec{y}$ are in different half planes. Since $|\vec{x}-\vec{y}^*| = |\vec{y}-\vec{x}^*|$ we obtain:

$$\begin{aligned}\Phi(\vec{y}) =& \frac{1}{4\pi\varepsilon_0}\int_{\text{free-space}} d^3\vec{x}\ \left[\frac{\rho(\vec{x})}{|\vec{x}-\vec{y}|} + \frac{-\rho(\vec{x})}{|\vec{x}-\vec{y}^*|}\right] \\ &- \frac{1}{4\pi}\int_S d\vec{S}\cdot\Phi(\vec{x})\vec{\nabla}\left[\frac{1}{|\vec{x}-\vec{y}|} - \frac{1}{|\vec{x}-\vec{y}^*|}\right] \\ =& \frac{1}{4\pi\varepsilon_0}\int_{\text{free-space}} d^3\vec{x}\ \left[\frac{\rho(\vec{x})}{|\vec{x}-\vec{y}|} + \frac{-\rho(\vec{x})}{|\vec{x}^*-\vec{y}|}\right] \\ &- \frac{1}{4\pi}\int_S d\vec{S}\cdot\Phi(\vec{x})\vec{\nabla}\left[\frac{1}{|\vec{x}-\vec{y}|} - \frac{1}{|\vec{x}^*-\vec{y}|}\right] \\ =& V + \frac{1}{4\pi\varepsilon_0}\int_{\text{free-space}} d^3\vec{x}\ \left[\frac{\rho(\vec{x})}{|\vec{x}-\vec{y}|} + \frac{-\rho(\vec{x})}{|\vec{x}^*-\vec{y}|}\right]\end{aligned}$$

where we used that the potential $\Phi(\vec{x})$ is constant on the surface S and takes a value of V. In case of discrete charges:

$$\rho(\vec{x}) = \sum_i q_i\delta(\vec{x}-\vec{x}_i) \quad \Rightarrow \quad \Phi(\vec{y}) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{|\vec{y}-\vec{x}|} + \frac{-q_i}{|\vec{y}-\vec{x}_i^*|}$$

3.3.2 Method of images

Consider a sphere of radius R around the origin. Place a charge $+q$ at a position $\vec{d}$ and a charge of opposite charge $-\frac{R}{d}q$ at a position $\frac{R^2}{d^2}\vec{d}$. Contribution of these two charges to the scalar potential at a position $\vec{r}$:

$$\Phi(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\left|\vec{r}-\vec{d}\right|} - \frac{\frac{R}{d}}{\left|\vec{r}-\frac{R^2}{d^2}\vec{d}\right|}\right] \quad , \quad \Phi(\vec{R}) = 0$$

On the surface of a sphere with radius R, the potential vanishes. Solution for the problem with the charge $+q$ at a distance d from the centre of a conductor of radius R:

$$\Phi(\vec{r}) = \begin{cases}\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{r}-\vec{d}|} - \frac{\frac{R}{d}}{\left|\vec{r}-\frac{R^2}{d^2}\vec{d}\right|}\right] & \forall\vec{r} : r \geq R \\ 0 & \forall\vec{r} : r < R\end{cases}$$

General solution:

$$\Phi(\vec{r}) = \frac{q}{4\pi\varepsilon_0}G_D(\vec{d},\vec{r}) \Rightarrow G_D(\vec{d},\vec{r}) = \frac{4\pi\varepsilon_0}{q}\Phi(\vec{r}) = \frac{1}{\left|\vec{r}-\vec{d}\right|} - \frac{\frac{R}{d}}{\left|\vec{r}-\frac{R^2}{d^2}\vec{d}\right|}$$

3.4 Green's functions from Laplacian eigenfunctions

Consider eigenfunctions $\psi_n(\vec{x})$ of the Laplace operator:

$$\vec{\nabla}^2\psi_n = \lambda_n\psi_n \quad , \quad \psi_n(\vec{x}) = 0\ \forall\vec{x} \in S(V)$$

with the condition that they vanish on the boundary. Eigenvalues are not degenerate and real. Orthogonality condition:

$$\int_V d^3\vec{x}\ \psi_m^*(\vec{x})\psi_n(\vec{x}) = \delta_{nm}$$

Eigenfunctions form a complete basis: Any other function $f(\vec{x})$ which vanishes on the boundary $S(V)$ can be written as a linear superposition of the Laplace eigenfunctions:

$$f(\vec{x}) = \sum_n c_n\psi_n(\vec{x}) \quad , \quad c_n = \int_V d^3\vec{x}\ \psi_n^*(\vec{x})f(\vec{x})$$

Completeness condition:

$$\sum_n \psi_n^*(\vec{x})\psi_n(\vec{x}) = \delta(\vec{x}-\vec{y})$$

Green's function:

$$G_D(\vec{x},\vec{y}) = -4\pi\sum_n\frac{\psi_n^*(\vec{y})\psi_n(\vec{x})}{\lambda_n}$$

3.4.1 Infinite Space

$$G_D(\vec{x},\vec{y}) = \frac{1}{|\vec{x}-\vec{y}|} \quad , \quad \psi_{\vec{k}}(\vec{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{i\vec{k}\cdot\vec{x}} \quad , \quad \lambda_{\vec{k}} = -\left|\vec{k}\right|^2$$

$$\int_{-\infty}^{\infty} dx \ e^{ixa} = 2\pi\delta(a)$$

3.4.2 Orthogonal Parallelepide

Consider an orthogonal parallelepide with sidelengths $a \times b \times c$.

$$\psi_{lmn} = \sqrt{\frac{8}{abc}} \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{m\pi x}{b}\right) \sin\left(\frac{n\pi x}{c}\right)$$

$$\lambda_{lmn} = -\pi^2 \left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)$$

$$G(\vec{x},\vec{y}) = -4\pi \sum_{l,m,n=1}^{\infty} \frac{\psi_{lmn}(\vec{x})\psi_{lmn}(\vec{y})}{\lambda_{lmn}}$$

3.5 Laplace operator and spherical symmetry

Spherical Laplace operator:

$$\vec{\nabla}^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{\hat{A}}{r^2} \quad , \quad \hat{A} = \frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial}{\partial\theta} + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial^2\phi}$$

Setting $x = \cos(\theta)$ we obtain:

$$\hat{A} = \frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x} + \frac{1}{1-x^2}\frac{\partial^2}{\partial^2\phi}$$

Separation of variables: $\psi(r,\theta,\phi) = \frac{R(r)}{r} Y(\theta,\phi)$

$$r\left[\frac{1}{R}\frac{\partial^2 R}{\partial r^2} - \lambda\right] = -\frac{1}{Y}\hat{A}Y = l(l+1)$$

$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2} - \frac{l(l+1)}{r} - \lambda = 0 \quad , \quad \hat{A}Y = -l(l+1)Y$$

3.5.1 Radial Differential Equation

Ansatz $R = r^a$ yiels $a = -l, l+1$, thus:

$$\psi(r,\theta,\phi) = \sum_l \frac{1}{r}\left(A_l r^{-l} + B_l r^{l+1}\right) Y_l(\theta,\phi)$$

In the case $r \to 0$ we approximate

$$r^{a-2}a(a-1) - l(l+1)r^{a-2} = \lambda r^a \approx 0$$

because r^a is of higher order than r^{a-2}.

3.5.2 Angular Differential Equation

Again separation of variables: $Y(\theta,\phi) = \Theta(\theta)\Phi(\phi)$

$$\frac{1+x^2}{\Theta}\left[\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x} + l(l+1)\right]\Theta = -\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = -m^2$$

$$\frac{\partial^2\Phi}{\partial\phi^2} = -m^2\Phi \quad \Rightarrow \quad \Phi(\phi) = e^{im\phi} \ , \ m \in \mathbb{Z}$$

Interlude Legendre Polynomials

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l = \frac{1}{2^l}\sum_{k=0}^{l}\binom{l}{k}^2 (x-1)^{l-k}(x+1)^k$$

Normalization condition: $P_l(1) = 1$. Legendre polynomials vanish when integrated with any other polynomial of a lesser degree in the range $[-1,1]$:

$$\int_{-1}^{1} dx \ x^k P_l(x) = 0 \ \forall k = 0,1,\ldots,(l-1)$$

Orthogonality:

$$\int_{-1}^{1} dx \ P_l(x)P_m(x) = \frac{2}{2l+1}\delta_{lm}$$

Legendre Polynomials form a basis for all continuous functions $f(x)$ in $[-1,1]$:

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x) \quad , \quad c_n = \frac{2n+1}{2}\int_{-1}^{1} dx \ P_n(x)f(x)$$

Completeness condition:

$$\sum_{n=0}^{\infty} P_n(x)P_n(y)\frac{2n+1}{2} = \delta(x-y)$$

Associated Legendre Polynomials

$$\begin{aligned} P_l^m(x) &= (-1)^m(1-x^2)^{\frac{m}{2}}\frac{\partial^m}{\partial x^m}P_l(x) \\ &= \frac{(-1)^m}{2^l l!}(1-x^2)^{\frac{m}{2}}\frac{\partial^{l+m}}{\partial x^{l+m}}(x^2-1)^l \end{aligned}$$

for $m = -l, -l+1, \ldots, 0, \ldots, l-1, l$.

$$P_l^{-m} = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x)$$

Orthogonality condition:

$$\int_{-1}^{1} dx \ P_k^m(x)P_l^m(x) = \frac{2(m+l)!}{(2l+1)(l-m)!}\delta_{kl}$$

$$\int_{-1}^{1} dx \ \frac{P_l^m(x)P_l^n}{1-x^2} = \frac{(l+m)!}{m(l-m)!}\delta_{mn} \quad \text{if } m \neq 0$$

Spherical Harmonics

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}e^{im\phi}P_l^m(\cos(\theta))$$

Orthogonality condition:

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \ \sin(\theta)Y^*_{l'm'}(\theta,\phi)Y^*_{lm}(\theta,\phi) = \delta_{l'l}\delta_{m'm}$$

Every function of the polar and azimutal angles can be written as a linear superposition of spherical harmonics:

$$\begin{aligned} f(\theta,\phi) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_{lm}Y_{lm}(\theta,\phi) \\ c_{lm} &= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \ \sin(\theta)Y^*_{lm}(\theta,\phi)f(\theta,\phi) \end{aligned}$$

Completeness identity:

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) = \delta(\phi'-\phi)\delta(\cos(\theta')-\cos(\theta))$$

Further $Y_{l,-m}(\theta,\phi) = (-1)^m Y^*_{l,m}(\theta,\phi)$. Now we have e general solution to the Laplace differential equation $\vec{\nabla}^2\Psi(r,\theta,\phi) = 0$:

$$\Psi(r,\theta,\phi) = \sum_l \sum_{m=-l}^{l}\left(A_{lm}r^{-l-1} + B_{lm}r^l\right)Y_{lm}(\theta,\phi)$$

3.5.3 Expansion of inverse distance in Legendre polynomials

Take two vectors $\vec{r}_L, \vec{r}_S$ with $r_L > r_S$, then:

$$|\vec{r}_L - \vec{r}_S| = \left[r_L^2 + r_S^2 - 2r_L r_S \cos(\theta)\right]^{\frac{1}{2}}$$
$$\frac{1}{|\vec{r}_L - \vec{r}_S|^a} = \sum_{l=0}^{\infty} \frac{(a,l)}{l!} \frac{r_S^l}{r_L^{l+a}} P_l(\cos(\theta)) \quad \text{with } (a,l) = \frac{\Gamma(a+l)}{\Gamma(a)}$$

3.6 Multipole expansion

Consider charge distribution $\rho(\vec{x})$ in a volume V'. We are interested in the potential that this charge distribution creates at a distance $\vec{r}$ outside of the region of the charge distribution.

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} d^3\vec{x} \frac{\rho(\vec{x})}{|\vec{x} - \vec{r}|} = \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \int_{V'} d^3\vec{x} \rho(\vec{x}) x^l P_l(\cos(\gamma))$$

with γ the angle between $\vec{r} \equiv (r,\theta,\phi)$ and $\vec{x} \equiv (x, \theta_x, \phi_x)$ and $d^3\vec{x} = x^2 dx d\Omega_x$. Addition Theorem:

$$P_l(\cos(\gamma)) = \frac{4\pi}{1+2l} \sum_{m=-l}^{l} Y_{lm}^*(\theta_x, \phi_x) Y_{lm}(\theta,\phi)$$

With this we obtain:

$$\Phi(\vec{r}) = \frac{1}{\varepsilon_0} \sum_{l=0}^{\infty} \frac{1}{1+2l} \frac{1}{r^{l+1}} \sum_{m=-l}^{l} \frac{q_{lm} Y_{lm}(\theta,\phi)}{r^{l+1}}$$

with

$$q_{lm} = \int_{V'} d^3\vec{x}\, Y_{lm}^*(\theta_x,\phi_x) \rho(\vec{x}) x^l$$

Examples:

$$q_{00} = \frac{Q}{\sqrt{4\pi}} = \sqrt{\frac{3}{4\pi}} p_3 \ , \ q_{11} = -\sqrt{\frac{3}{8\pi}} (p_1 - ip_2) \ , \ q_{21} = \frac{1}{2}\sqrt{\frac{5}{4\pi}} Q_{33}$$
$$q_{22} = \frac{1}{12}\sqrt{\frac{15}{2\pi}} (Q_{11} - 2iQ_{12} - Q_{22}) \quad , \quad q_{21} = -\frac{1}{3}\sqrt{\frac{15}{8\pi}} (Q_{13} - iQ_{23})$$

with $\vec{p}$ the dipole moment and Q_{ij} the quadrupole tensor.

$$\vec{p} = (p_1, p_2, p_3) = \int d^3\vec{x}\ \vec{x}\rho(\vec{x}) \ , \ Q_{ij} = \int d^3\vec{x}\ (x_i x_j - x^2 \delta_{ij}) \rho(\vec{x})$$

4 Magnetic Field

4.1 Currents

Define current density $\vec{j}$ as the charge transversing a surface $d\vec{S}$ per unit time and surface:

$$q N_{\text{escaping}} = \vec{J} \cdot d\vec{S} dt \quad \Rightarrow \quad \vec{J} = \rho \vec{v}$$

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

4.2 Magnetic field of steady currents

Assume charge density to be constant: $\frac{\partial \rho}{\partial t} = 0$. Therefore: $\vec{\nabla} \cdot \vec{J} = 0$. Further, the electric and magnetic field are constant in time:

$$\vec{\nabla} \cdot \vec{B} = 0 \quad , \quad c^2 \vec{\nabla} \times \vec{B} = \frac{\vec{J}}{\varepsilon_0}$$
$$\Rightarrow \int_{S(V)} \vec{B} \cdot d\vec{S} = 0 \quad , \quad c^2 \oint_{\Gamma} \vec{B} \cdot d\vec{l} = \frac{\int_{S(\Gamma)} \vec{J} \cdot d\vec{S}}{\varepsilon_0}$$

We define $I_{\text{through } \Gamma} \equiv \int_{S(\Gamma)} \vec{J} \cdot d\vec{S}$ the total charge passing through the closed loop Γ per unit time. We arrive at **Ampere's Law**:

$$\oint_{\Gamma} \vec{B} \cdot d\vec{l} = \frac{I_{\text{through } \Gamma}}{\varepsilon_0 c^2}$$

4.3 Vector potential

$\vec{\nabla} \cdot \vec{B} = 0$ follows automatically if $\vec{B} = \vec{\nabla} \times \vec{A}$. From $\vec{\nabla} \times \vec{B} = \frac{\vec{J}}{\varepsilon_0 c^2}$ we derive:

$$\vec{\nabla}^2 \vec{A} - \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) = -\frac{\vec{J}}{c^2 \varepsilon_0}$$

4.3.1 Gauge Invariance

If $\vec{\nabla} \cdot \left(\vec{\nabla} \cdot \vec{A}\right) = 0$ we would have a poisson equation. We can choose to eliminate this term by gauge invariance. Consider $\vec{A}$ and $\vec{A}'$ and a scalar functioin f.

$$\vec{A}' = \vec{A} + \vec{\nabla} f$$

Easy to see: $\vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} = \vec{B}$. Assume $\vec{\nabla} \cdot \vec{A}' \neq 0$, then:

$$\vec{\nabla}^2 f = -\vec{\nabla} \cdot \vec{A}' \quad \Rightarrow \quad f(\vec{r}) = \frac{1}{4\pi} \int d^3\vec{x} \frac{\left(\vec{\nabla} \cdot \vec{A}'\right)(\vec{x})}{|\vec{r} - \vec{x}|}$$

Then: $\vec{\nabla} \cdot \vec{A} = \vec{\nabla} \cdot \left(\vec{A}' + \vec{\nabla} f\right) = 0$. This is called **Coulomb gauge**. In that gauge:

$$\vec{\nabla}^2 \vec{A} = -\frac{\vec{J}}{c^2 \varepsilon_0} \quad \Rightarrow \quad \vec{A}(\vec{x}) = \frac{1}{4\pi\varepsilon_0 c^2} \int d^3\vec{y} \frac{\vec{J}(\vec{y})}{|\vec{x} - \vec{y}|}$$

4.4 Magnetic Dipole

Consider a steady current $\vec{J}$. We want to compute the vector potential and the magnetic field in a position $\vec{r}$ which is far from the current. First we approximate:

$$\frac{1}{|\vec{r} - \vec{x}|} = \frac{1}{r} + \frac{\vec{x} \cdot \vec{r}}{r^3} + \mathcal{O}\left(\frac{x^2}{r^3}\right)$$

First term gives a zero contribution to the vector potential because $\int d^3\vec{x} \vec{J} = 0$. Contribution of second term:

$$\vec{A} \approx \frac{1}{8\pi\varepsilon_0 c^2 r^3} \int d^3\vec{x} \left[\vec{J}(\vec{x} \cdot \vec{r}) - \left(\vec{J} \cdot \vec{r}\right) \vec{x}\right]$$
$$\vec{A} = \frac{1}{4\pi\varepsilon_0 c^2} \frac{\vec{m} \times \vec{r}}{r^3}$$

with $\vec{m} = \vec{\mu}$ the magnetic moment

$$\vec{\mu} = \frac{1}{2} \int d^3\vec{x} \left(\vec{x} \times \vec{J}\right)$$

Special case: steady current circulating anti-clockwise in a wire which lays on a plane:

$$\vec{\mu} = \oint \frac{I}{2} \vec{x} \times d\vec{l}$$

Notice: $d\vec{S} = \frac{1}{2}\vec{x} \times d\vec{l} \Rightarrow \vec{m} = I\vec{S}$ with $\vec{S} = \int d\vec{S}$ the total area of the loop.

Special case: current created by a single charge q and mass M moving along a closed loop. We have:

$$\vec{J} = \rho\vec{v} = q\delta\left(\vec{x} - \vec{r}(t)\right)\vec{v} \quad , \quad \vec{m} = \frac{1}{2} \int d^2\vec{x}\ \vec{x} \times \vec{J} = \frac{q}{2} \vec{r} \times \vec{v} = \frac{2}{2M} \vec{L}$$

with $\vec{L} = \vec{r} \times (M\vec{v})$ the angular momentum of the charge. Magnetic field of a magnetic dipole:

$$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \frac{\vec{m} \times \vec{r}}{r^3} = -\vec{\nabla} \times \left[\vec{m} \times \vec{\nabla}\frac{1}{r}\right] = \left(\vec{m} \cdot \vec{\nabla}\right) \vec{\nabla} \frac{1}{r}$$
$$= -\frac{\vec{m} - 3\hat{r}\left(\hat{r} \cdot \vec{m}\right)}{r^3}$$

4.5 Relativity of the electric and magnetic field

Consider a wire on the x-axis producing a magnetic field $B = \frac{1}{2\pi\varepsilon_0 c^2}\frac{I}{r}$ at a distance r. Assume induced electrons have inside have a velocity $\vec{v}$. Imagine another electron outside the wire to move parallel to the wire at a distance r from it with the same velocity $\vec{v}$. Force acting on electron outside:

$$F = qvB = \frac{1}{2\pi\varepsilon_0}\frac{qvI}{c^2 r} = \frac{qS}{2\pi\varepsilon_0}\frac{\rho_-}{r}\frac{v^2}{c^2}$$

where $I = S\rho_- v$ with S the cross-section area of the wire and ρ_- the electron charge density inside.

Now choose reference frame that moves with a relative velocity $\vec{v}$ along the wire. Here, electron outside is static ($v' = 0$). Here, magnetic component of the force must vanish. In original frame: $\rho = \rho_+ + \rho_- = 0$ and in the moving frame $\rho' = \rho'_+ + \rho'_- \neq 0$.

$$\rho_{\text{moving}} = \frac{\rho_{\text{rest}}}{\sqrt{1-\frac{v^2}{c^2}}} \quad \Rightarrow \quad \rho'_+ = \frac{\rho_+}{\sqrt{1-\frac{v^2}{c^2}}} \ , \ \rho'_- = \rho_-\sqrt{1-\frac{v^2}{c^2}}$$

Given that $\rho_+ = -\rho_-$ we have:

$$\rho' = -\rho_- \frac{\frac{v^2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

The new electric field and force are:

$$E' = \frac{1}{2\pi\varepsilon_0}S\frac{\rho'}{r} \quad , \quad F' = \frac{F}{\sqrt{1-\frac{v^2}{c^2}}}$$

5 Time varying electromagnetic fields

5.1 Charge conservation

When integrating the continuity equation over all of space, we obtain that the total charge in the universe has to be constant: $\frac{\partial Q_{\text{universe}}}{\partial t} = 0 \Rightarrow Q_{\text{universe}} = \text{const.}$

5.2 Vector and scalar potential

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \rightsquigarrow \vec{\nabla}\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = 0$$

This is satisfied if we introduce a scalar potential ϕ such that

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi$$

We define $\Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$. Then we have:

$$\begin{aligned} \Box\vec{A} + \vec{\nabla}\left[\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right] &= \frac{\vec{J}}{c^2\varepsilon_0} \\ \vec{\nabla}^2\phi - \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} &= \frac{\rho}{\varepsilon_0} \\ \Box\phi - \frac{\partial}{\partial t}\left[\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right] &= \frac{\rho}{\varepsilon_0} \end{aligned}$$

5.3 Gauge Invariance

Electric and magnetic field remain invariant under the following transformation of the scalar and vector potential:

$$\phi \to \phi' = \phi - \frac{\partial f}{\partial t} \qquad , \qquad \vec{A} \to \vec{A}' = \vec{A}' + \vec{\nabla} f$$

with $f = f(\vec{x}, t)$. We can not choose f such that the potentials $\vec{A}_L$, ϕ_L satisfy

$$\vec{\nabla}\cdot\vec{A}_L + \frac{1}{c^2}\frac{\partial\phi_L}{\partial t} = 0$$

This is called the "Lorentz gauge". Now we have:

$$\Box\phi_L = \frac{\rho}{\varepsilon_0} \qquad , \qquad \Box\vec{A}_L = \frac{\vec{J}}{c^2\varepsilon_0}$$

5.4 Electromagnetic waves in empty space

In empty space $\vec{J} = 0$ and $\rho = 0$. In empty space, all fields satisfy the same equation:

$$\Box f = 0 \quad , \quad f \in \left\{\phi, \vec{A}, \vec{E}, \vec{B}\right\}$$

A solution to this equation is

$$f(\vec{x}, t) = f(\hat{\eta}\cdot\vec{x} - ct)$$

with $\hat{\eta}^2 = 1$. In this solution, f depends on a single combination $u = \hat{\eta}\cdot\vec{x} - ct$. The solution f is a travelling wave with a speed equal to the speed of light c, moving along the direction of the unit vector $\hat{\eta}$. We can now write:

$$\vec{E} = \hat{e}E(\hat{\eta}\cdot\vec{x} - ct) \quad , \quad \vec{B} = \hat{b}B(\hat{\eta}\cdot\vec{x} - ct)$$

We find: $\vec{E}, \vec{B} \perp \hat{\eta}$ and $\vec{E} \perp \vec{B}$. Further: $|\vec{E}| = c\cdot|\vec{B}|$.

5.4.1 Spherical waves

If we are able to change the charge and current density at one point in the entire space only, we will generate an electromagnetic wave with spherical symmetry, i.e. no preferred direction. We still have:

$$\Box f = 0 \quad , \quad \forall f \in \left\{\vec{A}, \vec{B}, \vec{E}\right\}$$

Now with spherical symmetry we have: $f(\vec{r}, t) = f(r, t)$. So we have:

$$\begin{aligned} \Box f(r,t) &= \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} - \frac{1}{r}\frac{\partial^2(rf)}{\partial r^2} = 0 \\ &\Rightarrow \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2}\right](rf) = 0 \\ &\Rightarrow rf = A(r-ct) + B(r+ct) \\ &\Rightarrow f(r,t) = \frac{A(r-ct)}{r} + \frac{B(r+ct)}{r} \end{aligned}$$

The first term corresponds to a spherical wave propagating outwards and the second term is a spherical wave that propagates inwards. Typically only the outwards propagaion is realistic.

5.5 Moving charges in a homogeneous magnetic field

Consider a $\vec{B}$-field with $\vec{j}_{source}$ and far away at distance $\vec{r}_0$ a cloud of charges Ω with $\vec{j}(x) \neq 0$. The force of the $\vec{B}$-field acting on Ω is

$$\vec{F} = (\vec{m}\cdot\vec{\nabla})\vec{B}\Big|_{x=0} \quad , \quad \vec{m} = \frac{1}{2}\int_\Omega d^3\vec{x}\ \vec{x}\times\vec{j}$$

with $\vec{m}$ the magnetic dipole moment. The potential energy is then:

$$U = -\vec{m}\cdot\vec{B}$$

6 General solution of Maxwell equations with sources

We want to find the solution to Maxwell equations in the presence of sources ($\vec{J}, \rho \neq 0$). In the Lorentz gauge we have:

$$\Box \phi = \frac{\rho}{\varepsilon_0} \quad , \quad \Box \vec{A} = \frac{\vec{J}}{c^2 \varepsilon_0}$$

Suppose we know a solution ϕ_{sol} such that $\Box \phi_{sol} = \frac{\rho}{\varepsilon_0}$. Then $\phi' = \phi_{free} + \phi_{sol}$ with $\Box \phi_{free} = 0$ is also a solution.

6.1 Green's functions

To solve above differential equations we seek functions $G(\vec{x}, \vec{x}', t, t')$ which satisfy

$$\Box_{\vec{x},t} G(\vec{x}, \vec{x}', t, t') = \delta^{(3)}(\vec{x} - \vec{x}')\delta(t - t')$$

Then a solution for the scalar potential is:

$$\phi(\vec{x}, t) = \phi_{free}(\vec{r}, t) + \int_{-\infty}^{\infty} d^3\vec{x}' dt' \; G(\vec{x}, \vec{x}', t, t') \frac{\rho(\vec{x}', t')}{\varepsilon_0}$$

This violates causality because we need to know future states at times $t' > t$ to determine the one at time t. Therefore, we say

$$\Box = \lim_{\delta \to 0} \Box_\delta = \lim_{\delta \to 0} \left(\frac{1}{c} \frac{\partial}{\partial t} + \delta \right)^2 - \vec{\nabla}^2$$

6.2 Fourier transformation

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \frac{dt'}{\sqrt{2\pi}} e^{ikt'} f(t') \quad , \quad f(t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} e^{-ikt} \tilde{f}(k)$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \; e^{-i\omega t} = \delta(t) = \partial_t \Theta(t) = \partial_t \left(-\frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\omega + i\varepsilon} \right)$$

6.3 Fourier transformation and Green's functions

We want Green's functions which are only dependent on space-time differences

$$G(\vec{x}, \vec{x}', t, t') = G(\vec{x} - \vec{x}', t - t') = G(\Delta\vec{x}, \delta t)$$
$$\Box G(\Delta\vec{x}, \Delta t) = \delta(\Delta\vec{x})\delta(\Delta t)$$

If we calculate $\Box_\delta G(\Delta\vec{x}, \Delta t)$ and compare with the expected result, we obtain:

$$\tilde{G}(E, \vec{k}) = \frac{-c}{(E + i\delta)^2 - \vec{k}^2}$$

From this we calculate:

$$\begin{aligned} G(\Delta\vec{x}, \Delta t) &= \lim_{\delta \to 0^+} -c \int \frac{d^3\vec{k} dE}{(2\pi)^4} \frac{e^{-i(Ec\Delta t - \vec{k}\cdot\Delta\vec{x})}}{(E + i\delta)^2 - \vec{k}^2} \\ &= \frac{1}{4\pi |\vec{x} - \vec{x}'|} \delta\left(t - t' - \frac{|\vec{x} - \vec{x}'|}{c} \right) \Theta(t > t') \end{aligned}$$

So we obtain:

$$\phi(\vec{x}, t) = \frac{1}{4\pi\varepsilon_0} \int d^3\vec{x}' \frac{\rho\left(\vec{x}', t - \frac{|\vec{x} - \vec{x}'|}{c} \right)}{|\vec{x} - \vec{x}'|}$$

If $c \to \infty$ we obtain Coulomb's law. For the vector potential we obtain:

$$\vec{A}(\vec{x}, t) = \frac{1}{4\pi\varepsilon_0 c^2} \int d^3\vec{x}' \frac{\vec{J}\left(\vec{x}', t - \frac{|\vec{x} - \vec{x}'|}{c} \right)}{|\vec{x} - \vec{x}'|}$$

The retarded Green's function is given as:

$$G(\vec{x} - \vec{x}', t - t') = \frac{1}{2\pi} \delta\left((t - t')^2 - \frac{|\vec{x} - \vec{x}'|^2}{c^2} \right) \Theta(t > t')$$

6.4 Potential of a moving charge with a constant velocity

We calculate the scalar and vector potential of a point-like charge q, moving with a velocity $\vec{v}$. The charge density is

$$\rho(\vec{x}', t') = q\delta(\vec{x}' - \vec{v}t')$$

So the scalar potential is:

$$\Phi(\vec{x}, t) = \frac{q}{2\pi\varepsilon_0} \int dt' \delta\left((t - t')^2 - \frac{|\vec{x} - \vec{v}t|^2}{c^2} \right) \Theta(t > t')$$

We want to find the zeros of the argument of the delta funciton. Therefore, we decompose $\vec{x}$ into $\vec{x} = \vec{x}_\parallel + \vec{x}_\perp$.

$$0 = (t - t')^2 - \frac{|\vec{x} - \vec{v}t'|^2}{c^2} = t'^2 - 2t't + t^2 - \frac{(x_\parallel - vt')^2 + x_\perp^2}{c^2}$$

We can now define "boosted variables"

$$x_b = \gamma\left(x_\parallel - vt \right) \quad , \quad t_b = \gamma\left(t - \frac{x_\parallel v}{c^2} \right) \quad , \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

We introduce:

$$\tau^2 = c^2 t^2 - \left(x_\parallel^2 + x_\perp^2 \right) = c^2 t_b^2 - (x_b^2 + x_\perp^2) \quad , \quad r_b^2 = x_b^2 + x_\perp^2$$

Then:

$$\frac{t'^2}{\gamma^2} - 2\frac{t'}{\gamma} + \frac{\tau^2}{c^2} = 0 \quad , \quad \Delta = \frac{4(t_b^2 - \tau^2/c^2)}{\gamma^2} = \frac{4r_b^2}{\gamma^2 c^2}$$

Now the solutions are:

$$t_\pm = \gamma\left(t_b \pm \frac{r_b}{c} \right)$$

The delta function becomes:

$$\delta\left((t - t')^2 - \frac{|\vec{x} - \vec{v}t|^2}{c^2} \right) \Theta(t > t') = \frac{c\gamma}{2r_b} \Theta(t > t') \delta(t' - t_-)$$

In conclusion we obtain:

$$\Phi(\vec{x}, t) = \frac{q}{4\pi\varepsilon_0} \frac{\gamma}{r_b} = \frac{q}{4\pi\varepsilon_0} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{1}{\left[\left(\frac{x_\parallel - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \right)^2 + x_\perp^2 \right]^{\frac{1}{2}}}$$

$$\vec{A}(\vec{x}, t) = \frac{\vec{v}}{c^2} \Phi(\vec{x}, t)$$

In a reference frame where the charge is at rest, the scalar and vector potentials are:

$$\Phi(\vec{x}, t)|_{rest} = \frac{q}{4\pi\varepsilon_0} \frac{1}{\left[x_\parallel^2 + x_\perp^2 \right]} \quad , \quad \vec{A}(\vec{x}, t)|_{rest} = 0$$

We have the following transformations:

$$x_\parallel \to \gamma\left(x_\parallel - vt \right) \quad , \quad x_\perp \to x_\perp$$

And we define the following "four-vector":

$$\begin{pmatrix} \Phi \\ \vec{A} \end{pmatrix} \equiv A^\mu$$

7 Special Relativity

In special relativity different coordinate systems are related via Lorentz transformations:

$$x^\mu \to x^{\mu'} = \Lambda^\mu{}_\nu x^\nu + \rho^\mu \quad \text{with} \quad x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad , \quad \Lambda^\mu{}_\nu \equiv \frac{\partial x'^\mu}{\partial x^\nu}$$

$\Lambda^\mu{}_\nu$ satisfies: $\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma g_{\mu\nu} = g_{\rho\sigma}$ with $g_{\mu\nu}$ the metric:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

7.1 Proper Time

Lorentz transformations leave invariant "proper-time" intervals which are defined as

$$d\tau^2 \equiv c^2 dt^2 - d\vec{x}^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Further relations:

$$d\tau^2 = d\tau'^2 \quad , \quad g_{\rho\sigma} = g_{\mu\nu} \frac{\partial x^\mu}{\partial x_\rho} \frac{\partial x^\nu}{\partial x_\sigma}$$

$$\frac{\partial x'^\mu}{\partial x^\sigma} \frac{\partial x^\sigma}{\partial x'^\nu} = \delta_{\mu\nu} \quad , \quad \frac{\partial^2 x'^\mu}{\partial x^\varepsilon \partial x^\rho} = 0$$

7.2 Subgroups of Lorentz transformations

Group properties: $x^\mu \xrightarrow{(\Lambda_1,\rho_1)} x'^\mu \xrightarrow{(\Lambda_2,\rho_2)} x''^\mu$. Then for $x^\mu \xrightarrow{(\Lambda_3,\rho_3)} x''^\mu$ we have:

$$\Lambda^\mu_{3\rho} \equiv \Lambda^\mu_{2\nu} \Lambda^\nu_{1\rho} \quad , \quad \rho^\mu_3 = \Lambda^\mu_{2\nu} \rho^\nu_1 + \rho^\mu_2 \quad , \quad x''^\mu = \Lambda^\mu_{3\nu} x^\nu + \rho^\mu_3$$

The set of all Lorentz transformations is called the inhomogeneous Lorentz group or the Poincare' group. The subset of transformations with $\rho^\mu = 0$ is known as the homogeneous Lorentz group. Here:

$$(\Lambda^0{}_0)^2 - \sum_{i=1}^{3} (\Lambda^i{}_0)^2 = 1 \;\Rightarrow\; (\Lambda^0{}_0)^2 \geq 1$$

$$g = \Lambda^T g \Lambda \;\Rightarrow\; \det(\Lambda)^2 = 1 \;\Rightarrow\; \det(\Lambda) = \pm 1$$

The subgroup of transformations with $\det(\Lambda) = 1$ and $\Lambda^0{}_0 \geq 1$ is known as the proper group of Lorentz transformations. Transformations with $\Lambda^0_0 \geq 1$ are known as orthochronous Lorentz transformations (preserve right flow of time).

Assume a reference frame O in which a particle appears at rest and O' a reference frame in where the particle appears to move with a velocity $\vec{v}$. Then:

$$dx'^\mu = \Lambda^\mu{}_\nu dx^\nu = \Lambda^\mu{}_0 = ct \quad , \quad dt' = \Lambda^0{}_0 dt \quad , \quad dx'^i = \Lambda^i{}_0 cdt$$

$$v^i \equiv \frac{dx'^i}{dt'} = c \frac{\Lambda^i{}_0}{\Lambda^0{}_0} \;\Rightarrow\; \Lambda^i{}_0 = \frac{v^i}{c} \Lambda^0{}_0 \quad , \quad \Lambda^0{}_0 = \gamma \;\Rightarrow\; \Lambda^i{}_0 = \gamma \frac{v^i}{c}$$

For coordinate systems O and O' with parallel axes we find that

$$\Lambda^i{}_j = \delta^i_j + \frac{v^i v^j}{v^2}(\gamma - 1) \quad , \quad \Lambda^0{}_j = \gamma \frac{v^j}{c}$$

The group of Rotations:

$$\Lambda^0{}_0 = 1 \;,\; \Lambda^i{}_0 = \Lambda^0{}_i = 0 \;,\; \Lambda^i{}_j = R_{ij} \quad \text{with } \det(R) = 1 \;,\; R^T R = 1$$

7.3 Time dilation

Consider an inertial observer O which looks as a clock at rest. Here:

$$dt = \Delta t \quad , \quad d\vec{x} = 0 \quad , \quad d\tau = c\Delta t$$

A second observer sees the clock with velocity $\vec{v}$. Here:

$$dt' = \Delta t' \quad , \quad d\vec{x}' = \vec{v} dt' \quad , \quad d\tau' = c\Delta t' \sqrt{1 - \frac{\vec{v}^2}{c^2}}$$

So we find:

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}} = \gamma \Delta t$$

7.4 Doppler Effect

Take moving clock to be a lightsource with frequency $\omega = \frac{2\pi}{\Delta t}$. For observer where light source is moving with velocity $\vec{v}$: $dt' = \gamma \Delta t$ and the distance of the observer from the light source increased by $v_r dr'$ where v_r is the component of the velocity of the light source along the direction of sight of the observer. Time elapsing between the reception of two seccessive light wave fronts from the observer:

$$cdt_0 = cdt' + v_r dt'$$

Frequency measured by the observer: ($\star$ case if $v_r = v$)

$$\omega' = \frac{2\pi}{dt_0} = \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{v_r}{c}} \omega \quad \overset{\star}{=} \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} \omega$$

7.5 Particle dynamics

Question: How to compute relativistic force. Newtonian expressions for the force should be valid if a particle is at rest. An elegant solution is to define a relativistic force acting on a particle as

$$f^\mu = mc^2 \frac{d^2 x^\mu}{d\tau^2}$$

with m the mass of the particle. If the particle is at rest: $d\tau = cdt$. Therefore, in the restframe of the particle:

$$f^0_{\text{rest}} = mc \frac{d^2 t}{dt^2} = 0 \quad , \quad f^i_{\text{rest}} = m \frac{d^2 x^i}{dt^2} = F^i_{\text{Newton}} \quad , \quad i = 1,2,3$$

where $\vec{F}_{\text{Newton}}$ is the force-vector known from Newtonian mechanics. The time component vanishes. Under Lorentz transformation

$$f'^\mu = \Lambda^\mu{}_\nu f^\nu$$

Transformation from the rest frame of a particle to a frame where the particle moves with a velocity $\vec{v}$, we have

$$f^\mu = \Lambda^\mu{}_\nu(\vec{v}) f^\nu_{\text{rest}}$$

With

$$\Lambda^0{}_0(\vec{v}) = \gamma \quad , \quad \Lambda^i{}_0(\vec{v}) = \Lambda^0{}_i(\vec{v}) = \gamma \frac{v^i}{c} \quad , \quad \Lambda^i{}_j(\vec{v}) = \delta^i_j + \frac{v^i v^j}{v^2}(\gamma - 1)$$

Therefore:

$$\vec{f} = \vec{F}_{\text{Newton}} + (\gamma - 1) \frac{\vec{v}\left(\vec{F}_{\text{Newton}} \cdot \vec{v}\right)}{v^2} \quad , \quad f^0 = \gamma \frac{\vec{v} \cdot \vec{F}_{\text{Newton}}}{c} = \frac{\vec{v}}{c} \cdot \vec{f}$$

In Newtonian mechanics we can calculate the trajectory $\vec{x}(t)$ of a particle by solving a differential equation. Analogously we could solve for $x^\mu = x^\mu(\tau)$. Therefore, we need to find $\tau = \tau(x^0)$. A second constraint is:

$$\Omega \equiv g_{\mu\nu} \frac{d^2 x^\mu}{d\tau^2} \frac{dx^\nu}{d\tau} = \frac{2}{mc^2} g_{\mu\nu} f^\mu \frac{dx^\mu}{d\tau}$$

The rhs is a Lorentz invariant quantity. We can derive this in the rest frame. Here $x^\mu = (ct, \vec{0})$ and $f^\mu = (0, \vec{F}_{\text{Newton}})$.

$$\frac{d\Omega}{d\tau} = \frac{2}{mc^2} \left(f^0 \frac{dx^0}{d\tau} - \vec{f} \cdot \vec{x} \right) = 0$$

Therefore, Ω is always a constant. $\Omega(\tau) = \Omega(\tau) = \text{constant} = 1$.

7.6 Energy and momentum

The relativistic four-momentum is given as:

$$p^\mu = mc\frac{dx^\mu}{d\tau} \quad , \quad p^0 = m\gamma c \quad , \quad p^i = m\gamma v^i$$

with $d\tau = \frac{ct}{\gamma}$. We identify the relativistic energy of a particle with

$$E = cp^0 = m\gamma c^2$$

We obtain the relation:

$$E = \sqrt{\vec{p}^2 c^2 + m^2 c^4}$$

7.7 The inverse of a Lorentz transformation

We define the inverse of the metric matrix $g_{\mu\nu}$ as $g^{\mu\nu}$.

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu{}_\nu \quad \Rightarrow \quad g^{\mu\nu} = g_{\mu\nu} = \text{diag}(1,-1,-1,-1)$$

The inverse of a Lorentz transformation $\Lambda^\mu{}_\nu$ is $\Lambda_\mu{}^\nu$.

$$\Lambda_\mu{}^\nu \equiv g_{\mu\rho} g^{\nu\sigma} \Lambda^\rho{}_\sigma$$

If $\Lambda^\mu{}_\nu$ is a velocity $\vec{v}$ boost transformation, then

$$\Lambda_0{}^0(\vec{v}) = \gamma \;\; , \;\; \Lambda_i{}^0(\vec{v}) = \Lambda_0{}^i(\vec{v}) = -\gamma\frac{v^i}{c} \;\; , \;\; \Lambda_i{}^j(\vec{v}) = \delta_i^j + \frac{v^i v^j}{v^2}(\gamma - 1)$$

Therefore: $\Lambda_\mu{}^\nu(\vec{v}) = \Lambda^\mu{}_\nu(-\vec{v})$.

7.8 Vectors and Tensors

Contravariant vectors transform according to the rule:

$$V^\mu \to V'^\mu = \Lambda^\mu{}_\nu V^\nu$$

These are vectors that transform the same way as space-time coordinates x^μ.

Covariant vectors transform according to the rule

$$U_\mu = \Lambda_\mu{}^\nu U_\nu$$

Derivatives $\frac{\partial}{\partial x^\mu}$ transform in this way. These transformations transform according to the inverse Lorentz transformation.

Dual Vectors For every contravariant vector U^μ there is a dual vector which is covariant.

$$U_\mu = g_{\mu\nu} U^\nu \quad , \quad U^\rho = g^{\rho\mu} U_\mu$$

The scalar product of a contravariant and a covariant vector is invariant under Lorentz transformation.

$$A \cdot B \equiv A^\mu B_\mu = A_\mu B^\mu$$

The D'Alembert operator $\Box \equiv \partial^2 \equiv \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$ is invariant under Lorentz transformation.

Tensor We define a tensor with multiple "up" and/or "down" indices to be an object $T^{\mu_1\mu_2\ldots}_{\nu_1\nu_2\ldots}$ which transforms as

$$T^{\mu_1\mu_2\ldots}_{\nu_1\nu_2\ldots} \to \Lambda^{\mu_1}{}_{\rho_1}\Lambda^{\mu_2}{}_{\rho_2}\ldots\Lambda_{\nu_1}{}^{\sigma_1}\Lambda_{\nu_2}{}^{\sigma_2}\ldots T^{\rho_1\rho_2\ldots}_{\sigma_1\sigma_2\ldots}$$

7.9 Currents and densities

For n particles with charges q_n and positions $\vec{r}_n(t)$ the charge and current density are:

$$\rho(\vec{x},t) = \sum_n q_n \delta(\vec{x} - \vec{r}_n(t))$$

$$\vec{j} = \sum_n q_n \frac{d\vec{r}_n(t)}{dt}\delta(\vec{x} - \vec{r}_n(t)) = \sum_n q_n \frac{d\vec{x}}{dt}\delta(\vec{x} - \vec{r}_n(t))$$

We can combine the charge and current densities into one object:

$$j^\mu \equiv \left(c\rho, \vec{j}\right) \quad \text{with} \quad j^\mu(\vec{x},t) = \sum_n q_n \frac{dx^\mu}{dt}\delta(\vec{x} - \vec{r}_n(t))$$

This is a contravariant four-vector. δ-function in four-dimensions:

$$\delta(x^\mu - y^\mu) = \delta(x^0 - y^0)\delta(\vec{x} - \vec{y}) = \frac{1}{c}\delta(t_x - t_y)\delta(\vec{x} - \vec{y})$$

$$\delta(U'^\mu) = \delta(\Lambda^\mu{}_\nu U^\nu) = \frac{\delta(U^\nu)}{|\det(\Lambda)|} = \delta(U^\nu)$$

Current-density in integralform:

$$\begin{aligned} j^\mu(\vec{x},t) &= \sum_n q_n \int dt' \frac{dx^\mu}{dt'}\delta(\vec{x} - \vec{r}_n(t))\delta(t' - t) \\ &= c\sum_n q_n \int dt' \frac{dx^\mu}{dt'}\delta(x^\mu - r_n^\mu(t)) \\ &= c\sum_n q_n \int d\tau \frac{dx^\mu}{d\tau}\delta(x^\mu - r_n^\mu(\tau)) \end{aligned}$$

Continuity equation:

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0 \quad \to \quad \partial_\mu j^\mu = 0$$

7.10 Energy-Momentum tensor

For n particles at positions $\vec{r}_n(t)$ the energy density and "energy-current density" is:

$$\text{energy density} = \sum_n E_n(t)\delta(\vec{x} - \vec{r}_n(t))$$

$$\text{energy current density} = \sum_n E_n(t)\frac{d\vec{r}_n}{dt}\delta(\vec{x} - \vec{r}_n(t))$$

Combine the two:

$$\sum_n E_n(t)\frac{dr_n^\nu}{dt}\delta(\vec{x} - \vec{r}_n(t))$$

We define the "Energy-momentum Tensor" as

$$T^{\mu\nu} \equiv \sum_n p_n^\mu \frac{dr_n^\mu}{dt}\delta(\vec{x} - \vec{r}_n(t)) = \sum_n \int d\tau p_n^\mu \frac{dx_n^\nu}{d\tau}\delta(x^\rho - r_n^\rho(\tau))$$

It transforms as the product of two-four vectors under Lorentz transformation

$$T'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma T^{\rho\sigma}$$

It is symmetric: $T^{\mu\nu} = T^{\nu\mu}$ and takes the form:

$$T^{\mu\nu} \equiv \sum_n \frac{p_n^\mu p_n^\nu}{E_n}\delta(\vec{x} - \vec{r}_n(t))$$

We arrive at the equation:

$$\partial_\nu T^{\mu\nu} = G^\mu = \sum_n \frac{\partial p_n^\mu}{\partial t}\delta(\vec{x} - \vec{r}_n) = \sum_n \frac{\partial\tau}{\partial t} f_n^\mu(t)\delta(\vec{x} - \vec{r}_n)$$

with G^μ the "density of force". For free particles, p_n^μ =constant. Therefore, $\partial_\nu T^{\mu\nu} = 0$. Energy momentum Tensor is also conserved if the particles interact only at the points where they collide with each other. If the continuity equation is satisfied, then the following vector is conserved.

$$P^\mu \equiv \int d^3\vec{x}\, T^{\mu 0} = \text{const} = P^\mu = \sum_n \int d^3\vec{x}\, p_n^\mu \delta(\vec{x} - \vec{r}_n(t)) = \sum_n p_n^\mu$$

8 Relativistic formulation of Electrodynamics

From now on we will assume $\varepsilon_0 = c = 1$. The Maxwell Eq. become:

$$\vec{\nabla} \cdot \vec{E} = \rho \quad , \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad , \quad \vec{\nabla} \cdot \vec{B} = 0 \quad , \quad \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j}$$

We define the electromagnetic field tensor $R^{\mu\nu}$.

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}$$

It has the following properties:

$$F^{0i} = -E^i \quad , \quad F^{ij} = -\varepsilon_{ijk} B^k \quad , \quad F^{\mu\nu} = -F^{\nu\mu}$$
$$B^i = -\frac{1}{2}\varepsilon_{ijk} F^{jk} \quad , \quad -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{\vec{E}^2 - \vec{B}^2}{2}$$
$$\varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} = 8 \vec{E} \cdot \vec{B}$$

We define two four-vectors. One for the charge and current densities and the other for the scalar potential and the vector potential.

$$j^\mu = (\rho, \vec{j}) \quad , \quad A^\mu \equiv \left(\phi, \vec{A}\right) = \left(\phi, A^1, A^2, A^3\right)$$

We can simplify the Maxwell Eq. to:

$$\partial_\nu F^{\nu\mu} = j^\mu \quad , \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

We further have the followig identities:

$$\varepsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0 \quad , \quad \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$$

Lastly:

$$\partial^2 A^\nu - \partial^\nu \left(\partial_\mu A^\mu\right) = j^\mu$$

Gauge Invariance Gauge transformations of the vector and scalar potentials which leave Maxwell Eq. invariant are written in the following form with χ a scalar function.

$$A'_\mu = A_\mu + \partial_\mu \chi$$

The Lorentz gauge-fixing condition becomes

$$\vec{\nabla} \cdot \vec{A} + \frac{\partial \phi}{\partial t} = 0 \quad \rightsquigarrow \quad \partial_\mu A^\mu = 0$$

In the Lorentz gauge, the Maxwell Eq. become:

$$\partial^2 A^\mu = j^\mu$$

In relativistic notation, the solutions for the four-vector potential take the form:

$$A^\nu(x^\mu) = \frac{1}{2\pi} \int d^4x' j^\nu(x'^\mu) \delta\left(\left(x'^\mu - x^\mu\right)^2\right) \Theta\left(x^0 > x'^0\right)$$

The electromagnetic force acting on a particle with a charge q is:

$$f^\mu = q F^{\mu\nu} \frac{dx_\nu}{d\tau}$$

In the frame where the charge is moving with a velocity $\vec{v}$, the three dimensional force is:

$$\vec{f} = q\gamma\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

8.1 Energy-Momentum Tensor in the presence of an electromagnetic field

Consider a number of charges q_n which interact via the electromagnetic field. The energy-momentum tensor is not conserved:

$$\partial_\nu T^{\mu\nu} = G^\mu = F^{\mu\nu} j_\nu$$

We define the following symmetric and gauge-invariant tensor:

$$T^{\mu\nu}_{em} \equiv F^\mu{}_\rho F^{\rho\nu} + \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}$$

The components of the tensor are:

$$T^{00}_{em} = \frac{\vec{E}^2 + \vec{B}^2}{2} \quad , \quad T^{0i}_{em} = T^{i0}_{em} = \left(\vec{E} \times \vec{B}\right)_i$$

We find:

$$\partial_\nu T^{\mu\nu}_{em} = -F^{\mu\nu} j_\nu$$

We define:

$$\Theta^{\mu\nu} \equiv T^{\mu\nu} + T^{\mu\nu}_{em} = \sum_n \frac{p^\mu_n p^\mu_n}{E_n} \delta\left(\vec{x} - \vec{r}_n(t)\right) + F^{\mu\rho} F_\rho{}^\nu \frac{g^{\mu\nu}}{4} F_{\rho\sigma} F^{\rho\sigma}$$

This quantity safisfies the continuity equation:

$$\partial_\nu \Theta^{\mu\nu} = 0$$

The conserved four vector is:

$$P^\mu = \int d^3\vec{x}\, \Theta^{\mu 0} = \sum_n p^\mu_n + \int d^3\vec{x}\, T^{\mu 0}_{em} = P^\mu_{charges} + P^\mu_{em}$$

Where $P^\mu_{charges}$ is the four-momentum of the charges and P^μ_{em} is the momentum carried by the electromagnetic field itself. Neither of them is conserved on its own, only their sum.

The energy stored in the electromagnetic field is:

$$E_{em} = \int d^3\vec{x}\, T^{00}_{em}$$

Therefore, the energy density w of the electromagnetic field is:

$$w = T^{00}_{em} = \frac{\vec{E}^2 + \vec{B}^2}{2}$$

The three-momentum density $\vec{S}$ of the electromagnetic field is:

$$\vec{S} = T^{0i}_{em} = T^{i0}_{em} = \left(\vec{E} \times \vec{B}\right)_i$$

The vector $\vec{S} \equiv \vec{E} \times \vec{B}$ is known as the Poynting vector. For $\mu = 0$ we arrive at

$$\frac{\partial w}{\partial t} + \vec{\nabla} \cdot \vec{S} = -\vec{E} \cdot \vec{j}$$

9 Radiation from moving charges

Radiation is a physical phenomenon due to accelerated or decelerated electric charges.

9.1 The vector potential from a moving charge

A moving charge q has a current density

$$j^\mu(x) = \left(q\delta\left(\vec{x} - \vec{r}(t)\right), q\frac{d\vec{r}}{dt} \delta(\vec{x} - \vec{r}(t))\right)$$

In explicit covariant form:

$$j^\mu = q \int d\tau\; v^\mu \delta\left(x^\rho - r^\rho(\tau)\right)$$

with $v^\mu = \frac{dr^\mu}{d\tau}$. With this result we obtain:

$$A^\mu = \int d^4x' G_{ret}(x - x') j^\mu(x')$$

with G_{ret} the retarded Green's function:

$$G_{ret}(x-x') = \frac{1}{2\pi}\delta\left((x-x')^2\right)\Theta(x^0 > x'^0)$$

So we obtain:

$$A^\mu(x) = \frac{q}{2\pi}\int d\tau\ v^\mu(\tau)\delta\left((x-r(\tau))^2\right)\Theta\left(x^0 > r^0(\tau)\right)$$

Further:

$$\delta\left((x-r(\tau))^2\right) = \frac{\delta(\tau-\tau_0)}{\left|\frac{\partial}{\partial\tau}_{|\tau=\tau_0}(x-r(\tau))^2\right|}$$

We define $R \equiv x - r(\tau)$. We obtain:

$$\frac{\partial}{\partial t}R^2 = -2(x-r(\tau))\cdot v(\tau)$$

All together:

$$A^\mu(x) = \frac{q}{4\pi}\int d\tau\ \frac{\delta(\tau-\tau_0)v^\mu(\tau)}{(x-r(\tau))\cdot v(\tau)} = \frac{q}{4\pi}\frac{v^\mu}{v\cdot(x-r(\tau))}\Big|_{retarded}$$

The subscript *retarded* denotes that all quantities in this expression must be computed at a retarded proper time $\tau = \tau_0$ and not at the current time of the measurement.
In the restframe where $v^\mu = (1, \vec{0})$ we recover the Coulomb potential.

$$A^\mu(x) = \frac{q}{4\pi}\frac{(1,\vec{0})}{|\vec{x}-\vec{r}(\tau)|}$$

9.2 The electromagnetic field tensor from a moving charge

We define $R^\mu = x^\mu - r^\mu(\tau_0)$. Because the greens function requires $R^2 = 0$ and $R^0 > 0$ it follows that $R^0 = \left|\vec{R}\right|$. Therefore we can write $R^\mu \equiv (R^0, \vec{R}) = \left|\vec{R}\right|(1, \hat{n})$ with $\hat{n} = \frac{\vec{R}}{|\vec{R}|}$. It is also true that $\vec{B} = \vec{n}\times\vec{E}$ and

$$\vec{E} = \frac{q}{4\pi(1-\hat{n}\cdot\vec{v})^3}\cdot\left\{\frac{(1-\vec{v}^2)}{\left|\vec{R}\right|^2}(\hat{n}-\vec{v}) + \frac{1}{\left|\vec{R}\right|}\hat{n}\times\left[(\hat{n}-\vec{v})\times\dot{\vec{v}}\right]\right\}\Bigg|_{retarded}$$

9.3 Radiation from an accelerated charge in its rest frame: Larmor's formula

If $\dot{\vec{v}} \neq 0$ and $\vec{v} = 0$ (or very small) then:

$$\vec{E} = \frac{q}{4\pi}\left(\frac{1}{\left|\vec{R}\right|^2}\hat{n} + \frac{1}{\left|\vec{R}\right|}\hat{n}\times\left[\hat{n}\times\dot{\vec{v}}\right]\right)$$

The pointing vector becomes:

$$\vec{S} = \vec{E}\times\vec{B} = \left|\vec{E}\right|^2\hat{n} - \left(\vec{E}\cdot\hat{n}\right)\cdot\vec{E}$$

Inserting the E-field at the rest frame and expanding in $\frac{1}{|\vec{R}|}$:

$$\begin{aligned}\vec{S} &= \hat{n}\frac{q^2}{16\pi^2\left|\vec{R}\right|^2}\left|\hat{n}\times\left(\hat{n}\times\dot{\vec{v}}\right)\right|^2 + \mathcal{O}\left(\frac{1}{\left|\vec{R}\right|^3}\right)\\ &= \hat{n}\frac{q^2}{16\pi^2\left|\vec{R}\right|^2}\left|\dot{\vec{v}}\right|^2\sin^2(\Theta) + \mathcal{O}\left(\frac{1}{\left|\vec{R}\right|^3}\right)\end{aligned}$$

where Θ is the angle between $\hat{n}$ and $\dot{\vec{v}}$: $\dot{\vec{v}}\cdot\hat{n} = \left|\dot{\vec{v}}\right|\cos(\Theta)$.
The power (energy per unit time) dP emitted through a segment $d\vec{A}$ of a closed surface around the retarded position of the charge q is given by

$$dP \equiv \frac{dW}{dt} = \vec{S}\cdot d\vec{A}$$

For a segment of a sphere with radius $\left|\vec{R}\right|$ centered around the retarded position of the charge, we have:

$$d\vec{A} = \hat{n}\left|\vec{R}\right|^2 d\Omega \quad , \quad d\Omega = \sin(\Theta)d\Theta d\phi$$

and therefore:

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2}\left|\dot{\vec{v}}\right|^2\sin^2(\Theta) + \mathcal{O}\left(\frac{1}{\left|\vec{R}\right|}\right)$$

The radiation power, i.e. the power which is radiated at infinitely large distances, per solid angle $d\Omega$ is:

$$\frac{dP_{rad.}}{d\Omega} = \frac{q^2}{16\pi^2}\left|\dot{\vec{v}}\right|^2\sin^2(\Theta)$$

The total power radiated at all solid angles surrounding the retarded position of the charge is:

$$P_{rad.} = \int d\Omega\ \frac{dP_{rad}}{d\Omega} = \frac{q^2}{4\pi}\frac{2}{3}\left|\dot{\vec{v}}\right|^2 = \frac{q^2}{4\pi\varepsilon_0}\frac{2}{3}\frac{\left|\dot{\vec{v}}\right|^2}{c}$$

In the last equality we took $c \neq 1$ and $\varepsilon_0 \neq 1$. This is known as Larmor's formula.

9.4 Radiation from an accelerated charge with a relativistic velocity

The power of radiation Power = $\frac{\text{Energy}}{\text{Time}}$ is invariant under Lorentz transformation. Larmor's formula reads:

$$P_{rad.} = \frac{q^2}{4\pi}\frac{2}{3}\left|\dot{\vec{v}}\right|^2 = \frac{q^2}{4\pi}\frac{2}{3m^2}\left(\frac{d\vec{p}}{dt}\right)^2$$

Relativistic force in the rest frame: $\frac{dp^\mu}{d\tau} = \left(0, \frac{d\vec{p}}{dt}\right)$. Therefore: $\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau} = -\left(\frac{d\vec{p}}{dt}\right)^2$. Thus we can write:

$$P_{rad.} = -\frac{q^2}{4\pi}\frac{2}{3m^2}\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}$$

In the frame where the particle moves with a velocity $\vec{v}$:

$$P_{rad.} = \frac{q^2}{4\pi}\frac{2}{3}\gamma^6\left[\left|\dot{\vec{v}}\right|^2 - \left|\vec{v}\times\dot{\vec{v}}\right|^2\right]$$

9.4.1 Circular motion

In this case: $\vec{v}\perp\dot{\vec{v}}$. So we obtain:

$$P_{rad.} = \frac{q^2}{4\pi}\frac{2}{3}\gamma^4\left|\dot{\vec{v}}\right|^2$$

So a particle radiates constantly energy with $P_{rad.}\sim\gamma^4$.

9.4.2 Linear accelerators

In this case: $\vec{v}\parallel\dot{\vec{v}}$. So we obtain:

$$P_{rad.} = \frac{q^2}{4\pi}\frac{2}{3}\gamma^6\left|\dot{\vec{v}}\right|^2$$

So $P_{rad.}\sim\gamma^6$.

9.5 Angular distribution of radiation from a linearly accelerated relativistic charge

The electric field of an accelerated charge is:

$$\vec{E} = \frac{q}{4\pi\left(1-\hat{n}\cdot\vec{v}\right)^3}\frac{1}{|\vec{R}|}\hat{n}\times\left[\left(\hat{n}-\vec{v}\right)\times\dot{\vec{v}}\right] + \mathcal{O}\left(\frac{1}{|\vec{R}|^2}\right)\Bigg|_{retarded}$$

For $\vec{v} \parallel \dot{\vec{v}}$:

$$\vec{E} = \frac{q}{4\pi\left(1-\hat{n}\cdot\vec{v}\right)^3}\frac{1}{|\vec{R}|}\hat{n}\times\left[\hat{n}\times\dot{\vec{v}}\right] + \mathcal{O}\left(\frac{1}{|\vec{R}|^2}\right)\Bigg|_{retarded}$$

The Poynting vector is:

$$\vec{S} = \hat{n}\frac{q}{16\pi^2|\vec{R}|^2}\frac{\left|\hat{n}\times\left(\hat{n}\times\dot{\vec{v}}\right)\right|}{\left(1-\hat{n}\cdot\vec{v}\right)^6} + \mathcal{O}\left(\frac{1}{|\vec{R}|^3}\right)$$

The power of radiation through a solid angle $d\Omega$ at a retarded distance $|\vec{R}|$ is:

$$\frac{dP_{rad.}}{d\Omega} = \frac{q^2}{16\pi^2}|\dot{\vec{v}}|^2\frac{\sin^2(\Theta)}{\left(1-v\cos(\Theta)\right)^6}$$

where Θ is the angle between the direction of the emitted radiation and the velocity of the particle. In the case where $v \approx c = 1$ and $\Theta \to 0$ the denominator tends towards zero the thus the whole fraction tends to infinity. Therefore, radiation tends to be collinear.

$$1 - v\cos(\Theta) \approx \frac{1}{2\gamma^2}\left(1+\gamma^2\Theta^2\right)$$
$$\frac{dP_{rad.}}{d\Omega} \approx \frac{q^2}{16\pi^2}|\dot{\vec{v}}|^2\frac{(\gamma\Theta)^2}{\left(1+(\gamma\Theta)^2\right)^6}$$

The distribution vanishes for small and large values of $\gamma\Theta$. It is maximal for $\gamma\Theta = 1/\sqrt{5}$. We conclude that the radiation is emitted within a characteristic angle $\Theta \sim \frac{1}{\gamma}$.

10 Scattering

Consider an electric charge $+q$ which stricken by an incident electromagnetic field with a certain frequency ω. An electromagnetic force is then exerted on the charge which accelerates it. Consequently, the electric charge will emit radiation.

10.1 Thomson scattering

Consider a free charge q in its restframe $\vec{v} = 0$ (or $v \approx 0$) striken by an electromagnetic field $\vec{E} = \hat{e}E\cos\left(\omega t - \vec{k}\cdot\vec{x}\right)$. The force acting on the charge, placed at $\vec{x} = 0$ is:

$$\vec{F} = q\vec{E} = qE\cos(\omega t)\hat{e}$$

So the acceleration of the charge is:

$$\dot{\vec{v}} = \frac{\vec{F}}{m} = \frac{q}{m}E\cos(\omega t)\hat{e}$$

Due to its acceleration, the charge emits radiation:

$$\frac{dP_{rad.}}{d\Omega} = \frac{q^2}{16\pi^2}\dot{v}^2\sin^2(\Theta) = \frac{q^4}{16\pi^2m^2}\sin^2(\Theta)E^2\cos^2(\omega t)$$

Wher Θ is the angle between $\hat{e}$ and $\hat{n}$. Averaging over time:

$$\left\langle\frac{dP_{rad.}}{d\Omega}\right\rangle = \frac{q^4}{32\pi^2m^2}E^2\sin^2(\Theta)$$

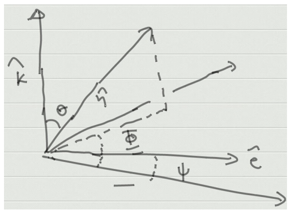
The average incoming flux is the time average of the poynting vector:

$$\langle S\rangle = \langle E^2\cos^2(\omega t)\rangle = \frac{E^2}{2}$$

The differential cross-section is defined as the ratio of the outgoing radiation powers per unit solid angle devided by the incoming flux of energy.

$$\frac{d\sigma}{d\Omega} \equiv \frac{\left\langle\frac{dP_{rad.}}{d\Omega}\right\rangle}{\langle S\rangle} = \frac{q^4}{16\pi^2m^2}\sin^2(\Theta)$$

The differential cross-section for unpolarised incoming photon beam:



$$\frac{d\sigma}{d\Omega} = \frac{q^4}{16\pi^2m^2}\frac{1+\cos^2(\theta)}{2}$$

The total cross-section is:

$$\sigma = \int d\Omega\frac{d\sigma}{d\Omega} = \underbrace{\frac{q^4}{16\pi^2m^2}}_{\equiv r_q^2}\frac{8\pi}{3} = \frac{8\pi}{3}r_q^2$$

We define the Thomson radius as:

$$r_e \equiv \frac{e^2}{4\pi\varepsilon_0 m_e c^2} \approx 2.8\cdot10^{-15}\text{m}$$

It can be understood as the effective radius of an electron if you assign to it a spherical shape.

10.2 Rayleigh scattering

Now we examine the scattering of an electromagnetic wave on an electric charge which is bound in an atom or molecule. The Equations of Motion and its solution are:

$$\frac{qE}{m} = \ddot{x} + \gamma\dot{x} + \omega_0^2 x \quad\Rightarrow\quad \vec{x} = \frac{\frac{q\vec{E}}{m}}{\omega_0^2 - \omega^2 + i\gamma\omega}$$

The acceleration is then given as:

$$\dot{\vec{v}} = \ddot{\vec{x}} = \frac{q\vec{E}}{m}\frac{-\omega^2}{\omega_0^2-\omega^2+i\gamma\omega} = \dot{\vec{v}}_{Thomson}\frac{-\omega^2}{\omega_0^2-\omega^2+i\gamma\omega}$$

where $\dot{\vec{v}}_{Thomson} = \frac{q\vec{E}}{m}$. The crosssection is:

$$\sigma = \sigma_{Thomson}\frac{\omega^4}{(\omega^2-\omega_0^2)^2+\gamma^2\omega^2} \quad\text{with}\quad \sigma_{Thomson} \equiv \frac{8\pi}{3}r_q^2$$

For $\omega \ll \omega_0$ this process is known as Rayleigh scattering and the crosssection becomes:

$$\sigma_{Rayleigh} = \sigma_{Thomson}\frac{\omega^4}{\omega_0^4}$$

11 Electromagnetics in a medium

We identify two types of motion of charged particles: fast currents within small distances, due to the motion of charges in atoms and molecules, and slow currents, which extend at distances much larger than the size of atoms due to free electrons or ions. In macroscopic measurements we are only interested in slow variations at the atomic level. We average over the currents within the atoms. We can write:

$$j^\mu \approx j^\mu_{slow} + \langle j^\mu_{atomic}\rangle$$

We introduce an averaging of a function over some distances via

$$\langle F(\vec{x},t)\rangle = \int d^3\vec{y} f(\vec{y})F(\vec{x}-\vec{y},t)$$

where $f(\vec{y})$ is the weighting factor / probability density function. It is well behaved, smooth, positive, peaks at 0 and has Norm 1.

11.1 Average of the atomic charge density

The charge density $j^0_{atomic} = \rho_{atomic}$ corresponds to the charge density of the charges inside molecules/atoms. We write

$$\rho_{atomic} = \sum_{n\in \text{ atoms}} \rho_{(n)} \quad , \quad \rho_{(n)} = \sum_{j\in(n)} q_j \delta(\vec{x} - \vec{x}_n - \vec{x}_j)$$

where $\rho_{(n)}$ is the charge density of the n-th atom/molecule, $\vec{x}_n$ is the position of the atom/molecule and $\vec{x}_j$ is the position of the charge with respect to the "centre" of the molecule.

$$\begin{aligned}\langle\rho_{(n)}\rangle &= \sum_{j\in(n)} q_j f(\vec{x} - \vec{x}_n - \vec{x}_j) \\ &= \sum_{j\in(n)} \left[q_j f(\vec{x} - \vec{x}_n) - (q_j\vec{x}_j)\cdot\vec{\nabla} f(\vec{x} - \vec{x}_n) + \dots\right] \\ &= q_n f(\vec{x} - \vec{x}_n) - \vec{p}_n \cdot \vec{\nabla} f(\vec{x} - \vec{x}_n) + \dots \\ &= q_n f(\vec{x} - \vec{x}_n) - \vec{\nabla}\left(\vec{p}_n \cdot f(\vec{x} - \vec{x}_n)\right) + \dots\end{aligned}$$

Where we taylor expanded in $\frac{|\vec{x}_j|}{|\vec{x}-\vec{x}_n|}$ and used that $\vec{\nabla}$ only differentiates with respect to $\vec{x}$. Summing over all atoms/molecules we can write:

$$\langle\rho_{atomic}\rangle \equiv \sum_n \langle\rho_{(n)}\rangle = \langle\rho_{eff/atom}\rangle - \vec{\nabla}\cdot\vec{P} + \dots \quad \text{with}$$
$$\langle\rho_{eff/atom}\rangle = \sum_{n\in \text{ atoms}} q_n f(\vec{x} - \vec{x}_n) \quad , \quad \vec{P} \equiv \sum_{n\in \text{ atoms}} \vec{p}_n \cdot f(\vec{x} - \vec{x}_n)$$

11.2 Average of atomic current density

We restrict ourselves to non-relativistic velocities. The current density in an atom/molecule can be written as

$$\vec{j}_{(n)} = \sum_{k\in n} q_k(\vec{v}_n + \vec{v}_k)\delta(\vec{x} - \vec{x}_n - \vec{x}_k)$$

where $\vec{v}_n$ is the velocity of the atom/molecule and $\vec{v}_k$ is the relative velocity of the charge to the center of the atom.

$$\langle\vec{j}_n\rangle = \sum_k q_k\left(\vec{v}_n + \vec{v}_k\right) f(\vec{x} - \vec{x}_n - \vec{x}_k)$$

For $|\vec{x}_k| \ll |\vec{x}_n|$ and $|\vec{v}_n| \ll |\vec{v}_k|$ we can expand:

$$f(\vec{x} - \vec{x}_n - \vec{x}_k) \approx f(\vec{x} - \vec{x}_n) - \vec{x}_k\cdot\vec{\nabla} f(\vec{x} - \vec{x}_n) + \dots$$

This yields:

$$\langle\vec{j}_n\rangle = \sum_k q_k\vec{v}_k f(\vec{x} - \vec{x}_n) + \sum_k q_k\vec{v}_n f(\vec{x} - \vec{x}_n) - \sum_k q_k\vec{v}_k\vec{x}_k\cdot\vec{\nabla} f(\vec{x} - \vec{x}_n) + \mathcal{O}\left(x_k^2, x_k v_n, v_n^2\right)$$

We can rewrite:

$$\sum_k q_k\vec{v}_k f(\vec{x} - \vec{x}_n) \approx \frac{d}{dt}\left(\vec{p}_n f(\vec{x} - \vec{x}_n)\right)$$
$$\sum_k q_k\vec{v}_k\vec{x}_k\cdot\vec{\nabla} f(\vec{x} - \vec{x}_n) \approx -\vec{\nabla}\times\left(\vec{m}_n f(\vec{x} - \vec{x}_n)\right)$$

With $\vec{m}_n$ the magnetic moment of the atom and $\vec{M}$ the magnetization of the material

$$\vec{m}_n = \sum_j \frac{q_j}{2}\left(\vec{x}_j\times\vec{v}_j\right) \quad , \quad \vec{M} = \sum_n\left(\vec{m}_n f(\vec{x} - \vec{x}_n)\right)$$

we can write the sum of the average contribution from all atoms as

$$\langle\vec{j}_{atomic}\rangle = \sum\langle\vec{j}_{(n)}\rangle = \vec{j}_{eff/atomic} + \frac{d\vec{P}}{dt} + \vec{\nabla}\times\vec{M}$$
$$\vec{j}_{eff/aotmic} = \sum_n q_n\vec{v}_n f(\vec{x} - \vec{x}_n)$$

11.3 Maxwell equations in a medium

We approximate the charge and current density as

$$j^\mu = j^\mu_{free} + j^\mu_{atomic} \approx j^\mu_{free} + \langle j^\mu_{atomic}\rangle$$
$$j^0 \approx \rho_{eff} - \vec{\nabla}\cdot\vec{P} + \cdots$$
$$\vec{j} \approx \vec{j}_{eff} + \frac{\partial\vec{P}}{\partial t} + \vec{\nabla}\times\vec{M} + \cdots$$

With $\vec{M}$ the magnetisation vector. If we substitute into the Maxwell equations we obtain

$$\vec{\nabla}\cdot\vec{B} = 0 \quad , \quad \vec{\nabla}\times\left(\vec{B} - \frac{\vec{M}}{c^2\varepsilon_0}\right) = \frac{\vec{j}_{eff}}{c^2\varepsilon_0} + \frac{1}{c^2}\frac{\partial}{\partial t}\left(\vec{E} + \frac{\vec{P}}{\varepsilon_0}\right)$$
$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad , \quad \vec{\nabla}\cdot\left(\vec{E} + \frac{\vec{P}}{\varepsilon_0}\right) = \frac{\rho_{eff}}{\varepsilon_0}$$

with $\rho_{eff} = \rho_{free} + \rho_{eff/atomic}$ and $\vec{j}_{eff} = \vec{j}_{free} + \vec{j}_{eff/atomic}$.

The $\vec{D}$ and $\vec{H}$ field We define:

$$\vec{D} \equiv \varepsilon_0\vec{E} + \vec{P} \quad , \quad \vec{H} \equiv \vec{B} - \frac{\vec{M}}{c^2\varepsilon_0}$$

Now the Maxwell equations reduce to:

$$\vec{\nabla}\cdot\vec{D} = \rho \quad , \quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad , \quad \vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{H} = \frac{\vec{j}_{eff}}{\varepsilon_0 c^2} + \frac{1}{\varepsilon_0 c^2}\frac{\partial\vec{D}}{\partial t}$$

11.4 Maxwell equations inside a dielectric material

Assume $\vec{M} = 0$ and $\vec{P} \neq 0$. We find that $\vec{P}$ and $\vec{E}$ are correlated. If we ignore non-linearities we obtain:

$$\vec{P} = \chi\varepsilon_0\vec{E}$$

where χ is the "electric susceptibility". We define:

$$c_m = \frac{c}{\sqrt{1+\chi}} \quad , \quad \varepsilon = (1+\chi)\varepsilon_0$$

Now we can write the Maxwell equations in a dielectric medium as:

$$\vec{\nabla}\cdot\vec{E} \quad , \quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad , \quad \vec{\nabla}\cdot\vec{B} = 0 \quad , \quad \vec{\nabla}\times\vec{B} = \frac{\vec{j}_{eff}}{\varepsilon c_m^2} + \frac{1}{c_m^2}\frac{\partial\vec{E}}{\partial t}$$

11.5 A model for the dielectric susceptibility χ

For a dipole we have the differential equation

$$q\vec{E} = m\left(\ddot{x} + \gamma\dot{x} + \omega_0^2 x\right)$$

A solution on the microscopic level is

$$\vec{x} = \frac{q\vec{E}/m}{\omega_0^2 - \omega^2 + i\omega\gamma}$$

So for the dipole moment we obtain:

$$\vec{P} = q\vec{x} = \frac{q^2/m}{\omega_0^2 - \omega^2 + i\omega\gamma}\vec{E}$$
$$\langle\vec{P}\rangle = N\vec{x} = \frac{Nq^2/m}{\omega_0^2 - \omega^2 + i\omega\gamma}\vec{E}$$

Where N is the density of charges. When comparing to $\vec{P} = \chi\varepsilon_0\vec{E}$ we obtain:

$$\chi = \chi(\omega) = \frac{\frac{Nq^2}{m\varepsilon_0}}{\omega_0^2 - \omega^2 + i\omega\gamma} = n^2 - 1$$

We see that χ is complex, which has physical consequences on ε and c_m.

11.6 Waves in a dielectric medium

If $\vec{j}_{eff} = \rho_{eff} = 0$ ina medium, then we obtain the wave equation from the Maxwell equations.

$$\left[\frac{1}{c_m^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right]\vec{E} = 0$$

A solution is

$$\vec{E} = \vec{E}_0 e^{i\left(\omega t - \vec{k}\cdot\vec{x}\right)}$$

with $k := \left|\vec{k}\right|$:

$$k^2 = \frac{\omega^2}{c_m^2} = \frac{\omega^2}{c^2}(1+\chi) \quad , \quad v_{phase} = \frac{\omega}{k} = \frac{c}{n} \quad , \quad n = \sqrt{1+\chi}$$

We see that n, the refraction index, is complex.

11.7 The complex index of refraction

We can separate the complex and the real part or n into a real and an imaginary part: $n = n_R - in_I$. We find that the plane wave propagating in the dielectric is:

$$\vec{E} = \vec{E}_0 e^{i\omega\left[t - n\hat{k}\cdot\vec{x}\right]} = \vec{E}_0 e^{i\omega\left[t - \frac{n_R}{c}\hat{k}\cdot\vec{x}\right]} e^{-\omega\frac{n_I}{c}\hat{k}\cdot\vec{x}} \quad \Rightarrow \quad \left|\vec{E}\right| = \left|\vec{E}_0\right| e^{-\omega\frac{n_I}{c}\hat{k}\cdot\vec{x}}$$

11.8 Waves in metals

In metals, electrons move freely at large distances. So we set $\omega_0 = 0$.

$$\chi(\omega) = \frac{Nq^2/\varepsilon_0 m}{-\omega^2 + i\omega\gamma}$$

The density N can be obtained from macroscopic properties of the metal and the constant γ is an intrinsic parameter of our model. It is related to the resistanve or its inverse, conductivity. If $\vec{E}$ is constant, we have a differential equation for $\vec{v}$.

$$q\vec{E} = m(\dot{\vec{v}} + \gamma\vec{v}) \quad \Rightarrow \quad \vec{v} = \vec{v}_{drift} + \vec{v}_0 e^{-\gamma t} \quad , \quad \vec{v}_{drift} = \frac{q\vec{E}}{m\gamma}$$

We further have the relation

$$\vec{J} = \sigma\vec{E} = Nq\vec{v}_{drift} = \frac{Nq^2}{m\gamma}\vec{E} \quad \Rightarrow \quad \gamma = \frac{Nq^2}{m\sigma}$$

11.8.1 Low frequency approximation

$\omega \to 0$, so $\omega\gamma \gg \omega^2$. Thus $n^2 = -i\frac{\sigma}{\varepsilon_0\omega} \quad \Rightarrow \quad \sqrt{\frac{\sigma}{2\varepsilon_0\omega}}(1-i)$ and $\left|\vec{E}\right| = \left|\vec{E}_0\right| e^{-\frac{x}{\delta}}$ where $\delta = \sqrt{\frac{2\varepsilon_0}{\sigma\omega}}c$.

11.8.2 High frequency approximation

$\omega^2 \gg \omega\gamma \Rightarrow n^2 \approx 1 - \frac{\omega_P^2}{\omega^2}$ where $\omega_P^2 = \frac{Nq^2}{m\varepsilon_0}$ is the so called "plasma frequency". If $\omega < \omega_P$, then $n \in i\mathbb{R}$ and therefore the waves die off after some length. If $\omega > \omega_P$, then $n \in \mathbb{R}$ and thus the metal becomes transparent to the electromagnetic wave.

11.9 Reflection and refraction

Consider two materials with refraction indices n_1 and n_2 separated by a boundary surface on the yz plane. For the electromagnetic fields we have:

$$\vec{B}_1 = \vec{B}_2 \quad , \quad \vec{E}_{1,\parallel} = \vec{E}_{2,\parallel} \quad , \quad \left(\varepsilon_0\vec{E}_1 + \vec{P}_1\right)_\perp = \left(\varepsilon_0\vec{E}_2 + \vec{P}_2\right)_\perp$$

11.9.1 Snell's law

Consider an incident electromagnetic plane-wave $(\vec{E}_I, \vec{B}_I)$ from a medium n_1 to a medium n_2, the reflected $(\vec{E}_R, \vec{B}_R)$ and transmitted electromagnetic field $(\vec{E}_T, \vec{B}_T)$. The following are true:

$$\vec{E}_1 = \vec{E}_I + \vec{E}_R \quad , \quad \vec{E}_2 = \vec{E}_T \quad , \quad \vec{B}_1 = \vec{B}_I + \vec{B}_R \quad , \quad \vec{B}_2 = \vec{B}_T$$

$$\vec{E}_I = \hat{e}_I E_I e^{i(\omega_I t - \vec{k}_I\cdot\vec{x})} \quad , \quad \vec{E}_R = \hat{e}_R E_R e^{i(\omega_R t - \vec{k}_R\cdot\vec{x})}$$

$$\vec{E}_T = \hat{e}_T E_T e^{i(\omega_T t - \vec{k}_T\cdot\vec{x})}$$

$$\vec{B}_I = \frac{\vec{k}_I \times \vec{E}_I}{\omega_I} \quad , \quad \vec{B}_R = \frac{\vec{k}_R \times \vec{E}_R}{\omega_R} \quad , \quad \vec{B}_T = \frac{\vec{k}_T \times \vec{E}_T}{\omega_T}$$

$$\vec{k}_I \cdot \hat{e}_I = \vec{k}_R \cdot \hat{e}_R = \vec{k}_T \cdot \hat{e}_T = 0$$

$$\frac{k_I}{\omega_I} = \frac{k_R}{\omega_R} = \frac{n_1}{c} \quad , \quad \frac{k_T}{\omega_T} = \frac{n_2}{c}$$

With some boundary conditions we obtain:

$$\omega_I = \omega_R = \omega_T = \omega \quad \text{and} \quad \vec{k}_{I,\parallel} = \vec{k}_{R,\parallel} = \vec{k}_{T,\parallel} = \vec{k}_\parallel$$

We also obtain Snell's law

$$\theta_I = \theta_R \quad , \quad \sin(\theta_T) = \frac{n_1}{n_2}\sin(\theta_I)$$