Written Assignment 1 — Solutions

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Question 1: Shortest Path Composition

Proof. When considering the path from a to c, suppose p_1 is not shortest from a to c. Then there exists an a to c path $\tilde{p}_1$ with $cost(\tilde{p}_1) < cost(p_1)$. Concatenate to get an a to b walk $\tilde{p}_1 \cdot p_2$ of cost

$$cost(\tilde{p}_1) + cost(p_2) < cost(p_1) + cost(p_2) = cost(p^*),$$

contradicting optimality of p^* as the cost of $\tilde{p}_1 \cdot p_2$ is less than p^*. Hence p_1 is shortest $a \to c$ and there is no such $\tilde{p}_1$ exists. The same argument with (c, b) shows p_2 is shortest $c \to b$. □

Question 2: Iterative Deepening returns shortest paths

Proof. Let $\ell(v)$ be the shortest-path distance from s to v, for some source node s and some vertex v. For some depth limit d such $d < \ell(v)$, depth-limited DFS cannot expose any s–v path. When $d = \ell(v)$, a shortest s–v path of length $\ell(v)$ is eligible and will be found during that pass $(d = \ell(v))$. Therefore the first time v is returned is at depth limit $\ell(v)$, with a shortest path.

□

Question 3: Diameter bound

Proof. For any $u \neq v$, any simple u–v path visits vertices at most once, hence uses at most $|V| - 1$ edges. A shortest u–v path is simple (otherwise remove a cycle to shorten it). So $\text{dist}(u, v) \leq |V| - 1$ for all pairs, and the maximum over pairs is at most $|V| - 1$. The diameter is at most $|V| - 1$. □

Question 4 — Dijkstra's Algorithm and Shortest Paths

Theorem 1. *Let $G = (V, E, w)$ be a directed graph with strictly positive edge weights $w(e) > 0$. When Dijkstra's algorithm is run from a source s and it returns a path to any vertex v, the returned path has length $\delta(s, v)$, the true shortest-path distance from s to v.*

Proof. Assume for contradiction that some vertex is *settled* with an incorrect label. Let v be the first such vertex extracted from the priority queue with $d[v] > \delta(s, v)$; thus, every previously settled u satisfies $d[u] = \delta(s, u)$.

Consider a shortest $s \to v$ path P, and let y be the first vertex on P that is not yet settled just before v is extracted; let x be the predecessor of y on P. By choice of y, x is settled. By the induction hypothesis for earlier settled vertices, $d[x] = \delta(s, x)$. When x was settled, the relaxation of edge (x, y) gave

$$d[y] \;\leq\; d[x] + w(x, y) \;=\; \delta(s, x) + w(x, y) \;=\; \delta(s, y),$$

and since y lies on a shortest $s \to v$ path, we have $\delta(s, y) \leq \delta(s, v)$. Hence

$$d[y] \;\leq\; \delta(s, v).$$

Because Dijkstra extracts the unsettled vertex with minimum key, it holds that

$$d[v] \;\leq\; d[y] \;\leq\; \delta(s, v),$$

contradicting the assumption $d[v] > \delta(s, v)$. Therefore no such v exists, and every vertex is settled with its true distance; in particular, the path returned to any v is a shortest path. $\square$