Module 6

Homework 3

due Friday, 10 December, 2021 at 11:59 pm

Problem 1 [7 points] A d-dimensional hypersphere is defined by the condition:

$$x_0^2 + x_1^2 + \cdots + x_{d-1}^2 \leq R^2$$

Its volume can be calculated analytically using the formula:

$$V_d^{ana}(R) = \frac{\pi^{d/2} R^d}{\Gamma(\frac{d}{2} + 1)}$$

Write a program to determine the volume of unit hypershperes (R = 1) using the Sample Mean Monte Carlo integration method for $d = 2, 3, 4, 5, 6$. We will call this $V_d^{MC}(R)$. Use a hypercube with size $2R = 2$ as the bounding volume for your Monte Carlo.

Produce the following using a "good" PRNG, like D1(A1$_r$) or the Mersenne Twister (which one did you use?):

a) [4.0 points] Plots of the accuracy of your numerical results ($V_d^{MC} - V_d^{ana}$) with error bars corresponding to the statistical uncertainty on V_d^{MC} vs. the number of points (N) generated.

b) [1.0 points] Does the accuracy vs N plot follow the expected $N^{-1/2}$ behavior? If not, what might be the cause of the discrepancy?

c) [2.0 points] Compare the results you obtain with your good PRNG with a PRNG that has obvious problems, like LCG(5,3,32). Describe how the "bad" PRNG's results differ from your "good" results.

Problem 2 [8 points] Evaluate the integral

$$\int_0^1 f(x)\,dx = \int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$$

using importance sampling with

$$p(x) = A(1-x) \qquad\qquad [x \geq 0]$$

and a "good" PRNG, like D1(A1$_r$) or the Mersenne Twister.

- **a)** [0.5 points] What is the appropriate value of A?
- **b)** [1.0 points] What transformation can be used to allow us to generate random numbers x according to the PDF, $p(x)$, in $x \in [0,1]$ from uniformly distributed random numbers, r in $[0,1]$?
- **c)** [2.5 points] Make a plot of the deviation of your numerical result (including statistical uncertainties) from the analytic result vs. the number of points you generate, N, for N in the range $10^2 - 10^9$.
- **d)** [1.0 points] What is the variance of $f(x)/p(x)$ in the interval $[0,1]$? Compare this with the variance of $f(x)$ over the same interval.
- **e)** [3.0 points] Compare the time it takes to compute this integral to an error of 10^{-4} using the importance sampling method you developed with the time it takes to compute the integral using the Sample Mean method. Do your results agree with the general expectations based on function variances? If not, discuss possible causes of the difference.