Problem 1

(a)

$$\begin{cases} -2x_1 + x_2 + x_3^3 = 0 \\ x_1 - x_2 - x_3^3 = 0 \\ -x_3 - x_3^3 = 0 \end{cases}$$

$$x_3^3 = 2x_1 - x_2 = x_1 - x_2 = -x_3$$

$$x_1 = x_2 = x_3 = 0$$

(b)

$$\begin{aligned} V(\mathbf{x}) &= x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 - 2x_2x_3 \\ &= (x_1^2 + 2x_1x_2 + x_2^2) + (x_2^2 - 2x_2x_3 + x_3^2) + x_2^2 + x_3^2 \\ &= (x_1 + x_2)^2 + (x_2 - x_3)^2 + x_2^2 + x_3^2 \end{aligned}$$

Thus we know $V(\mathbf{x}) = 0$ when $x_1 = x_2 = x_3 = 0$, and $V(\mathbf{x}) \geq 0$. To prove that $V(\mathbf{x}) > 0$ when $\mathbf{x} \neq \mathbf{0}$, consider a hypothetical contradicting case where $\mathbf{x} \neq \mathbf{0}$ and $V(\mathbf{x}) = 0$; in this case since all terms in the sum are non-negative, all terms must be exactly 0; therefore we know immediately that $x_2 = x_3 = 0$. Since $x_1 + x_2 = 0$ and $x_2 = 0$, $x_1 = 0$, so we have $x_1 = x_2 = x_3 = 0$. So it is not possible to construct the contradicting case, and thus $V(\mathbf{x})$ must be strictly greater than 0 when $\mathbf{x} \neq \mathbf{0}$.

(c)

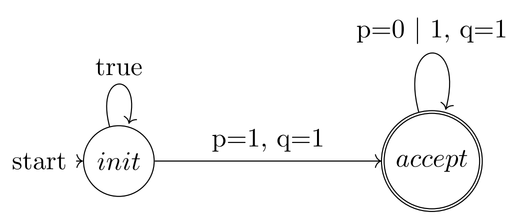
$$\frac{\partial V}{\partial \mathbf{x}} = \begin{bmatrix} 2x_1 + 2x_2 \\ 4x_2 + 2x_1 - 2x_3 \\ 4x_3 - 2x_2 \end{bmatrix}$$

$$\begin{aligned} \frac{\partial V}{\partial \mathbf{x}} \cdot \begin{bmatrix} -2x_1 + x_2 + x_3^3 \\ x_1 - x_2 - x_3^3 \\ -x_3 - x_3^3 \end{bmatrix} &= (2x_1 + 2x_2) * (-2x_1 + x_2 + x_3^3) + (4x_2 + 2x_1 - 2x_3) * (x_1 - x_2 - x_3^3) \\ &\quad + (4x_3 - 2x_2) * (-x_3 - x_3^3) \\ &= -2x_3^4 - 4x_3^2 - 2x_1x_3 + 4x_2x_3 - 2x_1^2 - 2x_2^2 \\ &= -(x_1^2 + 2x_1x_3 + x_3^2) - 2(x_2^2 - 2x_2x_3 + x_3^2) - x_1^2 - x_3^2 - 2x_3^4 \\ &= -(x_1 + x_3)^2 - 2(x_2 - x_3)^2 - x_1^2 - x_3^2 - 2x_3^4 \end{aligned}$$

All square and 4th-order terms must be non-negative, so the sum of their negations must be non-positive. And it's obvious that the equation equals 0 when $x_1 = x_2 = x_3 = 0$.

Problem 2

(a)



(If $q = 0$ while at the accepting state, there will be no corresponding enabled transition so the automaton will no longer accept.)

(b) 1. False; $(pq)^+ \models \mathbf{G}(p \vee q)$ but $(pq)^+ \not\models \mathbf{G}(p) \vee \mathbf{G}(q)$.[1]

2. True; $\mathbf{G}(p) \vee \mathbf{G}(p) \equiv \forall m \geq 0, (\exists a \geq m : \pi, a \models p) \vee (\exists b \geq m : \pi, b \models q) \implies \forall m \geq 0, \exists n \geq m : \pi, n \models$ (set n to either a or b) $(p \vee q) \equiv \mathbf{GF}(p \vee q)$; the reverse also holds since $\mathbf{GF}(p \vee q) \equiv \forall m \geq 0, \exists n \geq m : \pi, n \models (p \vee q) \implies \forall m \geq 0, (\exists a \geq m : \pi, a \models p) \vee (\exists b \geq m : \pi, b \models q)$ (set both a and b to n) $\equiv \mathbf{G}(p) \vee \mathbf{G}(p)$.

3. True; $(p \wedge q)\mathbf{U}r \equiv (\exists m \geq 0 : \pi, m \models r) \wedge (\forall n \text{ s.t. } 0 \leq n < m, \pi, n \models (p \wedge q)) \iff (\exists m \geq 0 : \pi, m \models r) \wedge (\forall a \text{ s.t. } 0 \leq a < m, \pi, a \models p) \wedge (\forall b \text{ s.t. } 0 \leq b < m, \pi, b \models q)$ (a and b span the same $\forall$ range as n, so the two can be combined to give $p \wedge q$ and vice versa) $\equiv (p\mathbf{U}r) \wedge (q\mathbf{U}r)$.

(c) Xiao's right, i.e. the automaton accepts traces that the formula cannot accept. $\mathbf{F}(p \wedge q)$ enforces that $\exists m \geq 0 : (p = 1 \wedge q = 1)$, but this is not enforced by the automaton; for example $qqqp^+$ can be accepted by the automaton, but cannot be accepted by the formula.[2]

Problem 3

(a)

	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5
$\mu = x - 0$	1	0	1	1	1	0	0	0
$\mathbf{F}_{[0,0.5]}\mu$	1	1	1	1	1	0	0	
$\mathbf{G}_{[[0,0.5]]}\mathbf{F}_{[0,0.5]}\mu$	1	1	1	1	0	0		

$\rho(\varphi, \mathbf{x}, 0) = 1 > 0$, and moreover, $\rho(\varphi, \mathbf{x}, [0 : 2.5]) \geq 0$[3], so $x \models \varphi$ for the interval $[0 : 2.5]$.

(b)

	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5
$\mu_1 = 0 - y$	-0.1	-0.3	0.1	3	0.1	-1	-2	0.1
$\mathbf{F}_{[1,1.5]}\mu_1$	3	3	0.1	-1	0.1			
$\mu_2 = x - 0.5$	0.5	-0.5	0.5	0.5	0.5	-0.5	-0.5	-0.5
$\neg\mu_2$	-0.5	0.5	-0.5	-0.5	-0.5	0.5	0.5	0.5
$\neg\mu_2 \vee \mathbf{F}_{[1,1.5]}\mu_1$	3	3	0.1	-0.5	0.1			
$\mathbf{G}_{[0,2]}(\neg\mu_2 \vee \mathbf{F}_{[1,1.5]}\mu_1)$	-0.5							

$\rho(\psi, \mathbf{x}, 0) = -0.5 < 0$, so $\mathbf{x} \not\models \psi$.

(c) Make $x = 0.5$ at $t = 1.5$.

(d) Yes; the violations are caused by extreme values that are on the two ends of the interpolation; values in the linear interpolation cannot change the extreme values, so violations cannot be undone.

[1] $(pq)^+$ means $p = 1, q = 0$, then $p = 0, q = 1$, and repeat this pattern infinitely.
[2] $qqqp^+$ means q=1 and p=0 for the first 3 steps, then q=0 and p=1 for all following steps.
[3] $\rho = 0$ is fine here since the requirement is $x \geq 0$.

Problem 4

(a) We are looking for probabilities of being in either q_0 or q_3 at time 0, 1 and 2.
At time 0, since initial probability distribution is assumed to be uniform, there's a 0.25 probability for being in each of q_0, q_1, q_2, q_3. $0.25 + 0.25 = 0.5$.
At time 1, $P(q_0) = \Sigma_q P(s_1 = q_0, s_0 = q) = \Sigma_q P(s_0 = q) * p(q, q_0) = \Sigma_q 0.25 * p(q, q_0)$, and same for q_1, q_2, q_3, thus:

$$P(q_0) = 0.25 * 0.1 + 0.25 * 0.5 + 0.25 * 0 + 0.25 * 0.3 = 0.225$$
$$P(q_1) = 0.25 * 0.2 + 0.25 * 0.5 + 0.25 * 0.9 + 0.25 * 0.2 = 0.45$$
$$P(q_2) = 0.25 * 0.3 + 0.25 * 0 + 0.25 * 0.1 + 0.25 * 0.5 = 0.225$$
$$P(q_3) = 0.25 * 0.4 + 0.25 * 0 + 0.25 * 0 + 0.25 * 0 = 0.1$$

$0.225 + 0.1 = 0.325$.
At time 2, we do the same calculations and get:

$$P(q_0) = 0.225 * 0.1 + 0.45 * 0.5 + 0.225 * 0 + 0.1 * 0.3 = 0.2775$$
$$P(q_1) = 0.225 * 0.2 + 0.45 * 0.5 + 0.225 * 0.9 + 0.1 * 0.2 = 0.4925$$
$$P(q_2) = 0.225 * 0.3 + 0.45 * 0 + 0.225 * 0.1 + 0.1 * 0.5 = 0.14$$
$$P(q_3) = 0.225 * 0.4 + 0.45 * 0 + 0.225 * 0 + 0.1 * 0 = 0.09$$

$0.2775 + 0.09 = 0.3675$.

(b) We are looking for the probability of being in either q_1 or q_2 at time 2 and not in state q_1 or q_3 at time 0 and 1. Later time steps are dependent on previous time steps, so we have to use conditional probabilities. $P(s_2 = q_1 \text{ or } q_2, s_1 = q_0 \text{ or } q_2, s_0 = q_0 \text{ or } q_2) = P(s_2 = q_1 \text{ or } q_2 \mid s_1 = q_0 \text{ or } q_2, s_0 = q_0 \text{ or } q_2) P(s_1 = q_0 \text{ or } q_2 \mid s_0 = q_0 \text{ or } q_2) P(s_0 = q_0 \text{ or } q_2)$.
$P(s_0 = q_0 \text{ or } q_2) = 0.5$.
$P(s_1 = q_0 \text{ or } q_2 \mid s_0 = q_0 \text{ or } q_2) = \frac{P(s_1 = q_0 \text{ or } q_2, s_0 = q_0 \text{ or } q_2)}{P(s_0 = q_0 \text{ or } q_2)} = \frac{0.25*0.1+0.25*0.3+0.25*0.0+0.25*0.1}{0.5} = 0.25$.
$P(s_2 = q_1 \text{ or } q_2 \mid s_1 = q_0 \text{ or } q_2, s_0 = q_0 \text{ or } q_2) = P(s_2 = q_1 \text{ or } q_2 \mid s_1 = q_0 \text{ or } q_2) = \frac{P(s_2 = q_1 \text{ or } q_2, s_1 = q_0 \text{ or } q_2)}{P(s_1 = q_0 \text{ or } q_2)}$
$= \frac{0.225*0.2+0.225*0.3+0.225*0.9+0.225*0.1}{0.225+0.225} = 0.75$.
$0.5 * 0.25 * 0.75 = 0.09375$.

Problem 5

(a) $(left, x = -1) \xrightarrow{\delta t = 7} (left, x = 2.5)$
$(left, x = 2.5) \xrightarrow{2.5 > 2} (right, x = 2.5)$
$(right, x = 2.5) \xrightarrow{\delta t = 5} (right, x = -2.5)$
$(right, x = 2.5) \xrightarrow{-2.5 < -2} (right, x = -2.5)$

(b) The largest possible value of x when entering $left$ is -2; the smallest is -4. The largest possible value of x when leaving $left$ is 4; the smallest is 2.

Maximum dwell time: $\frac{(4-(-4))}{0.5} = 16$.
Minimum dwell time: $\frac{(2-(-2))}{0.5} = 8$.

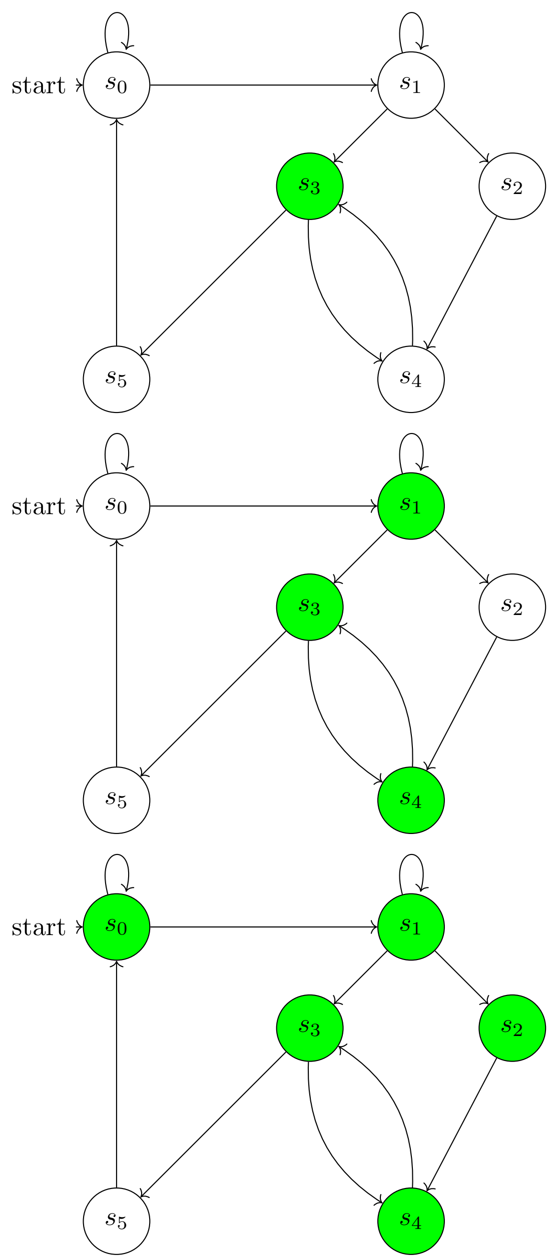
Problem 6

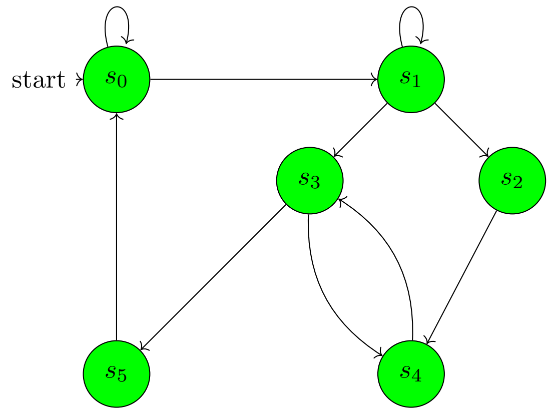
(a) No. Although the base case is trivially true because of initialization, we can find the counter-example $i' = 2, j' = 3, x' = true$ for the inductive case, which satisfies the expression since the precondition is false; in this case only T2 is enabled, so $(j' - 1) = 2$ while i', x' don't change and the precondition becomes true; however the postcondition $x = true \neq false$ is false, so the entire expression becomes false. Since it's not an inductive invariant, it cannot be used to prove safety invariant.

(b) No. Since $(j = 0)$ has to be true whenever $(i = 0) \wedge (j = 0)$ is true, T2 has to be enabled whenever T3 is enabled; if we always choose to execute T2 whenever T3 is enabled, we'll never reach a state $x = true$.

Problem 7

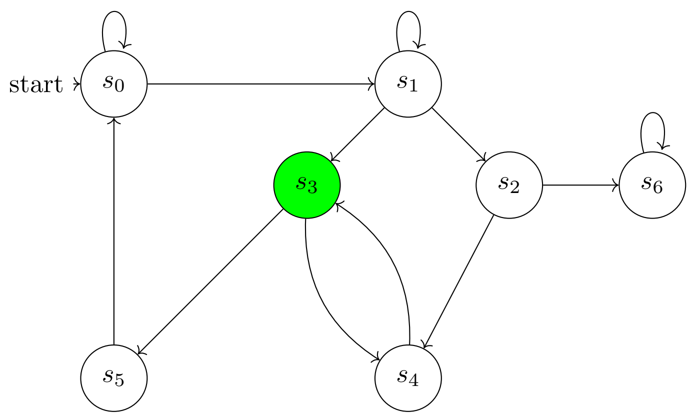
(a)

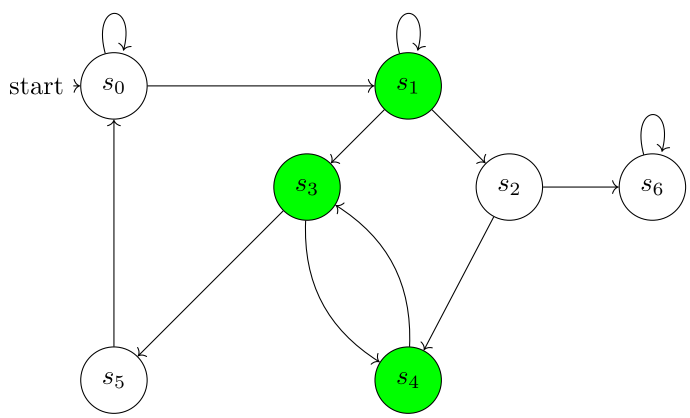


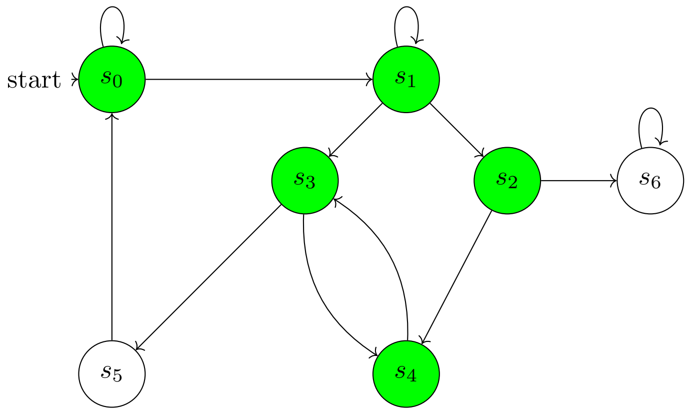


No state with unlabeled successor, so cannot remove any state. Satisfies **AGEF**$green$.

(b)



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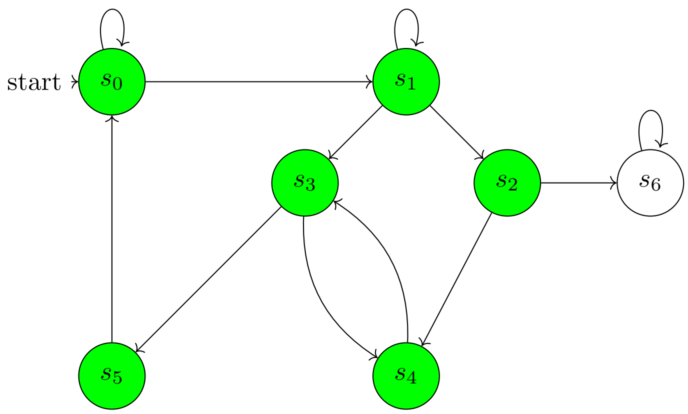
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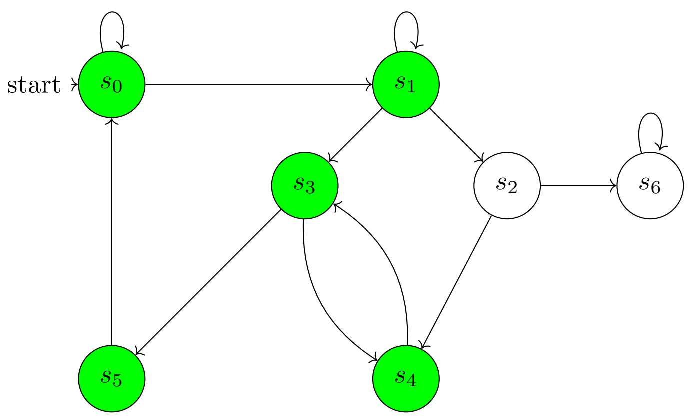
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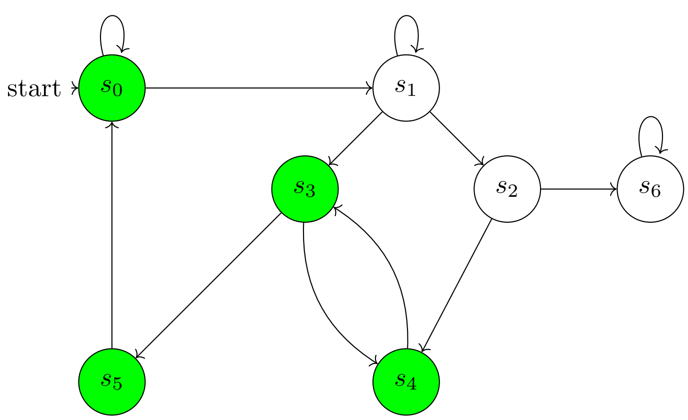
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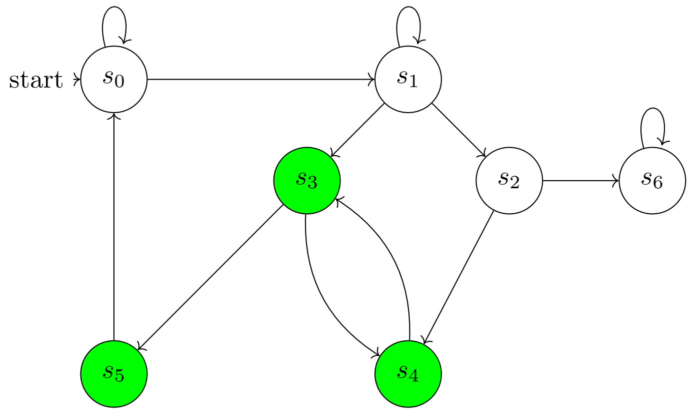
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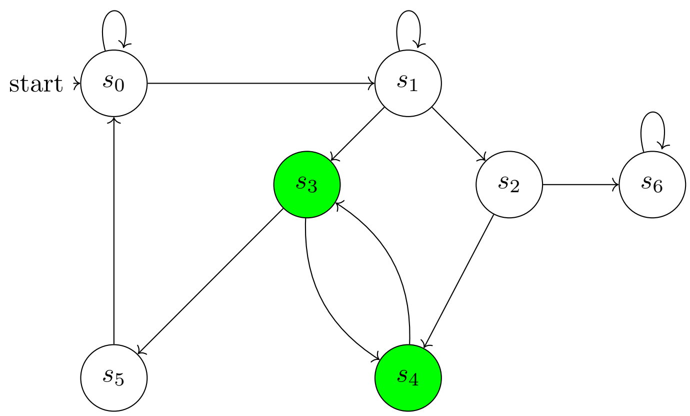
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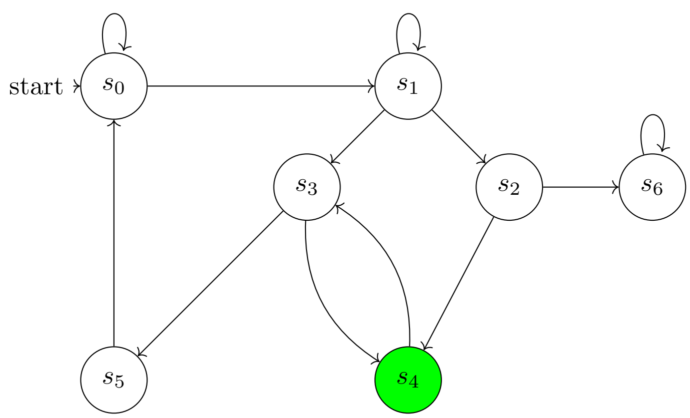
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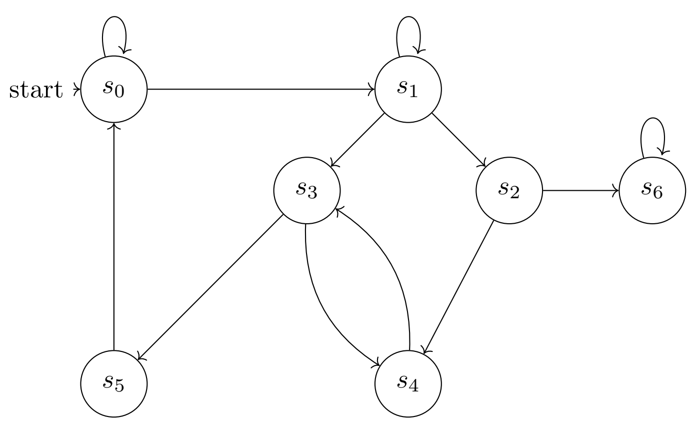
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No labeled state after running fixpoint; doesn't satisfy $\mathbf{AGEF}\mathit{green}$.