Mathematics problems

1.1.
$$\frac{z^{17}}{z^3 \cdot z^5} = \frac{z^{17}}{z^{3+5}} = z^{17-8} = \underline{\underline{z^9}}$$

1.2.
$$6^2 \cdot 6^x = 6^6$$
$$6^{2+x} = 6^6$$
$$2+x = 6$$
$$\underline{\underline{x=4}}$$

1.3. $xy = 5 \qquad x^3y^3 = ?$
$$x^3y^3 = (xy)^3 = 5^3 = \underline{\underline{125}}$$

1.4.
$$\frac{\sqrt{2^{10}}}{\sqrt{4^3}} = \frac{2^5}{\sqrt{64}} = \frac{32}{8} = \underline{\underline{4}}$$

1.5. (a) $x+y = y+x \rightarrow$ True

(b) $x(y+z) = xy + xz \rightarrow$ True

(c) $x^{y+z} = x^y + x^z \rightarrow$ False

It would be true if $x^{y+z} = x^y \cdot x^z$.

(d) $\frac{x^y}{x^z} = x^{y-z} \rightarrow$ True

1.6.
$$\frac{2x-5}{2} \geq 4$$
$$2x-5 \geq 8$$
$$2x \geq 13$$
$$\underline{\underline{x \geq \frac{13}{2}}}$$

2.1. $0°C = 32°F$

$100°C = 212°F$ $\rightarrow$ $1°C = 1,8°F$

$$°C = \frac{°F - 32}{1,8} \qquad °F = 1,8°C + 32$$

For these to be equal:

$$°C = 1,8°C + 32$$
$$-0,8°C = 32$$
$$°C = -40$$

-40 is measured by the same number on both scales.

2.2. $f(x) = 5x + 4$

$f(y) = 24$

$y = ?$

$$5y + 4 = 24$$
$$5y = 20$$
$$\underline{\underline{y = 4}}$$

2.3. $10^{x^2 - 2x + 2} = 100$

$10^2 = 100$, so $x^2 - 2x + 2 = 2$.

$$x^2 - 2x + 2 = 2$$
$$x^2 - 2x = 0$$
$$x(x-2) = 0$$

$\underline{\underline{x_1 = 0}}$ $\qquad$ $\underline{\underline{x_2 = 2}}$

2.4. annual GDP growth: 3%

doubling time?

$$1{,}03^{t^*} = 2$$

$$\downarrow$$

$$t^* \approx 23{,}45$$

It takes approximately 23 years for the economy to double its GDP.

2.5. $\ln\left(\frac{1}{e}\right) = -1$

3.1. $$\sum_{i=0}^{\infty}\left(\frac{1}{8^i} + 0{,}5^i\right) = \sum_{i=0}^{\infty}\left(\frac{1}{8}\right)^i + \sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i$$

$$\sum_{i=0}^{\infty}\left(\frac{1}{8}\right)^i \to |r| = \frac{1}{8} < 1 \to \frac{1}{1-r} = \frac{1}{\frac{7}{8}} = \frac{8}{7}$$

$$\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i \to |r| = \frac{1}{2} < 1 \to \frac{1}{1-r} = \frac{1}{\frac{1}{2}} = 2$$

$$\sum_{i=0}^{\infty}\left(\frac{1}{8^i} + 0{,}5^i\right) = \frac{8}{7} + 2 = \underline{\underline{\frac{22}{7}}}$$

3.2. $$\lim_{x \to 3} \frac{x-3}{2} = \frac{0}{2} = 0$$

3.3. $f(x) = x^2 - 4$

$(x_0, y_0) = (-1, -3)$

Calculating another point of the function:

$$x^2 - 4 = 5$$

$$x^2 = 9$$

$$x_1 = 3 \qquad x_2 = -3$$

$$\downarrow$$

$$(x_1, y_1) = (3, 5)$$

Calculating the slope:

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

$$m = \frac{5-(-3)}{3-(-1)} = \frac{8}{4} = \underline{\underline{2}}$$

3.4. $\left(\dfrac{x^2+3}{x+2}\right)'$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$f = x^2+3 \qquad g = x+2$$
$$f' = 2x \qquad g' = 1$$

$$\left(\frac{x^2+3}{x+2}\right)' = \frac{2x(x+2)-(x^2+3)\cdot 1}{(x+2)^2} = \frac{2x^2+4x-x^2-3}{(x+2)^2} =$$

$$= \frac{x^2+4x-3}{(x+2)^2}$$

3.5. $f = 4x^3+4$

$f' = 12x^2$

$\underline{\underline{f'' = 24x}}$

3.6. No, because $x=0$ is not part of the function's domain.

3.7. $f(x) = 3x^3 - 9x$

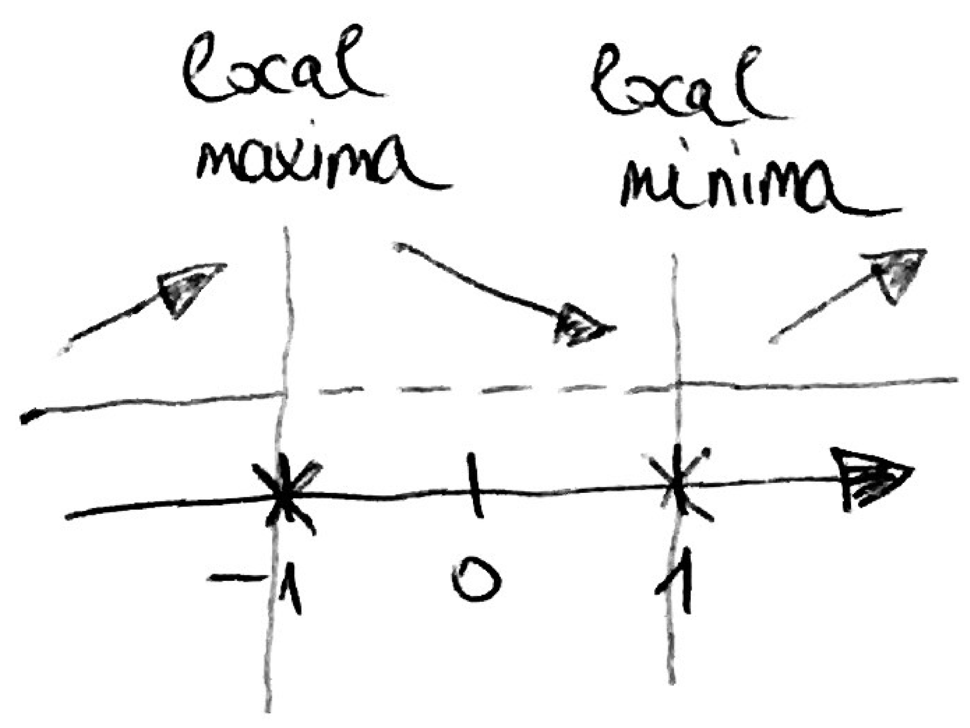
① $f'(x) = 9x^2 - 9$

$9x^2 - 9 = 0$

$9x^2 = 9$

$x^2 = 1$

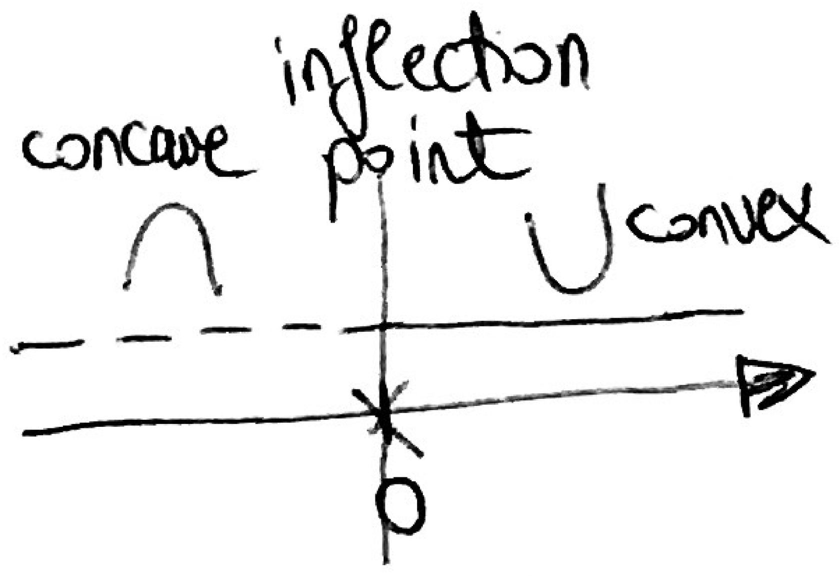
$x_1 = -1 \qquad x_2 = 1$



② $f''(x) = 18x$

$18x = 0$

$x = 0$



Local maxima: $x=-1, y=6$; local minima: $x=1, y=-6$; inflection point: $x=0, y=0$; the function is concave if $x<0$ and convex if $x>0$.

3.8.

$$f(x,y) = x^2y^3$$

$$f(2,3) = 2^2 \cdot 3^3 = \underline{108}$$

3.9.

$$f(x,y) = \ln(x-y)$$

constraint: $x-y > 0$

$$\underline{x > y}$$

3.10.

$$f = x^5 + xy^3$$

$$f'_x = 5x^4 + y^3$$

$$\underline{f''_{xx} = 20x^3}$$

3.11.

$$f(x,y) = \sqrt{xy} - 0{,}5x - 0{,}5y = x^{\frac{1}{2}} \cdot y^{\frac{1}{2}} - 0{,}5x - 0{,}5y$$

$$f'_x = \frac{1}{2}x^{-\frac{1}{2}} \cdot y^{\frac{1}{2}} - 0{,}5 = \frac{1}{2\sqrt{x}} \cdot \sqrt{y} - 0{,}5 = \frac{\sqrt{y}}{2\sqrt{x}} - 0{,}5$$

$$f'_y = x^{\frac{1}{2}} \cdot \frac{1}{2}y^{-\frac{1}{2}} - 0{,}5 = \frac{\sqrt{x}}{2\sqrt{y}} - 0{,}5$$

$$\frac{\sqrt{y}}{2\sqrt{x}} - 0{,}5 = 0$$

$$\sqrt{y} = 0{,}5(2\sqrt{x})$$

$$\sqrt{y} = \sqrt{x}$$

$$y = x$$

$$\frac{\sqrt{x}}{2\sqrt{y}} - 0{,}5 = 0$$

$$\sqrt{x} = 0{,}5(2\sqrt{y})$$

$$\sqrt{x} = \sqrt{y}$$

$$x = y$$

Since $y=x$, there's no local minima/maxima.

3.12.

$$\max\ x^2y^2 \qquad \text{s.t. } x+y=5$$

$$\mathcal{L}(x,y) = x^2y^2 - \lambda(x+y-5)$$

$$\mathcal{L}'_x = 2xy^2 - \lambda = 0$$

$$\mathcal{L}'_y = x^2 2y - \lambda = 0$$

$$\cancel{2}xy^2 = \cancel{2}yx^2$$

$$xy^2 = yx^2$$

$$x = y \longrightarrow x+y=5$$

$$2x = 5$$

$$\underline{\underline{x = \frac{5}{2}}} \longrightarrow \underline{\underline{y = \frac{5}{2}}}$$

4.1.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \\ 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 2\cdot1+3\cdot2 & 2\cdot4+3\cdot1 & 2\cdot1+3\cdot2 \\ 4\cdot1+1\cdot2 & 4\cdot4+1\cdot1 & 4\cdot1+1\cdot2 \\ 1\cdot1+2\cdot2 & 1\cdot4+2\cdot1 & 1\cdot1+2\cdot2 \end{bmatrix} = \begin{bmatrix} 8 & 11 & 8 \\ 6 & 17 & 6 \\ 5 & 6 & 5 \end{bmatrix}$$

4.2.

$$B \cdot A = \begin{bmatrix} 1\cdot2+4\cdot4+1\cdot1 & 1\cdot3+4\cdot1+1\cdot2 \\ 2\cdot2+1\cdot4+2\cdot1 & 2\cdot3+1\cdot1+2\cdot2 \end{bmatrix} = \begin{bmatrix} 19 & 9 \\ 10 & 11 \end{bmatrix}$$

4.3.

$$A^T = \begin{bmatrix} 3{,}3 & 2 & 4 \\ 5{,}1 & 6{,}1 & 5{,}76 \\ 4{,}7 & 1{,}23 & 0 \end{bmatrix}$$

4.4.

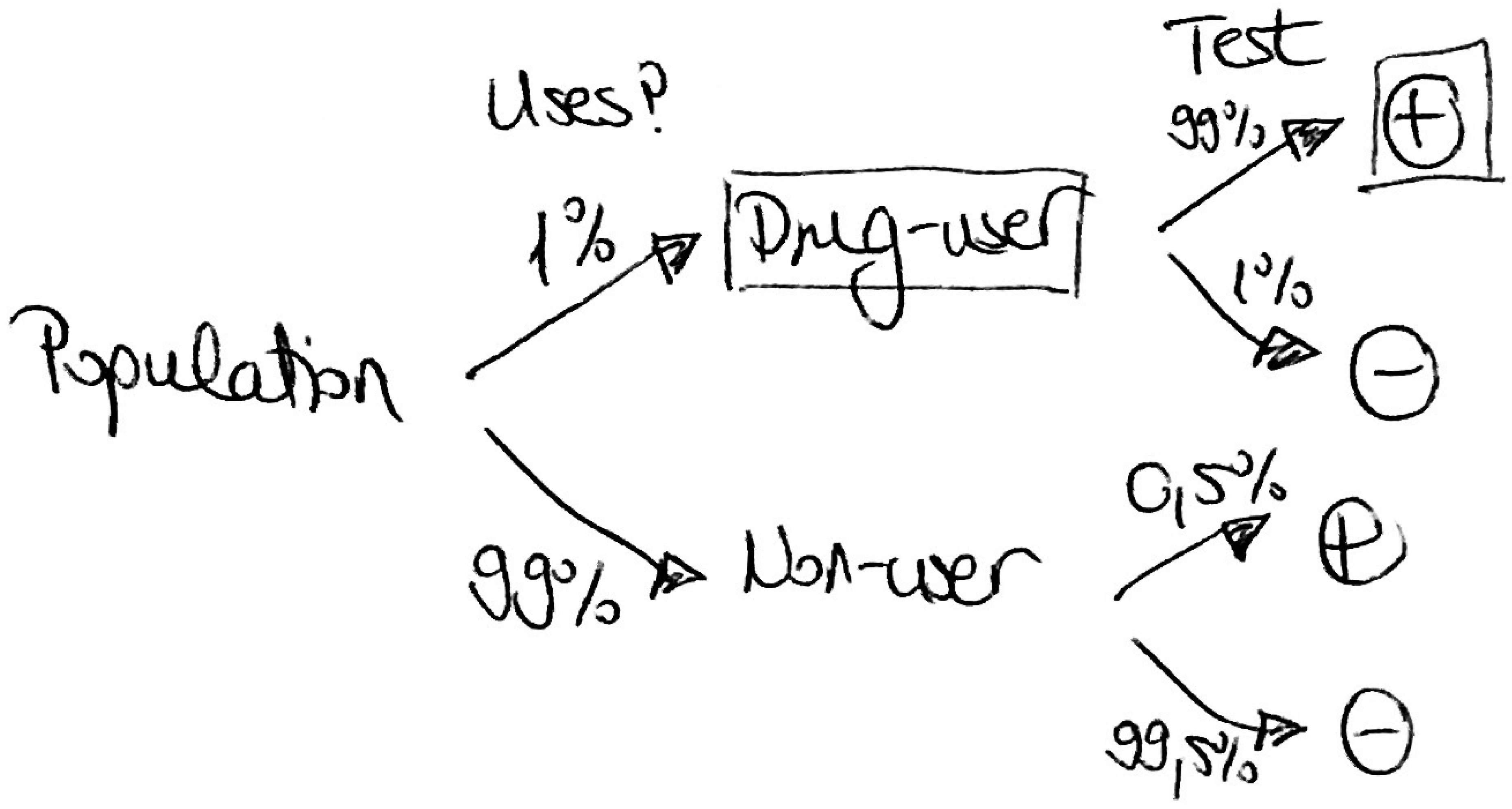
$$\det\left(\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\right) =$$

$$= 2\cdot5 - 3\cdot4 = 10-12 = \underline{\underline{-2}}$$

5.1.

The sample space for the experiment is $2^4 = 16$.

5.2.



Out of the population drug-user with a positive test: $1\% \cdot 99\% = \underline{\underline{0{,}99\%}}$.

5.3. X: sum of the two throws P(X): probability of the sum

X (sum)	2	3	4	5	6	7	8	9	10	11	12
P(X)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$E(X) = \sum x \cdot P(X) = 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + \ldots + 12\left(\frac{1}{36}\right) = \underline{\underline{7}}$$

The expected value of the sum is 7.