Matematika G3 - GY9

Szukcesszív approximáció

$y' = f(x;y)$ DE $\rightarrow$ iteratív megoldás
$y(x_0) = y_0$

$$y_{n+1} = y_0 + \int_{x_0}^{x} f(t; y_n(t))\, dt$$

4. feladat

$y' = y \qquad y(0) = 1$

$x_0 \quad y_0$

$$y_1 = y_0 + \int_{x_0}^{x} f(t; y_0(t))\, dt = 1 + \int_0^x 1\, dt = 1 + x$$

$$y_2 = y_0 + \int_{x_0}^{x} f(t; y_1(t))\, dt = 1 + \int_0^x 1 + t\, dt = 1 + x + \frac{x^2}{2}$$

$$y_3 = y_0 + \int_{x_0}^{x} f(t; y_2(t))\, dt = 1 + \int_0^x 1 + t + \frac{t^2}{2}\, dt = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$y_4 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

$$y_5 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$$

$\vdots$

$$y_k = y_{k-1} + \frac{x^k}{k!}$$

$\vdots$

$$y_\infty = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

5. feladat

$y' = xy \qquad y(0) = 1$

$$y_1 = 1 + \int_0^x t \cdot 1\, dt = 1 + \frac{x^2}{2}$$

$$y_2 = 1 + \int_0^x t\left(1 + \frac{t^2}{2}\right) dt = 1 + \frac{x^2}{2} + \frac{x^4}{8}$$

$$y_3 = 1 + \int_0^x t\left(1 + \frac{t^2}{2} + \frac{t^4}{8}\right) dt = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48}$$

$2^0 \cdot 0! \quad 2^1 \cdot 1! \quad 2^2 \cdot 2! \quad 2^3 \cdot 3!$

$$y_k = y_{k-1} + \frac{x^{2k}}{2^k k!}$$

$$y_\infty = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{x^2}{2}\right)^k = e^{x^2/2}$$

Szétválasztható DE

$$y' = f(x)\cdot g(y) \Rightarrow \frac{dy}{dx} = f(x)g(y) \Rightarrow \frac{dy}{g(y)} = f(x)dx \overset{\int}{\Rightarrow} \dots$$

6. feladat

a) $(2x+1)\,y' - 3y = 0$

$$(2x+1)\frac{dy}{dx} = 3y \Rightarrow \int \frac{dy}{3y} = \int \frac{dx}{2x+1} \Rightarrow \frac{1}{3}\ln y = \frac{1}{2}\ln(2x+1) + k$$

$$\ln y = \ln(2x+1)^{3/2} + K$$

$$y = C\cdot(2x+1)^{3/2}$$

b) $\sqrt{1+x^2}\,y' - \sqrt{1-y^2} = 0$

$$\sqrt{1+x^2}\,\frac{dy}{dx} = \sqrt{1-y^2} \Rightarrow \int \frac{dy}{\sqrt{1+y^2}} = \int \frac{dx}{\sqrt{1+x^2}} \Rightarrow \arcsin y = \operatorname{arsinh} x + C$$

$$y = \sin(\operatorname{arsinh} x + C)$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \int dt = t + C = \arcsin x + C$$

$x = \sin t \qquad dx = \cos t\, dt \qquad \sqrt{1-x^2} = \cos t$

$$\int \frac{dx}{\sqrt{1+x^2}} = \int dt = t + C = \operatorname{arsinh} x + C$$

$x = \sinh t \qquad dx = \cosh t\, dt \qquad \sqrt{1+x^2} = \cosh t$

$$\int \frac{dx}{1+x^2} = \int dt = t + C = \arctan x + C$$

$x = \tan t \qquad dx = \frac{dt}{\cos^2 t} \qquad 1+x^2 = \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$

$$\int \frac{dx}{1-x^2} = \int dt = t + C = \operatorname{arctanh} x + C$$

$x = \tanh t \qquad dx = \frac{dt}{\cosh^2 t} \qquad 1-x^2 = \frac{\cosh^2 t}{\cosh^2 t} - \frac{\sinh^2 t}{\cosh^2 t} = \frac{1}{\cosh^2 t}$

c) $y' = \frac{1-x-y}{2x+2y-3}$ $\qquad y' = f(ax+by+c) \Rightarrow u = ax+by+c$

$1-x-y = \frac{-1}{2}(2x+2y-3+1)$

$u = 2x+2y-3 \qquad y' = -\frac{1}{2}\cdot\frac{u+1}{u}$

$u' = 2+2y' = 2 - \frac{u+1}{u} = \frac{u-1}{u}$

- $u = 1 \rightarrow u' = 1 \rightarrow$ szinguláris MO

$2x+2y-3 = 1 \Rightarrow x+y = 2 \Rightarrow y = 2-x$

- $u \neq 1 \rightarrow \frac{du}{dx} = \frac{u-1}{u} \rightarrow \int \frac{u\,du}{u-1} = \int dx$

$$\int \frac{u-1+1}{u-1}\,du = \int dx$$

$$\int 1 + \frac{1}{u-1}\,du = \int dx$$

$$u + \ln|u-1| = x + C$$

$$2x+2y-3+\ln|2x+2y-4| = x+C$$

d) $xy' = y(1+\ln y - \ln x)$ $\qquad y = f(y/x) \Rightarrow u = y/x$

$y' = \frac{y}{x}\left(1+\ln\frac{y}{x}\right)$

$$\left.\begin{array}{ll} u = \frac{y}{x} & y' = u(1+\ln u) \\ y = ux & y' = u'x + u \end{array}\right\} \quad u'x + u = u + u\ln u$$

$\frac{du}{dx} = \frac{u\ln u}{x} \Rightarrow \frac{du}{u\ln u} = \frac{dx}{x}$

$$\ln|\ln u| = \ln|x| + K$$

$$\ln u = \pm C x$$

$$u = e^{\pm Cx}$$

$$y = ux = xe^{\pm Cx}$$

e) $2xyy' = x^2+y^2$

$$\left.\begin{array}{ll} y' = \frac{x^2+y^2}{2xy} = \frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right) = \frac{1}{2}\left(u+\frac{1}{u}\right) & \\ u = y/x \qquad y' = (ux)' = u'x+u & \end{array}\right\} \quad u'x + u = \frac{1}{2}\left(u+\frac{1}{u}\right)$$

$$u' = \frac{1}{2x}\left(\frac{1}{u} - u\right) = \frac{1}{2x}\left(\frac{1-u^2}{u}\right)$$

Szinguláris

$$1-u^2 = 0$$
$$u^2 = 1$$
$$u = \pm 1$$
$$y = \pm x$$

$$\int \frac{u\,du}{u^2-1} = -\int \frac{dx}{2x}$$

$$\frac{1}{2}\ln|u^2-1| = -\frac{1}{2}\ln|x| + K$$

$$u^2 - 1 = \frac{C}{x}$$

$$\frac{y^2}{x^2} - 1 = \frac{C}{x}$$

$$y^2 - x^2 = Cx$$

f, $y' = \dfrac{3x^2+4x+2}{2y-2}$ $\qquad y(0) = -1$

$$\int 2y-2\,dy = \int 3x^2+4x+2\,dx$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$

$$(-1)^2 - 2(-1) = 0^3 + 2\cdot 0^2 + 2\cdot 0 + C \Rightarrow C = 3$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3$$

g, $xy' = x\cdot e^{y/x} + y$ $\qquad y(1) = 0$

$$\left.\begin{aligned} y' &= e^{y/x} + y/x = e^u + u \\ u &= y/x \qquad y' = (ux)' = u'x + u \end{aligned}\right\} \quad e^u = u'x \Rightarrow e^{-u}du = \frac{dx}{x}$$

$$-e^{-u} = \ln|x| + C$$

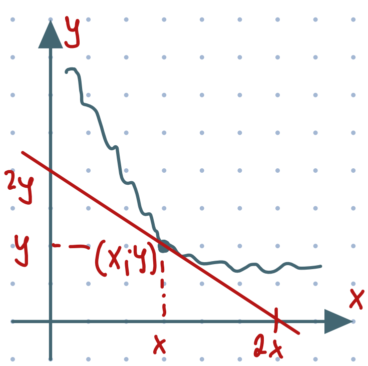
$$e^{-u} = -\ln|x| + C$$

$$e^{-0} = -\ln 1 + C \qquad C = 1$$

$$e^{-y/x} = -\ln|x| + 1$$

7. feladat

Adja meg azon görbét, amelynek bármely pontjában az érintő a koordináta-tengelyek közé eső részét az adott pontban felezi!



$$m = \frac{dy}{dx} = \frac{-y}{x} \Rightarrow \frac{dy}{-y} = \frac{dx}{x}$$

$$\ln|y| = -\ln|x| + K$$

$$y = \pm C\,\frac{1}{x}$$

8. feladat

8. Newton-törvénye értelmében ismert, hogy egy test hőmérsékletének változása a környezet hőmérsékletével való különbséggel arányos. Egy kenyeret a $t = 0$ időpillanatban kiveszünk a 200 °C-os sütőből, majd hűlni hagyjuk. 20 perc után 60 °C-ra hűl le. Mennyi idő múlva éri el a kenyér hőmérséklete a 30 °C-ot, ha a környezet hőmérséklete 20 °C?

$\dot{x} = \alpha(x - x_k)$

$x(t) \sim$ kenyér hőm

$x_k = 20°C \sim$ környezet hőm

$x(0) = 200°C$

$x(20\ min) = 60°C$

$\frac{dx}{x-x_k} = \alpha dt \Rightarrow \ln|x - x_k| = \alpha t + k \Rightarrow x(t) = x_k + Ce^{\alpha t}$

$x(0) = 200$

$200 = 20 + Ce^{\alpha \cdot 0} \Rightarrow C = 180$

$x(20) = 60$

$60 = 20 + 180 e^{\alpha 20}$

$\alpha = \frac{1}{20} \cdot \ln \frac{60-20}{180} = \frac{1}{20} \cdot \ln \frac{2}{9}$

$x(t_1) = 30$

$30 = 20 + 180 \exp\left(t \cdot \frac{1}{20} \ln \frac{2}{9}\right)$

$t = 20 \frac{\ln \frac{30-20}{180}}{\ln 2/9} \simeq 38{,}43\ min$

9. feladat

9. 100 kg 10 %-os sóoldatot tartalmazó edénybe másodpercenként 10 L tiszta víz áramlik be. Mikor lesz a sóoldat koncentrációja 5 %-os, ha a keveredés azonnal megtörténik, és ugyanilyen sebességgel folyik ki az edényből a keverék?

$x(t) \sim$ só $\frac{m}{m}\%$ $\quad x(0) = 10kg \quad x(t_1) = 5kg$

$\dot{x}_{be} = \dot{V}\rho_{be} \quad \dot{V} = 10L/s \quad \rho_{be} = 0 \quad \dot{x}_{be} = 0$

$\dot{x}_{ki}(t) = \dot{V}\rho_{ki}(t) = \dot{V}\frac{x(t)}{V_0} \quad \dot{x}_{ki} = 0{,}1\,x(t)$

Anyagmérleg: $\dot{x} = \dot{x}_{be} - \dot{x}_{ki}(t)$

$\frac{dx}{-0{,}1x} = dt \Rightarrow \ln|x| = -0{,}1t + k \Rightarrow x = Ce^{-0{,}1t}$

$x(0) = 10 \Rightarrow C = 10$

$x(t_1) = 5 \Rightarrow 5 = 10e^{-0{,}1t_1} \Rightarrow t_1 = \frac{1}{-0{,}1} \ln \frac{5}{10} = -10 \ln \frac{1}{2} = 10 \ln 2 = 6{,}93s$