

Operátorok, Potenciálosság

Matematika G3 – Vektoranalízis

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2.1. Skalármezők gradiense és Laplace

a) $\varphi(\mathbf{r}) = 6x^y + \sin e^z$

$$\text{grad } \varphi = \nabla \varphi = \left(\frac{\partial \varphi}{\partial x}; \frac{\partial \varphi}{\partial y}; \frac{\partial \varphi}{\partial z} \right)^T = \begin{bmatrix} 6yx^{y-1} \\ 6x^y \ln x \\ e^z \cos e^z \end{bmatrix}$$

$$\begin{aligned} \Delta \varphi &= \text{div grad } \varphi = \frac{\partial 6yx^{y-1}}{\partial x} + \frac{\partial 6x^y \ln x}{\partial y} + \frac{\partial e^z \cos e^z}{\partial z} \\ &= 6y(y-1)x^{y-2} + 6x^y \ln^2 x + e^z \cos e^z - e^{2z} \sin e^z \end{aligned}$$

b) $\psi(\mathbf{r}) = \mathbf{r}^2/2 = (x^2 + y^2 + z^2)/2$

$$\text{grad } \psi = \nabla \psi = \left(\frac{\partial \psi}{\partial x}; \frac{\partial \psi}{\partial y}; \frac{\partial \psi}{\partial z} \right)^T = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{r}$$

$$\Delta \psi = \text{div grad } \psi = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

c) $\chi(\mathbf{r}) = xy + xz + yz$

$$\text{grad } \chi = \nabla \chi = \left(\frac{\partial \chi}{\partial x}; \frac{\partial \chi}{\partial y}; \frac{\partial \chi}{\partial z} \right)^T = \begin{bmatrix} y+z \\ x+z \\ x+y \end{bmatrix}$$

$$\Delta \chi = \text{div grad } \chi = \frac{\partial(y+z)}{\partial x} + \frac{\partial(x+z)}{\partial y} + \frac{\partial(x+y)}{\partial z} = 0$$

d) $\omega(\mathbf{r}) = 2x^2y + xz^2 + 6y$

$$\text{grad } \omega = \nabla \omega = \left(\frac{\partial \omega}{\partial x}; \frac{\partial \omega}{\partial y}; \frac{\partial \omega}{\partial z} \right)^T = \begin{bmatrix} 4xy + z^2 \\ 2x^2 + 6 \\ 2xz \end{bmatrix}$$

$$\Delta \omega = \text{div grad } \omega = \frac{\partial 4xy + z^2}{\partial x} + \frac{\partial 2x^2 + 6}{\partial y} + \frac{\partial 2xz}{\partial z} = 4y + 2x$$

2.2. Vektormezők rotációja és divergenciája

a) $\mathbf{v}(\mathbf{r}) = \mathbf{r} = (x)\hat{\mathbf{i}} + (y)\hat{\mathbf{j}} + (z)\hat{\mathbf{k}}$

$$\operatorname{div} \mathbf{v} = \langle \nabla; \mathbf{v} \rangle = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

$$\operatorname{rot} \mathbf{v} = \nabla \times \mathbf{v} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 - 0 \\ 0 - 0 \\ 0 - 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

A vektormező sehol sem forrásmentes, de örvénymentes az egész értelmezési tartományán.

b) $\mathbf{w}(\mathbf{r}) = (3xy + z^2)\hat{\mathbf{i}} + (6e^z)\hat{\mathbf{j}} + (-5x^y)\hat{\mathbf{k}}$

$$\operatorname{div} \mathbf{w} = \langle \nabla; \mathbf{w} \rangle = \frac{\partial(3xy + z^2)}{\partial x} + \frac{\partial(6e^z)}{\partial y} + \frac{\partial(-5x^y)}{\partial z} = 3y + 0 + 0 = 3y$$

$$\operatorname{rot} \mathbf{w} = \nabla \times \mathbf{w} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} 3xy + z^2 \\ 6e^z \\ -5x^y \end{bmatrix} = \begin{bmatrix} -5x^y \ln x - 6e^z \\ 2z + 5yx^{y-1} \\ -3x \end{bmatrix}$$

A vektormező forrásmentes az $y = 0$ síkon, de sehol sem örvénymentes. (z koordináta: $x = 0$, x koordináta: $x \neq 0$, ez ellentmondás.)

c) $\mathbf{u}(\mathbf{r}) = (\ln(xy/z))\hat{\mathbf{i}} + (\ln(yz/x))\hat{\mathbf{j}} + (\ln(zx/y))\hat{\mathbf{k}}$

$$\begin{aligned} \operatorname{div} \mathbf{u} &= \langle \nabla; \mathbf{u} \rangle = \frac{\partial \ln(xy/z)}{\partial x} + \frac{\partial \ln(yz/x)}{\partial y} + \frac{\partial \ln(zx/y)}{\partial z} = \\ &= \frac{z}{xy} \cdot \frac{y}{z} + \frac{x}{yz} \cdot \frac{z}{x} + \frac{y}{zx} \cdot \frac{x}{y} = \frac{1}{x} + \frac{1}{y} - \frac{1}{z} \end{aligned}$$

$$\operatorname{rot} \mathbf{u} = \nabla \times \mathbf{u} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} \ln(xy/z) \\ \ln(yz/x) \\ \ln(zx/y) \end{bmatrix} = \begin{bmatrix} \frac{y}{xz} \left(\frac{-zx}{y^2} \right) - \frac{x}{yz} \left(\frac{y}{x} \right) \\ \frac{z}{xy} \left(\frac{-xy}{z^2} \right) - \frac{y}{xz} \left(\frac{z}{y} \right) \\ \frac{x}{yz} \left(\frac{-yz}{x^2} \right) - \frac{z}{xy} \left(\frac{x}{z} \right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{y} - \frac{1}{z} \\ -\frac{1}{z} - \frac{1}{x} \\ -\frac{1}{x} - \frac{1}{y} \end{bmatrix}$$

d) $\mathbf{s}(\mathbf{r}) = \mathbf{a}\|\mathbf{r}\| + \|\mathbf{a}\|\mathbf{r}$ $(\mathbf{a} \in \mathbb{R}^3)$

$$\begin{aligned} \operatorname{div} \mathbf{s} &= \langle \nabla; \mathbf{s} \rangle = \langle \nabla; \mathbf{a}\|\mathbf{r}\| \rangle + \langle \nabla; \|\mathbf{a}\|\mathbf{r} \rangle = \langle \mathbf{a}; \nabla\|\mathbf{r}\| \rangle + \|\mathbf{a}\| \langle \nabla; \mathbf{r} \rangle = \\ &= \left\langle \mathbf{a}; \frac{1}{2\|\mathbf{r}\|} \cdot 2\mathbf{r} \right\rangle + \|\mathbf{a}\| \cdot 3 = \frac{\langle \mathbf{a}; \mathbf{r} \rangle}{\|\mathbf{r}\|} + 3\|\mathbf{a}\| \end{aligned}$$

$$\begin{aligned} \operatorname{rot} \mathbf{s} &= \nabla \times \mathbf{s} = \nabla \times (\mathbf{a}\|\mathbf{r}\|) + \nabla \times (\|\mathbf{a}\|\mathbf{r}) = \nabla\|\mathbf{r}\| \times \mathbf{a} + \|\mathbf{a}\| \nabla \times \mathbf{r} = \\ &= \frac{\mathbf{r}}{\|\mathbf{r}\|} \times \mathbf{a} + \|\mathbf{a}\| \cdot \mathbf{0} = \frac{\mathbf{r} \times \mathbf{a}}{\|\mathbf{r}\|} \end{aligned}$$

Egyszerűsítések során felhasznált képletek:

$$\operatorname{grad} \mathbf{r} = \frac{\mathbf{r}}{\|\mathbf{r}\|}, \quad \operatorname{div} \mathbf{r} = 3, \quad \operatorname{rot} \mathbf{r} = \mathbf{0}.$$

2.3. Azonosságok bizonyítása

a) $\operatorname{rot} \operatorname{grad} \Phi \equiv \mathbf{0}$

$$\operatorname{rot} \operatorname{grad} \Phi = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} \partial_x \Phi \\ \partial_y \Phi \\ \partial_z \Phi \end{bmatrix} = \begin{bmatrix} \partial_y \partial_z \Phi - \partial_z \partial_y \Phi \\ \partial_z \partial_x \Phi - \partial_x \partial_z \Phi \\ \partial_x \partial_y \Phi - \partial_y \partial_x \Phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

b) $\operatorname{div} \operatorname{rot} \mathbf{v} \equiv 0$

$$\begin{aligned} \operatorname{div} \operatorname{rot} \mathbf{v} &= \left\langle \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix}; \begin{bmatrix} \partial_y v_z - \partial_z v_y \\ \partial_z v_x - \partial_x v_z \\ \partial_x v_y - \partial_y v_x \end{bmatrix} \right\rangle = \\ &= \frac{\partial^2 v_z}{\partial x \partial y} - \frac{\partial^2 v_y}{\partial x \partial z} + \frac{\partial^2 v_x}{\partial y \partial z} - \frac{\partial^2 v_z}{\partial y \partial x} + \frac{\partial^2 v_y}{\partial z \partial x} - \frac{\partial^2 v_x}{\partial z \partial y} = \\ &= \frac{\partial^2 v_x}{\partial y \partial z} - \frac{\partial^2 v_x}{\partial z \partial y} + \frac{\partial^2 v_y}{\partial z \partial x} - \frac{\partial^2 v_y}{\partial x \partial z} + \frac{\partial^2 v_z}{\partial x \partial y} - \frac{\partial^2 v_z}{\partial y \partial x} = 0 \end{aligned}$$

c) $\operatorname{grad}(\Phi \Psi) = \Phi \operatorname{grad} \Psi + \Psi \operatorname{grad} \Phi$

$$(\operatorname{grad}(\Phi \Psi))_i = \partial_i(\Phi \Psi) = \Phi \partial_i \Psi + \Psi \partial_i \Phi = (\Phi \operatorname{grad} \Psi + \Psi \operatorname{grad} \Phi)_i$$

d) $\Delta(\Phi \Psi) = (\Delta \Phi) \Psi + 2 \langle \operatorname{grad} \Phi; \operatorname{grad} \Psi \rangle + \Psi(\Delta \Phi)$

$$\begin{aligned} \Delta(\Phi \Psi) &= \sum_{i=1}^3 \partial_i^2(\Phi \Psi) = \sum_{i=1}^3 \partial_i(\Phi \partial_i \Psi + \Psi \partial_i \Phi) = \sum_{i=1}^3 (\Phi \partial_i^2 \Psi + 2(\partial_i \Phi)(\partial_i \Psi) + \Psi \partial_i^2 \Phi) = \\ &= \Phi \sum_{i=1}^3 \partial_i^2 \Psi + 2 \sum_{i=1}^3 (\partial_i \Phi)(\partial_i \Psi) + \Psi \sum_{i=1}^3 \partial_i^2 \Phi = \Phi(\Delta \Psi) + 2 \langle \operatorname{grad} \Phi; \operatorname{grad} \Psi \rangle + \Psi(\Delta \Phi) \end{aligned}$$

e) $\operatorname{div}(\Phi \mathbf{v}) = \langle \operatorname{grad} \Phi; \mathbf{v} \rangle + \Phi \operatorname{div} \mathbf{v}$

$$\operatorname{div}(\Phi \mathbf{v}) = \sum_{i=1}^3 \partial_i(\Phi v_i) = \sum_{i=1}^3 ((\partial_i \Phi) v_i + \Phi(\partial_i v_i)) = \langle \operatorname{grad} \Phi; \mathbf{v} \rangle + \Phi \operatorname{div} \mathbf{v}$$

f) $\operatorname{div}(\mathbf{v} \times \mathbf{w}) = \langle \operatorname{rot} \mathbf{v}; \mathbf{w} \rangle - \langle \mathbf{v}; \operatorname{rot} \mathbf{w} \rangle$

$$\begin{aligned} \operatorname{div}(\mathbf{v} \times \mathbf{w}) &= \frac{\partial(v_y w_z - v_z w_y)}{\partial x} + \frac{\partial(v_z w_x - v_x w_z)}{\partial y} + \frac{\partial(v_x w_y - v_y w_x)}{\partial z} \\ &= \frac{\partial v_y}{\partial x} w_z - \frac{\partial v_z}{\partial x} w_y + \frac{\partial v_z}{\partial y} w_x - \frac{\partial v_x}{\partial y} w_z + \frac{\partial v_x}{\partial z} w_y - \frac{\partial v_y}{\partial z} w_x \\ &\quad + \frac{\partial w_z}{\partial x} v_y - \frac{\partial w_y}{\partial x} v_z + \frac{\partial w_x}{\partial y} v_z - \frac{\partial w_z}{\partial y} v_x + \frac{\partial w_y}{\partial z} v_x - \frac{\partial w_x}{\partial z} v_y \\ &= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) w_x + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) w_y + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) w_z \\ &\quad - \left(\frac{\partial w_z}{\partial y} - \frac{\partial w_y}{\partial z} \right) v_x - \left(\frac{\partial w_x}{\partial z} - \frac{\partial w_z}{\partial x} \right) v_y - \left(\frac{\partial w_y}{\partial x} - \frac{\partial w_x}{\partial y} \right) v_z \\ &= \langle \operatorname{rot} \mathbf{v}; \mathbf{w} \rangle - \langle \mathbf{v}; \operatorname{rot} \mathbf{w} \rangle \end{aligned}$$

2.4. Potenciálfüggvények

a) $\mathbf{v}(\mathbf{r}) = (y+z)\hat{\mathbf{i}} + (x+z)\hat{\mathbf{j}} + (x+y)\hat{\mathbf{k}}$

- A vektormező skalárpotenciállos, ha rotációja zérus:

$$\text{rot } \mathbf{v} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} y+z \\ x+z \\ x+y \end{bmatrix} = \begin{bmatrix} \partial_y(x+y) - \partial_z(x+z) \\ \partial_z(y+z) - \partial_x(x+y) \\ \partial_x(x+z) - \partial_y(y+z) \end{bmatrix} = \begin{bmatrix} 1-1 \\ 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

A potenciálfüggvény:

$$\begin{aligned} \varphi(\mathbf{r}) &= \int_0^x v_x(\xi; y; z) d\xi + \int_0^y v_y(0; \eta; z) d\eta + \int_0^z v_z(0; 0; \zeta) d\zeta \\ &= \int_0^x (y+z) d\xi + \int_0^y (0+z) d\eta + \int_0^z (0+0) d\zeta \\ &= xy + xz + yz + C. \end{aligned}$$

A keresett potenciálfüggvény:

$$\varPhi(\mathbf{r}) = xy + xz + yz.$$

- A vektormező vektorpotenciállos, ha divergenciája zérus:

$$\text{div } \mathbf{v} = \frac{\partial \mathbf{v}}{\partial x} + \frac{\partial \mathbf{v}}{\partial y} + \frac{\partial \mathbf{v}}{\partial z} = 0 + 0 + 0 = 0.$$

A potenciálfüggvény:

$$\begin{aligned} V_x(\mathbf{r}) &= \int_0^z v_y(x; y; \zeta) d\zeta = \int_0^z (x+\zeta) d\zeta = xz + \frac{z^2}{2} + C_x, \\ V_y(\mathbf{r}) &= \int_0^x v_z(\xi; y; 0) d\xi - \int_0^z v_x(x; y; \zeta) d\zeta \\ &= \int_0^x (\xi+y) d\xi - \int_0^z (y+\zeta) d\zeta \\ &= \frac{x^2}{2} + xy - \frac{z^2}{2} - yz + C_y \end{aligned}$$

A keresett vektorpotenciál:

$$\mathbf{V}(\mathbf{r}) = (xz + \frac{z^2}{2})\hat{\mathbf{i}} + (\frac{x^2}{2} + xy - \frac{z^2}{2} - yz)\hat{\mathbf{j}} + (0)\hat{\mathbf{k}}.$$

b) $\mathbf{w}(\mathbf{r}) = (e^{x+\sin y})\hat{\mathbf{i}} + (e^{x+\sin y} \cos y)\hat{\mathbf{j}} + (0)\hat{\mathbf{k}}$

- Egy vektormező skalárpotenciállos, ha rotációja zérus:

$$\text{rot } \mathbf{w} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} e^{x+\sin y} \\ e^{x+\sin y} \cos y \\ 0 \end{bmatrix} = \begin{bmatrix} 0-0 \\ 0-0 \\ e^{x+\sin y} \cos y - e^{x+\sin y} \cos y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

A potenciálfüggvény:

$$\begin{aligned}\psi(\mathbf{r}) &= \int_0^x w_x(\xi; y; z) d\xi + \int_0^y w_y(0; \eta; z) d\eta + \int_0^z w_z(0; 0; \zeta) d\zeta \\ &= \int_0^x e^{\xi+\sin y} d\xi + \int_0^y e^{\sin y} \cos y d\eta + \int_0^z 0 d\zeta \\ &= (e^x - 1)e^{\sin y} + e^{\sin y} - 1 + 0 + C.\end{aligned}$$

A keresett potenciálfüggvény:

$$\Psi(\mathbf{r}) = e^{x+\sin y}.$$

Egy vektormező vektorpotenciálos, ha divergenciája zérus:

$$\operatorname{div} \mathbf{w} = \frac{\partial \mathbf{w}}{\partial x} + \frac{\partial \mathbf{w}}{\partial y} + \frac{\partial \mathbf{w}}{\partial z} = e^{x+\sin y} + e^{x+\sin y} (\cos^2 y - \sin y) \neq 0.$$

Mivel $\operatorname{div} \mathbf{w} \neq 0$, ezért nem létezik \mathbf{w} -nek vektorpotenciálja.

c) $\mathbf{u}(\mathbf{r}) = (2zx^3)\hat{\mathbf{i}} + (3z)\hat{\mathbf{j}} + (-3x^2z^2)\hat{\mathbf{k}}$

- Egy vektormező skalárpotenciálos, ha rotációja zérus.

$$\operatorname{rot} \mathbf{u} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} 2zx^3 \\ 3z \\ -3x^2z^2 \end{bmatrix} = \begin{bmatrix} 0 - 3 \\ 2x^3 + 6xz^2 \\ 0 - 0 \end{bmatrix} \neq \mathbf{0}$$

Mivel $\operatorname{rot} \mathbf{u} \neq \mathbf{0}$, ezért \mathbf{u} -nak nem létezik skalárpotenciálja.

- Egy vektormező vektorpotenciálos, ha divergenciája zérus.

$$\operatorname{div} \mathbf{u} = \frac{\partial \mathbf{u}}{\partial x} + \frac{\partial \mathbf{u}}{\partial y} + \frac{\partial \mathbf{u}}{\partial z} = 6x^2z + 0 - 6x^2z = 0.$$

A potenciálfüggvény:

$$\begin{aligned}U_x(\mathbf{r}) &= \int_0^z u_y(x; y; \zeta) d\zeta = \int_0^z (3\zeta) d\zeta = \frac{3z^2}{2} + C_x, \\ U_y(\mathbf{r}) &= \int_0^x u_z(\xi; y; 0) d\xi - \int_0^z u_x(x; y; \zeta) d\zeta \\ &= \int_0^x (0) d\xi - \int_0^z (2\zeta x^3) d\zeta = 0 - x^3 z^2 + C_y.\end{aligned}$$

A keresett vektorpotenciál:

$$\mathbf{U}(\mathbf{r}) = \left(\frac{3z^2}{2}\right) \hat{\mathbf{i}} + (-x^3 z^2) \hat{\mathbf{j}} + (0) \hat{\mathbf{k}}.$$