2 MO

Operátorok, Potenciálosság

Matematika G3 – Vektoranalízis
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2.1. Skalármezők gradiense és Laplace

a) $\varphi(\boldsymbol{r}) = 6x^y + \sin e^z$

$$\operatorname{grad}\varphi = \nabla\varphi = \left(\frac{\partial\varphi}{\partial x}; \frac{\partial\varphi}{\partial y}; \frac{\partial\varphi}{\partial z}\right)^{\mathsf{T}} = \begin{bmatrix} 6yx^{y-1} \\ 6x^y \ln x \\ e^z \cos e^z \end{bmatrix}$$

$$\begin{aligned}\Delta\varphi = \operatorname{div}\operatorname{grad}\varphi &= \frac{\partial 6yx^{y-1}}{\partial x} + \frac{\partial 6x^y \ln x}{\partial y} + \frac{\partial e^z \cos e^z}{\partial z} \\ &= 6y(y-1)x^{y-2} + 6x^y \ln^2 x + e^z \cos e^z - e^{2z} \sin e^z\end{aligned}$$

b) $\psi(\boldsymbol{r}) = \boldsymbol{r}^2/2 = (x^2 + y^2 + z^2)/2$

$$\operatorname{grad}\psi = \nabla\psi = \left(\frac{\partial\psi}{\partial x}; \frac{\partial\psi}{\partial y}; \frac{\partial\psi}{\partial z}\right)^{\mathsf{T}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \boldsymbol{r}$$

$$\Delta\psi = \operatorname{div}\operatorname{grad}\psi = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

c) $\chi(\boldsymbol{r}) = xy + xz + yz$

$$\operatorname{grad}\chi = \nabla\chi = \left(\frac{\partial\chi}{\partial x}; \frac{\partial\chi}{\partial y}; \frac{\partial\chi}{\partial z}\right)^{\mathsf{T}} = \begin{bmatrix} y+z \\ x+z \\ x+y \end{bmatrix}$$

$$\Delta\chi = \operatorname{div}\operatorname{grad}\chi = \frac{\partial(y+z)}{\partial x} + \frac{\partial(x+z)}{\partial y} + \frac{\partial(x+y)}{\partial z} = 0$$

d) $\omega(\boldsymbol{r}) = 2x^2y + xz^2 + 6y$

$$\operatorname{grad}\omega = \nabla\omega = \left(\frac{\partial\omega}{\partial x}; \frac{\partial\omega}{\partial y}; \frac{\partial\omega}{\partial z}\right)^{\mathsf{T}} = \begin{bmatrix} 4xy + z^2 \\ 2x^2 + 6 \\ 2xz \end{bmatrix}$$

$$\Delta\omega = \operatorname{div}\operatorname{grad}\omega = \frac{\partial 4xy + z^2}{\partial x} + \frac{\partial 2x^2 + 6}{\partial y} + \frac{\partial 2xz}{\partial z} = 4y + 2x$$

2.2. Vektormezők rotációja és divergenciája

a) $\boldsymbol{v}(\boldsymbol{r}) = \boldsymbol{r} = (x)\,\hat{\boldsymbol{\imath}} + (y)\,\hat{\boldsymbol{\jmath}} + (z)\,\hat{\boldsymbol{k}}$

$$\operatorname{div} \boldsymbol{v} = \langle \nabla; \boldsymbol{v} \rangle = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

$$\operatorname{rot} \boldsymbol{v} = \nabla \times \boldsymbol{v} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0-0 \\ 0-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

A vektormező sehol sem forrásmentes, de örvénymentes az egész értelmezési tartományán.

b) $\boldsymbol{w}(\boldsymbol{r}) = (3xy + z^2)\,\hat{\boldsymbol{\imath}} + (6e^z)\,\hat{\boldsymbol{\jmath}} + (-5x^y)\,\hat{\boldsymbol{k}}$

$$\operatorname{div} \boldsymbol{w} = \langle \nabla; \boldsymbol{w} \rangle = \frac{\partial(3xy + z^2)}{\partial x} + \frac{\partial(6e^z)}{\partial y} + \frac{\partial(-5x^y)}{\partial z} = 3y + 0 + 0 = 3y$$

$$\operatorname{rot} \boldsymbol{w} = \nabla \times \boldsymbol{w} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} 3xy + z^2 \\ 6e^z \\ -5x^y \end{bmatrix} = \begin{bmatrix} -5x^y \ln x - 6e^z \\ 2z + 5yx^{y-1} \\ -3x \end{bmatrix}$$

A vektormező forrásmentes az $y = 0$ síkon, de sehol sem örvénymentes. (z koordináta: $x = 0$, x koordináta: $x \neq 0$, ez ellentmondás.)

c) $\boldsymbol{u}(\boldsymbol{r}) = (\ln(xy/z))\,\hat{\boldsymbol{\imath}} + (\ln(yz/x))\,\hat{\boldsymbol{\jmath}} + (\ln(zx/y))\,\hat{\boldsymbol{k}}$

$$\operatorname{div} \boldsymbol{u} = \langle \nabla; \boldsymbol{u} \rangle = \frac{\partial \ln(xy/z)}{\partial x} + \frac{\partial \ln(yz/x)}{\partial y} + \frac{\partial \ln(zx/y)}{\partial z} =$$
$$= \frac{z}{xy} \cdot \frac{y}{z} + \frac{x}{yz} \cdot \frac{z}{x} + \frac{y}{zx} \cdot \frac{x}{y} = \frac{1}{x} + \frac{1}{y} - \frac{1}{z}$$

$$\operatorname{rot} \boldsymbol{u} = \nabla \times \boldsymbol{u} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} \ln(xy/z) \\ \ln(yz/x) \\ \ln(zx/y) \end{bmatrix} = \begin{bmatrix} \frac{y}{xz}\left(\frac{-zx}{y^2}\right) - \frac{x}{yz}\left(\frac{y}{x}\right) \\ \frac{z}{xy}\left(\frac{-xy}{z^2}\right) - \frac{y}{xz}\left(\frac{z}{y}\right) \\ \frac{x}{yz}\left(\frac{-yz}{x^2}\right) - \frac{z}{xy}\left(\frac{x}{z}\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{y} - \frac{1}{z} \\ -\frac{1}{z} - \frac{1}{x} \\ -\frac{1}{x} - \frac{1}{y} \end{bmatrix}$$

d) $\boldsymbol{s}(\boldsymbol{r}) = \boldsymbol{a}\|\boldsymbol{r}\| + \|\boldsymbol{a}\|\boldsymbol{r}$ $\qquad (\boldsymbol{a} \in \mathbb{R}^3)$

$$\operatorname{div} \boldsymbol{s} = \langle \nabla; \boldsymbol{s} \rangle = \langle \nabla; \boldsymbol{a}\|\boldsymbol{r}\| \rangle + \langle \nabla; \|\boldsymbol{a}\|\boldsymbol{r} \rangle = \langle \boldsymbol{a}; \nabla\|\boldsymbol{r}\| \rangle + \|\boldsymbol{a}\|\,\langle \nabla; \boldsymbol{r} \rangle =$$
$$= \left\langle \boldsymbol{a}; \frac{1}{2\|\boldsymbol{r}\|} \cdot 2\boldsymbol{r} \right\rangle + \|\boldsymbol{a}\| \cdot 3 = \frac{\langle \boldsymbol{a}; \boldsymbol{r} \rangle}{\|\boldsymbol{r}\|} + 3\|\boldsymbol{a}\|$$

$$\operatorname{rot} \boldsymbol{s} = \nabla \times \boldsymbol{s} = \nabla \times (\boldsymbol{a}\|\boldsymbol{r}\|) + \nabla \times (\|\boldsymbol{a}\|\boldsymbol{r}) = \nabla\|\boldsymbol{r}\| \times \boldsymbol{a} + \|\boldsymbol{a}\|\nabla \times \boldsymbol{r} =$$
$$= \frac{\boldsymbol{r}}{\|\boldsymbol{r}\|} \times \boldsymbol{a} + \|\boldsymbol{a}\| \cdot \boldsymbol{0} = \frac{\boldsymbol{r} \times \boldsymbol{a}}{\|\boldsymbol{r}\|}$$

Egyszerűsítések során felhasznált képletek:

$$\operatorname{grad} \boldsymbol{r} = \frac{\boldsymbol{r}}{\|\boldsymbol{r}\|}, \quad \operatorname{div} \boldsymbol{r} = 3, \quad \operatorname{rot} \boldsymbol{r} = \boldsymbol{0}.$$

2.3. Azonosságok bizonyítása

a) $\operatorname{rot}\operatorname{grad}\varPhi \equiv \boldsymbol{0}$

$$\operatorname{rot}\operatorname{grad}\varPhi = \begin{bmatrix}\partial_x\\ \partial_y\\ \partial_z\end{bmatrix} \times \begin{bmatrix}\partial_x\varPhi\\ \partial_y\varPhi\\ \partial_z\varPhi\end{bmatrix} = \begin{bmatrix}\partial_y\partial_z\varPhi - \partial_z\partial_y\varPhi\\ \partial_z\partial_x\varPhi - \partial_x\partial_z\varPhi\\ \partial_x\partial_y\varPhi - \partial_y\partial_x\varPhi\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} = \boldsymbol{0}$$

b) $\operatorname{div}\operatorname{rot}\boldsymbol{v} \equiv 0$

$$\begin{aligned}\operatorname{div}\operatorname{rot}\boldsymbol{v} &= \left\langle \begin{bmatrix}\partial_x\\ \partial_y\\ \partial_z\end{bmatrix}; \begin{bmatrix}\partial_y v_z - \partial_z v_y\\ \partial_z v_x - \partial_x v_z\\ \partial_x v_y - \partial_y v_x\end{bmatrix}\right\rangle = \\ &= \frac{\partial^2 v_z}{\partial x\,\partial y} - \frac{\partial^2 v_y}{\partial x\,\partial z} + \frac{\partial^2 v_x}{\partial y\,\partial z} - \frac{\partial^2 v_z}{\partial y\,\partial x} + \frac{\partial^2 v_y}{\partial z\,\partial x} - \frac{\partial^2 v_x}{\partial z\,\partial y} = \\ &= \frac{\partial^2 v_x}{\partial y\,\partial z} - \frac{\partial^2 v_x}{\partial z\,\partial y} + \frac{\partial^2 v_y}{\partial z\,\partial x} - \frac{\partial^2 v_y}{\partial x\,\partial z} + \frac{\partial^2 v_z}{\partial x\,\partial y} - \frac{\partial^2 v_z}{\partial y\,\partial x} = 0\end{aligned}$$

c) $\operatorname{grad}(\varPhi\varPsi) = \varPhi\operatorname{grad}\varPsi + \varPsi\operatorname{grad}\varPhi$

$$\big(\operatorname{grad}(\varPhi\varPsi)\big)_i = \partial_i(\varPhi\varPsi) = \varPhi\,\partial_i\varPsi + \varPsi\,\partial_i\varPhi = \big(\varPhi\operatorname{grad}\varPsi + \varPsi\operatorname{grad}\varPhi\big)_i$$

d) $\Delta(\varPhi\varPsi) = (\Delta\varPhi)\varPsi + 2\langle\operatorname{grad}\varPhi;\operatorname{grad}\varPsi\rangle + \varPsi(\Delta\varPhi)$

$$\begin{aligned}\Delta(\varPhi\varPsi) &= \sum_{i=1}^{3}\partial_i^2(\varPhi\varPsi) = \sum_{i=1}^{3}\partial_i\left(\varPhi\,\partial_i\varPsi + \varPsi\,\partial_i\varPhi\right) = \sum_{i=1}^{3}\left(\varPhi\,\partial_i^2\varPsi + 2(\partial_i\varPhi)(\partial_i\varPsi) + \varPsi\,\partial_i^2\varPhi\right) = \\ &= \varPhi\sum_{i=1}^{3}\partial_i^2\varPsi + 2\sum_{i=1}^{3}(\partial_i\varPhi)(\partial_i\varPsi) + \varPsi\sum_{i=1}^{3}\partial_i^2\varPhi = \varPhi(\Delta\varPsi) + 2\langle\operatorname{grad}\varPhi;\operatorname{grad}\varPsi\rangle + \varPsi(\Delta\varPhi)\end{aligned}$$

e) $\operatorname{div}(\varPhi\boldsymbol{v}) = \langle\operatorname{grad}\varPhi;\boldsymbol{v}\rangle + \varPhi\operatorname{div}\boldsymbol{v}$

$$\operatorname{div}(\varPhi\boldsymbol{v}) = \sum_{i=1}^{3}\partial_i(\varPhi v_i) = \sum_{i=1}^{3}\left((\partial_i\varPhi)v_i + \varPhi(\partial_i v_i)\right) = \langle\operatorname{grad}\varPhi;\boldsymbol{v}\rangle + \varPhi\operatorname{div}\boldsymbol{v}$$

f) $\operatorname{div}(\boldsymbol{v}\times\boldsymbol{w}) = \langle\operatorname{rot}\boldsymbol{v};\boldsymbol{w}\rangle - \langle\boldsymbol{v};\operatorname{rot}\boldsymbol{w}\rangle$

$$\begin{aligned}\operatorname{div}(\boldsymbol{v}\times\boldsymbol{w}) &= \frac{\partial(v_y w_z - v_z w_y)}{\partial x} + \frac{\partial(v_z w_x - v_x w_z)}{\partial y} + \frac{\partial(v_x w_y - v_y w_x)}{\partial z} \\ &= \frac{\partial v_y}{\partial x}w_z - \frac{\partial v_z}{\partial x}w_y + \frac{\partial v_z}{\partial y}w_x - \frac{\partial v_x}{\partial y}w_z + \frac{\partial v_x}{\partial z}w_y - \frac{\partial v_y}{\partial z}w_x \\ &\quad + \frac{\partial w_z}{\partial x}v_y - \frac{\partial w_y}{\partial x}v_z + \frac{\partial w_x}{\partial y}v_z - \frac{\partial w_z}{\partial y}v_x + \frac{\partial w_y}{\partial z}v_x - \frac{\partial w_x}{\partial z}v_y \\ &= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)w_x + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)w_y + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)w_z \\ &\quad - \left(\frac{\partial w_z}{\partial y} - \frac{\partial w_y}{\partial z}\right)v_x - \left(\frac{\partial w_x}{\partial z} - \frac{\partial w_z}{\partial x}\right)v_y - \left(\frac{\partial w_y}{\partial x} - \frac{\partial w_x}{\partial y}\right)v_z \\ &= \langle\operatorname{rot}\boldsymbol{v};\boldsymbol{w}\rangle - \langle\boldsymbol{v};\operatorname{rot}\boldsymbol{w}\rangle\end{aligned}$$

2.4. Potenciálfüggvények

a) $\boldsymbol{v}(\boldsymbol{r}) = (y+z)\,\hat{\boldsymbol{\imath}} + (x+z)\,\hat{\boldsymbol{\jmath}} + (x+y)\,\hat{\boldsymbol{k}}$

- A vektormező skalárpotenciálos, ha rotációja zérus:

$$\operatorname{rot}\boldsymbol{v} = \begin{bmatrix}\partial_x\\ \partial_y\\ \partial_z\end{bmatrix} \times \begin{bmatrix}y+z\\ x+z\\ x+y\end{bmatrix} = \begin{bmatrix}\partial_y(x+y)-\partial_z(x+z)\\ \partial_z(y+z)-\partial_x(x+y)\\ \partial_x(x+z)-\partial_y(y+z)\end{bmatrix} = \begin{bmatrix}1-1\\ 1-1\\ 1-1\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

A potenciálfüggvény:

$$\begin{aligned}\varphi(\boldsymbol{r}) &= \int_0^x v_x(\xi;y;z)\,\mathrm{d}\xi + \int_0^y v_y(0;\eta;z)\,\mathrm{d}\eta + \int_0^z v_z(0;0;\zeta)\,\mathrm{d}\zeta\\ &= \int_0^x (y+z)\,\mathrm{d}\xi + \int_0^y (0+z)\,\mathrm{d}\eta + \int_0^z (0+0)\,\mathrm{d}\zeta\\ &= xy+xz+yz+C.\end{aligned}$$

A kereseett potenciálfüggvény:

$$\Phi(\boldsymbol{r}) = xy+xz+yz.$$

- A vektormező vektorpotenciálos, ha divergenciája zérus:

$$\operatorname{div}\boldsymbol{v} = \frac{\partial\boldsymbol{v}}{\partial x} + \frac{\partial\boldsymbol{v}}{\partial y} + \frac{\partial\boldsymbol{v}}{\partial z} = 0+0+0 = 0.$$

A potenciálfüggvény:

$$\begin{aligned}V_x(\boldsymbol{r}) &= \int_0^z v_y(x;y;\zeta)\,\mathrm{d}\zeta = \int_0^z (x+\zeta)\,\mathrm{d}\zeta = xz + \frac{z^2}{2} + C_x,\\ V_y(\boldsymbol{r}) &= \int_0^x v_z(\xi;y;0)\,\mathrm{d}\xi - \int_0^z v_x(x;y;\zeta)\,\mathrm{d}\zeta\\ &= \int_0^x (\xi+y)\,\mathrm{d}\xi - \int_0^z (y+\zeta)\,\mathrm{d}\zeta\\ &= \frac{x^2}{2} + xy - \frac{z^2}{2} - yz + C_y\end{aligned}$$

A keresett vektorpotenciál:

$$\boldsymbol{V}(\boldsymbol{r}) = (xz+\frac{z^2}{2})\,\hat{\boldsymbol{\imath}} + (\frac{x^2}{2}+xy-\frac{z^2}{2}-yz)\,\hat{\boldsymbol{\jmath}} + (0)\,\hat{\boldsymbol{k}}.$$

b) $\boldsymbol{w}(\boldsymbol{r}) = (e^{x+\sin y})\,\hat{\boldsymbol{\imath}} + (e^{x+\sin y}\cos y)\,\hat{\boldsymbol{\jmath}} + (0)\,\hat{\boldsymbol{k}}$

- Egy vektormező skalárpotenciálos, ha rotációja zérus:

$$\operatorname{rot}\boldsymbol{w} = \begin{bmatrix}\partial_x\\ \partial_y\\ \partial_z\end{bmatrix} \times \begin{bmatrix}e^{x+\sin y}\\ e^{x+\sin y}\cos y\\ 0\end{bmatrix} = \begin{bmatrix}0-0\\ 0-0\\ e^{x+\sin y}\cos y - e^{x+\sin y}\cos y\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}.$$

A potenciálfüggvény:

$$\begin{aligned}\psi(\boldsymbol{r}) &= \int_0^x w_x(\xi; y; z)\,\mathrm{d}\xi + \int_0^y w_y(0; \eta; z)\,\mathrm{d}\eta + \int_0^z w_z(0; 0; \zeta)\,\mathrm{d}\zeta \\ &= \int_0^x e^{\xi+\sin y}\,\mathrm{d}\xi + \int_0^y e^{\sin\eta}\cos\eta\,\mathrm{d}\eta + \int_0^z 0\,\mathrm{d}\zeta \\ &= (e^x - 1)e^{\sin y} + e^{\sin y} - 1 + 0 + C.\end{aligned}$$

A keresett potenciálfüggvény:

$$\Psi(\boldsymbol{r}) = e^{x+\sin y}.$$

Egy vektormező vektorpotenciálos, ha divergenciája zérus:

$$\operatorname{div}\boldsymbol{w} = \frac{\partial \boldsymbol{w}}{\partial x} + \frac{\partial \boldsymbol{w}}{\partial y} + \frac{\partial \boldsymbol{w}}{\partial z} = e^{x+\sin y} + e^{x+\sin y}(\cos^2 y - \sin y) \neq 0.$$

Mivel $\operatorname{div}\boldsymbol{w} \neq 0$, ezért nem létezik $\boldsymbol{w}$-nek vektorpotenciálja.

c) $\boldsymbol{u}(\boldsymbol{r}) = (2zx^3)\,\hat{\boldsymbol{\imath}} + (3z)\,\hat{\boldsymbol{\jmath}} + (-3x^2z^2)\,\hat{\boldsymbol{k}}$

- Egy vektormező skalárpotenciálos, ha rotációja zérus.

$$\operatorname{rot}\boldsymbol{u} = \begin{bmatrix}\partial_x\\ \partial_y\\ \partial_z\end{bmatrix} \times \begin{bmatrix}2zx^3\\ 3z\\ -3x^2z^2\end{bmatrix} = \begin{bmatrix}0-3\\ 2x^3+6xz^2\\ 0-0\end{bmatrix} \neq \boldsymbol{0}$$

Mivel $\operatorname{rot}\boldsymbol{u} \neq \boldsymbol{0}$, ezért $\boldsymbol{u}$-nak nem létezik skalárpotenciálja.

- Egy vektormező vektorpotenciálos, ha divergenciája zérus.

$$\operatorname{div}\boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial x} + \frac{\partial \boldsymbol{u}}{\partial y} + \frac{\partial \boldsymbol{u}}{\partial z} = 6x^2z + 0 - 6x^2z = 0.$$

A potenciálfüggvény:

$$\begin{aligned}U_x(\boldsymbol{r}) &= \int_0^z u_y(x; y; \zeta)\,\mathrm{d}\zeta = \int_0^z (3\zeta)\,\mathrm{d}\zeta = \frac{3z^2}{2} + C_x, \\ U_y(\boldsymbol{r}) &= \int_0^x u_z(\xi; y; 0)\,\mathrm{d}\xi - \int_0^z u_x(x; y; \zeta)\,\mathrm{d}\zeta \\ &= \int_0^x (0)\,\mathrm{d}\xi - \int_0^z (2\zeta x^3)\,\mathrm{d}\zeta = 0 - x^3z^2 + C_y.\end{aligned}$$

A keresett vektorpotenciál:

$$\boldsymbol{U}(\boldsymbol{r}) = (\frac{3z^2}{2})\,\hat{\boldsymbol{\imath}} + (-x^3z^2)\,\hat{\boldsymbol{\jmath}} + (0)\,\hat{\boldsymbol{k}}.$$