

definitionhidealllines=true,leftline=true,linewidth=3pt,linecolor=primaryColor,frametilerule=true,frametitlebackgroundcolor=primaryColor,backgroundcolor=gray!10, frametitleaboveskip=2mm, frametitlebelowskip=2mm, innertopmargin=3mm,

definitionsection

theoremhidealllines=true,leftline=true,linewidth=3pt,linecolor=secondaryColor,frametilerule=true,frametitlebackgroundcolor=secondaryColor,backgroundcolor=gray!10, frametitleaboveskip=2mm, frametitlebelowskip=2mm, innertopmargin=3mm,

theoremsection

blueBoxhidealllines=true,leftline=true,backgroundcolor=cyan!10,linecolor=secondaryColor,linewidth=3pt, nertopmargin=.66em,innerbottommargin=.66em,

notehidealllines=true,leftline=true,backgroundcolor=yellow!10,linecolor=ternaryColor,linewidth=3pt, nertopmargin=.66em,innerbottommargin=.66em,

statementhidealllines=true,leftline=true,backgroundcolor=primaryColor!10,linecolor=primaryColor,linewidth=3pt,innertopmargin=.66em,innerbottommargin=.66em,singleextra=let 1=(P), 2=(O) in ((2,0)+0.5*(0,1)) node[rectangle, fill=primaryColor!10, draw=primaryColor, line width=2pt, overlay,] primaryColor!;

learnMoreTitle==Kiteintő calc,arrows,backgrounds excursus arrow/.style=line width=2pt, draw=secondaryColor, rounded corners=1ex, , excursus head/.style= font=, **anchor=base west, text=secondaryColor, inner sep=1.5ex, inner ysep=1ex,** ,

learnMoresingleextra=let 1=(P), 2=(O) in (2,1) coordinate (Q); let 1=(Q), 2=(O) in (1,2) coordinate (BL); let 1=(Q), 2=(P) in (2,1) coordinate (TR); [excursus head] (A) at ((Q) + (2.5em,0)) ; [excursus arrow, line width=2pt] ((BL) + (1pt,0)) |- ((Q) + (2em,0)); [excursus arrow, line width=2pt, fill=gray!10, -to] ((Q) + (1em,0)) -| (A.north west) -| (A.base east) - (TR) ; [excursus head] (A) at ((Q) + (2.5em,0)) ; , backgroundcolor=gray!10, middlelinewidth=0, hidealllines=true,topline=true, innertopmargin=2.5ex, innerbottommargin=1.5ex, innerrightmargin=2ex, innerleftmargin=2ex, skipabove=0.5nobreak=true,

examplehidealllines=true, leftline=true, backgroundcolor=magenta!10, linecolor=magenta!60!black, linewidth=3pt, innertopmargin=.66em, innerbottommargin=.66em,

Többszörös analízis BMETE94BG02 13

ft bg bg,main,ft

Matematika G2

Integrálás I

Utoljára frissítve: 2025. április 22.

0.1 Elméleti Áttekintő

[style=blueBox, nobreak=true,] **Integrálás téglatartományon:**

Téglatartomány esetén az integrálás sorrendje tetszőleges.

```
[thick, scale=5/4] [draw =
primaryColor, -to] (-0.35,0) –
(3.35,0) node[below]  $x$ ; [draw =
primaryColor, -to] (0,-0.35) –
(0,2.35) node[left]  $y$ ;
[ draw=secondaryColor, pattern =
north east lines, pattern color =
secondaryColor, ] (.75,.5)
rectangle (2.75,1.75);
[draw=ternaryColor, dashed]
(.75,.5) – (0,.5) node[left]  $y_1$ 
(.75,1.75) – (0,1.75) node[left]  $y_2$ 
(.75,.5) – (.75,0) node[below]  $x_1$ 
(2.75,.5) – (2.75,0) node[below]  $x_2$ 
;
```

$$I = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x; y) y x = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x; y) x y$$

[style=blueBox, nobreak=true,] **Integrálás normáltartományon:**

$[thick, scale=5/4] [draw =$
 $primaryColor, -to] (-0.35,0) -$
 $(3.35,0) node[below] x; [draw =$
 $primaryColor, -to] (0,-0.35) -$
 $(0,2.35) node[left] y;$
 $[draw=ternaryColor, dashed]$
 $(.75,2) - (.75,0) node[below] x_1$
 $(2.75,2) - (2.75,0) node[below] x_2 ;$
 $[draw = secondaryColor, name$
 $path = f1] plot[domain=0.75:2.75,$
 $smooth, samples=150] (,$
 $0.25+0.125*\sin(540*))+*0.125)$
 $node[right] g_1(x) ;$
 $[draw = secondaryColor,name$
 $path = f2] plot[domain=0.75:2.75,$
 $smooth, samples=150] (,$
 $1.75+0.125*\sin(540*))-*0.125)$
 $node[right] g_2(x) ;$
 $[of=f1 and f2, on layer = bg,]$
 $pattern = north east lines, pattern$
 $color = secondaryColor,$

$$I = \int_{x_1}^{x_2} \int_{g_1(x)}^{g_2(x)} f(x; y) y x = \int_{y_1}^{y_2} \int_{g_1^{-1}(y)}^{g_2^{-1}(y)} f(x; y) x y$$

$[thick, scale=5/4] [draw =$
 $primaryColor, -to] (-0.35,0) -$
 $(3.35,0) node[below] x; [draw =$
 $primaryColor, -to] (0,-0.35) -$
 $(0,2.35) node[left] y;$
 $[draw=ternaryColor, dashed]$
 $(3,.5) - (0,.5) node[left] y_1 (3,1.75)$
 $- (0,1.75) node[left] y_2 ;$
 $[draw = secondaryColor, name$
 $path = f1] plot[domain=1.75:0.5,$
 $smooth, samples=150]$
 $(0.75+0.125*\sin(540*))+*0.125,) ;$
 $[above] at (1,1.75) g_1(y);$
 $[draw = secondaryColor,name$
 $path = f2] plot[domain=0.5:1.75,$
 $smooth, samples=150]$
 $(2.75+0.125*\sin(540*))-*0.125,)$
 $node[above] g_2(y) ;$
 $[of=f1 and f2, on layer = bg,]$
 $pattern = north east lines, pattern$
 $color = secondaryColor,$

$$I = \int_{y_1}^{y_2} \int_{g_1(y)}^{g_2(y)} f(x; y) x y = \int_{x_1}^{x_2} \int_{g_1^{-1}(x)}^{g_2^{-1}(x)} f(x; y) y x$$

0.2 Feladatok

1. Számolja ki az alábbi függvények integrálját a megadott téglá tartományokon!

a) $f(x; y) = 2x^2 + 3xy + 4y^2$ $1 \leq x \leq 2$ $0 \leq y \leq 3$

b) $g(x; y) = xy \sin(x^2 + y^2)$ $0 \leq x \leq \pi/2$ $0 \leq y \leq \pi/2$

c) $h(x; y) = y \cos(2xy)$ $1 \leq x \leq 2$ $1 \leq y \leq 3$

2. Határozza meg az alábbi hármasintegrált! $\int_1^2 \int_0^3 \int_0^1 z x \sqrt{x^2 + y} xy z$

3. Határozza meg az alábbi integrálokat, majd írja fel az integrációs határokat, ha először az x , majd a y változó szerint integrálnánk! 2

a) $\int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + y^2) y x$

b) $\int_1^3 \int_0^{1/x} (2y + x + 2) y x$

4. Határozza meg az alábbi felületi integrálok értékét!

a) $T x^2 + y^2 T$

$$T = \{(x; y) \mid 0 \leq y \leq 1 \wedge y \leq x \leq 3 - y\}$$

```
[thick] [draw =
primaryColor, -to]
(-0.35,0) – (3.35,0)
node[below] x; [draw =
primaryColor, -to]
(0,-0.85) – (0,1.85)
node[left] y;
[draw=ternaryColor,
dashed] (1,1) – (0,1)
node[left] 1 (1,1) – (1,0)
node[below] 1 (2,1) – (2,0)
node[below] 2 ; [below] at
(3,0) 3; [below right] at
(0,0) 0;
[ draw = secondaryColor,
pattern = north east
lines, pattern color =
secondaryColor, ] (0,0) –
(1,1) – (2,1) – (3,0) –
cycle;
[ fill = white, fill
opacity=.5, text
opacity=1 ] at (1.5,0.5) T;
```

b) $_{Txy}T$

$$T = \{(x; y) \mid x^2 + y^2 \leq R^2 \wedge x; y \geq 0\}$$

c) $_{Tye^{(x-1)^2}}T$

$$T = \{(x; y) \mid x \leq 1 + y^2 \wedge y \geq 0 \wedge x \leq 5\}$$

```
[thick] [draw =
primaryColor, -to]
(-1.85,0) - (1.85,0)
node[below] x; [draw =
primaryColor, -to]
(0,-1.35) - (0,1.35)
node[left] y;
[ternaryColor] (0,0) circle
(1);
[ draw = secondaryColor,
pattern = north east
lines, pattern color =
secondaryColor, ] (0,0) -
(1,0) arc (0:90:1) - cycle;
[ fill = white, fill
opacity=.5, text
opacity=1 ] at (0.4,0.4) T;
[above right=-.5mm] at
(1,0) R; [above
right=-.5mm] at (0,1) R;
[thick] [draw =
primaryColor, -to]
(-0.85,0) - (2.85,0)
node[below] x; [draw =
primaryColor, -to]
(0,-0.85) - (0,1.85)
node[left] y;
[ draw = secondaryColor,
pattern = north east
lines, pattern color =
secondaryColor,
scale=1/2, ]
plot[domain=0:2, smooth,
samples=150] (1+*, )
coordinate(T) |- (0,0);
[ fill = white, fill
opacity=.5, text
opacity=1 ] at (1.7,0.35)
T;
[below] at (2.5,0) 5;
[below] at (0.5,0) 1;
[draw=ternaryColor,
dashed] (T) - ++(-2.5,0)
node[left] 2;
```

5. Adja meg az integrációs intervallumokat, ha az alábbi felületeken kell integrálni:
- a) R sugarú körfelület,
 - b) ha az $x = 0$, $y = x^2$ és $y = 2 - x$ görbék által határolt felület!
6. Adja meg a $f(x; y) = xy$ függvény a $P_1(1; 1)$, $P_2(4; 5)$ és $P_3(4; 2)$ pontok által meghatározott háromszög terület fölötti integrálját!