ME 495 Embedded Systems Homework 2

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1 Turtle Trajectories

1.1 The Kinematics

For a differential-drive robot, the control inputs are linear velocity v and angular velocity ω. The kinematic equations are:

$$\begin{aligned}
\dot{x} &= v\cos\theta \\
\dot{y} &= v\sin\theta \\
\dot{\theta} &= \omega
\end{aligned}$$

1.2 Differential Flatness

Taking advantage of the differential flatness of the system, we solve for the control inputs of v and ω. First, we solve the kinematic equations for $\theta = f(\dot{x}, \dot{y})$ by looking at $\frac{\dot{y}}{\dot{x}}$.

$$\begin{aligned}
\frac{\dot{y}}{\dot{x}} &= \frac{v\sin\theta}{v\cos\theta} \\
&= \tan\theta \\
\theta &= \tan^{-1}\frac{\dot{y}}{\dot{x}}
\end{aligned}$$

Next, we want to find $\omega = \dot{\theta} = f'(\dot{x}, \dot{y})$.

$$\begin{aligned}
\frac{d\theta}{dt} &= \frac{d}{dt}\left(\tan^{-1}\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) \\
&= \frac{1}{1+\frac{\frac{dy}{dt}}{\frac{dx}{dt}}^2}\left(\frac{d}{dt}\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) \\
&= \frac{1}{1+\frac{\frac{dy}{dt}}{\frac{dx}{dt}}^2}\left(\frac{\frac{d^2y}{dt^2}\frac{dx}{dt}-\frac{d^2x}{dt}\frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^2}\right) \\
&= \frac{\frac{d^2y}{dt}\frac{dx}{dt}-\frac{d^2x}{dt}\frac{dy}{dt}}{\left(\frac{dy}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2} \\
\omega &= \frac{\ddot{y}\dot{x}-\ddot{x}\dot{y}}{\dot{y}^2+\dot{x}^2}
\end{aligned}$$

Lastly, we want to solve for $v = g(\dot{x}, \dot{y}, \ddot{x}, \ddot{y})$ using the kinematic equations:

$$
\begin{aligned}
\dot{x}^2 + \dot{y}^2 &= (v\cos\theta)^2 + (v\sin\theta)^2 \\
&= v^2\cos^2\theta + v^2\sin^2\theta \\
&= v^2\left(\cos^2\theta + \sin^2\theta\right) \\
v &= \sqrt{\dot{x}^2 + \dot{y}^2}
\end{aligned}
$$

Now, we have solved for the controls of ω and v in terms of the vehicle's (x, y) position and derivatives. Since the robot follows a trajectory when $x(t) = x_d(t)$ and $y(t) = y_d(t)$, we take the necessary derivatives of:

$$
\begin{aligned}
x_d(t) &= \frac{W}{2}\sin\left(\frac{2\pi t}{T}\right) \\
y_d(t) &= \frac{H}{2}\sin\left(\frac{4\pi t}{T}\right)
\end{aligned}
$$

Which give us:

$$
\begin{aligned}
\dot{x}_d(t) &= \frac{W\pi}{T}\cos\left(\frac{2\pi t}{T}\right) \\
\dot{y}_d(t) &= \frac{2H\pi}{T}\cos\left(\frac{4\pi t}{T}\right) \\
\ddot{x}_d(t) &= -\frac{2W\pi^2}{T^2}\sin\left(\frac{2\pi t}{T}\right) \\
\ddot{y}_d(t) &= -\frac{8H\pi^2}{T^2}\sin\left(\frac{4\pi t}{T}\right)
\end{aligned}
$$

Once we get the derivatives, we substitute $\dot{x}_d$ and $\dot{y}_d$ into the expressions for v and ω.

2 Xacro ARM

For the two-link revolute joint arm, we want the end-effector to track the following trajectory:

$$
\begin{aligned}
x(t) &= 0.9\cos\left(\frac{2\pi t}{T}\right)\sqrt{(L_1 + L_2)^2 - h^2} \\
y(t) &= \frac{2}{3}(L_1 + L_2)
\end{aligned}
$$

Where $h = \frac{2}{3}(L_1 + L_2)$ so the trajectory is always achievable. We use inverse kinematics to solve for the joint values. The following equations have been taken from Lynch, Park, *Modern Robotics* Chapter 6.

$$
\begin{aligned}
\cos\theta_2 &= \frac{x^2 + y^2 - L_1^2 - L_2^2}{2L_1L_2} \\
&= D \\
\theta_2 &= \operatorname{atan2}\left(\sqrt{1 - D^2}, D\right) \\
\theta_1 &= \operatorname{atan2}(y, x) - \operatorname{atan2}(L_2\sin\theta_2, L_1 + L_2\cos\theta_2)
\end{aligned}
$$

These values for θ_1 and θ_2 are then published to the `joint_states` topic.