第 2 次习题课解答

第 1 部分 课堂内容回顾

1. 高阶偏导数

(1) **二阶偏导数可交换次序的充分条件:**

设 $\Omega \subset \mathbb{R}^n$ 为开集. 若 $f:\Omega \to \mathbb{R}$ 在 Ω 上有二阶偏导函数 $\frac{\partial^2 f}{\partial x_j \partial x_i}$, $\frac{\partial^2 f}{\partial x_i \partial x_j}$, 且当中一个在点 $X_0 \in \Omega$ 连续, 则 $\frac{\partial^2 f}{\partial x_j \partial x_i}(X_0) = \frac{\partial^2 f}{\partial x_i \partial x_j}(X_0)$.

(2) 设 $\Omega \subset \mathbb{R}^n$ 为开集, 而 $k \geqslant 0$ 为整数. 记 $\mathscr{C}^{(k)}(\Omega)$ 为 Ω 上具有 k 阶连续偏导数的所有函数的集合.

(3) 设 $k \geqslant 2$ 为整数. 若 $f \in \mathscr{C}^{(k)}(\Omega)$, 则对任意整数 r $(1 \leqslant r \leqslant k)$, 均有 $f \in \mathscr{C}^{(r)}(\Omega)$ 并且 f 的任意 r 阶偏导数均与求偏导的次序无关.

2. 向量值函数的微分

(1) **定义:** 向量值函数的微分, Jacobi 矩阵, Jacobi 行列式.

(2) **向量值函数微分的性质:** 微分的唯一性, 可微性蕴含连续性.

(3) **微分的链式法则 (矩阵表示):**

$$\begin{aligned}
\mathrm{d}(\vec{f} \circ \vec{g})(X_0) &= \mathrm{d}\vec{f}(\vec{g}(X_0)) \circ \mathrm{d}\vec{g}(X_0), \\
J_{\vec{f}\circ\vec{g}}(X_0) &= J_{\vec{f}}(\vec{g}(X_0)) \cdot J_{\vec{g}}(X_0), \\
\frac{\partial f_i(g_1, \ldots, g_m)}{\partial x_j} &= \frac{\partial f_i}{\partial y_1}(*)\frac{\partial g_1}{\partial x_j} + \frac{\partial f_i}{\partial y_2}(*)\frac{\partial g_2}{\partial x_j} + \cdots + \frac{\partial f_i}{\partial y_m}(*)\frac{\partial g_m}{\partial x_j}.
\end{aligned}$$

3. 隐函数定理、反函数定理及其应用

(1) **隐函数定理:**

(a) **两个变量的方程:** 设函数 $F(x,y)$ 为 $\mathscr{C}^{(1)}$ 类使得

$$F(x_0,y_0)=0,\ \frac{\partial F}{\partial y}(x_0,y_0) \neq 0.$$

则方程 $F(x,y)=0$ 在局部上有 $\mathscr{C}^{(1)}$ 类的解 $y=f(x)$, 并且

$$f'(x) = -\frac{\frac{\partial F}{\partial x}(x,f(x))}{\frac{\partial F}{\partial y}(x,f(x))}.$$

(b) **多个变量的方程:** 设函数 $F(x_1,x_2,\ldots,x_n,y)$ 为 $\mathscr{C}^{(1)}$ 类使得

$$F(X_0,y_0)=0,\ \frac{\partial F}{\partial y}(X_0,y_0) \neq 0.$$

则方程 $F(x_1,x_2,\ldots,x_n,y)=0$ 在局部上有 $\mathscr{C}^{(1)}$ 类解 $y=f(x_1,x_2,\ldots,x_n)$, 并且

$$\frac{\partial f}{\partial x_i}(X) = -\frac{\frac{\partial F}{\partial x_i}(X,f(X))}{\frac{\partial F}{\partial y}(X,f(X))}.$$

(c) **多个变量的方程组:** 设 $F_i(x_1,\dots,x_n,y_1,\dots,y_m)$ $(1\leqslant i\leqslant m)$ 为 $\mathscr{C}^{(1)}$ 类使得 $F_i(X_0,Y_0)=0$ $(1\leqslant i\leqslant m)$, $\frac{D(F_1,\dots,F_m)}{D(y_1,\dots,y_m)}(X_0,Y_0)\neq 0$. 则方程组

$$F_i(x_1,\dots,x_n,y_1,\dots,y_m)=0\ (1\leqslant i\leqslant m)$$

在局部上有 $\mathscr{C}^{(1)}$ 类解 $y_i=f_i(x_1,x_2,\dots,x_n)$ $(1\leqslant i\leqslant m)$, 且

$$J_{\vec f}(X)=-\left(\frac{\partial(F_1,\dots,F_m)}{\partial(y_1,\dots,y_m)}(X,\vec f(X))\right)^{-1}\cdot\frac{\partial(F_1,\dots,F_m)}{\partial(x_1,\dots,x_n)}(X,\vec f(X)).$$

(2) **反函数定理:** 设 $X=\vec g(Y)$ 为 $\mathscr{C}^{(1)}$ 类使得 $X_0=\vec g(Y_0)$ 且 $J_{\vec g}(Y_0)$ 可逆. 则局部上存在 $\mathscr{C}^{(1)}$ 反函数 $Y=\vec f(X)$, 并且 $J_{\vec f}(X)=\Big(J_{\vec g}(\vec f(X))\Big)^{-1}$, 也即

$$\frac{\partial(f_1,f_2,\dots,f_n)}{\partial(x_1,x_2,\dots,x_n)}(X)=\left(\frac{\partial(g_1,g_2,\dots,g_n)}{\partial(y_1,y_2,\dots,y_n)}(\vec f(X))\right)^{-1}.$$

4. 空间曲面的切平面与法线

(1) **曲面 $S: z=f(x,y)$ 在点 (x_0,y_0,z_0) 的切平面方程:**

$$z-z_0=\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)(y-y_0).$$

相应的法线方程为

$$\frac{x-x_0}{\frac{\partial f}{\partial x}(x_0,y_0)}=\frac{y-y_0}{\frac{\partial f}{\partial y}(x_0,y_0)}=\frac{z-z_0}{-1}.$$

(2) **曲面 $S:\begin{cases}x=f_1(u,v)\\ y=f_2(u,v)\\ z=f_3(u,v)\end{cases}$ 在参数 (u_0,v_0) 所对应点处的切平面方程为:**

$$\begin{pmatrix}x-x_0\\ y-y_0\\ z-z_0\end{pmatrix}=\frac{\partial(f_1,f_2,f_3)}{\partial(u,v)}(u_0,v_0)\begin{pmatrix}u-u_0\\ v-v_0\end{pmatrix},$$

该切平面也可以表示成:

$$\frac{D(f_2,f_3)}{D(u,v)}(u_0,v_0)(x-x_0)+\frac{D(f_3,f_1)}{D(u,v)}(u_0,v_0)(y-y_0)+\frac{D(f_1,f_2)}{D(u,v)}(u_0,v_0)(z-z_0)=0.$$

相应的法线方程为

$$\frac{x-x_0}{\frac{D(f_2,f_3)}{D(u,v)}(u_0,v_0)}=\frac{y-y_0}{\frac{D(f_3,f_1)}{D(u,v)}(u_0,v_0)}=\frac{z-z_0}{\frac{D(f_1,f_2)}{D(u,v)}(u_0,v_0)}.$$

(3) **曲面 $S: F(x,y,z)=0$ 在点 P_0 处的切平面方程为:**

$$\frac{\partial F}{\partial x}(P_0)(x-x_0)+\frac{\partial F}{\partial y}(P_0)(y-y_0)+\frac{\partial F}{\partial z}(P_0)(z-z_0)=0.$$

相应的的法线方程为

$$\frac{x-x_0}{\frac{\partial F}{\partial x}(P_0)}=\frac{y-y_0}{\frac{\partial F}{\partial y}(P_0)}=\frac{z-z_0}{\frac{\partial F}{\partial z}(P_0)}.$$

5. 空间曲线及切线和法平面

(1) **空间曲线 Γ 的参数表示:** $\begin{cases} x = x(t), \\ y = y(t), \quad t \in [\alpha, \beta]. \\ z = z(t), \end{cases}$ 若上述函数在点 $t = t_0$ 处可微, 则称曲线 Γ 在相应点 $P_0(x_0, y_0, z_0)$ 处可微, 相应的切线方程为

$$\begin{cases} x - x_0 = x'(t_0)(t - t_0), \\ y - y_0 = y'(t_0)(t - t_0), \\ z - z_0 = z'(t_0)(t - t_0). \end{cases}$$

该切线方程也可表述成

$$\frac{x - x_0}{x'(t_0)} = \frac{y - y_0}{y'(t_0)} = \frac{z - z_0}{z'(t_0)},$$

这里假设 $(x'(t_0), y'(t_0), z'(t_0))^T$ 不为零向量. 我们将过点 P_0 且与上述切线垂直的平面称为 Γ 在点 P_0 处的法平面, 其方程为

$$x'(t_0)(x - x_0) + y'(t_0)(y - y_0) + z'(t_0)(z - z_0) = 0.$$

(2) **空间曲线 Γ 的隐函数表示:** $\begin{cases} F_1(x, y, z) = 0, \\ F_2(x, y, z) = 0. \end{cases}$ 设 F_1, F_2 在点 $P_0(x_0, y_0, z_0)$ 可微且 $\mathrm{grad}F_1(P_0)$, $\mathrm{grad}F_2(P_0)$ 不为零, 则曲线 Γ 在该点的切线为

$$\begin{cases} \frac{\partial F_1}{\partial x}(P_0)(x - x_0) + \frac{\partial F_1}{\partial y}(P_0)(y - y_0) + \frac{\partial F_1}{\partial z}(P_0)(z - z_0) = 0, \\ \frac{\partial F_2}{\partial x}(P_0)(x - x_0) + \frac{\partial F_2}{\partial y}(P_0)(y - y_0) + \frac{\partial F_2}{\partial z}(P_0)(z - z_0) = 0. \end{cases}$$

该切线的方向为

$$\vec{T} = \mathrm{grad}F_1(P_0) \times \mathrm{grad}F_2(P_0) = \begin{pmatrix} \frac{D(F_1, F_2)}{D(y, z)}(P_0) \\ \frac{D(F_1, F_2)}{D(z, x)}(P_0) \\ \frac{D(F_1, F_2)}{D(x, y)}(P_0) \end{pmatrix}.$$

只有当 $\vec{T} \neq \vec{0}$ 时, 上述方程组才能定义一条直线. 此时 Jacobi 矩阵 $\frac{\partial(F_1, F_2)}{\partial(x, y, z)}(P_0)$ 的秩等于 2. 借助 $\vec{T}$, 我们也可以得到上述切线方程的另外一个表述:

$$\frac{x - x_0}{\frac{D(F_1, F_2)}{D(y, z)}(P_0)} = \frac{y - y_0}{\frac{D(F_1, F_2)}{D(z, x)}(P_0)} = \frac{z - z_0}{\frac{D(F_1, F_2)}{D(x, y)}(P_0)}.$$

6. Taylor 公式

设 $X_0 \in \mathbb{R}^n$, $r > 0$, $f \in \mathscr{C}^{(2)}(B(X_0, r))$, 而 $X \in B(X_0, r)$.

(1) **一阶带 Lagrange 余项的 Taylor 公式:**

$$\begin{aligned} f(X) &= f(X_0) + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(X_0)(x_j - x_j^{(0)}) + \frac{1}{2!} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(X_\theta)(x_i - x_i^{(0)})(x_j - x_j^{(0)}) \\ &= f(X_0) + J_f(X_0)\, \Delta X + \frac{1}{2!}(\Delta X)^T H_f(X_\theta)\, \Delta X, \end{aligned}$$

其中 $\Delta X = X - X_0$, $H_f(X_\theta) = (\frac{\partial^2 f}{\partial x_i \partial x_j}(X_\theta))_{1 \leqslant i,j \leqslant n}$, $X_\theta = X_0 + \theta(X - X_0)$, $\theta \in (0, 1)$.

(2) **带 Lagrange 余项的一般 Taylor 展式:** 若 f 为 $\mathscr{C}^{(m+1)}(B(X_0,r)$ 类, 则

$$f(X)=\sum_{k=0}^{m}\frac{1}{k!}\Big(\sum_{j=1}^{n}(x_j-x_j^{(0)})\frac{\partial}{\partial x_j}\Big)^k f(X_0)+\frac{1}{(m+1)!}\Big(\sum_{j=1}^{n}(x_j-x_j^{(0)})\frac{\partial}{\partial x_j}\Big)^{m+1}f(X_\theta).$$

(3) **带 Peano 余项的二阶 Taylor 展式:** 当 $X\to X_0$ 时, 我们有

$$f(X)=f(X_0)+J_f(X_0)\,\Delta X+\frac{1}{2!}(\Delta X)^T H_f(X_0)\Delta X+o(\|\Delta X\|^2),$$

其中 $\Delta X=X-X_0$.

第 2 部分 题目解答

1. (微分形式的不变性) 设 $z = f(u,v)$, $u = u(x,y)$, $v = v(x,y)$ 均为连续可微函数. 将 z 看成是 x,y 的函数. 求证:

$$\mathrm{d}z = \frac{\partial z}{\partial x}\,\mathrm{d}x + \frac{\partial z}{\partial y}\,\mathrm{d}y = \frac{\partial f}{\partial u}\,\mathrm{d}u + \frac{\partial f}{\partial v}\,\mathrm{d}v.$$

证明: 由复合求导法则可知

$$\begin{aligned}
\mathrm{d}z &= \frac{\partial z}{\partial x}\,\mathrm{d}x + \frac{\partial z}{\partial y}\,\mathrm{d}y = \big(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\big)\,\mathrm{d}x + \big(\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}\big)\,\mathrm{d}y \\
&= \frac{\partial f}{\partial u}\big(\frac{\partial u}{\partial x}\,\mathrm{d}x + \frac{\partial u}{\partial y}\,\mathrm{d}y\big) + \frac{\partial f}{\partial v}\big(\frac{\partial v}{\partial x}\,\mathrm{d}x + \frac{\partial v}{\partial y}\,\mathrm{d}y\big) \\
&= \frac{\partial f}{\partial u}\,\mathrm{d}u + \frac{\partial f}{\partial v}\,\mathrm{d}v.
\end{aligned}$$

2. 设 $z = x^3 f\big(xy, \frac{y}{x}\big)$, 其中 f 为可微函数. 求 $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

解: 方法 1.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= 3x^2 f\big(xy, \frac{y}{x}\big) + x^3 \partial_1 f\big(xy, \frac{y}{x}\big)\cdot y + x^3 \partial_2 f\big(xy, \frac{y}{x}\big)\cdot\big(-\frac{y}{x^2}\big) \\
&= 3x^2 f\big(xy, \frac{y}{x}\big) + x^3 y \partial_1 f\big(xy, \frac{y}{x}\big) - xy\partial_2 f\big(xy, \frac{y}{x}\big), \\
\frac{\partial z}{\partial y} &= x^3 \partial_1 f\big(xy, \frac{y}{x}\big)\cdot x + x^3 \partial_2 f\big(xy, \frac{y}{x}\big)\cdot\big(\frac{1}{x}\big) \\
&= x^4 \partial_1 f\big(xy, \frac{y}{x}\big) + x^2 \partial_2 f\big(xy, \frac{y}{x}\big).
\end{aligned}$$

方法 2.

$$\begin{aligned}
\mathrm{d}z &= \mathrm{d}\Big(x^3 f\big(xy, \frac{y}{x}\big)\Big) = 3x^2 f\big(xy, \frac{y}{x}\big)\mathrm{d}x + x^3\mathrm{d}\Big(f\big(xy, \frac{y}{x}\big)\Big) \\
&= 3x^2 f\big(xy, \frac{y}{x}\big)\mathrm{d}x + x^3\Big(\partial_1 f\big(xy, \frac{y}{x}\big)\mathrm{d}(xy) + \partial_2 f\big(xy, \frac{y}{x}\big)\mathrm{d}\big(\frac{y}{x}\big)\Big) \\
&= 3x^2 f\big(xy, \frac{y}{x}\big)\mathrm{d}x + x^3\partial_1 f\big(xy, \frac{y}{x}\big)\big(y\mathrm{d}x + x\mathrm{d}y\big) \\
&\quad + x^3\partial_2 f\big(xy, \frac{y}{x}\big)\big(\frac{1}{x}\mathrm{d}y - \frac{y}{x^2}\mathrm{d}x\big) \\
&= \Big(3x^2 f\big(xy, \frac{y}{x}\big) + x^3 y\partial_1 f\big(xy, \frac{y}{x}\big) - xy\partial_2 f\big(xy, \frac{y}{x}\big)\Big)\mathrm{d}x \\
&\quad + \Big(x^4\partial_1 f\big(xy, \frac{y}{x}\big) + x^2\partial_2 f\big(xy, \frac{y}{x}\big)\Big)\mathrm{d}y.
\end{aligned}$$

由此立刻可得

$$\begin{aligned}
\frac{\partial z}{\partial x} &= 3x^2 f\big(xy, \frac{y}{x}\big) + x^3 y\partial_1 f\big(xy, \frac{y}{x}\big) - xy\partial_2 f\big(xy, \frac{y}{x}\big), \\
\frac{\partial z}{\partial y} &= x^4\partial_1 f\big(xy, \frac{y}{x}\big) + x^2\partial_2 f\big(xy, \frac{y}{x}\big).
\end{aligned}$$

3. 设函数 $z=f(x,y)$ 在点 (a,a) 处可微, 并且 $f(a,a)=a$,

$$\frac{\partial f}{\partial x}(a,a)=\frac{\partial f}{\partial y}(a,a)=b.$$

令 $\varphi(x)=\big(f(x,f(x,f(x,x)))\big)^2$. 求 $\varphi'(a)$.

解: 由题设可得

$$\begin{aligned}
&\varphi'(x)=2f(x,f(x,f(x,x)))\frac{\mathrm{d}f(x,f(x,f(x,x)))}{\mathrm{d}x}\\
&=2f(x,f(x,f(x,x)))\Big(\frac{\partial f}{\partial x}(x,f(x,f(x,x)))\\
&\quad+\frac{\partial f}{\partial y}(x,f(x,f(x,x)))\frac{\mathrm{d}f(x,f(x,x))}{\mathrm{d}x}\Big)\\
&=2f(x,f(x,f(x,x)))\Big(\frac{\partial f}{\partial x}(x,f(x,f(x,x)))\\
&\quad+\frac{\partial f}{\partial y}(x,f(x,f(x,x)))\big(\frac{\partial f}{\partial x}(x,f(x,x))\\
&\quad+\frac{\partial f}{\partial y}(x,f(x,x))\frac{\mathrm{d}f(x,x)}{\mathrm{d}x}\big)\Big)\\
&=2f(x,f(x,f(x,x)))\bigg(\frac{\partial f}{\partial x}(x,f(x,f(x,x)))\\
&\quad+\frac{\partial f}{\partial y}(x,f(x,f(x,x)))\cdot\Big(\frac{\partial f}{\partial x}(x,f(x,x))\\
&\quad+\frac{\partial f}{\partial y}(x,f(x,x))\big(\frac{\partial f}{\partial x}(x,x)+\frac{\partial f}{\partial y}(x,x)\frac{\mathrm{d}x}{\mathrm{d}x}\big)\Big)\bigg),
\end{aligned}$$

于是我们有 $\varphi'(a)=2a\big(b+b\big(b+b(b+b)\big)\big)=2ab(1+b+2b^2)$.

4. $\forall(x,y)\in\mathbb{R}^2$, 定义

$$f(x,y)=\begin{cases}(x^2+y^2)\sin\frac{1}{x^2+y^2}, & 若\ (x,y)\neq(0,0),\\ 0, & 若\ (x,y)=(0,0).\end{cases}$$

问 f 是否有二阶偏导数?

解: 由于 f 在 $\mathbb{R}^2\setminus\{(0,0)\}$ 上为初等函数, 故它在该集合上有二阶偏导数. 另外, $\forall(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$, 我们有

$$\frac{\partial f}{\partial x}(x,y)=2x\sin\frac{1}{x^2+y^2}-\frac{2x}{x^2+y^2}\cos\frac{1}{x^2+y^2}.$$

又由偏导数的定义可得 $\frac{\partial f}{\partial x}(0,0)=0$. 注意到极限

$$\lim_{x\to0}\frac{\frac{\partial f}{\partial x}(x,0)-\frac{\partial f}{\partial x}(0,0)}{x}=\lim_{x\to0}\Big(2\sin\frac{1}{x^2}-\frac{2}{x^2}\cos\frac{1}{x^2}\Big)$$

不存在, 因此 $\frac{\partial^2 f}{\partial x^2}(0,0)$ 不存在.

5. 设 $D=[0,a]\times[0,b]$, 而函数 $F:D\to\mathbb{R}$ 关于第二个变量的偏导数 $\frac{\partial F}{\partial y}$ 存在. 求证: 存在函数 $g:[0,a]\to\mathbb{R}$, $h:[0,b]\to\mathbb{R}$ 使得 $\forall(x,y)\in D$, 我们均有 $F(x,y)=g(x)+h(y)$ 当且仅当 $\forall(x,y)\in D$, 均有 $\frac{\partial^2 F}{\partial x\partial y}(x,y)=0$.

证明: 必要性. 假设存在两个 $g:[0,a]\to\mathbb{R}$, $h:[0,b]\to\mathbb{R}$ 使得 $\forall(x,y)\in D$, 均有 $F(x,y)=g(x)+h(y)$, 则 $\frac{\partial F}{\partial y}(x,y)=h'(y)$, 进而可得 $\frac{\partial^2 F}{\partial x\partial y}(x,y)=0$.

充分性. $\forall(x,y)\in D$, 定义 $\varphi(x,y)=F(x,y)-F(0,y)$. 由 Lagrange 中值定理可知, 存在 ξ 介于 $0,x$ 之间使得

$$\begin{aligned}\frac{\partial\varphi}{\partial y}(x,y) &= \frac{\partial F}{\partial y}(x,y)-\frac{\partial F}{\partial y}(0,y)\\ &= x\frac{\partial^2 F}{\partial x\partial y}(\xi,y)=0.\end{aligned}$$

同样由 Lagrange 中值定理可知, 存在 η 介于 $0,y$ 之间使得

$$\begin{aligned}\big(F(x,y)-F(0,y)\big)-\big(F(x,0)-F(0,0)\big) &= \varphi(x,y)-\varphi(x,0)\\ &= y\frac{\partial\varphi}{\partial y}(x,\eta)=0,\end{aligned}$$

于是 $F(x,y)=\big(F(x,0)-F(0,0)\big)+F(0,y)$. 故所证结论成立.

6. 假设 $D=[0,a]\times[0,b]$, 而 $u\in\mathscr{C}^{(2)}(D)$ 使得 $\forall(x,y)\in D$, 均有 $u(x,y)\neq 0$. 求证: 存在 $f:[0,a]\to\mathbb{R}$, $g:[0,b]\to\mathbb{R}$ 使 $\forall(x,y)\in D$, $u(x,y)=f(x)g(y)$ 当且仅当在 D 上, 成立 $\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}=u\frac{\partial^2 u}{\partial x\partial y}$.

证明: 由于 D 为连通集, 而 u 在 D 上连续且恒不为零, 则由连续函数介值定理可知 u 在 D 上恒为正或恒为负. 不失一般性, 我们可以假设 u 在 D 上恒为正, 否则可以考虑 $-u$.

必要性. 假设存在两个函数 $f:[0,a]\to\mathbb{R}$, $g:[0,b]\to\mathbb{R}$ 使得 $\forall(x,y)\in D$, 均有 $u(x,y)=f(x)g(y)$. 由于 $u\in\mathscr{C}^{(2)}(D)$, 则 f,g 可导且 $\forall(x,y)\in D$, 成立

$$\frac{\partial u}{\partial x}(x,y)\frac{\partial u}{\partial y}(x,y)=f'(x)g(y)f(x)g'(y)=u(x,y)\frac{\partial^2 u}{\partial x\partial y}(x,y).$$

充分性. 假设 $\forall(x,y)\in D$, 均有 $\frac{\partial u}{\partial x}(x,y)\frac{\partial u}{\partial y}(x,y)=u(x,y)\frac{\partial^2 u}{\partial x\partial y}(x,y)$. $\forall(x,y)\in D$, 定义 $F(x,y)=\log u(x,y)$. 则 $F\in\mathscr{C}^{(2)}(D)$ 且 $\forall(x,y)\in D$, 均有

$$\begin{aligned}\frac{\partial^2 F}{\partial x\partial y}(x,y) &= \frac{\partial}{\partial x}\big(\frac{1}{u(x,y)}\frac{\partial u}{\partial y}(x,y)\big)\\ &= \frac{1}{(u(x,y))^2}\Big(u(x,y)\frac{\partial^2 u}{\partial x\partial y}(x,y)-\frac{\partial u}{\partial x}(x,y)\frac{\partial u}{\partial y}(x,y)\Big)=0.\end{aligned}$$

故存在 $p:[0,a]\to\mathbb{R}$, $q:[0,b]\to\mathbb{R}$ 使得 $\forall(x,y)\in D$, $F(x,y)=p(x)+q(y)$. 此时令 $f(x)=e^{p(x)}$, $g(y)=e^{q(y)}$, 则 $u(x,y)=e^{F(x,y)}=f(x)g(y)$.

7. 设 $u(x,y)=\varphi(x+y)+\varphi(x-y)+\int_{x-y}^{x+y}\psi(t)\,\mathrm{d}t$, 其中 φ 为二阶可导, 而 ψ 为一阶可导. 求证: $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}$.

证明: 由题设可得

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \varphi'(x+y)+\varphi'(x-y)+\psi(x+y)-\psi(x-y),\\
\frac{\partial^2 u}{\partial x^2} &= \varphi''(x+y)+\varphi''(x-y)+\psi'(x+y)-\psi'(x-y),\\
\frac{\partial u}{\partial y} &= \varphi'(x+y)-\varphi'(x-y)+\psi(x+y)+\psi(x-y),\\
\frac{\partial^2 u}{\partial y^2} &= \varphi''(x+y)+\varphi''(x-y)+\psi'(x+y)-\psi'(x-y).
\end{aligned}$$

由此我们立刻可得 $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}$.

8. 假设 $f:(0,+\infty)\to\mathbb{R}$ 二阶可导且 $z=f(\sqrt{x^2+y^2})$ 满足 $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$.
(1) 验证 $f''(u)+\frac{f'(u)}{u}=0$; (2) 若 $f(1)=0$, $f'(1)=1$, 求 f 的表达式.

解: (1) 由题设可知 $\frac{\partial z}{\partial x}=\frac{xf'(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}$, 则

$$\frac{\partial^2 z}{\partial x^2}=\frac{f'(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}+\frac{x^2f''(\sqrt{x^2+y^2})}{x^2+y^2}-\frac{x^2f'(\sqrt{x^2+y^2})}{(x^2+y^2)^{\frac{3}{2}}}.$$

由对称性可得

$$\frac{\partial^2 z}{\partial y^2}=\frac{f'(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}+\frac{y^2f''(\sqrt{x^2+y^2})}{x^2+y^2}-\frac{y^2f'(\sqrt{x^2+y^2})}{(x^2+y^2)^{\frac{3}{2}}}.$$

于是我们有

$$\begin{aligned}
0 &= \frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{f'(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}+\frac{x^2f''(\sqrt{x^2+y^2})}{x^2+y^2}-\frac{x^2f'(\sqrt{x^2+y^2})}{(x^2+y^2)^{\frac{3}{2}}}\\
&\quad+\frac{f'(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}+\frac{y^2f''(\sqrt{x^2+y^2})}{x^2+y^2}-\frac{y^2f'(\sqrt{x^2+y^2})}{(x^2+y^2)^{\frac{3}{2}}}\\
&= \frac{f'(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}+f''(\sqrt{x^2+y^2}).
\end{aligned}$$

特别地, $\forall u>0$, 若取 $x=u$, $y=0$, 则我们有 $f''(u)+\frac{f'(u)}{u}=0$.

(2) 由一阶齐次线性常微分方程的通解公式可知 $f'(u)=\frac{C}{u}$, 其中 C 为任意的常数. 又 $f'(1)=1$, 故 $C=1$, 从而 $f'(u)=\frac{1}{u}$, 进而可得

$$f(u)=f(1)+\log u=\log u.$$

9. 设函数 $f(u)$ 为二阶可导且 $z=\frac{1}{x}f(xy)+yf(x+y)$, 求 $\frac{\partial^2 z}{\partial x\partial y}$.

解: 由题设可知

$$\begin{aligned}\frac{\partial z}{\partial y} &= f'(xy)+f(x+y)+yf'(x+y),\\ \frac{\partial^2 z}{\partial x\partial y} &= yf''(xy)+f'(x+y)+yf''(x+y).\end{aligned}$$

10. 设 $z=\arctan\frac{x+y}{x-y}$, 求 $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial x\partial y}$.

解: 由题设可知

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{1}{1+(\frac{x+y}{x-y})^2}\frac{\partial}{\partial x}\Big(\frac{x+y}{x-y}\Big)\\ &= \frac{1}{1+(\frac{x+y}{x-y})^2}\cdot\frac{(x-y)-(x+y)}{(x-y)^2}=-\frac{y}{x^2+y^2},\\ \frac{\partial z}{\partial y} &= \frac{1}{1+(\frac{x+y}{x-y})^2}\frac{\partial}{\partial y}\Big(\frac{x+y}{x-y}\Big)\\ &= \frac{1}{1+(\frac{x+y}{x-y})^2}\cdot\frac{(x-y)+(x+y)}{(x-y)^2}=\frac{x}{x^2+y^2},\\ \frac{\partial^2 z}{\partial x\partial y} &= \frac{(x^2+y^2)-x\cdot(2x)}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}.\end{aligned}$$

11. 设 $g(x)=f(x,\varphi(x^2,x^2))$, 其中 f,φ 均为二阶连续可导, 求 $g''(x)$.

解: 由题设可知

$$\begin{aligned}g'(x) &= \partial_1 f(x,\varphi(x^2,x^2))+\partial_2 f(x,\varphi(x^2,x^2))\frac{\partial\big(\varphi(x^2,x^2)\big)}{\partial x}\\ &= \partial_1 f(x,\varphi(x^2,x^2))+2x\partial_2 f(x,\varphi(x^2,x^2))\big(\partial_1\varphi(x^2,x^2)+\partial_2\varphi(x^2,x^2)\big),\end{aligned}$$

由此立刻可得

$$\begin{aligned}g''(x) &= \partial_{11} f(x,\varphi(x^2,x^2))+2x\partial_{21} f(x,\varphi(x^2,x^2))\big(\partial_1\varphi(x^2,x^2)+\partial_2\varphi(x^2,x^2)\big)\\ &\quad +2\partial_2 f(x,\varphi(x^2,x^2))\big(\partial_1\varphi(x^2,x^2)+\partial_2\varphi(x^2,x^2)\big)\\ &\quad +2x\partial_{12} f(x,\varphi(x^2,x^2))\big(\partial_1\varphi(x^2,x^2)+\partial_2\varphi(x^2,x^2)\big)\\ &\quad +4x^2\partial_{22} f(x,\varphi(x^2,x^2))\big(\partial_1\varphi(x^2,x^2)+\partial_2\varphi(x^2,x^2)\big)^2\\ &\quad +4x^2\partial_2 f(x,\varphi(x^2,x^2))\big(\partial_{11}\varphi(x^2,x^2)+2\partial_{21}\varphi(x^2,x^2)+\partial_{22}\varphi(x^2,x^2)\big).\end{aligned}$$

12. 设 $z=f\big(xy,\frac{x}{y}\big)$, 其中 f 为二阶连续可导, 求 $\frac{\partial^2 z}{\partial x^2}$.

解: 由题设可得

$$\frac{\partial z}{\partial x} = y\partial_1 f\big(xy,\frac{x}{y}\big)+\frac{1}{y}\partial_2 f\big(xy,\frac{x}{y}\big),$$

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= y^2\partial_{11}f(xy,\frac{x}{y})+\partial_{21}f(xy,\frac{x}{y})\\ &\quad+\partial_{12}f(xy,\frac{x}{y})+\frac{1}{y^2}\partial_{22}f(xy,\frac{x}{y})\\ &= y^2\partial_{11}f(xy,\frac{x}{y})+2\partial_{21}f(xy,\frac{x}{y})+\frac{1}{y^2}\partial_{22}f(xy,\frac{x}{y}).\end{aligned}$$

13. 设函数 $u(x,y)$ 为二阶连续可导且

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=0,\ u(x,2x)=x,\ u'_x(x,2x)=x^2,$$

求 $u''_{xx}(x,2x)$, $u''_{xy}(x,2x)$, $u''_{yy}(x,2x)$.

解: 将等式 $u(x,2x)=x$ 两边对 x 求导可得 $u'_x(x,2x)+2u'_y(x,2x)=1$. 于是我们有 $u'_y(x,2x)=\frac{1}{2}(1-x^2)$. 进而可得

$$u''_{xy}(x,2x)+2u''_{yy}(x,2x)=-x.$$

将等式 $u'_x(x,2x)=x^2$ 两边对 x 求导可得

$$u''_{xx}(x,2x)+2u''_{yx}(x,2x)=2x.$$

又 u 为二阶连续可导, 故 $u''_{xy}=u''_{yx}$. 由题设可知 $u''_{xx}=u''_{yy}$, 于是

$$u''_{xx}(x,2x)=u''_{yy}(x,2x)=-\frac{4}{3}x,\ u''_{xy}(x,2x)=u''_{yx}(x,2x)=\frac{5}{3}x.$$

14. 考虑三元方程 $xy-z\log y+e^{xz}=1$, 由隐函数定理, 存在点 $(0,1,1)$ 的某个邻域使得在此邻域内, 该方程 (D)

(A) 只能确定一个连续可导的隐函数 $z=z(x,y)$;

(B) 可确定两个连续可导的隐函数 $y=y(x,z)$ 和 $z=z(x,y)$;

(C) 可确定两个连续可导的隐函数 $x=x(y,z)$ 和 $z=z(x,y)$;

(D) 可确定两个连续可导的隐函数 $x=x(y,z)$ 和 $y=y(x,z)$.

解: $\forall(x,y,z)\in\mathbb{R}^3$, 定义 $F(x,y,z)=xy-z\log y+e^{xz}-1$. 则

$$\begin{aligned}\frac{\partial F}{\partial x}(0,1,1) &= (y+ze^{xz})\Big|_{(0,1,1)}=2,\\ \frac{\partial F}{\partial y}(0,1,1) &= \big(x-\frac{z}{y}\big)\Big|_{(0,1,1)}=-1,\\ \frac{\partial F}{\partial z}(0,1,1) &= (-\log y+xe^{xz})\Big|_{(0,1,1)}=0.\end{aligned}$$

于是由隐函数定理知, 由方程 $F(x,y,z)=0$ 在点 $(0,1,1)$ 的某个邻域内只能确定两个连续可导的隐函数 $x=x(y,z)$ 和 $y=y(x,z)$.

15. 通过曲面 S: $e^{xyz}+x-y+z=3$ 上的点 $(1,0,1)$ 的切平面 (B).
(A) 通过 y 轴; (B) 平行于 y 轴; (C) 垂直于 y 轴; (D) A, B, C 都不对.

解: 曲面在点 $(1,0,1)$ 的法向量为

$$\vec{n}=\begin{pmatrix} yze^{xyz}+1 \\ xze^{xyz}-1 \\ xye^{xyz}+1 \end{pmatrix}\Big|_{(1,0,1)}=\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

该向量与 y 轴垂直, 故曲面在点 $(1,0,1)$ 处的切平面与 y 轴平行, 其方程为

$$(x-1)+(z-1)=0,$$

也即 $x+z-2=0$, 故该切平面不经过 y轴.

16. 求证: 方程 $xyz+\sqrt{x^2+y^2+z^2}=\sqrt{2}$ 在点 $(1,0,-1)$ 的某个邻域内可确定一个隐函数 $z=z(x,y)$, 并在该点处求微分 $\mathrm{d}z$.

解: $\forall(x,y,z)\in\mathbb{R}^3\setminus\{(0,0,0\}$, 定义

$$F(x,y,z)=xyz+\sqrt{x^2+y^2+z^2}-\sqrt{2}.$$

则 F 为初等函数, 因此连续可导且我们有

$$\begin{aligned}
\frac{\partial F}{\partial x}(1,0,-1) &= \Big(yz+\frac{x}{\sqrt{x^2+y^2+z^2}}\Big)\Big|_{(1,0,-1)}=\frac{\sqrt{2}}{2}, \\
\frac{\partial F}{\partial y}(1,0,-1) &= \Big(xz+\frac{y}{\sqrt{x^2+y^2+z^2}}\Big)\Big|_{(1,0,-1)}=-1, \\
\frac{\partial F}{\partial z}(1,0,-1) &= \Big(xy+\frac{z}{\sqrt{x^2+y^2+z^2}}\Big)\Big|_{(1,0,-1)}=-\frac{\sqrt{2}}{2}\neq 0,
\end{aligned}$$

于是由隐函数定理知方程 $F(x,y,z)=0$ 在点 $(1,0,-1)$ 的某个邻域内可确定一个连续可导的隐函数 $z=z(x,y)$, 并且

$$\frac{\partial z}{\partial x}\Big|_{(1,0,-1)}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}\Big|_{(1,0,-1)}=1,\quad \frac{\partial z}{\partial y}\Big|_{(1,0,-1)}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}\Big|_{(1,0,-1)}=-\sqrt{2},$$

从而在点 $(1,0,-1)$ 处的所求微分为 $\mathrm{d}z=\mathrm{d}x-\sqrt{2}\,\mathrm{d}y$.

17. 假设由方程组 $\begin{cases} F(y-x,y-z)=0, \\ G(xy,\frac{z}{y})=0, \end{cases}$ 可确定隐函数 $x=x(y)$, $z=z(y)$, 其中 F,G 均为连续可导. 求 $\frac{\mathrm{d}x}{\mathrm{d}y}$, $\frac{\mathrm{d}z}{\mathrm{d}y}$.

解: 将方程组两边关于 y 求导可得

$$\partial_1F(y-x,y-z)\Big(1-\frac{\mathrm{d}x}{\mathrm{d}y}\Big)+\partial_2F(y-x,y-z)\Big(1-\frac{\mathrm{d}z}{\mathrm{d}y}\Big)=0,$$
$$\partial_1G\Big(xy,\frac{z}{y}\Big)\Big(y\frac{\mathrm{d}x}{\mathrm{d}y}+x\Big)+\partial_2G\big(xy,\frac{z}{y}\big)\Big(\frac{1}{y}\frac{\mathrm{d}z}{\mathrm{d}y}-\frac{z}{y^2}\Big)=0.$$

出于简便, 将 $\partial_1 F(y-x,y-z), \partial_2 F(y-x,y-z), \partial_1 G\big(xy,\frac{z}{y}\big), \partial_2 G\big(xy,\frac{z}{y}\big)$ 分别简记为 $\partial_1 F, \partial_2 F, \partial_1 G, \partial_2 G$, 则我们有

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}y} &= \frac{y\partial_1 F\partial_2 G + (y-z)\partial_2 F\partial_2 G + xy^2\partial_2 F\partial_1 G}{y(\partial_1 F\partial_2 G - y^2\partial_2 F\partial_1 G)},\\ \frac{\mathrm{d}z}{\mathrm{d}y} &= \frac{-(x+y)y^2\partial_1 F\partial_1 G + z\partial_1 F\partial_2 G - y^3\partial_2 F\partial_1 G}{y(\partial_1 F\partial_2 G - y^2\partial_2 F\partial_1 G)}.\end{aligned}$$

18. 若隐函数 $y=y(x)$ 由 $ax+by=f(x^2+y^2)$ 确定, 而 a,b 为常数. 求 $\frac{\mathrm{d}y}{\mathrm{d}x}$.

解: 将方程 $ax+by=f(x^2+y^2)$ 两边对 x 求导可得

$$a+by' = f'(x^2+y^2)\cdot(2x+2yy'),$$

于是我们有 $y' = \frac{a-2xf'(x^2+y^2)}{2yf'(x^2+y^2)-b}$.

19. 设 $x=x(z)$, $y=y(z)$ 由 $\begin{cases} x^2+y^2+z^2-1=0 \\ x^2+2y^2-z^2-1=0 \end{cases}$ 确定, 求 $\frac{\mathrm{d}x}{\mathrm{d}z}$, $\frac{\mathrm{d}y}{\mathrm{d}z}$.

解: 由于方程组确定了两个函数 $x=x(z)$, $y=y(z)$, 将方程组对 z 求导可得

$$2x\frac{\mathrm{d}x}{\mathrm{d}z} + 2y\frac{\mathrm{d}y}{\mathrm{d}z} + 2z = 0,\ 2x\frac{\mathrm{d}x}{\mathrm{d}z} + 4y\frac{\mathrm{d}y}{\mathrm{d}z} - 2z = 0.$$

由此立刻可得 $\frac{\mathrm{d}x}{\mathrm{d}z} = -\frac{3z}{x}$, $\frac{\mathrm{d}y}{\mathrm{d}z} = \frac{2z}{y}$.

20. 设 $z=z(x,y)$ 由方程 $x^2+y^2+z^2=a^2$ 确定, 求 $\frac{\partial^2 z}{\partial x\partial y}$.

解: 将方程 $x^2+y^2+z^2=a^2$ 两边关于 x,y 求偏导数可得

$$2x+2z\frac{\partial z}{\partial x} = 0,\ 2y+2z\frac{\partial z}{\partial y} = 0,$$

于是 $\frac{\partial z}{\partial x} = -\frac{x}{z}$, $\frac{\partial z}{\partial y} = -\frac{y}{z}$, 进而可得 $\frac{\partial^2 z}{\partial x\partial y} = \frac{y}{z^2}\frac{\partial z}{\partial x} = -\frac{xy}{z^3}$.

21. 求曲面 $S:\ 2x^2-2y^2+2z=1$ 上的所有点使过这些点的切平面与直线

$$L:\ \begin{cases} 3x-2y-z=5 \\ x+y+z=0 \end{cases}$$

平行.

解: 曲面 S 上的点 $P(x,y,z)$ 处的法向量为 $\vec{n} = \begin{pmatrix} 4x \\ -4y \\ 2 \end{pmatrix}$, 而 L 的方向为

$$\vec{T} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ 5 \end{pmatrix},$$

则所求点 P 满足 $\vec{n}\perp\vec{T}$, 即 $\vec{n}\cdot\vec{T}=-4x+16y+10=0$, 故所求点的轨迹为

$$\begin{cases} 2x^2-2y^2+2z=1, \\ -2x+8y+5=0, \end{cases}$$

这是一条空间曲线.

22. 过直线

$$\begin{cases} 10x+2y-2z=27 \\ x+y-z=0 \end{cases}$$

作曲面 $3x^2+y^2-z^2=27$ 的切平面, 求该切平面的方程.

解: 方法 1. 设所求切平面的切点为 $P_0(x_0,y_0,z_0)$. 曲面在该点的法向量为

$$\vec{n}=\begin{pmatrix} 6x_0 \\ 2y_0 \\ -2z_0 \end{pmatrix},$$

从而相应切平面方程为 $6x_0(x-x_0)+2y_0(y-y_0)-2z_0(z-z_0)=0$, 也即

$$3x_0x+y_0y-z_0z=3x_0^2+y_0^2-z_0^2=27.$$

点 $(\frac{27}{8},0,\frac{27}{8})$, $(\frac{27}{8},-\frac{27}{8},0)$ 属于题设直线, 因此也属于上述切平面. 将之代入切平面方程得 $3x_0-z_0=8$, $3x_0-y_0=8$, 于是 $y_0=z_0=3x_0-8$. 代入曲面方程得 $3x_0^2=27$, 故 $x_0=\pm 3$, 进而知所求切点为 $(3,1,1)$ 或 $(-3,-17,-17)$, 相应的切平面方程为 $9x+y-z=27$ 或 $9x+17y-17z=-27$.

方法 2. 设所求切平面的切点为 $P_0(x_0,y_0,z_0)$. 曲面在该点的法向量为

$$\vec{n}=\begin{pmatrix} 6x_0 \\ 2y_0 \\ -2z_0 \end{pmatrix},$$

从而相应切平面方程为 $6x_0(x-x_0)+2y_0(y-y_0)-2z_0(z-z_0)=0$, 也即

$$3x_0x+y_0y-z_0z=3x_0^2+y_0^2-z_0^2=27.$$

该切平面包含题给直线当且仅当题给方程组的解也为上述切平面方程的解, 这表明切平面方程是题给直线的两个方程的线性组合, 也即 $\exists\lambda,\mu\in\mathbb{R}$ 使得

$$3x_0x+y_0y-z_0z-27=\lambda(10x+2y-2z-27)+\mu(x+y-z).$$

比较两边的系数可得

$$3x_0=10\lambda+\mu,\ y_0=2\lambda+\mu,\ z_0=2\lambda+\mu,\ 27=27\lambda,$$

也即 $\lambda=1$, $x_0=\frac{10}{3}+\frac{1}{3}\mu$, $y_0=z_0=2+\mu$. 带入曲面方程可得

$$3\Big(\frac{10}{3}+\frac{1}{3}\mu\Big)^2=27,$$

故 $\mu=-1$ 或 -19, 从而所求切点为 $(3,1,1)$ 或 $(-3,-17,-17)$, 相应切平面方程为 $9x+y-z=27$ 或 $9x+17y-17z+27=0$.

23. 求螺线

$$\begin{cases} x=a\cos t \\ y=a\sin t \quad (a>0,\ c>0) \\ z=ct \end{cases}$$

在点 $M\big(\frac{a}{\sqrt{2}},\frac{a}{\sqrt{2}},\frac{\pi c}{4}\big)$ 处的切线与法平面.

解: 点 $M\big(\frac{a}{\sqrt{2}},\frac{a}{\sqrt{2}},\frac{\pi c}{4}\big)$ 所对应的参数为 $t=\frac{\pi}{4}$, 则螺线在该点的切线方向为

$$\begin{pmatrix} -a\sin t \\ a\cos t \\ c \end{pmatrix}\Bigg|_{t=\frac{\pi}{4}}=\begin{pmatrix} -\frac{a}{\sqrt{2}} \\ \frac{a}{\sqrt{2}} \\ c \end{pmatrix}.$$

故所求切线方程为 $\frac{x-\frac{a}{\sqrt{2}}}{-\frac{a}{\sqrt{2}}}=\frac{y-\frac{a}{\sqrt{2}}}{\frac{a}{\sqrt{2}}}=\frac{z-\frac{\pi c}{4}}{c}$, 相应的法平面方程为

$$-\frac{a}{\sqrt{2}}\Big(x-\frac{a}{\sqrt{2}}\Big)+\frac{a}{\sqrt{2}}\Big(y-\frac{a}{\sqrt{2}}\Big)+c\Big(z-\frac{\pi c}{4}\Big)=0.$$

24. 求曲线

$$\begin{cases} x^2+y^2+z^2-6=0 \\ z-x^2-y^2=0 \end{cases}$$

在点 $M(1,1,2)$ 处的切线与法平面.

解: 由题设可知, 曲线在点 $M(1,1,2)$ 处的切线方向为

$$\begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}\Bigg|_{(1,1,2)}\times\begin{pmatrix} -2x \\ -2y \\ 1 \end{pmatrix}\Bigg|_{(1,1,2)}=\begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}\times\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}=\begin{pmatrix} 10 \\ -10 \\ 0 \end{pmatrix},$$

故切线方程为 $\frac{x-1}{10}=\frac{y-1}{-10}=\frac{z-2}{0}$, 相应的法平面方程为 $10(x-1)-10(y-1)=0$, 也即我们有 $x-y=0$.

25. 求曲线 $\begin{cases} x = t \\ y = t^2 \\ z = t^3 \end{cases}$ 上的点使曲线在该点的切线平行于平面 $x+2y+z=4$.

解: 设所求曲线上的点为 (t_0, t_0^2, t_0^3), 曲线在该点的切线方向为 $\begin{pmatrix} 1 \\ 2t_0 \\ 3t_0^2 \end{pmatrix}$, 则该切线与平面 $x+2y+z=4$ 平行当且仅当

$$0 = \begin{pmatrix} 1 \\ 2t_0 \\ 3t_0^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 1 + 4t_0 + 3t_0^2,$$

也即 $t_0 = -1$ 或 $-\frac{1}{3}$. 则所求点为 $(-1, 1, -1)$ 或 $\left(-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}\right)$.

26. $\forall (x,y) \in (0,+\infty) \times \mathbb{R}$, 定义 $f(x,y) = x^y$. 求函数 f 在点 $(1,0)$ 处带 Peano 余项的二阶 Taylor 展式.

解: 方法 1. 由于 f 为初等函数, 从而在 $(0,+\infty)\times\mathbb{R}$ 上为 $\mathscr{C}^{(2)}$ 类函数, 故

$$\begin{aligned}
\frac{\partial f}{\partial x}(1,0) &= yx^{y-1}\Big|_{(1,0)} = 0, \\
\frac{\partial f}{\partial y}(1,0) &= \frac{\partial}{\partial y}(e^{y\log x})\Big|_{(1,0)} = x^y \log x\Big|_{(1,0)} = 0, \\
\frac{\partial^2 f}{\partial x^2}(1,0) &= y(y-1)x^{y-2}\Big|_{(1,0)} = 0, \\
\frac{\partial^2 f}{\partial y \partial x}(1,0) &= \big(x^{y-1} + yx^{y-1}\log x\big)\Big|_{(1,0)} = 1, \\
\frac{\partial^2 f}{\partial y^2}(1,0) &= x^y(\log x)^2\Big|_{(1,0)} = 0.
\end{aligned}$$

于是函数 f 在点 $(1,0)$ 处带 Peano 余项的二阶 Taylor 展式为

$$\begin{aligned}
f(x,y) &= f(1,0) + \frac{\partial f}{\partial x}(1,0)(x-1) + \frac{\partial f}{\partial y}(1,0)y + \frac{1}{2!}\Big(\frac{\partial^2 f}{\partial x^2}(1,0)(x-1)^2 \\
&\quad + 2\frac{\partial^2 f}{\partial y \partial x}(1,0)(x-1)y + \frac{\partial^2 f}{\partial y^2}(1,0)y^2\Big) + o\big((x-1)^2 + y^2\big) \\
&= 1 + (x-1)y + o((x-1)^2 + y^2), \quad (x,y) \to (1,0).
\end{aligned}$$

方法 2. 当 $(x,y) \to (1,0)$ 时, 我们有 $y\log x = y(x-1)(1+o(1))$, 从而

$$\begin{aligned}
f(x,y) &= e^{y\log x} = 1 + y\log x + y(\log x)o(1) \\
&= 1 + y\big((x-1) - \frac{1}{2}(x-1)^2(1+o(1)) + y(x-1)o(1) \\
&= 1 + y(x-1) - \frac{1}{2}y(x-1)^2(1+o(1)) + y(x-1)o(1), \\
&= 1 + (x-1)y + o((x-1)^2 + y^2),
\end{aligned}$$

因为 $y(x-1)^2 = y(x-1)o(1)$, 而 $|y(x-1)| \leqslant \frac{1}{2}(y^2 + (x-1)^2)$.

27. 求 $f(x,y)=\frac{\cos x}{1+y}$ 在点 $(0,0)$ 处带 Lagrange 余项的一阶 Taylor 展式.

解: 由于 f 为初等函数, 故为 $\mathscr{C}^{(2)}$ 类函数, 从而

$$\begin{aligned}
\frac{\partial f}{\partial x}(0,0) &= -\frac{\sin x}{1+y}\Big|_{(0,0)}=0,\\
\frac{\partial f}{\partial y}(0,0) &= -\frac{\cos x}{(1+y)^2}\Big|_{(0,0)}=-1,\\
\frac{\partial^2 f}{\partial x^2}(x,y) &= -\frac{\cos x}{1+y},\\
\frac{\partial^2 f}{\partial y\partial x}(x,y) &= \frac{\sin x}{(1+y)^2},\\
\frac{\partial^2 f}{\partial y^2}(x,y) &= \frac{2\cos x}{(1+y)^3}.
\end{aligned}$$

于是 f 在点 $(0,0)$ 处带 Lagrange 余项的一阶 Taylor 展式为 (其中 $\theta\in(0,1)$):

$$\begin{aligned}
f(x,y) &= f(0,0)+\frac{\partial f}{\partial x}(0,0)x+\frac{\partial f}{\partial y}(0,0)y\\
&\quad+\frac{1}{2!}\Big(\frac{\partial^2 f}{\partial x^2}(\theta x,\theta y)x^2+2\frac{\partial^2 f}{\partial y\partial x}(\theta x,\theta y)xy+\frac{\partial^2 f}{\partial y^2}(\theta x,\theta y)y^2\Big)\\
&= 1-y+\frac{1}{2}\Big(-\frac{\cos(\theta x)}{1+\theta y}\cdot x^2+\frac{2\sin(\theta x)}{(1+\theta y)^2}\cdot xy+\frac{2\cos(\theta x)}{(1+\theta y)^3}\cdot y^2\Big).
\end{aligned}$$

28. 求 $f(x,y)=\sin(xy)$ 在点 $(1,1)$ 处的二阶 Taylor 多项式.

解: **方法 1.** 由于 f 为初等函数, 故为 $\mathscr{C}^{(2)}$ 类函数, 从而

$$\begin{aligned}
\frac{\partial f}{\partial x}(1,1) &= y\cos(xy)\Big|_{(1,1)}=\cos 1,\\
\frac{\partial f}{\partial y}(1,1) &= x\cos(xy)\Big|_{(1,1)}=\cos 1,\\
\frac{\partial^2 f}{\partial x^2}(1,1) &= -y^2\sin(xy)\Big|_{(1,1)}=-\sin 1,\\
\frac{\partial^2 f}{\partial y\partial x}(1,1) &= \big(\cos(xy)-xy\sin(xy)\big)\Big|_{(1,1)}=\cos 1-\sin 1,\\
\frac{\partial^2 f}{\partial y^2}(1,1) &= -x^2\sin(xy)\Big|_{(1,1)}=-\sin 1.
\end{aligned}$$

于是函数 f 在点 $(1,1)$ 处的二阶 Taylor 多项式为

$$\begin{aligned}
&\sin 1+(\cos 1)(x-1)+(\cos 1)(y-1)\\
&+\frac{1}{2}\big(-(\sin 1)(x-1)^2+2(\cos 1-\sin 1)(x-1)(y-1)-(\sin 1)(y-1)^2\big).
\end{aligned}$$

方法 2. 令 $u=x-1$, $v=y-1$, 则当 $(u,v)\to(0,0)$ 时, 我们有

$$f(x,y) = \sin(u+1)(v+1)=\sin(uv+u+v+1)$$

$$\begin{aligned}
&= (\sin 1)\cos(uv+u+v)+(\cos 1)\sin(uv+u+v)\\
&= (\sin 1)\big(1-\frac{1}{2}(uv+u+v)^2)(1+o(1))\big)\\
&\quad +(\cos 1)\big((uv+u+v)+(uv+u+v)^2o(1)\big)\\
&= (\sin 1)\big(1-\frac{1}{2}(u^2+v^2+2uv)(1+o(1))\big)\\
&\quad +(\cos 1)(uv+u+v)+(u^2+v^2)o(1)\\
&= \sin 1+(\cos 1)(u+v)+\frac{1}{2}\big(-(\sin 1)(u^2+v^2)\\
&\quad +2(\cos 1-\sin 1)uv\big)+(u^2+v^2)o(1).
\end{aligned}$$

于是所求二阶 Taylor 多项式为

$$\begin{aligned}
&\sin 1+(\cos 1)(x-1)+(\cos 1)(y-1)\\
&+\frac{1}{2}\big(-(\sin 1)(x-1)^2+2(\cos 1-\sin 1)(x-1)(y-1)-(\sin 1)(y-1)^2\big).
\end{aligned}$$

29. 求证*: 方程 $x+y+z+xyz^3=0$ 在点 $(0,0,0)$ 的邻域内确定一个 $\mathscr{C}^{(2)}$ 类隐函数 $z=z(x,y)$, 并计算它在点 $(0,0)$ 处二阶带 Peano 余项的 Taylor 展式.*

证明: $\forall (x,y,z)\in\mathbb{R}^3$, 令 $F(x,y,z)=x+y+z+xyz^3$, 则 F 为初等函数, 故为 $\mathscr{C}^{(2)}$ 类. 又 $\frac{\partial F}{\partial z}(0,0,0)=(1+3xyz^2)\big|_{(0,0,0)}=1$, 由隐函数定理可知,隐函数方程 $F(x,y,z)=0$ 在原点邻域内可确定一个 $\mathscr{C}^{(2)}$ 类隐函数 $z=z(x,y)$.

下面来求上述隐函数在点 $(0,0)$ 处二阶带 Peano 余项的 Taylor 展式.

方法 1. 由隐函数定理可知

$$\begin{aligned}
&\frac{\partial z}{\partial x}(0,0)=-\frac{1+yz^3}{1+3xyz^2}\Big|_{(0,0,0)}=-1,\\
&\frac{\partial z}{\partial y}(0,0)=-\frac{1+xz^3}{1+3xyz^2}\Big|_{(0,0,0)}=-1,\\
&\frac{\partial^2 z}{\partial x^2}(0,0)=-\frac{3yz^2\frac{\partial z}{\partial x}(1+3xyz^2)-(1+yz^3)\big(3yz^2+6xyz\frac{\partial z}{\partial x}\big)}{(1+3xyz^2)^2}\Big|_{(0,0,0)}=0,\\
&\frac{\partial^2 z}{\partial y\partial x}(0,0)=-\frac{\big(z^3+3yz^2\frac{\partial z}{\partial x}\big)(1+3xyz^2)-(1+yz^3)\big(3xz^2+6xyz\frac{\partial z}{\partial x}\big)}{(1+3xyz^2)^2}\Big|_{(0,0,0)}=0,\\
&\frac{\partial^2 z}{\partial y^2}(0,0)=-\frac{3xz^2\frac{\partial z}{\partial x}(1+3xyz^2)-(1+xz^3)\big(3xz^2+6xyz\frac{\partial z}{\partial x}\big)}{(1+3xyz^2)^2}\Big|_{(0,0,0)}=0,
\end{aligned}$$

故隐函数 $z=z(x,y)$ 在点 $(0,0)$ 处二阶带 Peano 余项的 Taylor 展式为

$$z(x,y)=-x-y+o(x^2+y^2),\quad (x,y)\to(0,0).$$

方法 2. 因 $z=z(x,y)$ 在 $(0,0)$ 连续且 $z(0,0)=0$, 则 $(x,y)\to(0,0)$ 时, 我们有 $z(x,y)=o(1)$, 进而可知所求二阶带 Peano 余项的 Taylor 展式为

$$z(x,y)=-x-y-xyz^3=-x-y+xyo(1)=-x-y+o(x^2+y^2).$$