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Assignment 2

Question 1 to 2

True. True.

Question 3

1.

To use Hopfield network to retrieve noisy images, we can use the following steps:

1. suppose the noisy image is $x \in R^n$
2. each time, we update the entry x_i of x by $x_i = sign(\sum_{j=1}^{n} w_{ij}x_j)$ (i=1,2,...,n)
3. When the update process converges, we get the denoised image.

To use Hopfield network to retrieve masked images, we can use the following steps:

1. suppose the masked image is $x \in R^n$, and the masked part is $x_i (i \in I)$
2. each time, we update the entry x_i of x by $x_i = sign(\sum_{j=1}^{n} w_{ij}x_j)$ (i=$\{1, 2, ..., n\} - I$)
3. When the update process converges, we get the unmasked image.

2.

We can calculate that $\forall x$,

$$E(x) = -\frac{1}{2N} x^T (\sum_p y_p y_p^T) x = -\frac{1}{2N} \sum_p x^T y_p (y_p^T x) = -\frac{1}{2N} \sum_p ||y_p^T x||^2 = -\frac{||x||^2}{2N} = -\frac{1}{2}$$

Thus, all the patterns have the same energy, that is, it memorizes 2^N patterns.

Question 4

(Prove of the gradient of the Boltzmann machine)

$$\nabla_W L(W) = \nabla_W(-\frac{1}{|P|}\sum_{v\in P} log(\sum_h \mathbb{P}(v,h)))$$

$$= \nabla_W(-\frac{1}{|P|}\sum_{v\in P} log(\sum_h \frac{\exp(y^TWy)}{\sum_{y'}\exp(y'^TWy')}))$$

$$= \nabla_W(-\frac{1}{|P|}\sum_{v\in P} log(\sum_h \exp(y^TWy))) + \nabla_W(log\sum_{y'}\exp(y'^TWy'))$$

$$= \nabla_W(-\frac{1}{|P|}\sum_{v\in P} log(\sum_h \exp(y^TWy))) + \nabla_W(log\sum_{y'}\exp(y'^TWy'))$$

$$= -\frac{1}{|P|}\sum_{v\in P}\nabla_W log(\sum_h \exp(y^TWy)) + \nabla_W(log\sum_{y'}\exp(y'^TWy'))$$

$$= -\frac{1}{|P|}\sum_{v\in P}\frac{\sum_h \exp(y^TWy)(y^Ty)}{\sum_h \exp(y^TWy)} + \sum_y \frac{y^Ty\exp(y^TWy)}{\sum_{y'}\exp(y'^TWy')}$$

$$= -\frac{1}{|P|}\sum_{v\in P}\sum_h y^Ty\mathbb{P}(v,h) + \sum_y y^Ty\mathbb{P}(y)$$

$$= -\frac{1}{|P|}\sum_{v\in P}(\mathbb{E}_{h|v}[yy^T] - \mathbb{E}_{y'}[y'y'^T])$$

Question 5

(Gaussian RBM)
Energy:

$$\mathcal{E}_{W,b}(v,h) = \frac{1}{2}(v-b)^T(v-b) - v^TWh \tag{1}$$

1.

$$\mathbb{P}(v|h) = \frac{1}{Z_h}\exp(-\mathcal{E}_{W,b}(v,h))$$

$$Z_h = \int_v \exp(-\mathcal{E}_{W,b}(v,h))dv$$

$$= \int_v \exp(-\frac{1}{2}(v-b)^T(v-b) + v^TWh)dv$$

$$= \int_v \exp(-\frac{1}{2}v^Tv + (v+b)^TWh)dv$$

$$= e^{b^TWh}\int_v \exp(-\frac{1}{2}v^Tv + v^TWh)dv$$

$$= e^{b^T Wh + \frac{(Wh)^T Wh}{2}} \int_v \exp(-\frac{1}{2}(v - Wh)^T(v - Wh))dv$$

$$= e^{(b+\frac{Wh}{2})^T Wh} \int_v \exp(-\frac{1}{2}v^2)dv$$

$$= e^{(b+\frac{Wh}{2})^T Wh}(\sqrt{2\pi})^{N_v}$$

So,

$$\mathbb{P}(v|h) = \frac{1}{Z}\exp(-\mathcal{E}_{W,b}(v,h)) = \frac{1}{(\sqrt{2\pi})^{N_v}}\exp(-\frac{1}{2}(v-b)^T(v-b)+v^T Wh-(b+\frac{Wh}{2})^T Wh)$$

$$= \frac{1}{(\sqrt{2\pi})^{N_v}}\exp(-\frac{1}{2}(v-b)^T(v-b)+v^T Wh-(b^T Wh+\frac{(Wh)^T Wh}{2}))$$

$$= \frac{\exp(-\frac{1}{2}(v-b)^T(v-b)+v^T Wh-b^T Wh-\frac{(Wh)^T Wh}{2})}{(\sqrt{2\pi})^{N_v}}$$

$$= \frac{\exp(-\frac{1}{2}||v-b-Wh||^2)}{(\sqrt{2\pi})^{N_v}}$$

2. Similarly to Problem 4.2, we can get

$$\nabla_b L(W,b) = -\frac{1}{|P|}\sum_{v\in P}(\mathbb{E}_{h|v}[\nabla_b\mathcal{E}] - \mathbb{E}[\nabla_b\mathcal{E}])$$

since

$$\nabla_b\mathcal{E} = v - b$$

we know

$$\nabla_b L(W,b) = -\frac{1}{|P|}\sum_{v\in P}(\mathbb{E}_{h|v}[v-b] - \mathbb{E}[v-b])$$

$$= -\mathbb{E}_{v\in P}[v] + \mathbb{E}_v[v]$$

Question 6

1. D and F are not independent generally, since the joint distribution of D and F is

$$\mathbb{P}(D|F) = \frac{\mathbb{P}(D,F)}{\mathbb{P}(F)} = \frac{\mathbb{P}(D,F)}{\sum_{D'}\mathbb{P}(D,F)}$$

$$= \frac{f_{AD}(A,D)f_{AE}(A,E)f_{EF}(E,F)}{\sum_{D'} f_{AD}(A,D')f_{AE}(A,E)f_{EF}(E,F)}$$

which generally doesn't equal to $\mathbb{P}(D)$.

2.

$$\mathbb{P}(B,E|A) = \frac{\mathbb{P}(BEA)}{\mathbb{P}(A)}$$

$$= \frac{\sum_{C'} \mathbb{P}(BEAC')}{\sum_{C'} \mathbb{P}(AC')}$$

$$= \frac{\sum_{C'} f_{BC}(B,C')f_{AE}(A,E)f_{AC}(A,C')}{\sum_{C} f_{AC}(A,C')}$$

B and E are conditionally independent given A, because

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(BA)}{\mathbb{P}(A)} = \frac{\sum_{C'} \mathbb{P}(BAC')}{\sum_{C'} \mathbb{P}(AC')}$$

$$= \frac{\sum_{C'} f_{BC}(B,C')f_{AC}(A,C')}{\sum_{C'} f_{AC}(A,C')}$$

$$\mathbb{P}(E|A) = \frac{\mathbb{P}(EA)}{\mathbb{P}(A)} = \frac{\sum_{C'} \mathbb{P}(EAC')}{\sum_{C'} \mathbb{P}(AC')}$$

$$= \frac{\sum_{C'} f_{AE}(A,E)f_{AC}(A,C')}{\sum_{C'} f_{AC}(A,C')}$$

thus,

$$\mathbb{P}(B|A)\mathbb{P}(E|A)$$

$$= \frac{\sum_{C'} f_{BC}(B,C')f_{AC}(A,C')}{\sum_{C'} f_{AC}(A,C')} \frac{\sum_{C'} f_{AE}(A,E)f_{AC}(A,C')}{\sum_{C'} f_{AC}(A,C')}$$

$$= \frac{\sum_{C'} f_{BC}(B,C')f_{AC}(A,C')f_{AE}(A,E)f_{AC}(A,C')}{(\sum_{C'} f_{AC}(A,C'))^2}$$

$$= \frac{\sum_{C'} f_{BC}(B,C')f_{AE}(A,E)f_{AC}(A,C')}{\sum_{C'} f_{AC}(A,C')}$$

$$= \frac{\sum_{C'} f_{BC}(B,C')f_{AE}(A,E)f_{AC}(A,C')}{\sum_{C} f_{AC}(A,C')}$$

$$= \mathbb{P}(B,E|A)$$

3. That's because the joint probability

$$\mathbb{P}(A,B,C,D,E,F) = \frac{1}{Z} f_{BC}(B,C)f_{AC}(A,C)f_{AD}(A,D)f_{AE}(A,E)f_{EF}(E,F)$$

So the energy function $\mathcal{E}(A,B,C,D,E,F)$ can be expressed by $\mathcal{E}(A,C), \mathcal{E}(A,D), \mathcal{E}(A,E)$, $\mathcal{E}(B,C), \mathcal{E}(E,F)$ (just let the energy function be the minus log of the joint probability)