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Homework 2

1 True or False Questions

Problem 1

True.

Problem 2

True.

2 Q & A

Problem 3

(1) For noisy images, we can set all neurons into the given values and then run evolution until convergence to get the reconstructed image.

For the masked images, we can set the neurons corresponding to the given parts of the image into the given values, then run the evolution process until convergence to get the image.

(2) Let the matrix Y be such that $Y_{ij} = (y_i)_j$, then since y_i are orthogonal, we know that Y is an orthogonal matrix. Thus, we have

$$W_{ij} = \frac{1}{N}\sum_{p=1}^{N}(y_p)_i(y_p)_j = \frac{1}{N}(Y^TY)_{ij} = \frac{1}{N}\delta_{ij}.$$

This implies that for any y, the energy is

$$E = -\frac{1}{2}y^TWy = -\frac{1}{2N}y^Ty = -\frac{1}{2},$$

so all patterns are memorized.

Problem 4

The gradient is

$$
\begin{aligned}
\nabla_W L(W) &= -\frac{1}{|P|}\sum_{v\in P}\frac{\nabla_W \mathbb{P}(v)}{\mathbb{P}(v)} \\
&= -\frac{1}{|P|}\sum_{v\in P}\frac{1}{\mathbb{P}(v)}\sum_h \mathbb{P}(v,h)(\nabla_W \log \mathbb{P}(v,h)) \\
&= -\frac{1}{|P|}\sum_{v\in P}\frac{1}{\mathbb{P}(v)}\sum_h \mathbb{P}(v,h)\left(yy^T - \frac{\sum_{y'}\exp(y'^T W y')y'y'^T}{\sum_{y'}\exp(y'^T W y')}\right) \\
&= -\frac{1}{|P|}\sum_{v\in P}\left(\mathbb{E}_{h|v}[yy^T] - \sum_{y'}\frac{\exp(y'^T W y')}{\sum_{y''}\exp(y''^T W y'')}y'y'^T\right) \\
&= -\frac{1}{|P|}\sum_{v\in P}\left(\mathbb{E}_{h|v}[yy^T] - \mathbb{E}_{y'}[y'y'^T]\right),
\end{aligned}
$$

so we are done.

Problem 5

(1) We have the conditional distribution being

$$
\begin{aligned}
\mathbb{P}(v|h) &= \frac{\mathbb{P}(v,h)}{\int_v \mathbb{P}(v,h)dv} \\
&= \frac{\exp(-\frac{1}{2}(v-b)^T(v-b)+(v-b)^T Wh)}{\int_v \exp(-\frac{1}{2}(v-b)^T(v-b)+(v-b)^T Wh)dv}.
\end{aligned}
$$

Now, we let $x = v - b$, $c = Wh$ and do the integration:

$$
\begin{aligned}
\int \exp\left(-\frac{1}{2}x^T x + x^T c\right)dx &= \int\int\cdots\int \exp\left(-\frac{1}{2}\sum_{i=1}^{N_v} x_i^2 + \sum_{i=1}^{N_v} x_i c_i\right)dx_1 dx_2\cdots dx_n \\
&= \exp\left(\frac{1}{2}\sum_{i=1}^{N_v} c_i^2\right)\cdot (2\pi)^{\frac{N_v}{2}}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\mathbb{P}(v|h) &= (2\pi)^{-\frac{N_v}{2}}\exp\left(-\frac{1}{2}h^T W^T W h - \frac{1}{2}(v-b)^T(v-b)+(v-b)^T Wh\right) \\
&= (2\pi)^{-\frac{N_v}{2}}\exp\left(-\frac{1}{2}\left|\left|v-b-Wh\right|\right|^2\right).
\end{aligned}
$$

(2) The loss function is

$$\begin{aligned}
L(b) &= -\frac{1}{|P|}\sum_{v\in P}\log\left(\sum_h \mathbb{P}(v,h)\right)\\
&= -\frac{1}{|P|}\sum_{v\in P}\log\left(\frac{1}{Z}\sum_h \exp\left(-\frac{1}{2}(v-b)^T(v-b)+v^TWh\right)\right)\\
&= -\frac{1}{|P|}\sum_{v\in P}\log\left(\sum_h \exp\left(-\frac{1}{2}(v-b)^T(v-b)+v^TWh\right)\right)+\log Z
\end{aligned}$$

Then we can compute the gradient:

$$\begin{aligned}
\nabla_b L(b) = &-\frac{1}{|P|}\sum_{v\in P}\frac{\sum_h \exp\left(-\frac{1}{2}(v-b)^T(v-b)+v^TWh\right)(v-b)}{\sum_h \exp\left(-\frac{1}{2}(v-b)^T(v-b)+v^TWh\right)}\\
&+\frac{\sum_{v,h}\exp(-\frac{1}{2}(v-b)^T(v-b)+v^TWh)(v-b)}{Z}\\
&= -\frac{1}{|P|}\sum_{v\in P}(v-b)+\mathbb{E}_v[v-b]\\
&= -\frac{1}{|P|}\sum_{v\in P}v+\mathbb{E}_v[v]
\end{aligned}$$

Problem 6

(1) No. We can first find that

$$\begin{aligned}
\mathbb{P}(D,F) &= \frac{1}{Z}\int_{ABCE} f_{AD}(A,D)f_{AC}(A,C)f_{AE}(A,E)f_{BC}(B,C)f_{EF}(E,F)dAdBdCdE\\
&= \frac{1}{Z}\int_{AE} g(A)f_{AD}(A,D)f_{AE}(A,E)f_{EF}(E,F)dAdE,
\end{aligned}$$

where $g(A)$ is a function of A. Next,

$$\begin{aligned}
\mathbb{P}(D) &= \frac{1}{Z}\int_{AEF} g(A)f_{AD}(A,D)f_{AE}(A,E)f_{EF}(E,F)dAdEdF\\
&= \frac{1}{Z}\int_{A} g(A)h(A)f_{AD}(A,D)dA,
\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}(F) &= \frac{1}{Z}\int_{ADE} g(A)f_{AD}(A,D)f_{AE}(A,E)f_{EF}(E,F)dAdDdE \\ &= \frac{1}{Z}\int_{E} k(E)f_{EF}(E,F)dE,\end{aligned}$$

where $h(A), k(E)$ are functions of A, E, respectively. Thus, we can see that $\mathbb{P}(D,F) \neq \mathbb{P}(D)\mathbb{P}(F)$ in general.

(2) Yes. Ignoring the normalizing factor, we have

$$\begin{aligned}\mathbb{P}(B,E|A) &= \frac{1}{Z}\int_{CDF} f_{AD}(A,D)f_{AC}(A,C)f_{AE}(A,E)f_{BC}(B,C)f_{EF}(E,F)dCdDdF \\ &= \frac{1}{Z}f_{AE}(A,E)l(A,B)m(A)h(E),\end{aligned}$$

where m, h are functions only depending on A, E, respectively. We then find that B and E are independent given A.

(3) We have

$$\log\mathbb{P}(A,B,C,D,E,F) = \mathcal{E}(A,B,C,D,E,F) + C,$$

where C is a constant independent of $A, B, ..., F$. Thus, we can have

$$\begin{aligned}&\mathcal{E}(A,B,C,D,E,F) \\ &= C_1 - \log f_{AD}(A,D) - \log f_{AC}(A,C) - \log f_{AE}(A,E) - \log f_{BC}(B,C) - \log f_{EF}(E,F) \\ &= \mathcal{E}_{AD}(A,D) + \mathcal{E}_{AC}(A,C) + \mathcal{E}_{AE}(A,E) + \mathcal{E}_{BC}(B,C) + \mathcal{E}_{EF}(E,F).\end{aligned}$$