1. 设 $z=\arctan\frac{u}{v}$，$u=x^2+y^2$，$v=xy$，求 $\frac{\partial z}{\partial x}$，$\frac{\partial z}{\partial y}$，$\frac{\partial^2 z}{\partial x^2}$，$\frac{\partial^2 z}{\partial y^2}$，$\frac{\partial^2 z}{\partial x\partial y}$

解：

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\frac{1}{v}}{1+\frac{u^2}{v^2}}\cdot 2x+\frac{-\frac{u}{v^2}}{1+\frac{u^2}{v^2}}y=\frac{2xv-yu}{u^2+v^2}=\frac{y(x^2-y^2)}{u^2+v^2}$$

$$=\frac{y(x^2-y^2)}{x^4+3x^2y^2+y^4}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{2vy-ux}{u^2+v^2}=\frac{x(y^2-x^2)}{x^4+3x^2y^2+y^4}$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{y(x^2-y^2)}{x^4+3x^2y^2+y^4}\right)=\frac{y[2x(x^4+3x^2y^2+y^4)-(x^2-y^2)\cdot x(4x^2+6y^2)]}{(x^4+3x^2y^2+y^4)^2}=\frac{xy[-2x^4+4x^2y^2+8y^4]}{(x^4+3x^2y^2+y^4)^2}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{y(x^2-y^2)}{x^4+3x^2y^2+y^4}\right)=\frac{(x^2-3y^2)(x^4+3x^2y^2+y^4)-y(x^2-y^2)\cdot y(4y^2+6x^2)}{[x^4+3x^2y^2+y^4]^2}$$

$$=\frac{x^6-6x^4y^2-6x^2y^4+y^6}{(x^4+3x^2y^2+y^4)^2}$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{x(y^2-x^2)}{x^4+3x^2y^2+y^4}\right)=\frac{x[2y(y^4+3y^2x^2+x^4)-(y^2-x^2)\cdot y(4y^2+6x^2)]}{(x^4+3x^2y^2+y^4)^2}=\frac{xy(-2y^4+4x^2y^2+8x^4)}{(x^4+3x^2y^2+y^4)^2}$$

2. 已知 $u=f(x,y)$，$x=r\cos\theta$，$y=r\sin\theta$，f 可微，证明：

$$\left[\frac{\partial u}{\partial r}(r,\theta)\right]^2+\left[\frac{1}{r}\frac{\partial u}{\partial \theta}(r,\theta)\right]^2=\left(\frac{\partial f}{\partial x}(r\cos\theta,r\sin\theta)\right)^2+\left(\frac{\partial f}{\partial y}(r\cos\theta,r\sin\theta)\right)^2$$

证明：

$$\frac{\partial u}{\partial r}(r,\theta)=\frac{\partial u(x,y)}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u(x,y)}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial f}{\partial x}\Big|_{(r\cos\theta,r\sin\theta)}\cos\theta+\frac{\partial f}{\partial y}\Big|_{(r\cos\theta,r\sin\theta)}\sin\theta$$

$$\frac{\partial u}{\partial \theta}(r,\theta)=\frac{\partial u(x,y)}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u(x,y)}{\partial y}\frac{\partial y}{\partial \theta}=\frac{\partial f}{\partial x}\Big|_{(r\cos\theta,r\sin\theta)}(-r\sin\theta)+\frac{\partial f}{\partial y}\Big|_{(r\cos\theta,r\sin\theta)}(r\cos\theta)$$

故 $$\left[\frac{\partial u}{\partial r}(r,\theta)\right]^2+\left[\frac{1}{r}\frac{\partial u}{\partial \theta}(r,\theta)\right]^2=\left(\frac{\partial f}{\partial x}\cos\theta+\frac{\partial f}{\partial y}\sin\theta\right)^2+\left(-\sin\theta\frac{\partial f}{\partial x}+\cos\theta\frac{\partial f}{\partial y}\right)^2$$

$$=\left[\frac{\partial f}{\partial x}(r\cos\theta,r\sin\theta)\right]^2+\left[\frac{\partial f}{\partial y}(r\cos\theta,r\sin\theta)\right]^2$$

3. 设 f 满足 Laplace 方程 $\partial_{11}f+\partial_{22}f=0$，证明：$u(x,y)=f\left(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2}\right)$ 也满足 Laplace 方程

证明：

$$\frac{\partial u}{\partial x}=\partial_1 f\cdot\frac{y^2-x^2}{(x^2+y^2)^2}+\partial_2 f\cdot\frac{-y\cdot 2x}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y}=\partial_1 f\cdot\frac{-x\cdot 2y}{(y^2+x^2)^2}+\partial_2 f\cdot\frac{x^2-y^2}{(y^2+x^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left[\partial_1 f\left(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2}\right)\frac{y^2-x^2}{(x^2+y^2)^2}+\partial_2 f\left(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2}\right)\frac{-2xy}{(x^2+y^2)^2}\right]$$

$$=\left[\partial_{11}f\cdot\frac{y^2-x^2}{(x^2+y^2)^2}+\partial_{21}f\,\frac{-2xy}{(x^2+y^2)^2}\right]\frac{y^2-x^2}{(x^2+y^2)^2}+\partial_1 f\,\frac{x^2-3y^2}{(x^2+y^2)^3}\cdot 2x$$

$$+\left[\partial_{12}f\cdot\frac{y^2-x^2}{(x^2+y^2)^2}+\partial_{22}f\,\frac{-2xy}{(x^2+y^2)^2}\right]\frac{-2xy}{(x^2+y^2)^2}+\partial_2 f\cdot\frac{(y^2-3x^2)}{(x^2+y^2)^3}\cdot(-2y)$$

通过完全一样计算 ~~交换x与y~~，可得

$$\frac{\partial^2 u}{\partial y^2} = \left[\partial_{22} f \cdot \frac{x^2-y^2}{(x^2+y^2)^2} + \partial_{12} f \, \frac{-2xy}{(x^2+y^2)^2}\right] \frac{x^2-y^2}{(x^2+y^2)^2} + \partial_2 f \, \frac{y^2-3x^2}{(y^2+x^2)^3} \cdot (-2y)$$

$$+ \left[\partial_{21} f \cdot \frac{x^2-y^2}{(x^2+y^2)^2} + \partial_{11} f \, \frac{-2xy}{(x^2+y^2)^2}\right] \frac{-2xy}{(x^2+y^2)^2} + \partial_1 f \, \frac{x^2-3y^2}{(x^2+y^2)^3} (-2x)$$

相加得出 $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial_{11} f + \partial_{22} f}{(x^2+y^2)^2} = 0$ 证毕.

4. 设向量值函数 $Y=f(U)$, $U=g(X)$ 可微，求复合函数 $Y=f\circ g(X)$ 的 Jacobi 矩阵与全微分，其中

$$\begin{cases} y_1 = u_1 + u_2 \\ y_2 = u_1 u_2 \\ y_3 = \frac{u_2}{u_1} \end{cases} \qquad \begin{cases} u_1 = \frac{x}{x^2+y^2} \\ u_2 = \frac{y}{x^2+y^2} \end{cases}$$

解：$$\frac{\partial(y_1,y_2,y_3)}{\partial(x,y)} = \frac{\partial(y_1,y_2,y_3)}{\partial(u_1,u_2)}(u_1(x,y),u_2(x,y)) \frac{\partial(u_1,u_2)}{\partial(x,y)}$$

$$= \begin{bmatrix} 1 & 1 \\ u_2 & u_1 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1} \end{bmatrix} \begin{bmatrix} \frac{y^2-x^2}{(x^2+y^2)^2} & \frac{-2xy}{(x^2+y^2)^2} \\ \frac{-2xy}{(x^2+y^2)^2} & \frac{x^2-y^2}{(x^2+y^2)^2} \end{bmatrix} = \begin{bmatrix} \frac{y^2-2xy-x^2}{(x^2+y^2)^2} & \frac{x^2-2xy-y^2}{(x^2+y^2)^2} \\ \frac{(y^2-3x^2)y}{(x^2+y^2)^3} & \frac{x(x^2-3y^2)}{(x^2+y^2)^3} \\ -\frac{y}{x^2} & \frac{1}{x} \end{bmatrix}$$

全微分为 $$\begin{cases} dy_1 = \frac{y^2-2xy-x^2}{(x^2+y^2)^2}dx + \frac{x^2-2xy-y^2}{(x^2+y^2)^2}dy \\ dy_2 = \frac{y(y^2-3x^2)}{(x^2+y^2)^3}dx + \frac{x(x^2-3y^2)}{(x^2+y^2)^3}dy \\ dy_3 = -\frac{y}{x^2}dx + \frac{1}{x}dy \end{cases}$$

5. 问方程 $e^{-(x+y+z)} = x+y+z$ 在哪些点附近可确定一个隐函数 $z=z(x,y)$，并求相应的 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

解：令 $F = e^{-(x+y+z)} - x - y - z$. 则 ~~...~~ F 为连续一阶可微

存在隐函数充要条件为 $\frac{\partial F}{\partial z} \neq 0$，即 $-e^{-(x+y+z)} - 1 \neq 0$，恒成立

因此，在任一点均可确定 $z = z(x,y)$

$$\frac{\partial z}{\partial(x,y)} = -\left[\frac{\partial F}{\partial z}\right]^{-1} \frac{\partial F}{\partial(x,y)} = -\frac{1}{-e^{-(x+y+z)}-1}\left[-e^{-(x+y+z)}-1, \ -e^{-(x+y+z)}-1\right]$$

$$= [-1, -1]$$

即 $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = -1$

6. 问方程组 $\begin{cases} x+y+z+z^2=0 \\ x+y^2+z+z^3=0 \end{cases}$ 在点 $P=(-1,1,0)$ 的附近能否确定一个向量值函数 $\begin{pmatrix} y \\ z \end{pmatrix} = \vec{f}(x)$？若能，求 $y'(-1)$, $z'(-1)$

解：令 $F_1 = x+y+z+z^2$, $F_2 = x+y^2+z+z^3$，二者均连续一阶可微.

验证有 $F_1(-1,1,0) = F_2(-1,1,0) = 0$. 同时,

$$\frac{\partial(F_1,F_2)}{\partial(y,z)} = \begin{bmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 & 1+2z \\ 2y & 1+3z^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

故 $$\frac{\partial(y,z)}{\partial(x)} = -\left[\frac{\partial(F_1,F_2)}{\partial(y,z)}\right]^{-1} \frac{\partial(F_1,F_2)}{\partial x} = -\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

因此，$\begin{bmatrix} y'(-1) \\ z'(-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$，即 $y'(-1)=0$，$z'(-1)=1$.

7. 求向量值函数 $\begin{cases} u=x^2-y^2 \\ v=2xy \end{cases}$ 的逆映射的 Jacobi 矩阵与 Jacobi 行列式

解：$\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$，其行列式为 $4(x^2+y^2)$，因此该变换在除原点外处可逆. 其逆映射 Jacobi 矩阵为 $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{2(x^2+y^2)} \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$

而 Jacobi 行列式为 $\dfrac{D(x,y)}{D(u,v)} = \left| \dfrac{\partial(x,y)}{\partial(u,v)} \right| = \dfrac{1}{4(x^2+y^2)}$.

8. 求下列曲面在给定点处的切平面方程与法线方程

(1) $z=x^2+y^2$，点 $P(1,2,5)$

解：$\dfrac{\partial z}{\partial x} = 2x = 2$，$\dfrac{\partial z}{\partial y} = 2y = 4$. 因此切平面方程为

$$z-5 = 2(x-1)+4(y-2)，\text{即 } z=2x+4y-5.$$

法向量为 $\vec{n} = (2,4,-1)$，因此法线方程为

$$\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-5}{-1}.$$

(2) $(2a^2-z^2)x^2 = a^2y^2$，点 $P(a,a,a)$，其中 $a>0$

解：令 $F=(2a^2-z^2)x^2-a^2y^2$，则 F 一阶连续可微.

$\dfrac{\partial F}{\partial x} = (2a^2-z^2)\cdot 2x = 2a^3$，$\dfrac{\partial F}{\partial y} = -2a^2y = -2a^3$，$\dfrac{\partial F}{\partial z} = -2z\cdot x^2 = -2a^3$

因此切平面方程为

$$2a^3(x-a)-2a^3(y-a)-2a^3(z-a)=0$$

即 $y+z=x+a$.

法线方程为

$$\frac{x-a}{2a^3} = \frac{y-a}{-2a^3} = \frac{z-a}{-2a^3}，\text{即 } x-a=-y+a=-z+a.$$

(3). $\begin{cases} x=u+v \\ y=u^2+v^2 \\ z=u^3+v^3 \end{cases}$，点 $(u,v)=(1,2)$

解：$\dfrac{\partial(x,y,z)}{\partial(u,v)} = \begin{bmatrix} 1 & 1 \\ 2u & 2v \\ 3u^2 & 3v^2 \end{bmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 12 \end{pmatrix}$

因此，~~法线~~ 切平面方程为

$$(x-x_0)\cdot\begin{vmatrix}2&4\\3&12\end{vmatrix}+(y-y_0)\cdot\begin{vmatrix}3&12\\1&1\end{vmatrix}+(z-z_0)\begin{vmatrix}1&1\\2&4\end{vmatrix}=0$$

即 $(x-x_0)\cdot 12+(y-y_0)\cdot(-9)+(z-z_0)\cdot 2=0$

而 $x_0=3.\ y_0=5.\ z_0=9$

故 $12x-9y+2z=9$.

法线方程为 $\frac{x-3}{12}=\frac{y-5}{-9}=\frac{z-9}{2}$.

9. 在椭球面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ 上求点P使过该点的法线与坐标轴的正方向成等角

解：题意即可该法线与向量$(\frac{1}{\sqrt3},\frac{1}{\sqrt3},\frac{1}{\sqrt3})$平行. 设$P(x_0,y_0,z_0)$，则

$$\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}=1$$

令$F=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$，则F一阶连续可微.

$$\frac{\partial F}{\partial x}=\frac{2x}{a^2},\ \frac{\partial F}{\partial y}=\frac{2y}{b^2},\ \frac{\partial F}{\partial z}=\frac{2z}{c^2}$$

因此法线与向量$(\frac{2x_0}{a^2},\frac{2y_0}{b^2},\frac{2z_0}{c^2})$平行.

由此可知 $\frac{x_0}{a^2}=\frac{y_0}{b^2}=\frac{z_0}{c^2}$ 解得

$$P=(x_0,y_0,z_0)=\left(\frac{a^2}{\sqrt{a^2+b^2+c^2}},\frac{b^2}{\sqrt{a^2+b^2+c^2}},\frac{c^2}{\sqrt{a^2+b^2+c^2}}\right)$$

或 $\left(-\frac{a^2}{\sqrt{a^2+b^2+c^2}},-\frac{b^2}{\sqrt{a^2+b^2+c^2}},-\frac{c^2}{\sqrt{a^2+b^2+c^2}}\right)$

10. 求曲面 $x^2+2y^2+3z^2=21$ 上平行于 $x+4y+6z=0$ 的切平面

解：设切点为(x_0,y_0,z_0)，则 $x_0^2+2y_0^2+3z_0^2=21$.

令$F=x^2+2y^2+3z^2-21$，则F一阶连续可微

$$\frac{\partial F}{\partial x}=2x=2x_0,\ \frac{\partial F}{\partial y}=4y=4y_0,\ \frac{\partial F}{\partial z}=6z=6z_0$$

切平面方程为 $2x_0(x-x_0)+4y_0(y-y_0)+6z_0(z-z_0)=0$

因此 $\frac{2x_0}{1}=\frac{4y_0}{4}=\frac{6z_0}{6}$，故 $2x_0=y_0=z_0$

代入解得 $(x_0,y_0,z_0)=(1,2,2)$ 或 $(-1,-2,-2)$

因此切平面为 $x+4y+6z=21$ 或 $x+4y+6z=-21$.

11. 求曲线L： $\begin{cases} x^2+y^2+z^2=6 \\ x+y+z=0 \end{cases}$ 在P(1,-2,1)的切线与法平面方程

解： $F_1 = x^2+y^2+z^2-6$，$F_2 = x+y+z$，均为一阶连续可微.

$\frac{\partial F_1}{\partial x} = 2x = 2 \quad \frac{\partial F_1}{\partial y} = 2y = -4 \quad \frac{\partial F_1}{\partial z} = 2z = 2.$

$\frac{\partial F_2}{\partial x} = \frac{\partial F_2}{\partial y} = \frac{\partial F_2}{\partial z} = 1.$

因此切线方向为 $(2,-4,2)\times(1,1,1) = (-6,0,6)$

故切线方程为 $\begin{cases} \frac{x-1}{-6} = \frac{z-1}{6} \\ y+2=0 \end{cases}$ 即 $\begin{cases} x+z=2. \\ y=-2. \end{cases}$

法平面方程为 $-6(x-1)+6(z-1)=0$，即 $x=z$.

12. 证明：螺旋线 $\begin{cases} x=a\cos t \\ y=a\sin t \\ z=bt \end{cases}$ 的切线与z轴成定角.

证明： $\frac{dx}{dt} = -a\sin t$，$\frac{dy}{dt} = a\cos t$，$\frac{dz}{dt} = b$

因此切线的切向量为 $(-a\sin t, a\cos t, b)$.

其与z轴夹角 α 满足 $\cos\alpha = \frac{b}{\sqrt{a^2\sin^2 t+a^2\cos^2 t+b^2}} = \frac{b}{\sqrt{a^2+b^2}}$

因此 α 为定角.

13. 对 $\forall (x,y)\in\mathbb{R}^2$，定义 $f(x,y)=e^{x^2-y^2}$. 求f在原点一阶带Lagrange余项和二阶带Peano余项的Taylor公式

解： $\frac{\partial f}{\partial(x,y)} = \left(e^{x^2-y^2}\cdot 2x,\ e^{x^2-y^2}\cdot(-2y)\right).$

$$H_f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x\partial y} \\ \frac{\partial^2 f}{\partial x\partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} e^{x^2-y^2}\cdot(2+4x^2) & e^{x^2-y^2}(-4xy) \\ e^{x^2-y^2}(-4xy) & e^{x^2-y^2}(-2+4y^2) \end{bmatrix}$$

因此：一阶Lagrange余项为：$\exists\theta\in(0,1)$

$$f(x,y) = 1 + e^{(\theta x)^2-(\theta y)^2}\left[(x^2-y^2)+2\theta^2(x^2-y^2)^2\right]$$

二阶Peano余项为，当 $(x,y)\to(0,0)$

$$f(x,y) = 1 + x^2-y^2 + o(x^2+y^2)$$

14. 研究下列函数的极值.

(1) $z=e^{2x}(x+y^2+2y)$

解：~~驻点~~ z 为关于 x,y 的二阶连续可微函数.

取驻点，得 $0=\frac{\partial z}{\partial x}=e^{2x}(2x+2y^2+4y+1)$

$0=\frac{\partial z}{\partial y}=e^{2x}(2y+2)$

解出 $y=-1,\ x=\frac{1}{2}$

计算 $(\frac{1}{2},-1)$ 处海塞矩阵：

$$\frac{\partial^2 z}{\partial x^2}=e^{2x}(4x+4y^2+8y+2+2)=2e$$

$$\frac{\partial^2 z}{\partial y^2}=e^{2x}\cdot 2=2e$$

$$\frac{\partial^2 z}{\partial x\partial y}=2e^{2x}(2y+2)=0$$

因此 $H_z(\frac{1}{2},-1)=\begin{bmatrix}2e & 0\\ 0 & 2e\end{bmatrix}$ 正定

故这一点为极小值点.

综上知 此函数只有极小值，对应 $(x,y)=(\frac{1}{2},-1)$，$z=-\frac{e}{2}$

(2) $z=x_1+\frac{x_2}{x_1}+\frac{x_3}{x_2}+\cdots+\frac{x_n}{x_{n-1}}+\frac{2}{x_n}\quad(x_i>1,\ 1\le i\le n)$

解：z 在给定区域内为关于 $x_1\cdots x_n$ 的二阶连续可微函数

取驻点，得 $0=\frac{\partial z}{\partial x_1}=1-\frac{x_2}{x_1^2}$

$0=\frac{\partial z}{\partial x_i}=\frac{1}{x_{i-1}}-\frac{x_{i+1}}{x_i^2}\quad,i=2,3\cdots n-1$

$0=\frac{\partial z}{\partial x_n}=\frac{1}{x_{n-1}}-\frac{2}{x_n^2}$

故 $x_2=x_1^2,\quad x_{i+1}=\frac{x_i^2}{x_{i-1}}\ (i=2,3\cdots n-1),$

$x_n^2=2x_{n-1}.$

因此 $x_i=x_1^i\ (i=1,2\cdots n)$，由此得

$x_1^{2n}=2x_1^{n-1}$，得 $x_1=2^{\frac{1}{n+1}}$

此驻点为 $(x_1,\cdots,x_n)=(2^{\frac{1}{n+1}},2^{\frac{2}{n+1}},\cdots,2^{\frac{n}{n+1}})$

在这里 $z=2^{\frac{1}{n+1}}\cdot(n+1)$

另一方面，因 $x_i>1$，由算术几何不等式知

$z\ge(n+1)(2)^{\frac{1}{n+1}}$

因此该处取得全局最小值，故必为局部极小值.

综上所知，该函数只有极小值，对应

$(x_1,\cdots,x_n)=(2^{\frac{1}{n+1}},2^{\frac{2}{n+1}},\cdots,2^{\frac{n}{n+1}})$，$z=(n+1)2^{\frac{1}{n+1}}$

15. *求由方程 $x^2+\frac{y^2}{4}+\frac{1}{9}z^2-\frac{2}{3}z=0$ 所确定的隐函数 $z=z(x,y)$ 的极值.

解. 令 $F=x^2+\frac{y^2}{4}+\frac{1}{9}z^2-\frac{2}{3}z$，则 F 为二阶连续可微，因此 $z=z(x,y)$ 二阶连续可微

$$\frac{\partial z}{\partial(x,y)}=-\left(\frac{\partial F}{\partial z}\right)^{-1}\frac{\partial F}{\partial(x,y)}$$

$$=-\left(\frac{2}{9}z-\frac{2}{3}\right)^{-1}\left(2x,\frac{1}{2}y\right)$$ 故 $\begin{cases}\frac{\partial z}{\partial x}=\frac{-9x}{z-3}\\ \frac{\partial z}{\partial y}=\frac{-9y}{4(z-3)}\end{cases}$

取驻点，有 $x=y=0$. 代入得 $z=0$ 或 6

在驻点处考虑海森矩阵. 因为此时 $\frac{\partial z(x,y)}{\partial x}=\frac{\partial z(x,y)}{\partial y}=0$，有

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{-9x}{z(x,y)-3}\right)=\frac{-9}{z-3}$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{-9y}{4(z(x,y)-3)}\right)=\frac{-9}{4(z-3)}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{-9x}{z(x,y)-3}\right)=0$$

因此，在 $x=y=0$，$z=0$ 时，海森矩阵 $\begin{bmatrix}3 & 0\\ 0 & \frac{3}{4}\end{bmatrix}$ 正定；

$z=6$ 时海森矩阵 $\begin{bmatrix}-3 & 0\\ 0 & -\frac{3}{4}\end{bmatrix}$ 负定.

因此，$z=z(x,y)$ 只有极大值 6，极小值 0.