微积分 A (2)

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第 5 讲

第 4 讲回顾: 方向导数

- 方向导数的定义, 方向导数存在并不意味着偏导数存在.
- 若沿某一个坐标轴的偏导数存在, 则沿该轴正、反两方向的方向导数存在且互负.
- 函数在一点处沿任意的方向均有方向导数, 并不意味着函数在该点可微.
- 方向导数的表达式 (借助微分或偏导数).

回顾：数量场的梯度

- 梯度的定义及其意义.
- 当函数为可微时，其梯度可由偏导数构成的列向量表示，而方向导数则可为梯度与指示方向的单位向量的内积.
- 常值函数的梯度等于零；梯度满足与单变量函数求导类似的四则运算及复合法则.
- 典型问题：求函数在一点处的梯度与最大的方向导数，以及沿某一向量的方向导数.

回顾：高阶偏导数

- 二阶偏导数: $\frac{\partial^2 f}{\partial x_j \partial x_i} = \partial_{ji} f = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$, $\frac{\partial^2 f}{\partial x_i^2}$.
- k 阶偏导数: $\frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}$.
- 求偏导数一般不能交换次序.
- 设 $\Omega \subset \mathbb{R}^n$ 为开集. 若 $f : \Omega \to \mathbb{R}$ 在 Ω 上有二阶偏导函数 $\frac{\partial^2 f}{\partial x_j \partial x_i}$, $\frac{\partial^2 f}{\partial x_i \partial x_j}$, 并且当中一个在点 $X_0 \in \Omega$ 连续, 则 $\frac{\partial^2 f}{\partial x_j \partial x_i}(X_0) = \frac{\partial^2 f}{\partial x_i \partial x_j}(X_0)$.

回顾: 函数空间 $\mathscr{C}^{(k)}(\Omega)$

- 空间 $\mathscr{C}^{(k)}(\Omega)$ ($k \geqslant 0$ 为整数).
- 若 $f \in \mathscr{C}^{(k)}(\Omega)$, 则称之在 Ω 上为 k 阶连续可导或 k 阶连续可微.
- 设 $k \geqslant 2$ 为整数. 若 $f \in \mathscr{C}^{(k)}(\Omega)$, 则对任意整数 $1 \leqslant r \leqslant k$, 均有 $f \in \mathscr{C}^{(r)}(\Omega)$ 并且 f 的任意 r 阶偏导数均与求偏导的次序无关.

回顾: 向量值函数的微分

- 线性映射与矩阵的关系.
- $\vec{f}(X)=\vec{o}(|g(X)|)=|g(X)|\vec{o}(1)\ (X\to X_0)$.
- 定义: 微分, Jacobi 矩阵, Jacobi 行列式.
- 微分的唯一性. 可微性蕴含连续性.
- 向量值函数的微分与其各分量函数的微分之间的关系.

第 5 讲

可微复合向量值函数的微分

回顾: 矩阵的范数. 令 $A=(a_{ij})_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}}$. 定义

$$\|A\|=\Big(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\Big)^{\frac{1}{2}},$$

称为矩阵 A 的范数. $\forall X=(x_1,\ldots,x_n)^T\in\mathbb{R}^n$, 令 $Y=AX=(y_1,\ldots,y_m)^T$, 则我们有

$$y_i=\sum_{j=1}^{n}a_{ij}x_j,$$

由此可立刻导出

$$\|Y\|_m^2 = \sum_{i=1}^{m} |y_i|^2 = \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} a_{ij}x_j\Big)^2$$

$$\leqslant \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} |a_{ij}||x_j|\Big)^2 \leqslant \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} |a_{ij}|^2\Big)\Big(\sum_{j=1}^{n} |x_j|^2\Big)$$

$$= \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} |a_{ij}|^2\Big)\|X\|_n^2 = \|A\|^2\|X\|_n^2,$$

从而我们有 $\|AX\|_m = \|Y\|_m \leqslant \|A\| \cdot \|X\|_n$.

定理 1. 假设 $\Omega_1 \subseteq \mathbb{R}^n$, $\Omega_2 \subseteq \mathbb{R}^m$ 均为非空开集, $X_0 \in \Omega_1$, 而映射 $\vec{g} : \Omega_1 \to \Omega_2$ 在点 X_0 处可微, $\vec{f} : \Omega_2 \to \mathbb{R}^k$ 在点 $Y_0 = \vec{g}(X_0)$ 处可微, 则 $\vec{f} \circ \vec{g}$ 在点 X_0 处可微, 并且

$$\mathrm{d}(\vec{f} \circ \vec{g})(X_0) = \mathrm{d}\vec{f}(Y_0) \circ \mathrm{d}\vec{g}(X_0).$$

证明: 令 $A = \mathrm{d}\vec{g}(X_0)$, $B = \mathrm{d}\vec{f}(Y_0)$, 则我们有

$$\vec{g}(X) - \vec{g}(X_0) = A(X - X_0) + \vec{o}(\|X - X_0\|_n) \ (X \to X_0),$$
$$\vec{f}(Y) - \vec{f}(Y_0) = B(Y - Y_0) + \vec{o}(\|Y - Y_0\|_m) \ (Y \to Y_0).$$

于是当 $X \to X_0$ 时, 我们有

$$\begin{aligned}
&\|\vec{g}(X) - \vec{g}(X_0)\|_m \\
&= \|A(X - X_0) + \vec{o}(\|X - X_0\|_n)\|_m \\
&\leqslant \|A(X - X_0)\|_m + \|\|X - X_0\|_n \vec{o}(1)\|_m \\
&\leqslant \|A\| \cdot \|X - X_0\|_n + \|X - X_0\|_n o(1) \\
&= \|X - X_0\|_n O(1).
\end{aligned}$$

$$\begin{aligned}
&\|B\big(\vec{o}(\|X - X_0\|_n)\big)\|_k \leqslant \|B\| \cdot \|\|X - X_0\|_n \vec{o}(1)\|_m \\
&\leqslant \|B\| \cdot \|X - X_0\|_n o(1) = \|X - X_0\|_n o(1).
\end{aligned}$$

从而当 $X \to X_0$ 时, 我们有

$$\begin{aligned}
\vec{f}\circ\vec{g}(X) - \vec{f}\circ\vec{g}(X_0) &= B(\vec{g}(X) - \vec{g}(X_0)) \\
&\qquad + \vec{o}(\|\vec{g}(X) - \vec{g}(X_0)\|_m) \\
&= B\big(A(X - X_0) + \vec{o}(\|X - X_0\|_n)\big) \\
&\qquad + \|\vec{g}(X) - \vec{g}(X_0)\|_m \vec{o}(1) \\
&= B\circ A(X - X_0) + \|X - X_0\|_n \vec{o}(1) \\
&\qquad + \|X - X_0\|_n O(1)\vec{o}(1) \\
&= B\circ A(X - X_0) + \|X - X_0\|_n \vec{o}(1).
\end{aligned}$$

由微分的定义可知 $\vec{f}\circ\vec{g}$ 在点 X_0 可微且其微分为 $B\circ A$, 即 $\mathrm{d}(\vec{f}\circ\vec{g})(X_0) = \mathrm{d}\vec{f}(Y_0)\circ\mathrm{d}\vec{g}(X_0)$.

可微复合向量值函数微分的矩阵表示

- $J_{\vec{f}\circ\vec{g}}(X_0) = J_{\vec{f}}(\vec{g}(X_0)) \cdot J_{\vec{g}}(X_0)$.
- 记 $\vec{g} = (g_1, \ldots, g_m)^T$, $\vec{f} = (f_1, \ldots, f_k)^T$, 则

$$\frac{\partial(f_1 \circ \vec{g}, \ldots, f_k \circ \vec{g})}{\partial(x_1, \ldots, x_n)}\Big|_{X_0} = \frac{\partial(f_1, \ldots, f_k)}{\partial(y_1, \ldots, y_m)}\Big|_{\vec{g}(X_0)} \cdot \frac{\partial(g_1, \ldots, g_m)}{\partial(x_1, \ldots, x_n)}\Big|_{X_0}.$$

- 当 $k = 1$ 时, 我们有

$$\begin{aligned} \frac{\partial(f \circ \vec{g})}{\partial(x_1, \ldots, x_n)} &= (\frac{\partial f \circ \vec{g}}{\partial x_1}, \ldots, \frac{\partial f \circ \vec{g}}{\partial x_n}), \\ \frac{\partial(f)}{\partial(y_1, \ldots, y_m)} &= (\frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_m}), \end{aligned}$$

再注意到

$$\frac{\partial(g_1,\ldots,g_m)}{\partial(x_1,\ldots,x_n)}=\begin{pmatrix}\frac{\partial g_1}{\partial x_1}(X_0) & \cdots & \frac{\partial g_1}{\partial x_n}(X_0)\\ \vdots & \cdots & \vdots\\ \frac{\partial g_m}{\partial x_1}(X_0) & \cdots & \frac{\partial g_m}{\partial x_n}(X_0)\end{pmatrix},$$

于是对任意整数 $1\leqslant i\leqslant n$, 我们有

$$\frac{\partial(f\circ\vec{g})}{\partial x_i}(X_0)=\sum_{j=1}^{m}\frac{\partial f}{\partial y_j}(\vec{g}(X_0))\frac{\partial g_j}{\partial x_i}(X_0).$$

$$\partial_i(f\circ\vec{g})(X_0)=\sum_{j=1}^{m}\partial_j f(\vec{g}(X_0))\partial_i g_j(X_0).$$

也即我们有

$$\frac{\partial f(g_1,\ldots,g_m)}{\partial x_i}(X_0)=\sum_{j=1}^{m}\frac{\partial f}{\partial y_j}(Y_0)\frac{\partial g_j}{\partial x_i}(X_0)$$

$$=\frac{\partial f}{\partial y_1}(Y_0)\frac{\partial g_1}{\partial x_i}(X_0)+\frac{\partial f}{\partial y_2}(Y_0)\frac{\partial g_2}{\partial x_i}(X_0)+\cdots+\frac{\partial f}{\partial y_m}(Y_0)\frac{\partial g_m}{\partial x_i}(X_0),$$

$$\frac{\partial f(g_1,\ldots,g_m)}{\partial x_i}=\sum_{j=1}^{m}\frac{\partial f}{\partial y_j}\frac{\partial g_j}{\partial x_i}$$

$$=\frac{\partial f}{\partial y_1}\frac{\partial g_1}{\partial x_i}+\frac{\partial f}{\partial y_2}\frac{\partial g_2}{\partial x_i}+\cdots+\frac{\partial f}{\partial y_m}\frac{\partial g_m}{\partial x_i}.$$

例 2. 假设 $z = f(u,v) = u^2 v - uv^2$, $u = x\sin y$, $v = x\cos y$. 求 $\frac{\partial z}{\partial x}$.

解: 由题设可得

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial (x\sin y)}{\partial x} + \frac{\partial f}{\partial (x\cos y)}\frac{\partial v}{\partial x} \quad \text{(严重错误!)} \\
&= \frac{\partial f}{\partial u}\frac{\partial (x\sin y)}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial (x\cos y)}{\partial x} \\
&= (2uv - v^2)\sin y + (u^2 - 2uv)\cos y \\
&= (2x^2\sin y\cos y - x^2\cos^2 y)\sin y \\
&\quad +(x^2\sin^2 y - 2x^2\sin y\cos y)\cos y \\
&= \frac{3}{2}x^2(\sin y - \cos y)\sin(2y).
\end{aligned}$$

例 3. 设 $z=f(xy,x^2-y^2)$, f 可微. 求 $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

解: 由题设可知

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= \partial_1 f(xy,x^2-y^2)\frac{\partial(xy)}{\partial x} \\
&\quad +\partial_2 f(xy,x^2-y^2)\frac{\partial(x^2-y^2)}{\partial x} \\
&= y\partial_1 f(xy,x^2-y^2)+2x\partial_2 f(xy,x^2-y^2). \\
\frac{\partial z}{\partial y} &= \partial_1 f(xy,x^2-y^2)\frac{\partial(xy)}{\partial y} \\
&\quad +\partial_2 f(xy,x^2-y^2)\frac{\partial(x^2-y^2)}{\partial y} \\
&= x\partial_1 f(xy,x^2-y^2)-2y\partial_2 f(xy,x^2-y^2).
\end{aligned}
$$

例 4. 设 $z=\frac{y}{x}+xyf(\frac{y}{x})$, f 可微, 求 $\frac{\partial z}{\partial x}$.

解: 由题设可得

$$\begin{aligned}\frac{\partial z}{\partial x} &= -\frac{y}{x^2}+yf(\frac{y}{x})+xy\cdot f'(\frac{y}{x})\cdot(-\frac{y}{x^2}) \\ &= -\frac{y}{x^2}+yf(\frac{y}{x})-\frac{y^2}{x}f'(\frac{y}{x}).\end{aligned}$$

例 5. 设 $z=f(g_1(x_1,\ldots,x_n),\ldots,g_m(x_1,\ldots,x_n))$, $f,g_1,\ldots,g_m$ 二阶可微, 求 $\frac{\partial^2 z}{\partial x_i\partial x_j}$ $(1\leqslant i,j\leqslant n)$.

解:
$$\frac{\partial^2 z}{\partial x_i\partial x_j}=\frac{\partial}{\partial x_i}\Big(\frac{\partial z}{\partial x_j}\Big)=\frac{\partial}{\partial x_i}\Big(\sum_{k=1}^m\frac{\partial f}{\partial y_k}(*)\frac{\partial g_k}{\partial x_j}\Big)$$
$$=\sum_{k=1}^m\Big[\frac{\partial}{\partial x_i}\Big(\frac{\partial f}{\partial y_k}(*)\Big)\frac{\partial g_k}{\partial x_j}+\frac{\partial f}{\partial y_k}(*)\frac{\partial}{\partial x_i}\Big(\frac{\partial g_k}{\partial x_j}\Big)\Big]$$
$$=\sum_{k=1}^m\Big[\Big[\sum_{l=1}^m\frac{\partial}{\partial y_l}\Big(\frac{\partial f}{\partial y_k}\Big)(*)\frac{\partial g_l}{\partial x_i}\Big]\frac{\partial g_k}{\partial x_j}+\frac{\partial f}{\partial y_k}(*)\frac{\partial^2 g_k}{\partial x_i\partial x_j}\Big]$$
$$=\sum_{k=1}^m\Big[\sum_{l=1}^m\frac{\partial^2 f}{\partial y_l\partial y_k}(*)\frac{\partial g_l}{\partial x_i}\frac{\partial g_k}{\partial x_j}+\frac{\partial f}{\partial y_k}(*)\frac{\partial^2 g_k}{\partial x_i\partial x_j}\Big].$$

例 6. (Laplace 方程) 定义 $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, $r = \sqrt{x^2 + y^2 + z^2}$. 求证: 在 $\mathbb{R}^3 \setminus \{(0,0,0)\}$ 上,

$$\Delta\left(\frac{1}{r}\right) = \frac{\partial^2\left(\frac{1}{r}\right)}{\partial x^2} + \frac{\partial^2\left(\frac{1}{r}\right)}{\partial y^2} + \frac{\partial^2\left(\frac{1}{r}\right)}{\partial z^2} = 0.$$

证明: 在 $\mathbb{R}^3 \setminus \{(0,0,0)\}$ 上, 我们有

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{\partial r}{\partial x} = -\frac{1}{r^2}\frac{x}{\sqrt{x^2+y^2+z^2}} = -\frac{x}{r^3}.$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3x}{r^4}\frac{\partial r}{\partial x} = -\frac{1}{r^3} + \frac{3x^2}{r^5}.$$

于是由对称性可得

$$\begin{aligned}\Delta\Big(\frac{1}{r}\Big) &= \Big(-\frac{1}{r^3}+\frac{3x^2}{r^5}\Big)+\Big(-\frac{1}{r^3}+\frac{3y^2}{r^5}\Big)\\&\quad+\Big(-\frac{1}{r^3}+\frac{3z^2}{r^5}\Big)\\&= -\frac{3}{r^3}+\frac{3(x^2+y^2+z^2)}{r^5}=0.\end{aligned}$$

作业题: 第 1.5 节第 54 页第 3 题第 (1) 小题, 第 5 题, 第 7 题, 第 9 题第 (1) 小题.

§6. 隐 (向量值) 函数、反 (向量值) 函数的存在性及其微分

问题: 如何解方程 $F(x, y) = 0$? 具体来说, 如何从方程 $F(x, y) = 0$ 出发来求解 $y = y(x)$?

线性的情形: 假设 $F(x, y) = ax + by + c$. 此时可从 $F(x, y) = 0$ 解出 y 当且仅当 $\frac{\partial F}{\partial y} = b \neq 0$, 这时我们有 $y = -\frac{1}{b}(ax + c)$.

圆周: 现在考虑方程 $F(x,y) := x^2 + y^2 - 1 = 0$.
此时我们有 $y = \pm\sqrt{1-x^2}$.

- 当 $y > 0$ 时, $y = \sqrt{1-x^2}$, $\dfrac{\partial F}{\partial y} = 2y > 0$.
- 当 $y < 0$ 时, $y = -\sqrt{1-x^2}$, $\dfrac{\partial F}{\partial y} = 2y < 0$.
- 在 $(1,0)$ 的附近, 无法求 y, 而 $\dfrac{\partial F}{\partial y}(1,0) = 0$.

启示: 方程 $F(x,y) = 0$ 有解 $y = y(x)$ 与 $\dfrac{\partial F}{\partial y}$ 是否等于零有关?

隐函数定理

定理 1. 设 $X_0=(x_0,y_0)\in\mathbb{R}^2$, $r>0$, 而数量值函数 $F:B(X_0,r)\to\mathbb{R}$ 为 $\mathscr{C}^{(1)}$ 类的函数使得 $F(x_0,y_0)=0$, $\frac{\partial F}{\partial y}(x_0,y_0)\neq 0$. 则 $\exists\delta,\eta>0$ 使得 $B(x_0,\delta)\times B(y_0,\eta)\subset B(X_0,r)$ 且 $\forall x\in B(x_0,\delta)$, $\exists! y\in B(y_0,\eta)$ 使得 $F(x,y)=0$. 定义 $f(x)=y$. 则 $f:B(x_0,\delta)\to B(y_0,\eta)$ 为 $\mathscr{C}^{(1)}$ 类函数, 并且 $\forall x\in B(x_0,\delta)$, 均有 $f'(x)=-\frac{\frac{\partial F}{\partial x}(x,f(x))}{\frac{\partial F}{\partial y}(x,f(x))}$.

证明: 不失一般性, 我们可假设 $\frac{\partial F}{\partial y}(x_0, y_0) > 0$. 否则考虑函数 $-F$.

存在性: 由题设可知 $\frac{\partial F}{\partial y}$ 连续, 则 $\exists \eta > 0$ 使得 $\forall (x, y) \in B(X_0, \sqrt{2}\eta) \subsetneq B(X_0, r)$, $\frac{\partial F}{\partial y}(x, y) > 0$.

$\forall (x, y) \in B(X_0, \sqrt{2}\eta)$, 我们令 $g_x(y) = F(x, y)$. 则对于每个固定的 $x \in [x_0 - \eta, x_0 + \eta]$, 函数 g_x 在 $[y_0 - \eta, y_0 + \eta]$ 上可导且 $\frac{\mathrm{d}g_x}{\mathrm{d}y}(y) = \frac{\partial F}{\partial y}(x, y) > 0$, 从而 g_{x_0} 为严格递增函数. 又 $g_{x_0}(y_0) = 0$, 故

$$\begin{aligned} F(x_0, y_0 - \eta) &= g_{x_0}(y_0 - \eta) < g_{x_0}(y_0) = 0 \\ &< g_{x_0}(y_0 + \eta) = F(x_0, y_0 + \eta). \end{aligned}$$

注意到 F 连续, 于是由连续函数的保号性知, $\exists\delta\in(0,\eta)$ 使得 $\forall x\in(x_0-\delta,x_0+\delta)$, 均有

$$\begin{aligned} g_x(y_0-\eta) &= F(x,y_0-\eta)<0, \\ g_x(y_0+\eta) &= F(x,y_0+\eta)>0. \end{aligned}$$

$\forall y\in[y_0-\eta,y_0+\eta]$, 均有 $\frac{\mathrm{d}g_x}{\mathrm{d}y}(y)=\frac{\partial F}{\partial y}(x,y)>0$, 因此 g_x 在 $[y_0-\eta,y_0+\eta]$ 上严格递增且连续, 由连续函数介值定理, $\exists! y\in(y_0-\eta,y_0+\eta)$ 使得 $F(x,y)=g_x(y)=0$. 令 $f(x)=y$. 则 f 为所求.

连续性: 由前面讨论知, $\forall \varepsilon \in (0,\eta)$, $\exists \delta' \in (0,\varepsilon)$ 使 $\forall x \in B(x_0,\delta')$, $\exists! y \in B(y_0,\varepsilon)$ 使 $F(x,y)=0$, 此时 $y=f(x)$, 也即当 $|x-x_0|<\delta'$ 时, 我们有 $|f(x)-f(x_0)|<\varepsilon$. 故函数 f 在点 x_0 处连续.

取 $x_1 \in B(x_0,\delta)$, $y_1=f(x_1)$, 则 $F(x_1,y_1)=0$ 且 $(x_1,y_1)\in B((x_0,y_0),\sqrt{2}\eta)$, 故 $\frac{\partial F}{\partial y}(x_1,y_1)>0$. 由前面的讨论可知, 存在 $\delta_1\in(0,\delta), \eta_1\in(0,\eta)$ 以及在 x_1 连续的函数 $g: B(x_1,\delta_1)\to B(y_1,\eta_1)$ 使 $F(x,g(x))=0$. 另外可设 $B(x_1,\delta_1)\subset B(x_0,\delta)$.

由唯一性知 $\forall x \in B(x_1, \delta_1)$, 均有 $f(x) = g(x)$, 故 f 在点 x_1 处连续.

可导性: 取 $x \in B(x_0, \delta)$, $h \in \mathbb{R}$ 使 $x+h \in B(x_0, \delta)$. 令 $y = f(x)$, $\Delta y = f(x+h) - f(x)$. 由 Lagrange 中值定理可知, $\exists \theta_1, \theta_2 \in (0,1)$ 使得

$$\begin{aligned} 0 &= F(x+h, y+\Delta y) - F(x, y) \\ &= \big(F(x+h, y+\Delta y) - F(x, y+\Delta y)\big) \\ &\quad + \big(F(x, y+\Delta y) - F(x, y)\big) \\ &= \frac{\partial F}{\partial x}(x+\theta_1 h, y+\Delta y)h + \frac{\partial F}{\partial y}(x, y+\theta_2 \Delta y)\Delta y. \end{aligned}$$

由于 $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ 均连续, 于是由夹逼原理以及复合函数极限法则可知

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= -\lim_{h\to 0} \frac{\frac{\partial F}{\partial x}(x+\theta_1 h, y+\Delta y)}{\frac{\partial F}{\partial y}(x, y+\theta_2\Delta y)} \\ &= -\frac{\frac{\partial F}{\partial x}(x,y)}{\frac{\partial F}{\partial y}(x,y)} = -\frac{\frac{\partial F}{\partial x}(x,f(x))}{\frac{\partial F}{\partial y}(x,f(x))}. \end{aligned}$$

上式同时表明 f' 为连续函数, 故 f 连续可导.

定理 2. 设 $X_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}$, $r > 0$, 而数量值函数 $F: B((X_0, y_0), r) \to \mathbb{R}$ 为 $\mathscr{C}^{(1)}$ 类使 $F(X_0, y_0) = 0$, $\frac{\partial F}{\partial y}(X_0, y_0) \neq 0$. 则 $\exists \delta, \eta > 0$ 使得我们有

$$B(X_0, \delta) \times B(y_0, \eta) \subset B((X_0, y_0); r),$$

且 $\forall X \in B(X_0, \delta)$, $\exists! y \in B(y_0, \eta)$ 使 $F(X, y) = 0$. 令 $f(X) = y$. 则 $f: B(X_0, \delta) \to B(y_0, \eta)$ 为 $\mathscr{C}^{(1)}$ 类且 $\forall X \in B(X_0, \delta)$ 与任意整数 $1 \leqslant i \leqslant n$, 均有

$$\frac{\partial f}{\partial x_i}(X) = -\frac{\frac{\partial F}{\partial x_i}(X, f(X))}{\frac{\partial F}{\partial y}(X, f(X))}.$$

评注

上述最后一个等式可由对恒等式

$$F(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) = 0$$

求偏导数而得. 事实上, 对 x_i 求偏导数可得

$$\frac{\partial F}{\partial x_i}(X, f(X)) + \frac{\partial F}{\partial y}(X, f(X))\frac{\partial f}{\partial x_i}(X) = 0,$$

由此我们可立刻导出

$$\frac{\partial f}{\partial x_i}(X) = -\frac{\frac{\partial F}{\partial x_i}(X, f(X))}{\frac{\partial F}{\partial y}(X, f(X))}.$$

定理 3. 设 $X_0\in\mathbb{R}^n$, $Y_0\in\mathbb{R}^m$, $r>0$, 向量值函数 $\vec{F}=(F_1,\ldots,F_m)^T\colon B((X_0,Y_0),r)\to\mathbb{R}^m$ 为 $\mathscr{C}^{(1)}$ 类使得 $\vec{F}(X_0,Y_0)=\vec{0}$, $\frac{\partial(F_1,\ldots,F_m)}{\partial(y_1,\ldots,y_m)}(X_0,Y_0)$ 可逆. 那么 $\exists\delta,\eta>0$ 使 $B(X_0,\delta)\times B(Y_0,\eta)\subset B((X_0,Y_0);r)$ 且 $\forall X\in B(X_0,\delta)$, $\exists! Y\in B(Y_0,\eta)$ 使 $\vec{F}(X,Y)=0$. 令 $\vec{f}(X)=Y$. 则 $\vec{f}\colon B(X_0,\delta)\to B(Y_0,\eta)$ 为 $\mathscr{C}^{(1)}$ 类, 并且 $\forall X\in B(X_0,\delta)$, 我们均有

$$J_{\vec{f}}(X)=-\left(\frac{\partial(F_1,\ldots,F_m)}{\partial(y_1,\ldots,y_m)}(X,\vec{f}(X))\right)^{-1}\cdot\frac{\partial(F_1,\ldots,F_m)}{\partial(x_1,\ldots,x_n)}(X,\vec{f}(X)).$$

评注

- 上述定理也可表述成: $\forall X \in B(X_0, \delta)$ 以及 $\forall Y \in B(Y_0, \eta)$, 等式 $\vec{F}(X, Y) = \vec{0}$ 成立当且仅当我们有 $Y = \vec{f}(X)$.
- 若将 $\mathscr{C}^{(1)}$ 换成 $\mathscr{C}^{(k)}$ $(k \geqslant 1)$, 定理依然成立.
- 将 $F_i(X, \vec{f}(X)) = 0$ 对 x_j 求偏导可得

$$\frac{\partial F_i}{\partial x_j}(X, \vec{f}(X)) + \sum_{l=1}^{m} \frac{\partial F_i}{\partial y_l}(X, \vec{f}(X)) \frac{\partial f_l}{\partial x_j}(X) = 0,$$

进而我们可以导出

$$\frac{\partial(F_1,\ldots,F_m)}{\partial(x_1,\ldots,x_n)}(X,\vec{f}(X)) \\ +\frac{\partial(F_1,\ldots,F_m)}{\partial(y_1,\ldots,y_m)}(X,\vec{f}(X))\cdot\frac{\partial(f_1,\ldots,f_m)}{\partial(x_1,\ldots,x_n)}(X)=\vec{0},$$

于是我们有

$$\frac{\partial(f_1,\ldots,f_m)}{\partial(x_1,\ldots,x_n)}(X)=-\left(\frac{\partial(F_1,\ldots,F_m)}{\partial(y_1,\ldots,y_m)}(X,\vec{f}(X))\right)^{-1} \\ \cdot\frac{\partial(F_1,\ldots,F_m)}{\partial(x_1,\ldots,x_n)}(X,\vec{f}(X)).$$

例 1. $\forall (x,y,z)\in\mathbb{R}^3$, 定义

$$F(x,y,z)=x(1+yz)+e^{x+y+z}-1.$$

问方程 $F(x,y,z)=0$ 是否能在原点的附近确定一个隐函数 $z=f(x,y)$? 如果能, 求该隐函数在点 $(0,0)$ 处的偏导数.

解: 由题设可知 F 为初等函数, 从而为 $\mathscr{C}^{(1)}$ 类并且我们还有 $F(0,0,0)=0$, $\frac{\partial F}{\partial z}=xy+e^{x+y+z}$. 于是 $\frac{\partial F}{\partial z}(0,0,0)=1\neq 0$, 因此方程 $F(x,y,z)=0$ 能在原点附近确定一个隐函数 $z=f(x,y)$.

另外, 我们还有

$$
\begin{aligned}
\frac{\partial f}{\partial x}(0,0) &= -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}\Bigg|_{(0,0,0)} \\
&= -\frac{1+yz+e^{x+y+z}}{xy+e^{x+y+z}}\Bigg|_{(0,0,0)} = -2.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial f}{\partial y}(0,0) &= -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}\Bigg|_{(0,0,0)} \\
&= -\frac{xz+e^{x+y+z}}{xy+e^{x+y+z}}\Bigg|_{(0,0,0)} = -1.
\end{aligned}
$$

例 2. 设 F 为 $\mathscr{C}^{(2)}$ 类, 则由方程 $F(x,y,z)=0$ 确定的隐函数 $z=f(x,y)$ 为 $\mathscr{C}^{(2)}$ 类, 求 $\frac{\partial^2 z}{\partial y\partial x}$.

解: 令 $u=\frac{\partial F}{\partial z}(x,y,z(x,y))\neq 0$. 由题设可得

$$
\begin{aligned}
\frac{\partial^2 z}{\partial y\partial x} &= \frac{\partial}{\partial y}\Bigg(-\frac{\frac{\partial F}{\partial x}(x,y,z(x,y))}{\frac{\partial F}{\partial z}(x,y,z(x,y))}\Bigg)\\
&= -\frac{1}{u^2}\Big[\frac{\partial}{\partial y}\Big(\frac{\partial F}{\partial x}(x,y,z(x,y))\Big)\frac{\partial F}{\partial z}(x,y,z(x,y))\\
&\quad -\frac{\partial F}{\partial x}(x,y,z(x,y))\frac{\partial}{\partial y}\Big(\frac{\partial F}{\partial z}(x,y,z(x,y))\Big)\Big]
\end{aligned}
$$

$$\frac{\partial^2 z}{\partial y \partial x} = -\frac{1}{u^2}\Big[\frac{\partial}{\partial y}\Big(\frac{\partial F}{\partial x}(x,y,z(x,y))\Big)\frac{\partial F}{\partial z} - \frac{\partial F}{\partial x}\frac{\partial}{\partial y}\Big(\frac{\partial F}{\partial z}(x,y,z(x,y))\Big)\Big]$$

$$= -\frac{1}{u^2}\Big[\Big(\frac{\partial^2 F}{\partial y \partial x} + \frac{\partial^2 F}{\partial z \partial x}\frac{\partial z}{\partial y}\Big)\frac{\partial F}{\partial z} - \frac{\partial F}{\partial x}\Big(\frac{\partial^2 F}{\partial y \partial z} + \frac{\partial^2 F}{\partial z^2}\frac{\partial z}{\partial y}\Big)\Big]$$

$$= -\frac{1}{u^2}\Big[\Big[\frac{\partial^2 F}{\partial y \partial x} + \frac{\partial^2 F}{\partial z \partial x}\Big(-\frac{\frac{\partial F}{\partial y}}{u}\Big)\Big]\frac{\partial F}{\partial z} - \frac{\partial F}{\partial x}\Big[\frac{\partial^2 F}{\partial y \partial z} + \frac{\partial^2 F}{\partial z^2}\Big(-\frac{\frac{\partial F}{\partial y}}{u}\Big)\Big]\Big]$$

$$= -\frac{1}{u^3}\Big[\Big(\frac{\partial F}{\partial z}\Big)^2\frac{\partial^2 F}{\partial y \partial x} - \frac{\partial F}{\partial y}\frac{\partial F}{\partial z}\frac{\partial^2 F}{\partial z \partial x} - \frac{\partial F}{\partial x}\frac{\partial F}{\partial z}\frac{\partial^2 F}{\partial y \partial z} + \frac{\partial F}{\partial x}\frac{\partial F}{\partial y}\frac{\partial^2 F}{\partial z^2}\Big]$$

$$= -\frac{\left(\frac{\partial F}{\partial z}\right)^2\frac{\partial^2 F}{\partial y \partial x} - \frac{\partial F}{\partial y}\frac{\partial F}{\partial z}\frac{\partial^2 F}{\partial z \partial x} - \frac{\partial F}{\partial x}\frac{\partial F}{\partial z}\frac{\partial^2 F}{\partial y \partial z} + \frac{\partial F}{\partial x}\frac{\partial F}{\partial y}\frac{\partial^2 F}{\partial z^2}}{\left(\frac{\partial F}{\partial z}\right)^3}.$$

例 3. 求证: 下述方程组

$$\begin{cases} F_1(x,y,u,v) = 3x^2 + y^2 + u^2 + v^2 - 1 = 0, \\ F_2(x,y,u,v) = x^2 + 2y^2 - u^2 + v^2 - 1 = 0, \end{cases}$$

在点 $P_0(0, \frac{1}{2}, \sqrt{\frac{1}{8}}, \sqrt{\frac{5}{8}})$ 的某邻域内确定了一个向量值函数 $\begin{pmatrix} u \\ v \end{pmatrix} = \vec{f}(x,y)$, 并计算该向量值函数 $\vec{f}$ 在点 $(0, \frac{1}{2})$ 处的 Jacobi 矩阵与微分.

解: 由于 F_1, F_2 均为初等函数, 因此为 $\mathscr{C}^{(1)}$ 类.

又由题设可知 $F_1(P_0) = F_2(P_0) = 0$, 并且

$$\frac{D(F_1, F_2)}{D(u, v)}(P_0) = \left|\begin{vmatrix} 2u & 2v \\ -2u & 2v \end{vmatrix}\right|_{P_0} = 8uv\Big|_{P_0} = \sqrt{5},$$

从而 $\frac{\partial(F_1,F_2)}{\partial(u,v)}(P_0)$ 为可逆矩阵, 于是在点 P_0 的邻域内, 上述方程组可确定一个向量值函数

$$\begin{pmatrix} u \\ v \end{pmatrix} = \vec{f}(x, y),$$

进而可知所求 Jacobi 矩阵为

$$\frac{\partial(u,v)}{\partial(x,y)}(0,\frac{1}{2}) = -\Big(\frac{\partial(F_1,F_2)}{\partial(u,v)}(P_0)\Big)^{-1}\frac{\partial(F_1,F_2)}{\partial(x,y)}(P_0)$$

$$= -\left(\begin{array}{cc} 2u & 2v \\ -2u & 2v \end{array}\right)^{-1}\Bigg|_{P_0}\left(\begin{array}{cc} 6x & 2y \\ 2x & 4y \end{array}\right)\Bigg|_{P_0}$$

$$= -\left(\begin{array}{cc} 2\sqrt{\frac{1}{8}} & 2\sqrt{\frac{5}{8}} \\ -2\sqrt{\frac{1}{8}} & 2\sqrt{\frac{5}{8}} \end{array}\right)^{-1}\left(\begin{array}{cc} 0 & 1 \\ 0 & 2 \end{array}\right)$$

$$= -\frac{1}{\sqrt{5}}\left(\begin{array}{cc} 2\sqrt{\frac{5}{8}} & -2\sqrt{\frac{5}{8}} \\ 2\sqrt{\frac{1}{8}} & 2\sqrt{\frac{1}{8}} \end{array}\right)\left(\begin{array}{cc} 0 & 1 \\ 0 & 2 \end{array}\right) = \left(\begin{array}{cc} 0 & \frac{\sqrt{2}}{2} \\ 0 & -\frac{3\sqrt{10}}{10} \end{array}\right).$$

于是所求微分为

$$\mathrm{d}\vec{f}(0,\frac{1}{2}) = \begin{pmatrix} \mathrm{d}u \\ \mathrm{d}v \end{pmatrix}\Big|_{(0,\frac{1}{2})}$$

$$= \begin{pmatrix} 0 & \frac{\sqrt{2}}{2} \\ 0 & -\frac{3\sqrt{10}}{10} \end{pmatrix}\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2}\,\mathrm{d}y \\ -\frac{3\sqrt{10}}{10}\,\mathrm{d}y \end{pmatrix}.$$

作业题: 第 1.6 节第 65 页第 2 题第 (2) 小题, 第 66 页第 6 题.

反函数定理

问题: 向量值函数 $\vec{g}:\Omega\subset\mathbb{R}^n\to\mathbb{R}^n$ 是否能有反函数 $\vec{g}^{-1}$? 这等价于问方程 $X=\vec{g}(Y)$ 是否有解 $Y=\vec{g}^{-1}(X)$, 也即方程

$$F(X,Y):=\vec{g}(Y)-X=\vec{0}$$

是否有隐函数解 $Y=\vec{g}^{-1}(X)$?

定理 4. 设 $k \geqslant 1$ 为整数, $\Omega \subset \mathbb{R}^n$ 为非空开集, $Y_0 \in \Omega$, $\vec{g}: \Omega \to \mathbb{R}^n$ 为 $\mathscr{C}^{(k)}$ 类使得 $J_{\vec{g}}(Y_0)$ 可逆. 令 $X_0 = \vec{g}(Y_0)$. 则 $\exists \delta, \eta > 0$ 使得 $B(Y_0, \eta) \subset \Omega$, 且存在 $\vec{f}: B(X_0, \delta) \to B(Y_0, \eta)$ 为 $\mathscr{C}^{(k)}$ 类使得 $\forall X \in B(X_0, \delta), \forall Y \in B(Y_0, \eta)$, 等式 $X = \vec{g}(Y)$ 成立当且仅当 $Y = \vec{f}(X)$. 另外, $\forall X \in B(X_0, \delta)$,

$$J_{\vec{f}}(X) = \left(J_{\vec{g}}(\vec{f}(X))\right)^{-1}.$$

注: 该定理意味着 $\vec{g}$ 在点 Y_0 处 “局部可逆” .

证明: 选取 $r>0$ 使得 $B((X_0,Y_0);r)\subset\mathbb{R}^n\times\Omega$. $\forall(X,Y)\in B((X_0,Y_0);r)$, 定义

$$\vec{F}(X,Y)=\vec{g}(Y)-X.$$

则 $\vec{F}$ 为 $\mathscr{C}^{(k)}$ 类. 记 $\vec{F}=(F_1,\ldots,F_n)^T$, 那么

$$\vec{F}(X_0,Y_0)=\vec{0},\ \frac{\partial(F_1,\ldots,F_n)}{\partial(y_1,\ldots,y_n)}(X_0,Y_0)=J_{\vec{g}}(Y_0)$$

为可逆矩阵. 由隐函数定理知, $\exists\delta,\eta>0$ 使得

$$B(X_0,\delta)\times B(Y_0,\eta)\subset B((X_0,Y_0);r),$$

且 $\forall X \in B(X_0, \delta)$, $\exists! Y \in B(Y_0, \eta)$ 使得我们有 $\vec{F}(X, Y) = 0$. 令 $\vec{f}(X) = Y$, 那么 $\vec{g}(\vec{f}(X)) = X$, 并且 $\vec{f} : B(X_0, \delta) \to B(Y_0, \eta)$ 为 $\mathscr{C}^{(k)}$ 类向量值函数使得 $\forall X \in B(X_0, \delta)$, 我们均有

$$\begin{aligned} J_{\vec{f}}(X) &= -\Big(\frac{\partial(F_1, \ldots, F_n)}{\partial(y_1, \ldots, y_n)}(X, \vec{f}(X))\Big)^{-1} \\ &\quad \cdot \frac{\partial(F_1, \ldots, F_n)}{\partial(x_1, \ldots, x_n)}(X, \vec{f}(X)) \\ &= \Big(J_{\vec{g}}(\vec{f}(X)\Big)^{-1}. \end{aligned}$$

评注

- 上述定理意味着, $\forall X \in B(X_0, \delta)$, 我们均有 $Y = \vec{f}(X) \in B(Y_0, \eta)$, 且满足 $\vec{g}(\vec{f}(X)) = X$. 反过来, $\forall Y \in B(Y_0, \eta)$, 我们并不知道是否也有 $X = \vec{g}(Y) \in B(X_0, \delta)$, 故 $\vec{f}$ 并不一定是 $\vec{g}$ 真正的反函数.

- 若定义 $U = B(X_0, \delta)$, $V = \vec{f}(B(X_0, \delta))$, 那么 $\vec{f}: U \to V$, $\vec{g}: V \to U$ 互为逆映射:

  $\forall X \in U$, 均有 $\vec{f}(X) \in V$ 并且 $\vec{g}(\vec{f}(X)) = X$.

  又 $\forall Y \in V$, $\exists X \in B(X_0, \delta)$ 使得 $Y = \vec{f}(X)$,

  从而 $Y \in B(Y_0, \eta)$, 且 $\vec{g}(Y) = \vec{g}(\vec{f}(X)) = X$,

  于是我们有 $\vec{f}(\vec{g}(Y)) = \vec{f}(X) = Y$.

例 4. (极坐标变换) 令 $D = (0, +\infty) \times (-\pi, \pi)$.

$\forall (\rho, \varphi) \in D$, 定义

$$\vec{f}(\rho, \varphi) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \rho\cos\varphi \\ \rho\sin\varphi \end{pmatrix}.$$

则 $\vec{f}$ 为 $\mathscr{C}^{(\infty)}$ 类向量值函数且

$$J_{\vec{f}}(\rho, \varphi) = \begin{pmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{pmatrix},$$

从而 Jacobi 行列式 $\det J_{\vec{f}}(\rho,\varphi)=\rho>0$. 于是 $\vec{f}$ 为局部可逆, 其逆映射 $\vec{f}^{-1}$ 也为 $\mathscr{C}^{(\infty)}$ 类且

$$\begin{aligned} J_{\vec{f}^{-1}}(x,y) &= \begin{pmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{pmatrix}^{-1} \\ &= \frac{1}{\rho}\begin{pmatrix} \rho\cos\varphi & \rho\sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}. \end{aligned}$$

作业题: 第 1.6 节第 66 页第 9 题第 (1) 小题.

例 5. 已知函数 $z = z(x, y)$ 由参数方程

$$\begin{cases} x = u\cos v \\ y = u\sin v \\ z = uv \end{cases}$$

确定, 其中 $u > 0$. 试求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

解: 由于 $\frac{D(x,y)}{D(u,v)} = \begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix} = u > 0$, 因此

存在反函数, 可将 u, v 看成是 x, y 的函数并且

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{pmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{pmatrix}^{-1} = \frac{1}{u}\begin{pmatrix} u\cos v & u\sin v \\ -\sin v & \cos v \end{pmatrix}.$$

于是我们有

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \\
&= v\cos v + u \cdot \left(-\frac{1}{u}\sin v\right) \\
&= v\cos v - \sin v, \\
\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \\
&= v\sin v + u \cdot \left(\frac{1}{u}\cos v\right) \\
&= v\sin v + \cos v.
\end{aligned}
$$

例 6. 设隐函数 $u = u(x, y)$ 由方程组

$$\begin{cases} u = f(x, y, z, t) \\ g(y, z, t) = 0 \\ h(z, t) = 0 \end{cases}$$

确定, 求 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$.

解: 由题设可知, 利用方程组 $\begin{cases} g(y, z, t) = 0 \\ h(z, t) = 0 \end{cases}$

可将 z, t 确定为 y 的函数, 由此可得

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x}(x, y, z, t).$$

由隐函数定理可知

$$\begin{pmatrix} \frac{\mathrm{d}z}{\mathrm{d}y} \\ \frac{\mathrm{d}t}{\mathrm{d}y} \end{pmatrix} = -\begin{pmatrix} \frac{\partial g}{\partial z} & \frac{\partial g}{\partial t} \\ \frac{\partial h}{\partial z} & \frac{\partial h}{\partial t} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial y} \end{pmatrix}$$

$$= -\left(\left|\frac{\partial(g,h)}{\partial(z,t)}\right|\right)^{-1} \begin{pmatrix} \frac{\partial h}{\partial t} & -\frac{\partial g}{\partial t} \\ -\frac{\partial h}{\partial z} & \frac{\partial g}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial g}{\partial y} \\ 0 \end{pmatrix},$$

于是 $\frac{\mathrm{d}z}{\mathrm{d}y} = -\frac{\frac{\partial h}{\partial t}\cdot\frac{\partial g}{\partial y}}{\left|\frac{\partial(g,h)}{\partial(z,t)}\right|}$, $\frac{\mathrm{d}t}{\mathrm{d}y} = \frac{\frac{\partial h}{\partial z}\cdot\frac{\partial g}{\partial y}}{\left|\frac{\partial(g,h)}{\partial(z,t)}\right|}$, 进而可得

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\cdot\frac{\mathrm{d}z}{\mathrm{d}y} + \frac{\partial f}{\partial t}\cdot\frac{\mathrm{d}t}{\mathrm{d}y} = \frac{\partial f}{\partial y} + \frac{\left|\frac{\partial(h,f)}{\partial(z,t)}\right|}{\left|\frac{\partial(g,h)}{\partial(z,t)}\right|}\cdot\frac{\partial g}{\partial y}.$$

§7. 曲面与曲线的表示法　切平面与法线

回顾: 三维空间中的直线与平面

取 $P_0(x_0, y_0, z_0) \in \mathbb{R}^3$. 设 $\vec{e} = (a, b, c)^T \in \mathbb{R}^3$ 为非零向量. 过 P_0 沿方向 $\vec{e}$ 的直线 Γ 的方程为

$$\begin{cases} x = x_0 + at, \\ y = y_0 + bt, \\ z = z_0 + ct. \end{cases}$$

该直线也可以表示成

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}.$$

过 P_0 并且与 Γ 垂直的平面 S 称为 Γ 过 P_0 的法平面, 它的方程为 $\begin{pmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{pmatrix}\cdot\begin{pmatrix} a \\ b \\ c \end{pmatrix}=0$,

也就是说 $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$.

我们称 $\vec{e}$ 为平面 S 的法向量, Γ 为 S 的法线.

设过点 P_0 的平面 S 的参数方程为

$$\begin{cases} x = x_0 + a_1 u + b_1 v, \\ y = y_0 + a_2 u + b_2 v, \\ z = z_0 + a_3 u + b_3 v, \end{cases} \text{其中} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \text{线性无关,}$$

也即 $\begin{pmatrix} x - x_0 & a_1 & b_1 \\ y - y_0 & a_2 & b_2 \\ z - z_0 & a_3 & b_3 \end{pmatrix} \begin{pmatrix} -1 \\ u \\ v \end{pmatrix} = \vec{0}$, 进而可得

$$\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} (x - x_0) + \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} (y - y_0) + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} (z - z_0) = 0.$$

于是该平面的法向量为

$$\left(\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right).$$

从而平面 S 过点 P_0 的法线方程为

$$\frac{x - x_0}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} = \frac{y - y_0}{\begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix}} = \frac{z - z_0}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

曲面及其切平面和法线

(1) 曲面的显函数表示法: 曲面 S: $z = f(x, y)$, 其中 $(x, y) \in D \subset \mathbb{R}^2$. 假设 f 在点 (x_0, y_0) 处可微. 令 $z_0 = f(x_0, y_0)$. 当 $(x, y) \to (x_0, y_0)$ 时,

$$\begin{aligned} f(x, y) - z_0 &= \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) \\ &\quad + o(\sqrt{(x - x_0)^2 + (y - y_0)^2}). \end{aligned}$$

则曲面 S 在点 (x_0, y_0, z_0) 处的切平面方程为

$$z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

于是该切平面的法向量为

$$\vec{n} = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) \\ \frac{\partial f}{\partial y}(x_0, y_0) \\ -1 \end{pmatrix}.$$

相应的法线方程为

$$\frac{x - x_0}{\frac{\partial f}{\partial x}(x_0, y_0)} = \frac{y - y_0}{\frac{\partial f}{\partial y}(x_0, y_0)} = \frac{z - z_0}{-1}.$$

(2) 曲面的参数表示法:

考虑曲面 S: $\begin{cases} x = f_1(u,v), \\ y = f_2(u,v), \\ z = f_3(u,v), \end{cases}$ $(u,v) \in D \subset \mathbb{R}^2$.

设 $(u_0,v_0) \in D$, f_1, f_2, f_3 在点 (u_0,v_0) 可微. 令

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} f_1(u_0,v_0) \\ f_2(u_0,v_0) \\ f_3(u_0,v_0) \end{pmatrix}.$$

当 $(u,v)\to(u_0,v_0)$ 时, 我们有

$$\begin{pmatrix} f_1(u,v)-x_0 \\ f_2(u,v)-y_0 \\ f_3(u,v)-z_0 \end{pmatrix} = \frac{\partial(f_1,f_2,f_3)}{\partial(u,v)}(u_0,v_0)\begin{pmatrix} u-u_0 \\ v-v_0 \end{pmatrix} + \vec{o}\left(\sqrt{(u-u_0)^2+(v-v_0)^2}\right).$$

当矩阵 $\frac{\partial(f_1,f_2,f_3)}{\partial(u,v)}(u_0,v_0)$ 的秩等于 2 时, 曲面 S 在点 (x_0,y_0,z_0) 处有切平面

$$\begin{pmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{pmatrix} = \frac{\partial(f_1,f_2,f_3)}{\partial(u,v)}(u_0,v_0)\begin{pmatrix} u-u_0 \\ v-v_0 \end{pmatrix}.$$

该切平面也可以表示成

$$\frac{D(f_2,f_3)}{D(u,v)}(u_0,v_0)(x-x_0)+\frac{D(f_3,f_1)}{D(u,v)}(u_0,v_0)(y-y_0)$$
$$+\frac{D(f_1,f_2)}{D(u,v)}(u_0,v_0)(z-z_0)=0.$$

从而曲面 S 在点 (x_0,y_0,z_0) 处的法线方程为

$$\frac{x-x_0}{\frac{D(f_2,f_3)}{D(u,v)}(u_0,v_0)}=\frac{y-y_0}{\frac{D(f_3,f_1)}{D(u,v)}(u_0,v_0)}=\frac{z-z_0}{\frac{D(f_1,f_2)}{D(u,v)}(u_0,v_0)}.$$

(3) 曲面的隐函数表示法: 考虑 $S: F(x,y,z)=0$.

设 $P_0(x_0,y_0,z_0)\in S$, 而 F 在点 P_0 处可微. 则

当 $S \ni P(x,y,z) \to P_0$ 时, 我们有

$$\begin{aligned} 0 &= F(x,y,z) - F(x_0,y_0,z_0) \\ &= \frac{\partial F}{\partial x}(P_0)(x-x_0) + \frac{\partial F}{\partial y}(P_0)(y-y_0) \\ &\quad + \frac{\partial F}{\partial z}(P_0)(z-z_0) + o(\|P-P_0\|). \end{aligned}$$

从而当 $J_F(P_0) \neq \vec{0}$ 时, 曲面在点 P_0 有切平面

$$\frac{\partial F}{\partial x}(P_0)(x-x_0)+\frac{\partial F}{\partial y}(P_0)(y-y_0)+\frac{\partial F}{\partial z}(P_0)(z-z_0)=0.$$

于是曲面 S 在点 P_0 处的法向量为

$$\vec{n}=\begin{pmatrix} \frac{\partial F}{\partial x}(P_0) \\ \frac{\partial F}{\partial y}(P_0) \\ \frac{\partial F}{\partial z}(P_0) \end{pmatrix}=\mathrm{grad}F(P_0),$$

相应的法线方程为 $\frac{x-x_0}{\frac{\partial F}{\partial x}(P_0)}=\frac{y-y_0}{\frac{\partial F}{\partial y}(P_0)}=\frac{z-z_0}{\frac{\partial F}{\partial z}(P_0)}$.

换一种视点来处理上述情形. 设 F 为 $\mathscr{C}^{(1)}$ 类且 $\frac{\partial F}{\partial z}(P_0) \neq 0$. 由隐函数定理知, 局部上我们有 $z = f(x, y)$. 于是在点 P_0 处的切平面方程为

$$\frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

$$-\frac{\frac{\partial F}{\partial x}(P_0)}{\frac{\partial F}{\partial z}(P_0)}(x - x_0) - \frac{\frac{\partial F}{\partial y}(P_0)}{\frac{\partial F}{\partial z}(P_0)}(y - y_0) - (z - z_0) = 0.$$

$$\frac{\partial F}{\partial x}(P_0)(x - x_0) + \frac{\partial F}{\partial y}(P_0)(y - y_0) + \frac{\partial F}{\partial z}(P_0)(z - z_0) = 0.$$

例 1. 设曲面 S 的参数方程为

$$\begin{cases} x = u + e^{u+v}, \\ y = u + v, \\ z = e^{u-v}. \end{cases}$$

求 S 在 $u_0 = 1, v_0 = -1$ 处的切平面与法线.

解: 由题设可知

$$\frac{\partial(x,y,z)}{\partial(u,v)} = \begin{pmatrix} 1+e^{u+v} & e^{u+v} \\ 1 & 1 \\ e^{u-v} & -e^{u-v} \end{pmatrix}.$$

于是我们有

$$\frac{\partial(x,y,z)}{\partial(u,v)}\Big|_{(1,-1)} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ e^2 & -e^2 \end{pmatrix}.$$

由此立刻知矩阵 $\frac{\partial(x,y,z)}{\partial(u,v)}\Big|_{(1,-1)}$ 的秩等于 2, 从而所求切平面的参数方程为

$$\begin{pmatrix} x-2 \\ y \\ z-e^2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ e^2 & -e^2 \end{pmatrix} \begin{pmatrix} u-1 \\ v+1 \end{pmatrix}.$$

该切平面也可以表示成

$$-2e^2(x-2)+3e^2y+(z-e^2)=0.$$

进而可知所求法线方程为

$$\frac{x-2}{-2e^2}=\frac{y}{3e^2}=\frac{z-e^2}{1}.$$

作业题: 第 1.7 节第 78 页第 1 题第 (1), (3), (6) 小题, 其中第 (3) 中假设 $a>0$, 第 2, 3 题.

例 2. 若两曲面在交线上每点的法线互相垂直, 则称二者正交. 求证: 曲面 $S_1: F_1(x,y,z)=0$ 和曲面 $S_2: F_2(x,y,z)=0$ 正交的充分必要条件是对于交线上的每点 $P_0(x_0,y_0,z_0)$, 均有

$$\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial x}+\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial y}+\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial z}=0.$$

证明: 在上述两曲面的交点 P_0, 它们的法向量分别为 $\operatorname{grad}F_1(P_0)$, $\operatorname{grad}F_2(P_0)$, 二者正交当且仅当 $\operatorname{grad}F_1(P_0)\cdot\operatorname{grad}F_2(P_0)=0$. 由此得证.

例 3. 求证: 球面 $S_1: \ x^2+y^2+z^2=R^2$ 与锥面 $S_2: \ x^2+y^2=a^2z^2$ 正交.

证明: $\forall (x,y,z)\in\mathbb{R}^3$, 定义

$$F_1(x,y,z)=x^2+y^2+z^2-R^2,$$
$$F_2(x,y,z)=x^2+y^2-a^2z^2.$$

于是 $\forall P(x,y,z)\in S_1\cap S_2$, 我们有

$$\begin{aligned} & \mathrm{grad}F_1(P)\cdot\mathrm{grad}F_2(P) \\ = \ & (2x)\cdot(2x)+(2y)\cdot(2y)+(2z)\cdot(-2a^2z) \\ = \ & 4(x^2+y^2-a^2z^2)=4F_2(P)=0. \end{aligned}$$

故所证结论成立.

空间曲线及其切线和法平面

(1) 空间曲线的参数表示法:

$$\Gamma: \begin{cases} x = x(t), \\ y = y(t), \quad t \in [\alpha, \beta]. \\ z = z(t), \end{cases}$$

若上述函数在点 $t = t_0$ 处可微, 则称曲线 Γ 在相应点 $P_0(x_0, y_0, z_0)$ 处可微, 相应切线方程为

$$\begin{cases} x - x_0 = x'(t_0)(t - t_0), \\ y - y_0 = y'(t_0)(t - t_0), \\ z - z_0 = z'(t_0)(t - t_0). \end{cases}$$

该切线也可表述成

$$\frac{x-x_0}{x'(t_0)}=\frac{y-y_0}{y'(t_0)}=\frac{z-z_0}{z'(t_0)},$$

这里需要假设 $(x'(t_0),y'(t_0),z'(t_0))$ 不为零向量.

我们将经过点 P_0 并且与上述切线垂直的平面称为 Γ 在点 P_0 处的法平面, 其方程为

$$x'(t_0)(x-x_0)+y'(t_0)(y-y_0)+z'(t_0)(z-z_0)=0.$$

(2) 空间曲线的隐函数表示法:

$$\Gamma : \begin{cases} F_1(x, y, z) = 0, \\ F_2(x, y, z) = 0. \end{cases}$$

设 F_1, F_2 在点 $P_0(x_0, y_0, z_0)$ 可微且 $\text{grad}F_1(P_0)$, $\text{grad}F_2(P_0)$ 不为零, 则曲线 Γ 在该点的切线为

$$\begin{cases} \frac{\partial F_1}{\partial x}(P_0)(x - x_0) + \frac{\partial F_1}{\partial y}(P_0)(y - y_0) + \frac{\partial F_1}{\partial z}(P_0)(z - z_0) = 0, \\ \frac{\partial F_2}{\partial x}(P_0)(x - x_0) + \frac{\partial F_2}{\partial y}(P_0)(y - y_0) + \frac{\partial F_2}{\partial z}(P_0)(z - z_0) = 0. \end{cases}$$

该切线的方向为

$$\vec{T} = \mathrm{grad} F_1(P_0) \times \mathrm{grad} F_2(P_0)$$

$$= \begin{pmatrix} \frac{\partial F_1}{\partial x}(P_0) \\ \frac{\partial F_1}{\partial y}(P_0) \\ \frac{\partial F_1}{\partial z}(P_0) \end{pmatrix} \times \begin{pmatrix} \frac{\partial F_2}{\partial x}(P_0) \\ \frac{\partial F_2}{\partial y}(P_0) \\ \frac{\partial F_2}{\partial z}(P_0) \end{pmatrix} = \begin{pmatrix} \frac{D(F_1,F_2)}{D(y,z)}(P_0) \\ \frac{D(F_1,F_2)}{D(z,x)}(P_0) \\ \frac{D(F_1,F_2)}{D(x,y)}(P_0) \end{pmatrix}.$$

只有当 $\vec{T} \neq \vec{0}$ 时, 上述方程组才的确给出一条直线. 此时 Jacobi 矩阵 $\frac{\partial(F_1,F_2)}{\partial(x,y,z)}(P_0)$ 的秩等于 2. 借助 $\vec{T}$, 我们也可得到切线的另外一个表述:

$$\frac{x - x_0}{\frac{D(F_1,F_2)}{D(y,z)}(P_0)} = \frac{y - y_0}{\frac{D(F_1,F_2)}{D(z,x)}(P_0)} = \frac{z - z_0}{\frac{D(F_1,F_2)}{D(x,y)}(P_0)}.$$

例 4. 求曲线

$$\begin{cases} F_1(x,y,z) = x^2+y^2+z^2-9=0 \\ F_2(x,y,z) = xy-z=0 \end{cases}$$

在点 $P_0(1,2,2)$ 处的切线方程与法平面方程.

解: 由题设可知

$$\frac{\partial(F_1,F_2)}{\partial(x,y,z)}(P_0) = \begin{pmatrix} 2x & 2y & 2z \\ y & x & -1 \end{pmatrix}\Bigg|_{P_0} = \begin{pmatrix} 2 & 4 & 4 \\ 2 & 1 & -1 \end{pmatrix}.$$

于是所求切线的方程为

$$\begin{cases} 2(x-1)+4(y-2)+4(z-2)=0, \\ 2(x-1)+(y-2)-(z-2)=0. \end{cases}$$

进而可知所求切线的方向为

$$\vec{T}=\begin{pmatrix}2\\4\\4\end{pmatrix}\times\begin{pmatrix}2\\1\\-1\end{pmatrix}=\begin{pmatrix}-8\\10\\-6\end{pmatrix}.$$

于是所求切线的方程为 $\frac{x-1}{-8}=\frac{y-2}{10}=\frac{z-2}{-6}$, 相应法平面方程为

$$-8(x-1)+10(y-2)-6(z-2)=0.$$

作业题: 第 1.7 节第 79 页第 5 题, 第 6 题.

谢谢大家!