Estimation theory – Laboratory 1.

Agnieszka Szkutek, 208619
Marta Frankowska, 208581

October 19, 2017

Contents

1 Exercise 1

We generate a vector of $Y \sim N(\mu = 2, \sigma^2 = 4)$ with length $N = 1000$:

```
Y <- rnorm(n = 1000, mean = 2, sd = 2)
# transform Y to Y1
Y1 <- 3 * (Y - 1)
```

Y_1 has normal distribution, because it is a linear combination of Y. We can calculate analytical mean μ_1 and variance σ_1^2:

$$\mu_1 = E(Y_1) = E(3(Y-1)) = 3E(Y-1) = 3E(Y) - 3 = 3 \cdot 2 - 3 = 3$$

$$\sigma_1^2 = Var(Y_1) = Var(3(Y-1)) = 9Var(Y) = 9 \cdot 4 = 36$$

We can compare numerical and analytical results:

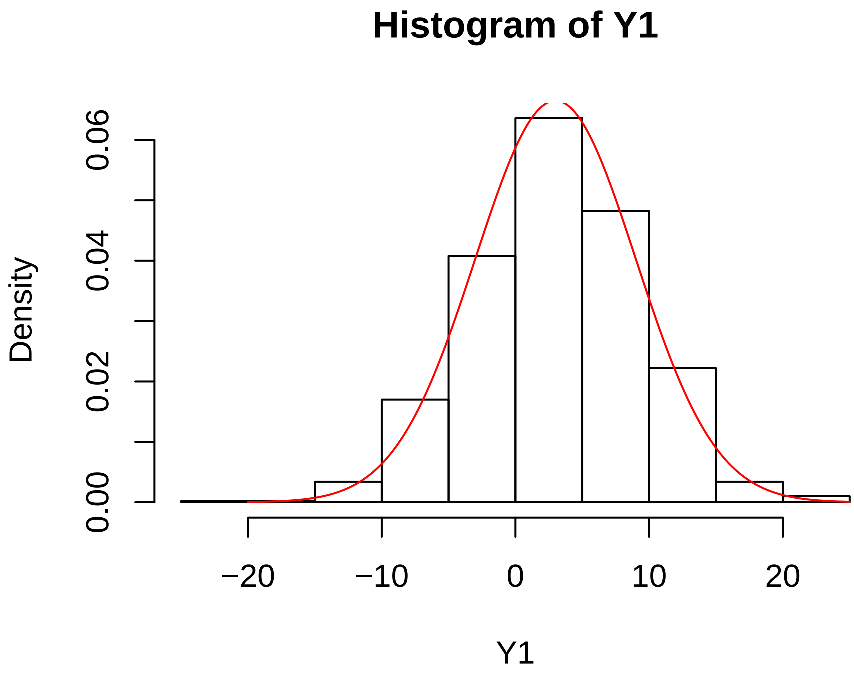
```
Y <- rnorm(n = 1000, mean = 2, sd = 2)
Y1 <- 3 * (Y - 1)
mean(Y1)

## [1] 2.65175

var(Y1)

## [1] 34.12698
```

Now we can plot the frequency historam of Y_1 and analytical normal density with $\mu = 3$ and $\sigma^2 = 36$.



Next, we create variable $Y_2 = \left(\frac{Y-2}{2}\right)^2$. Y_2 is a quadratic function of Y, so we know that distribution of Y_2 is not normal. What is more,

$$\frac{Y-2}{2} \sim N(0,1).$$

It means that Y_2, as a sum of squared standard normally distributed variables, has χ^2 distribution with 1 degree of freedom.

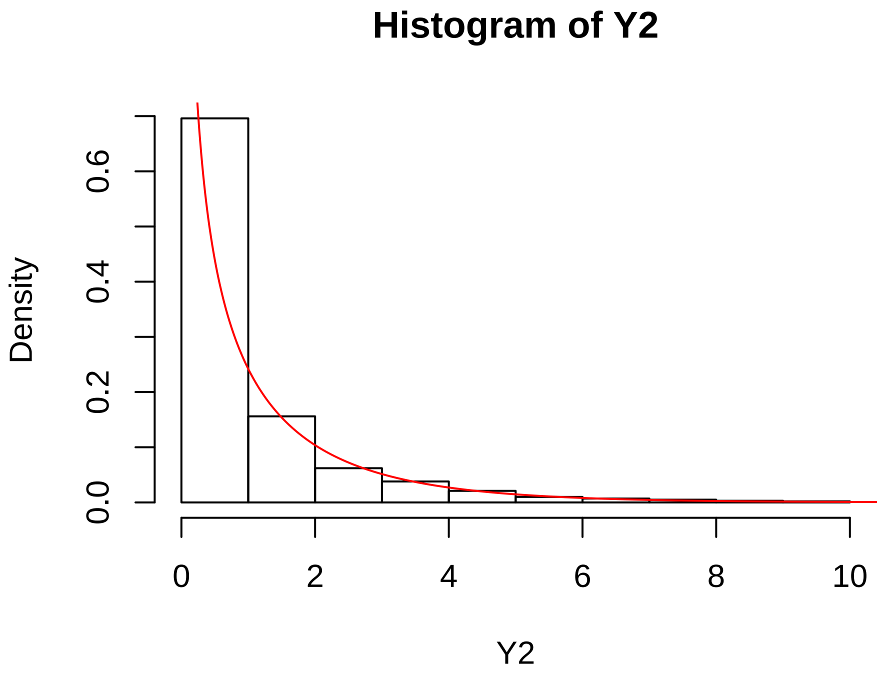


Figure 1: Histogram and theoretical probability density function of Y_2

Next we will compute a sequence of means m_n and a sequence of variances σ_n^2 for the variable Y, where

$$m_n = \frac{1}{n}\sum_{i=1}^{n} Y_i,$$

$$\sigma_n = \frac{1}{n}\sum_{i=1}^{n}(Y_i - m_n)^2$$

and plot the results.

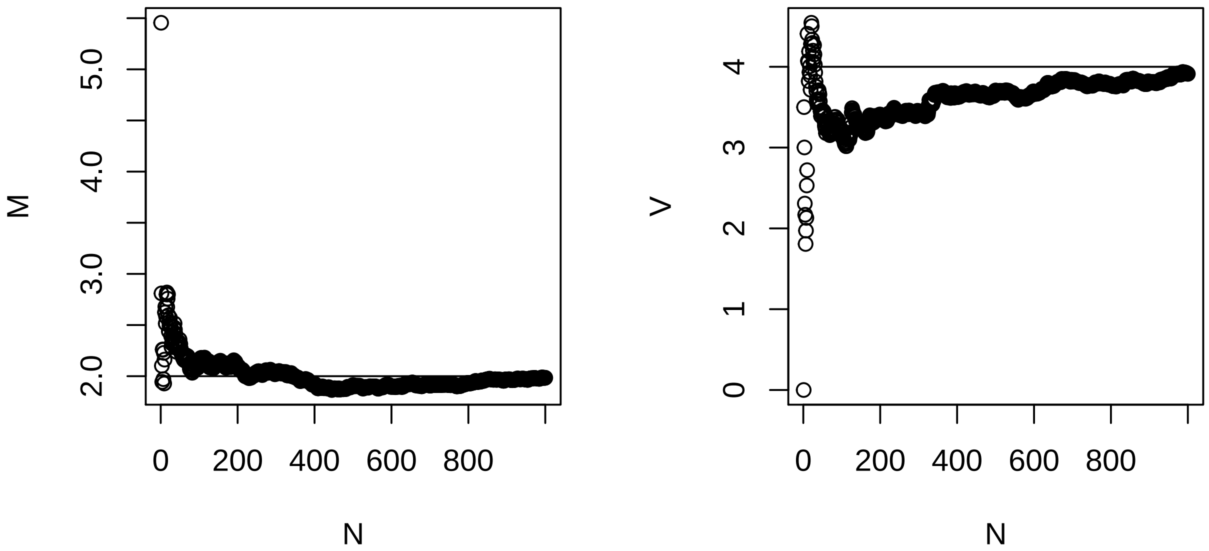


Figure 2: Sequence of means and variances and their respective analytical values.

The sequences m_n and σ_n^2 converge to theoretical mean and variance, respectively. To examine the variability of the sequences we can calculate relative errors for both values.

$$err_{m_n} = \left| \frac{m_n - \mu_n}{\mu_n} \right|, \qquad err_{\sigma_n^2} = \left| \frac{\sigma_n^2 - \sigma_n}{\sigma_n} \right|$$

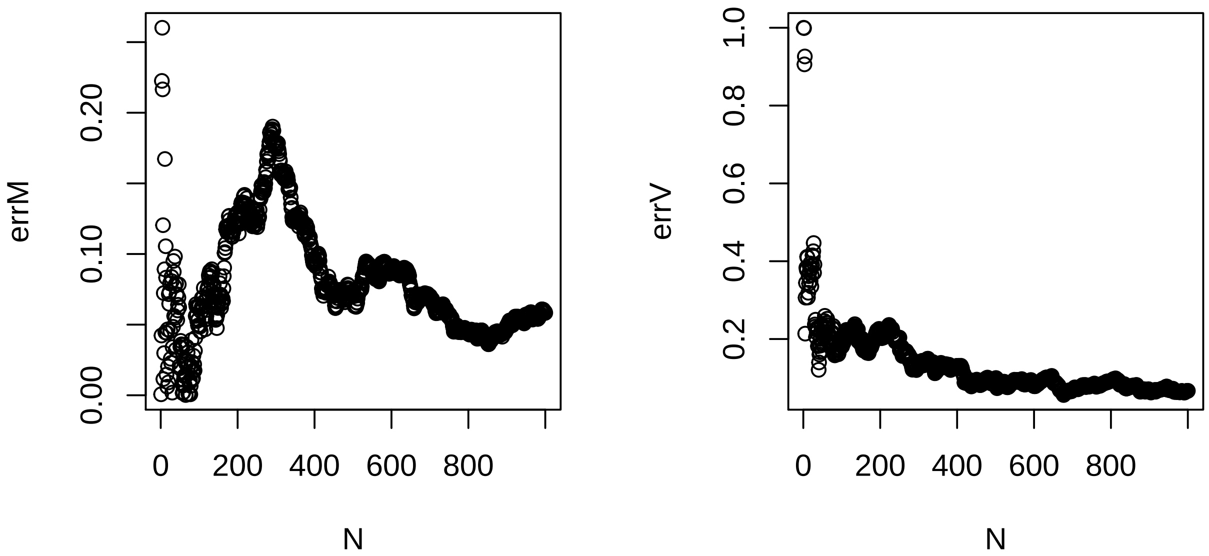


Figure 3: Relative errors for sequences of means and variances

For $N > 200$ value of the error is less than 10% of theoretical mean.

2 Exercise 2

We simulate 10000 times and then plot 2-dimensional random variable $X \sim N(0, I_2)$.

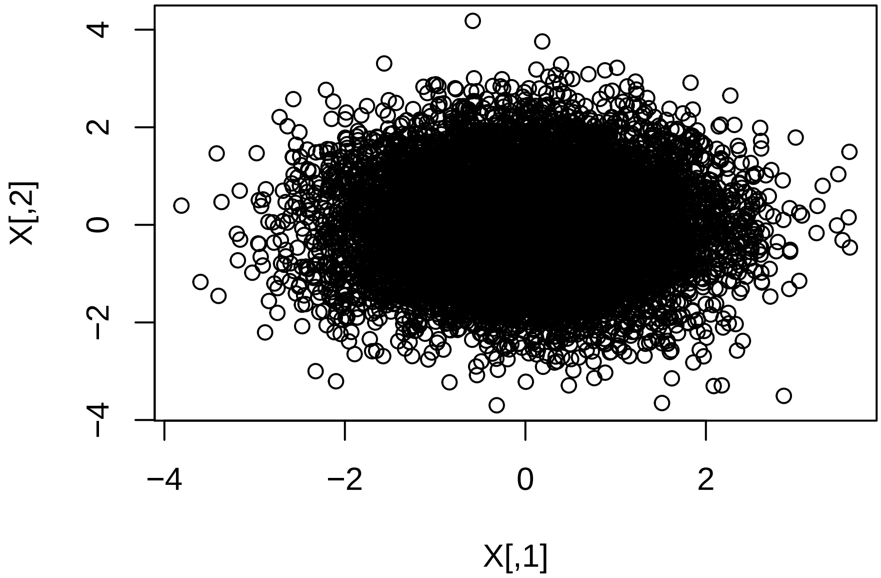


Figure 4: 2-dimensional random variable X of $N(0, I_2)$ distribution.

We have to transform variable X into variable $Y \sim N(\mu, \Sigma)$, where

$$\mu = [0,\ 1] \quad \text{and} \quad \Sigma = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 2 \end{bmatrix}$$

It means that we should find vector a and matrix A, such that $Y = AX + a$. Expected value of Y is equal to

$$E(Y) = E(AX + a) = AE(X) + a = a,$$

and variance

$$Var(Y) = Var(AX + a) = Var(AX) = AVar(X)A' = AIA' = AA' = \Sigma$$

We know that Σ is a symmetric and positive definite matrix, so we can use Cholesky decomposition (chol() function in R) to obtain the value of A:

```
Sigma <- matrix(c(2, 0.5, 0.5, 2), 2, 2)
chol(Sigma)

##          [,1]      [,2]
## [1,] 1.414214 0.3535534
## [2,] 0.000000 1.3693064
```

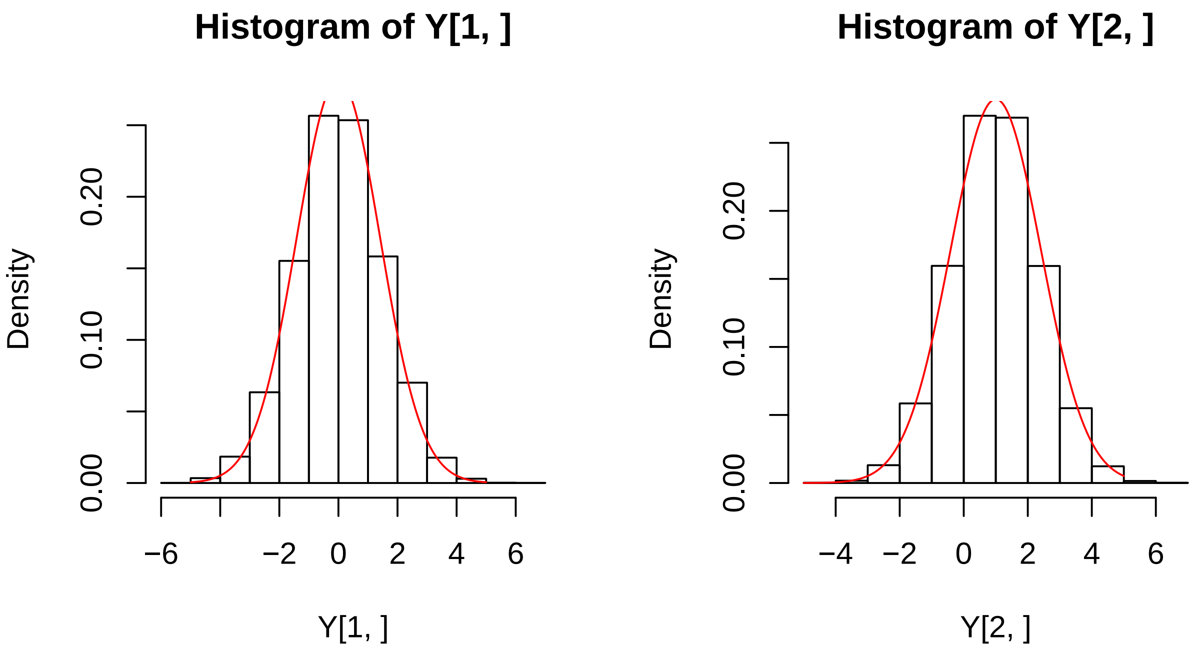


Figure 5: Histograms and probability density functions of Y

Now we can plot the 3D histogram of the random variable $Y = \begin{bmatrix} 1.414214 & 0.3535534 \\ 0.0 & 1.3693064 \end{bmatrix} X + \mu$.

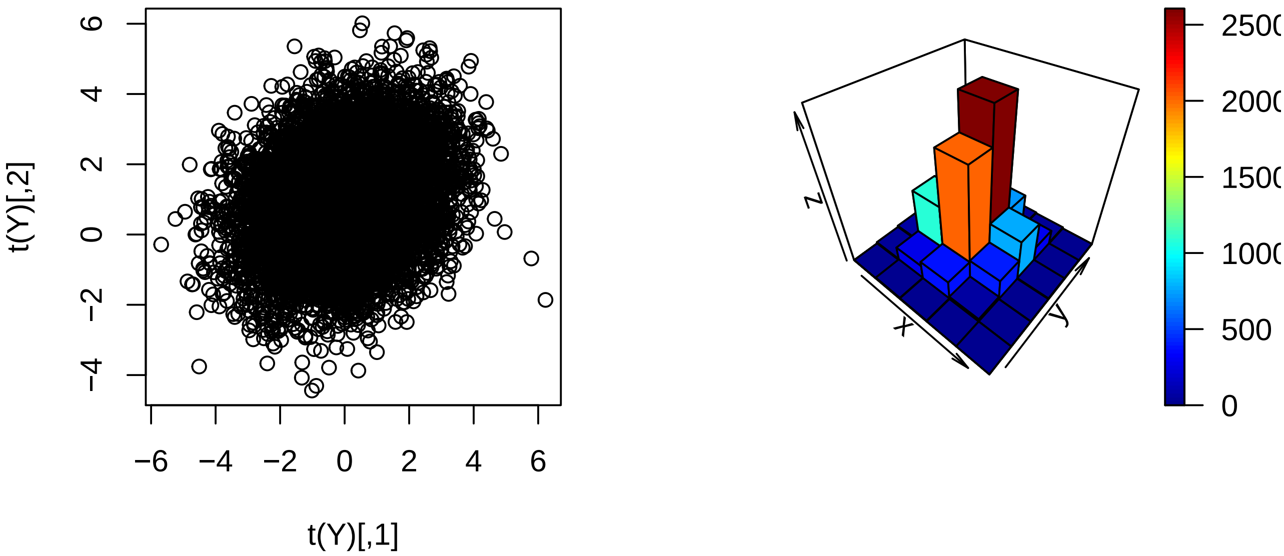


Figure 6: Random variable Y and its 3D histogram with 30 bins and bars colored according to height.

We are going to transform the variable Y into the variable $Z = (Y-\mu)'\Sigma^{-1}(Y-\mu)$, and Σ is a non-singular matrix.

$$Z = (Y-\mu)'\Sigma^{-1}(Y-\mu) = (Y-\mu)'\Sigma^{-0.5}\Sigma^{-0.5}(Y-\mu) = \left(\Sigma^{-0.5}(Y-\mu)\right)'\left(\Sigma^{-0.5}(Y-\mu)\right)$$

Let's take $\Sigma^{-0.5}(Y-\mu) = B$. We know that $B \sim N(0, I)$, so $B'B \sim \chi^2(k)$, where $k = 2$.

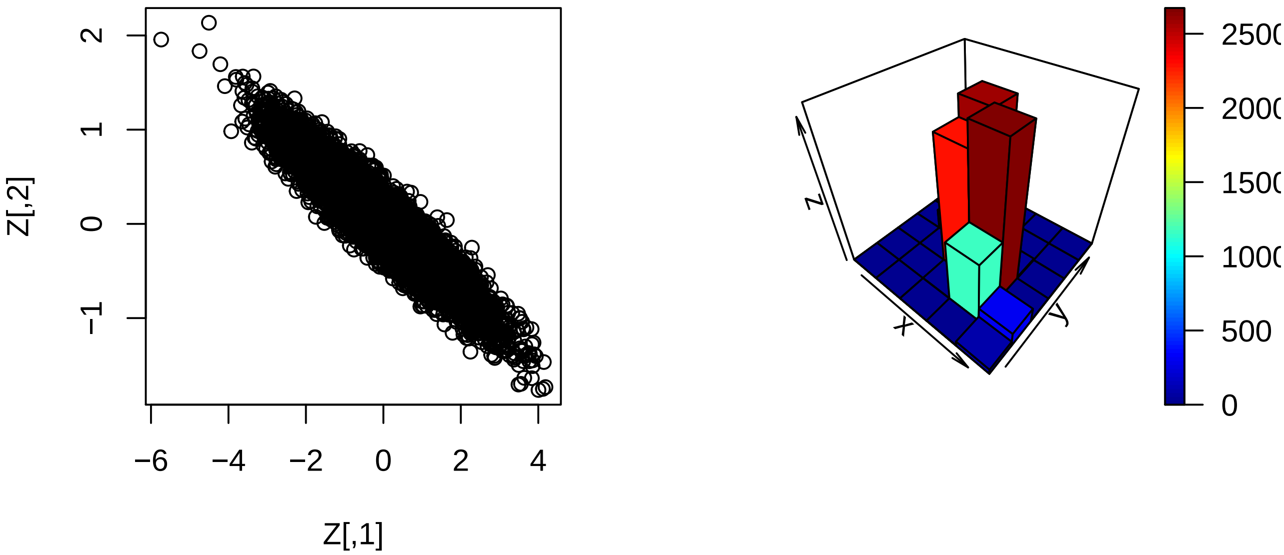


Figure 7: Random variable Z and its 3D histogram with 30 bins and bars colored according to height.

3 Exercise 3

Let $\hat{\beta}$ be a sequence of estimators of a $(K \times 1)$ vector β, which is asymptotically normal with

$$\sqrt{N}(\hat{\beta} - \beta) \to^d N(0, \Sigma).$$

3.1 Part 1

If $R \neq 0$ is an $(M \times K)$ matrix, the asymptotic distribution of $\sqrt{N}(R\hat{\beta} - R\beta) \to^? N(\mu, \sigma)$. We will compute the mean and the variance of $\sqrt{N}(R\hat{\beta} - R\beta)$.

$$\mu = E\left(\sqrt{N}(R\hat{\beta} - R\beta)\right) = E\left(\sqrt{N}R(\hat{\beta} - \beta)\right) = R\ E\left(\sqrt{N}(\hat{\beta} - \beta)\right) = 0$$

and

$$\sigma = Var\left(\sqrt{N}(R\hat{\beta} - R\beta)\right) = Var\left(\sqrt{N}R(\hat{\beta} - \beta)\right) =^{1} R\ Var\left(\sqrt{N}(\hat{\beta} - \beta)\right) R' = R\Sigma R',$$

so

$$\sqrt{N}(R\hat{\beta} - R\beta) \to^d N(0, R\Sigma R'),\ \text{for } R \neq 0.$$

[1]From task 2 point 2.

3.2 Part 2

If $p\lim \hat{A} = A$ then the asymptotic distribution of $\sqrt{N}\hat{A}(\hat{\beta}-\beta) \to^{?} N(\mu,\sigma)$. We know that if $x_n \to^d x$ and $p\lim y_n = c$, $x_n y_n \to^d cx$, so

$$\sqrt{N}\hat{A}(\hat{\beta}-\beta) \to^d \sqrt{N}A(\hat{\beta}-\beta) \to^d N(0, A\Sigma A').$$

3.3 Part 3

We will prove that $N\left(\hat{\beta}-\beta\right)'\hat{\Sigma}^{-1}\left(\hat{\beta}-\beta\right) \to^d \chi^2(K)$ if Σ is a non-singular matrix and $p\lim\hat{\Sigma} = \Sigma$.

$$\begin{gathered}
\sqrt{N}\left(\hat{\beta}-\beta\right)'\hat{\Sigma}^{-0.5}\hat{\Sigma}^{-0.5}\left(\hat{\beta}-\beta\right)\sqrt{N} \\
\downarrow^d \\
\sqrt{N}\left(\hat{\beta}-\beta\right)'\Sigma^{-0.5}\Sigma^{-0.5}\left(\hat{\beta}-\beta\right)\sqrt{N} \\
= \\
\left(\sqrt{N}\Sigma^{-0.5}\left(\hat{\beta}-\beta\right)\right)'\left(\sqrt{N}\Sigma^{-0.5}\left(\hat{\beta}-\beta\right)\right).
\end{gathered}$$

Let

$$C = \left(\sqrt{N}\Sigma^{-0.5}\left(\hat{\beta}-\beta\right)'\right) \to {}^{2} N(0, \underbrace{\Sigma^{-0.5}\Sigma\Sigma^{-0.5}}_{I_K})$$

We have $C'C$, where $C \to^d N(0,\ I_K)$, so

$$C'C \to^d \chi^2(K).$$

[2] From task 3 part 1.