Economathematics

Problem Sheet 1

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1. Calculate Ee^X where X is a gaussian random variable with mean μ and volatility $\sigma > 0$.

2. Verify that
$$\int_0^t W_s^2 dW_s = \frac{1}{3}W_t^3 - \int_0^t W_s ds,$$
where W is a Wiener process.

3. Find the stochastic differential equation (SDE) which is satisfied by the process
$$X(t) = e^{\sigma W_t}$$
for the volatility $\sigma > 0$.

4. Show that the process
$$X_t = (1-t)\int_0^t \frac{dW_s}{1-s}, \qquad t \in [0,1],$$
is the solution of
$$dX_t = \frac{-X_t}{1-t}dt + dW_t, \qquad X_0 = 0.$$

5. Show that $X_t = \sinh(t + W_t)$ is the solution of
$$dX_t = \left(\sqrt{1 + X_t^2} + \frac{1}{2}X_t\right)dt + \sqrt{1 + X_t^2}dW_t, \qquad X_0 = 0.$$
Recall that $\sinh(x) = \frac{e^x - e^{-x}}{2}$.

6. Find the value μ, such that the process $X_t = \mu t + W_t$, where W_t is the Wiener process, is a martingale.

7. Assets of a company can be described by the Brownian motion $X_t = X_0 + \mu t + \sigma W_t$ with $\mu = 1,5$ and $\sigma = 4$. Find the initial capital X_0 such that the probability of bankruptcy of the company in the first year is less than $0,05$.

8. Assume that the price of one-year future contract on gold is $F = 500$USD for ounce, spot price $S = 450$USD, interest rate $r = 7\%$, storage costs $U = 2$USD per year. How much can an investor earn?

9. Let $r > 0$ be the risk-free interest rate. The process of the stock price S_t is the binomial tree. From the node in which the process has value S, in the next moment it jumps up or down to the next node, where it takes values:

$$S \exp\{\mu\delta t + \sigma\sqrt{\delta t}\}$$

if the price increases and

$$S \exp\{\mu\delta t - \sigma\sqrt{\delta t}\}$$

if the price decreases with probabilities p and $1 - p$, accordingly.

   (i) Find S_t. (Hint: use the random variable X_n which is the number of jumps up among first n jumps).

   (ii) Calculate EX_n and $\mathrm{Var}X_n$.

   (iii) Calculate $\lim_{\delta t \downarrow 0} S_t$ (limit in distribution).

10. Suppose that X is an integrable random variable and $\{\mathcal{F}_t\}$ is chosen filtration. Prove that $M(t) = E[X|\mathcal{F}_t]$ is uniformly integrable martingale.

11. Find dZ_t when:

   (i)
$$Z_t = e^{\alpha W_t};$$

   (ii) $Z_t = X_t^2$, where X solves the following SDE:
$$dX_t = \alpha X_t dt + \sigma X_t dW_t.$$

12. Using Feynman-Kac formula solve the following PDE:

$$\begin{aligned}
&\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2}(t,x) = 0;\\
&F(T,x) = x^2.
\end{aligned}$$

13. Prove that

$$X_t = e^{\alpha t} x_0 + \sigma \int_0^t e^{\alpha(t-s)} dW_s$$

solves the following SDE:

$$\begin{aligned}
dX_t &= \alpha X_t dt + \sigma dW_t,\\
X_0 &= x_0.
\end{aligned}$$

14. Consider the standard Black-Scholes model. An innovative company, Z, has produced the derivative "the Golden Logarithm", henceforth abbreviated as the GL. The holder of a GL with maturity time T, denoted as GL(T), will, at time T, obtain the sum $\log S_T$. Note that if $S(T) < 1$ this means that the holder has to pay a positive amount to Z. Determine the arbitrage free price process for the GL(T).

15. Consider the standard Black-Scholes model. Find the arbitrage free price for $X = (S_T)^\beta$ where T is a maturity date.

16. A so called binary option is a claim which pays a certain amount if the stock price at a certain date T falls within some prespecified interval $[\alpha, \beta]$. Otherwise nothing will be paid out. Determine the arbitrage free price.

17. Find the arbitrage free price of $X = S_T / S_{T_0}$ for Black-Scholes market with expiry date T.

18. Do the same for $X = \frac{1}{T - T_0} \int_{T_0}^{T} S_u \, du$.

19. Consider corporation Ideal inc., whose stocks in Euro are given by the following SDE:

$$dS_t = \alpha S_t dt + \sigma S_t dW_t^1.$$

The exchange rate Y_t PLN/Euro is described by:

$$dY_t = \beta S_t dt + \delta Y_t dW_t^2,$$

where W^1 and W^2 are independent Wiener processes. Broker Ideal Inc. creates the derivative

$$X = \log \left[Z_T^2 \right]$$

with maturity date T, where Z is the stock price given in złoty. Find the arbitrage free price X (in PLN) assuming that r is a spot rate of złoty.

20. Consider the following financial market:

$$dB_t = rB_t dt, \qquad B_0 = 1,$$
$$dS(t) = \alpha S_t dt + \sigma S_t dW(t) + \delta S_{t-} dN_t,$$

where N is a Poisson process with intensity λ which is independent of the Wiener process W.

(i) Is this market arbitrage free ?

(ii) Is it complete ?

(iii) Does exist unique martingale measure ?

(iv) Suppose that we want to replicate European call option with the maturity date T. Is it possible to hedge it using portfolio consisting of the risk-free instrument B, the basic instrument S and European call option with expiry date $T - \delta$ for fixed $\delta > 0$?

21. Prove the following theorem.
Consider the following financial market:

$$dB_t = rB_t dt, \qquad B_0 = 1,$$
$$dS_t = \alpha(t, S_t) S_t dt + \sigma(t, S_t) S_t dW(t) + \delta S_{t-} dN_t$$

and the claim

$$X = \Phi(S_T, Z_T)$$

with the expiry date T and

$$Z_t = \int_0^t g(u, S_u)\, du.$$

Then X can be replicated in the following way:

$$\phi_t^1 = \frac{F(t, S_t, Z_t) - S_t F_s(t, S_t, Z_t)}{F(t, S_t, Z_t)},$$
$$\phi_t^1 = \frac{S_t F_s(t, S_t, Z_t)}{F(t, S_t, Z_t)},$$

where F solves the following boundary problem:

$$\begin{cases} F_t + srF_s + \frac{1}{2}s^2\sigma^2 F_{ss} + gF_z - rF & = & 0, \\ F(T, s, z) & = & \Phi(s, z). \end{cases}$$

The value process equals

$$F(t, s, z) = e^{-r(T-t)} E^Q_{t,s,z} \left[\Phi(S_t, Z_t) \right],$$

where Q-dynamics is described by the following SDEs:

$$\begin{aligned} &dS_u = rS_u du + S_u \sigma(u, S_u) dW_u, \\ &S_t = s, \\ &dZ_u = g(u, S_u) du, \\ &Z_t = z. \end{aligned}$$

22. Consider the Black-Scholes model and the derivative asset:

$$X = \begin{cases} K & S_T \leqslant A, \\ K + A - S_T & A < S_T < K + A, \\ 0 & S_T > K + A. \end{cases}$$

Replicate this derivative using portfolio consisting of bond, asset S and European call option. Find the arbitrage free price for X.

23. Do the same for

$$X = \begin{cases} K - S_T & 0 < S_T \leqslant K, \\ S_T - K & K < S_T. \end{cases}$$

24. Do the same for

$$X = \begin{cases} B & S_T > B, \\ S_T & A \leqslant S_T \leqslant B, \\ A & S_T < A. \end{cases}$$

25. Do the same for

$$X = \begin{cases} 0 & S_T < A, \\ S_T - A & A \leqslant S_T \leqslant B, \\ C - S_T & B \leqslant S_T \leqslant C, \\ 0 & S_T > C, \end{cases}$$

where $B = (A + C)/2$.

26. Consider the Black-Scholes model. A two-leg ratchet call option has the following features. At time $t = 0$ an initial strike K_0 is set. At time T_1, the strike is reset to K_1 being the asset value at time T_1. At the time $T > T_1$, the holder receives the payoff of a call with strike K_1 and the amount $(K_1 - K_0)^+$. Find the value option at time t depending if $t \leqslant T_1$ or $t > T_1$.

27. Let the stock prices S^1 and S^2 be given as the solutions to the following system of SDEs:

$$\begin{aligned} dS_t^1 &= \alpha S_t^1 dt + \delta S_t^1 dW_t^1, \qquad S_0^1 = s_1, \\ dS_t^2 &= \beta S_t^1 dt + \gamma S_t^2 dW_t^2, \qquad S_0^3 = s_2. \end{aligned}$$

The Wiener processes W^1 and W^2 are assumed to be independent. The parameters $\alpha, \delta, \beta, \gamma$ are assumed to be known and constant. Your task is to price *a minimum option*. This claim is defined by

$$X = \min\left[S_T^1, S_T^2\right].$$

The pricing function for a European call option in the Black-Scholes model is assumed to be known, and is denoted by $C(s, t, K, \sigma, r)$ where σ is the volatility, K is the strike price and r is the short rate. You are allowed to express your answer in terms of this function, with properly derived values for K, σ and r.

28. Consider two dates, T_0 and T, with $T_0 < T$. *A forward-start call option* is a contract in which the holder receives, at time T_0 (at no additional cost), a European call option with expiry date T and exercise price equal to S_{T_0}. Write down the terminal payoff, i.e. the payoff at time T, of a forward-start call option and then determine its arbitrage free price at time $t \in [0, T_0]$.