Optimization Theory

Applied Mathematics

Problem set 1

1. ([Be99]) Show that 2-dimensional function $f(x,y) = (x^2-4)^2 + y^2$ has two global minima and one stationary point, which is neither a local maximum nor a local minimum.
2. ([Be99]) Find all local minima and all local maxima of the 2-dimensional function $f(x,y) = \sin x + \sin y + \sin(x+y)$ within the set
$$\{(x,y) \in \mathbb{R}^2 : x \in (0,2\pi), y \in (0,2\pi)\}.$$
3. ([Be99]) For each value of the scalar β, find the set of all stationary points of the following function of the two variables x and y
$$f(x,y) = x^2 + y^2 + \beta xy + x + 2y.$$
Which of those stationary points are local minima? Which are global minima and why? Does this function have a global maximum for some value of β?
4. ([Be99]) Show that the 2-dimensional function $f(x,y) = (y-x^2)^2 - x^2$ has only one stationary point, which is neither a local minimum nor a local maximum. Next, consider the minimization of the function f subject to the constraint $-1 \leq y \leq 1$. Show that now there exists at least one global minimum and find all global minima.
Hint: Rewrite the function f as a sum of a quadratic function and a function of y, then find a lower bound on this expression and a pair of x and y giving the value of f equal to this lower bound.
5. Show that the n-dimensional function
$$f(x_1,\dots,x_n) = \left(\|(1-x_1, x_1-x_2, x_2-x_3, \dots, x_{n-1}-x_n, x_n-1)\|_2\right)^2$$
has exactly one stationary point which is a global minimum. Compute this minimum for $n=4$.
6. Show that the 2-dimensional function $f(x,y) = -(x-2y)^2 + x$ is concave. Find its global minimum subject to linear constraints
$$\begin{cases} x + 2y \leq 8 \\ -3x + 2y \leq 8 \\ -4x - y \leq 7 \\ 2x - y \leq 1 \end{cases}$$
7. Show that if $f : A \to \mathbb{R}$ is a convex function and A is a convex set, then:
   (a) Any local minimum of f over A is a global minimum of f.
   (b) If A is open, then $x^* \in A$ is a global minimum of f iff it is a stationary point.

   Is it possible that f has no global minimum if A is bounded? Is it possible that f has a global minimum which is not a stationary point of f if A is not open?
8. ([Be99]) Show that the following statements are true:
   (a) Any vector norm is a convex function.
   (b) The weighted sum of convex functions, with positive weights, is convex.
   (c) If I is an index set, $C \in \mathbb{R}^n$ is a convex set, and $f_i : C \to \mathbb{R}$ is convex for each $i \in I$, then the function $h : C \to \mathbb{R} \cup \{+\infty\}$ defined by
   $$h(x) = \sup_{i \in I} f_i(x)$$
   is also convex.
9. ([Be99]) Show that the following statements are true:
   (a) For any collection $\{C_i : i \in I\}$ of convex sets, their intersection $\bigcap_{i\in I} C_i$ is also convex.
   (b) The vector sum $\{x_1 + x_2 : x_1 \in C_1, x_2 \in C_2\}$ of two convex sets C_1 and C_2 is convex.
   (c) If C is a convex set and $f : C \to \mathbb{R}$ is a convex function, the level sets $\{x \in C : f(x) \leq \alpha\}$ and $\{x \in C : f(x) < \alpha\}$ are convex for all scalars α.

References:

[Be99] D.P. Bertsekas, Nonlinear Programming, Athena Scientific, Belmont, MA: 1999.