Estimation theory – Report 5

Marta Frankowska, 208581
Agnieszka Szkutek, 208619

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1 Exercise 1

We consider data from file *data-Lab5.xlsx*, which consists of two variables X and Y. We assume that Y is generated by the process

$$y_i = \alpha_0 + \alpha_1 x_i + \alpha_2 x_i^2 + e_i,$$

where e_i is white noise and $e_i \sim N(0, \sigma^2)$.

1.1 Part 1

We will verify hypothesis $H_0: \ \sigma^2 = 1$ versus the alternative $H_1: \ \sigma^2 > 1$ with three tests.

The restriction function is $h(\theta) = R\theta - r$, where $\theta = [\alpha_0, \alpha_1, \alpha_2, \sigma^2]'$, $R = [0, 0, 0, 1]$ and $r = 1$. Additionally $H(\theta) = h'(\theta) = R$.

From W. Green *Econometric analysis*, for independent $x_1, \ldots, x_p$ and $y = X\beta + \varepsilon$, where $\varepsilon_i \sim N(0, \sigma^2)$ we get the following ML estimators

$$\hat{\beta}_{ML} = (X'X)^{-1}X'y \qquad \text{and} \qquad \hat{\sigma}^2_{ML} = \frac{\hat{e}'\hat{e}}{N},$$

where $\hat{e} = y - X\hat{\beta}$ is estimator of residuals. The log-likelihood function is equal to

$$\ln L(\theta) = -\frac{N}{2}\ln 2\pi - \frac{N}{2}\ln \sigma^2 - \frac{(y - X\beta)'(y - X\beta)}{2\sigma^2}$$

and the inverse information matrix is as follows

$$I^{-1}(\hat{\theta}) = \begin{bmatrix} \sigma^2(X'X)^{-1} & 0 \\ 0' & \frac{2\sigma^4}{N} \end{bmatrix}, \quad \text{where} \qquad \theta = [\beta, \sigma^2]'.$$

In our case $\beta = [\alpha_0, \alpha_1, \alpha_2]'$.

1.1.1 Wald test

In general

$$W = h^T(\hat{\theta}) \left(H(\hat{\theta}) I^{-1}(\hat{\theta}) H^T(\hat{\theta}) \right)^{-1} h(\hat{\theta})$$

and in our case

$$W = [R\hat{\theta} - 1]' \left[R \; I^{-1}(\hat{\theta}) \; R^T \right]^{-1} [R\hat{\theta} - 1] \to^d \chi^2(1)$$

```
## alpha = 0.05
## p-value = 0.3297193
## Critical value = 3.841459
## Wald test statistic = 0.4598706
```

We can't reject the null hypothesis, because *p-value* $> \alpha$ and $|W| <$ *critical value.*

1.1.2 Likelihood ratio test

In general

$$LR = 2 \left(\ln L(\hat{\theta}) - \ln L(\hat{\theta}_R) \right),$$

where $\hat{\theta}_R$ is the restricted estimator of θ and is equal to $\hat{\theta}_R = [\hat{\alpha}_0, \hat{\alpha}_1, \hat{\alpha}_2, 1]$. In our case

$$LR = 2 \left(-\frac{N}{2} \ln 2\pi - \frac{N}{2} \ln \hat{\sigma}^2 - \frac{(y - X\hat{\beta})'(y - X\hat{\beta})}{2\hat{\sigma}^2} + \frac{N}{2} \ln 2\pi + \frac{(y - X\hat{\beta})'(y - X\hat{\beta})}{2} \right)$$

```
## alpha = 0.05
## p-value = 0.3297193
## Critical value = 3.841459
## LR test statistic = 0.4418897
```

We can't reject the null hypothesis, because *p-value* $> \alpha$ and $|LR| <$ *critical value.*

1.1.3 Lagrange multiplier test

In general

$$LM = Dl(\hat{\theta}_R)' I^{-1}(\hat{\theta}_R) Dl(\hat{\theta}_R).$$

In our case

$$Dl(\hat{\theta}_R) = \begin{bmatrix} X'(y - X\hat{\beta}) \\ -\frac{N}{2} + \frac{(y - X\hat{\beta})'(y - X\hat{\beta})}{2} \end{bmatrix}$$

and

$$I^{-1}(\hat{\theta}_R) = \begin{bmatrix} (X'X)^{-1} & 0 \\ 0' & \frac{2}{N} \end{bmatrix}.$$

```
## alpha = 0.05
## p-value = 0.3297193
## Critical value = 3.841459
## LM test statistic = 0.4331969
```

We can't reject the null hypothesis, because *p-value* $> \alpha$ and $|LM| <$ *critical value*.

1.2 Part 2

We want to test if the polynomial $\alpha_0 + \alpha_1 x_i + \alpha_2 x_i^2$ has exactly one root. We know that it has only one root if and only if $\alpha_1^2 - 4\alpha_0\alpha_2 = 0$. So

$$H_0: \; h(\theta) = \alpha_1^2 - 4\alpha_0\alpha_2 = 0.$$

Then

$$H(\theta) = h'(\theta) = [-4\alpha_2, \; 2\alpha_1, \; -4\alpha_0, \; 0].$$

The Wald test statistic is equal to

$$W = h^T(\hat{\theta}) \left(H(\hat{\theta}) I^{-1}(\hat{\theta}) H^T(\hat{\theta}) \right)^{-1} h(\hat{\theta}).$$

In our case it is equal to

$$W = \frac{1}{\hat{\sigma}^2} \frac{(\alpha_1^2 - 4\alpha_0\alpha_2)^2}{[-4\alpha_2, 2\alpha_1, -4\alpha_0, 0](X'X)^{-1}[-4\alpha_2, 2\alpha_1, -4\alpha_0, 0]'} \to \chi^2(1).$$

Asymptotic distribution of the parameters

$$\sqrt{N} h(\hat{\theta}) \to N\left(0, H(\theta_0) i^{-1}(\theta_0) H^T(\theta_0)\right)$$

Other test statistics are equal to

$$LR = 2\left(\ln L(\hat{\theta}) - \ln L(\hat{\theta}_R)\right) \quad \text{and} \quad LM = Dl(\hat{\theta}_R)' I^{-1}(\hat{\theta}_R) Dl(\hat{\theta}_R),$$

where

$$\hat{\theta}_R = \left[\frac{\hat{\alpha}_1^2}{4\hat{\alpha}_2}, \hat{\alpha}_1, \hat{\alpha}_2, \hat{\sigma}^2 \right].$$

1.3 Part 3

We use Wald test statistic to verify the null hypothesis from the previous task. The results are as follows

```
## alpha = 0.05
## p-value = 0.3297193
## Critical value = 3.841459
## Wald test statistic = 1.422491
```

We can't reject the null hypothesis, because *p-value* $> \alpha$ and $|W| <$ *critical value*.

2 Exercise 2

We want to verify if in the model

$$y_t = \alpha x_t + e_t$$

the parameter α is different than zero. We will test the hypothesis with a t-Student test. In order to verify the test performance we will conduct a small Monte Carlo experiment.

We define number of MC iterations $N = 1000$ and the sample size $T = 200$. We will compute the frequency of rejections for $\alpha = 2$ and $\alpha_0 \in [1.8, 2.2]$.

We generate N different realizations of the stationary exogenous variable x_t from normal distribution $N(5, 1)$.

The t-statistic is as follows

$$\frac{|\hat{\alpha} - \alpha_0|}{\mathrm{SE}_{\hat{\alpha}}} \sim t(T-2) \quad \text{and} \quad \mathrm{SE}_{\hat{\alpha}} = \frac{\sqrt{\frac{1}{T-2}\sum_{i=1}^{T}(y_i - \hat{y}_i)^2}}{\sqrt{\sum_{i=1}^{T}(x_i - \bar{x})^2}},$$

where

$$y_i - \hat{y}_i = y_i - \hat{\alpha} x_i \quad \text{and} \quad \hat{\alpha} = (X'X)^{-1}X'Y.$$

2.1 Test properties

To test properties for stationary residuals for each MC iteration we generate white noise $e_t \sim N(0, \sigma^2)$, $\sigma = 1$. Then with $\alpha = 2$ we generate y_t and compute t-statistic and frequency of rejections using the following functions.

```
Monte_Carlo_test_normal <- function(alpha0,alpha,N,T)
{
  a <- 0.05
  critical.val <- qt(1 - a, df = T-2)
  rejecting = 0
  for (i in 1:N)
  {
    x <- rnorm(200,mean = 5, sd = 1)
    e <- rnorm(200,mean = 0, sd = 1)
    y <- alpha*x + e
    X <- as.matrix(x)
    Y <- as.matrix(y)
    alpha_est = solve(t(X)%*%X)%*%t(X)%*%Y
    SE_alpha_est = sqrt(sum((y-alpha_est*x)^2)/(T-2))/sqrt(sum((x-mean(x))^2))
    t_student_test = abs(alpha_est - alpha0)/SE_alpha_est
    if (abs(t_student_test)>=critical.val)
    {
      rejecting = rejecting + 1
    }
  }
  return(rejecting/N)
}
```

```
Monte_Carlo_test_random_walk <- function(alpha0,alpha,N,T)
{
  a <- 0.05
  critical.val <- qt(1 - a, df = T-2)
  rejecting = 0
  for (i in 1:N)
  {
    x <- cumsum(rnorm(200,mean = 5, sd = 1))
    e <- rnorm(200,mean = 0, sd = 1)
    y <- alpha*x + e
    X <- as.matrix(x)
    Y <- as.matrix(y)
    alpha_est = solve(t(X)%*%X)%*%t(X)%*%Y
    SE_alpha_est = sqrt(sum((y-alpha_est*x)^2)/(T-2))/sqrt(sum((x-mean(x))^2))
    t_student_test = abs(alpha_est - alpha0)/SE_alpha_est
    if (abs(t_student_test)>=critical.val)
    {
      rejecting = rejecting + 1
    }
  }
  return(rejecting/N)
}
```

We get the following results

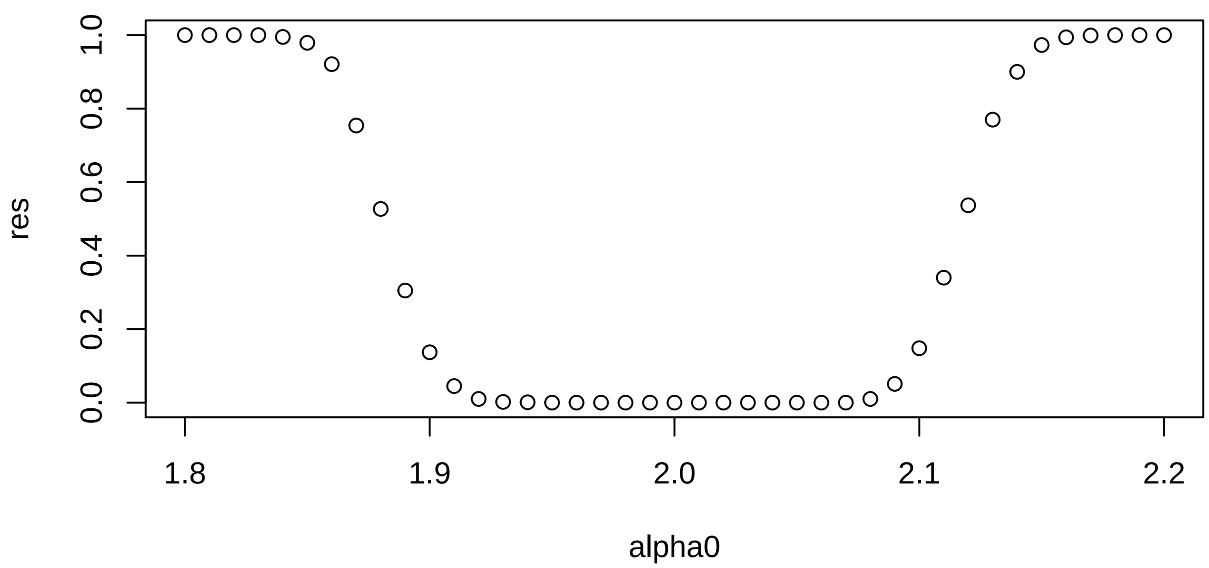

Figure 1: Frequency of null rejection with residuals from N(0,1)

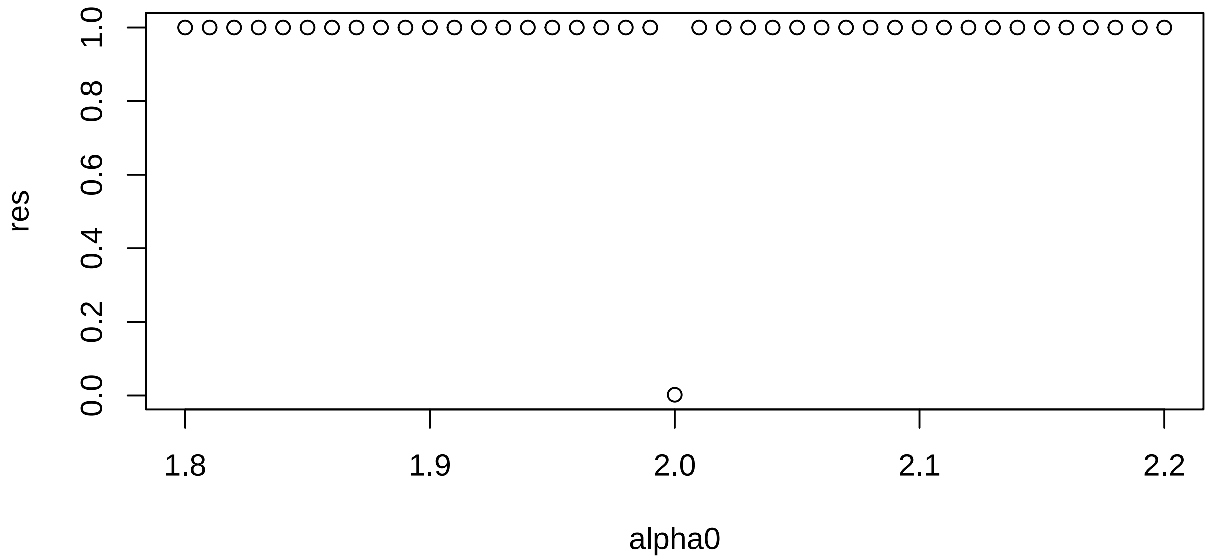

Figure 2: Frequency of null rejection with residuals from random walk

In both cases the size of the test is close to 0

```
## Size for residuals from N(0,1):  0
## Size for residuals from random walk:  0
```

And the powers are as follows

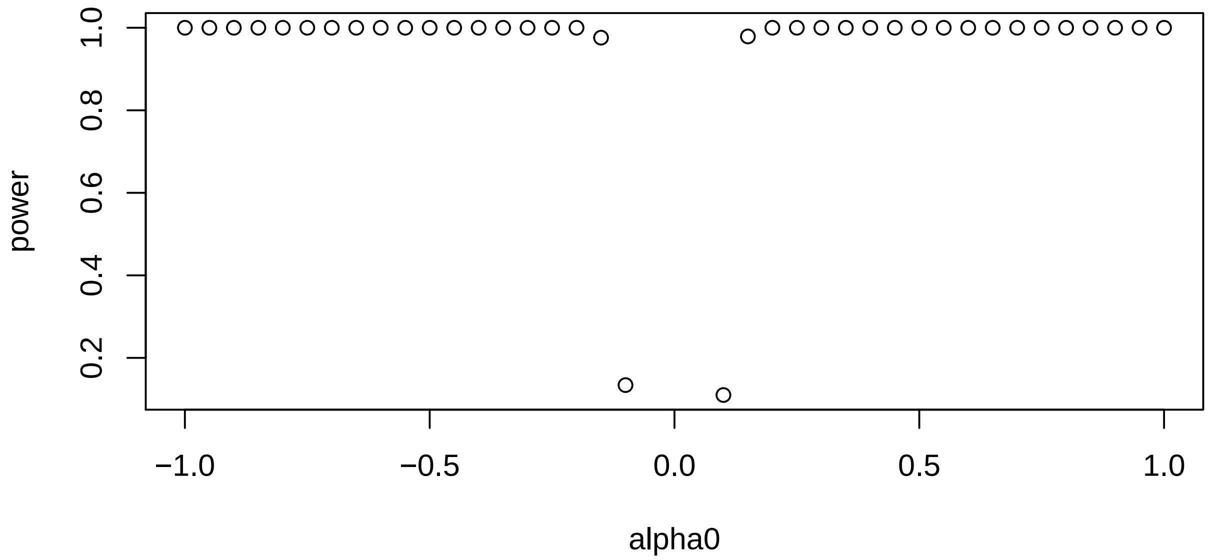

Figure 3: Power of the test with residuals from N(0,1)

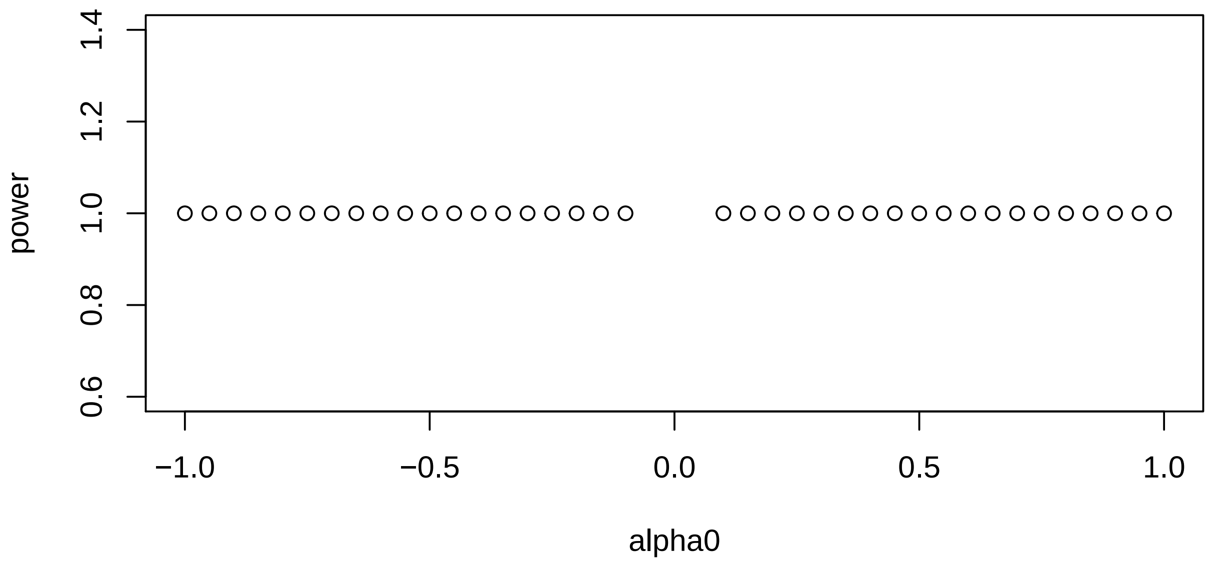

Figure 4: Power of the test with residuals from random walk

In the case of random walk even if H_0 is very close to the true value, the test will reject it.