Estimation theory – Report 2

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Contents

1 Exercise 1

We have data file with 100 rows and 4 columns. We take the first column as a column vector y and the remaining 3 columns as a matrix X, where y depends on X. We assume that the model for our data is as follows

$$y = X\alpha + u. \tag{1}$$

We will use regression model function in R to compute the parameters and compare them with the results we obtain manually.

```
data <- read.table('data_lab_2.csv', sep = ",", dec = ",", header = FALSE)
attach(data)

N <- 100
K <- 3
X <- as.matrix(data[, -1])
y <- as.matrix(data[, 1])
```

```r
# linear regression model using lm()
model <- lm(V1 ~ . - 1, data)

summary(model)

##
## Call:
## lm(formula = V1 ~ . - 1, data = data)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -2.68919 -0.47894  0.08483  0.47957  2.06775
##
## Coefficients:
##    Estimate Std. Error t value Pr(>|t|)
## V2  2.05860    0.06862   30.00   <2e-16 ***
## V3  1.07476    0.06720   15.99   <2e-16 ***
## V4  0.88974    0.07506   11.85   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.8956 on 97 degrees of freedom
## Multiple R-squared:  0.9951,Adjusted R-squared:  0.9949
## F-statistic:  6517 on 3 and 97 DF,  p-value: < 2.2e-16
```

1.1 Part 1

We want to express the loss function

$$L = \sum_{n=1}^{N} u_n^2$$

as a function of y, X and α. From (1) we have $u = y - X\alpha$. Then

$$\begin{aligned} L = \sum_{n=1}^{N} u_n^2 = u'u = (y - X\alpha)'(y - X\alpha) = (y' - \alpha' X')(y - X\alpha) = \\ = y'y - y'X\alpha - \alpha' X' y + \alpha' X' X\alpha = y'y - 2y'X\alpha + \alpha' X' X\alpha. \end{aligned}$$

1.2 Part 2

Next we will use the following equalities

$$\frac{\partial A\beta}{\partial \beta'} = A \qquad \text{and} \qquad \frac{\partial \beta' A\beta}{\partial \beta'} = \beta'(A + A')$$

to calculate the first derivative of L with respect to α.

$$\frac{\partial L(\alpha)}{\partial \alpha'} = 0 - 2y'X + \alpha'(X'X + X'X) = -2y'X + 2\alpha' X'X.$$

1.3 Part 3

Now, to minimize the L function, we will solve the first order condition equation $\frac{\partial L(\alpha)}{\partial \alpha'} = 0$.

$$\begin{aligned}
\frac{\partial L(\alpha)}{\partial \alpha'} &= 0 \\
-2y'X + 2\alpha' X'X &= 0 \\
\alpha' X'X &= y'X \quad / \cdot (X'X)^{-1} \\
\alpha' &= y'X(X'X)^{-1} \\
\alpha &= (X'X)^{-1}X'y
\end{aligned}$$

So $\hat{\alpha} = (X'X)^{-1}X'y$ is the LS estimator of the model parameter.

The first vector is the theoretical estimator of α and the second is the estimator obtained with R's linear regression model:

		theoretical	using lm()
1	alpha_1	2.05859906656869	2.05859906656869
2	alpha_2	1.07475575539487	1.07475575539487
3	alpha_3	0.889740635967315	0.889740635967315

Table 1: Estimator of alpha

1.4 Part 4

We are using unbiased estimator for the variance of residuals

$$\hat{\sigma}^2 = \frac{u'u}{N-K},$$

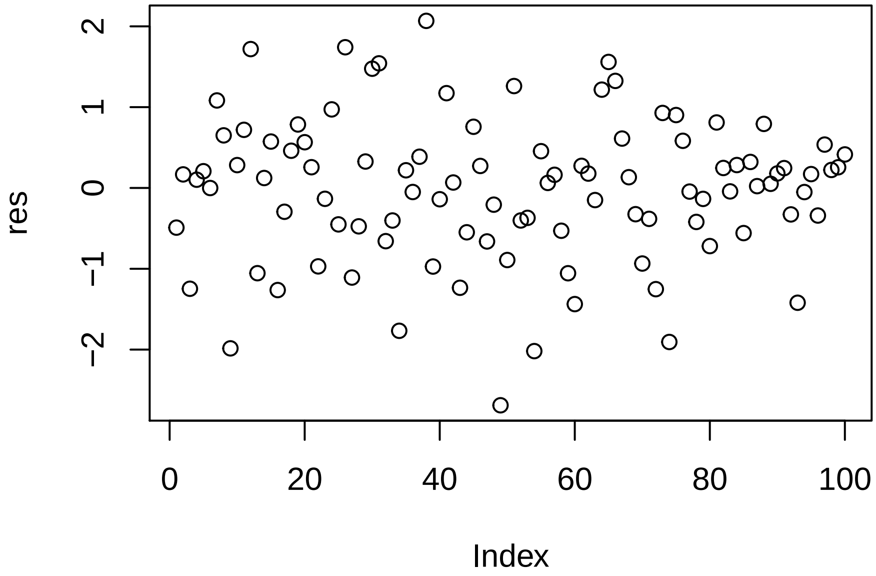
where in our case $N = 100$ and $K = 3$.

	theoretical	using lm()
1	0.80202	0.80210

Table 2: Variance of residuals

The first row is the theoretical estimator of σ and the second is the squared residual standard error obtained with R's linear regression model.

1.5 Part 5



We assume that the residuals are uncorrelated and homoscedastic. The variance-covariance matrix of LS estimator is

$$\hat{\Sigma}_{\hat{\alpha}} = \hat{\sigma}^2 (X'X)^{-1},$$

where $\hat{\sigma}^2$ is the variance of the residuals.

We can calculate the variance-covariance matrix using R:

```
##              V2           V3           V4
## V2  0.004708172 -0.002019469 -0.002544603
## V3 -0.002019469  0.004516371 -0.002430459
## V4 -0.002544603 -0.002430459  0.005633808
##              V2           V3           V4
## V2  0.004708172 -0.002019469 -0.002544603
## V3 -0.002019469  0.004516371 -0.002430459
## V4 -0.002544603 -0.002430459  0.005633808
```

where the first result is the theoretical estimator of Σ and the second is the one obtained with R's linear regression model.

Variance-covariance matrix for $\sqrt{N}\hat{\alpha}$ is equal to:

$$\hat{\Sigma}_{\sqrt{N}\hat{\alpha}} = N\hat{\sigma}^2 (X'X)^{-1}$$

```
##            V2         V3         V4
## V2  0.4708172 -0.2019469 -0.2544603
## V3 -0.2019469  0.4516371 -0.2430459
## V4 -0.2544603 -0.2430459  0.5633808
```

1.6 Part 6

The t-statistic tests the hypothesis $H_0 : \ \alpha_i = 0$, $H_1 : \ \alpha_i \neq 0$. The t-ratio is the ratio of the sample regression coefficient to its standard error. So

$$t_{\hat{\alpha}_i} = \frac{\hat{\alpha}_i}{\sqrt{Var\hat{\alpha}_i}},$$

where $t_{\hat{\alpha}_i} \sim t(N-K) = t(100-3) = t(97)$.

		theoretical	using lm()
1	V1	30.0016825108455	30.0016825108456
2	V2	15.9924488168616	15.9924488168616
3	V3	11.8539317172235	11.8539317172235

Table 3: t-ratios of the parameters

2 Exercise 2

In this exercise we assume that $\alpha_1 + \alpha_2 + \alpha_3 = 0$ and $\alpha_2 - \alpha_3 = 0$.

2.1 Part 1

We know that the restriction matrix R satisfies equation $R\alpha = r$. In this case

$$R \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Rank of the restriction matrix is $rank(R) = 2$, because there are two linearly independent vectors in matrix R, $[1,1,1]$ and $[0,1,-1]$.

2.2 Part 2

To express vector α and the loss function L as functions of α_3, we will first express α_1 and α_2 as functions of α_3.

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 0 \\ \alpha_2 - \alpha_3 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = -\alpha_2 - \alpha_3 \\ \alpha_2 = \alpha_3 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = -2\alpha_3 \\ \alpha_2 = \alpha_3 \end{cases}$$

So

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} -2\alpha_3 \\ \alpha_3 \\ \alpha_3 \end{bmatrix} = \alpha_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

and

$$L(\alpha_3) = y'y - 2y'X\alpha + \alpha'X'X\alpha = y'y - 2y'X\alpha_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} -2 & 1 & 1 \end{bmatrix} X'X\alpha_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} =$$

$$y'y - 2\alpha_3 y'X \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + \alpha_3^2 \begin{bmatrix} -2 & 1 & 1 \end{bmatrix} X'X \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

2.3 Part 3

Estimator of α_3 is equal to

$$\hat{\alpha}_3 = \arg\min_{\alpha_3} L(\alpha_3)$$

Again, we will use the formula $\frac{\partial A\beta}{\partial \beta'} = A$ to calculate

$$\frac{\partial L(\alpha_3)}{\partial \alpha_3} = 0 - 2y'X \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + 2\alpha_3 \begin{bmatrix} -2 & 1 & 1 \end{bmatrix} X'X \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

From F.O.C and the above formula we obtain

$$\hat{\alpha}_3 \begin{bmatrix} -2 & 1 & 1 \end{bmatrix} X'X \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = y'X \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\hat{\alpha}_3 = y'X \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \left(\begin{bmatrix} -2 & 1 & 1 \end{bmatrix} X'X \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right)^{-1}$$

3 Exercise 3

Next, we consider the following model

$$y_n = \alpha_1 X_{1n} + \alpha_2 X_{2n} + \varepsilon_n$$

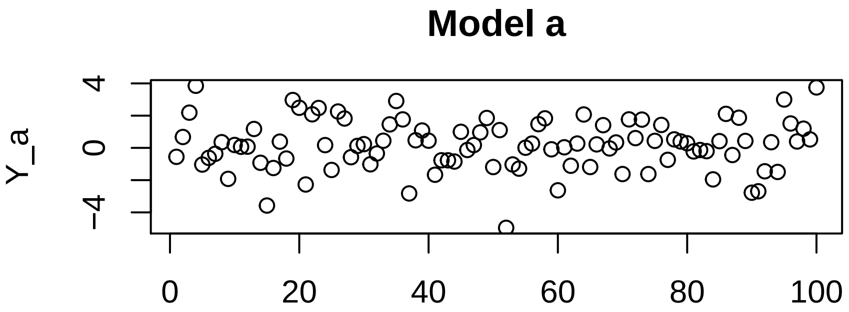
with $\alpha_1 = \alpha_2 = 1$ and $\varepsilon_n \sim N(0, 1)$.

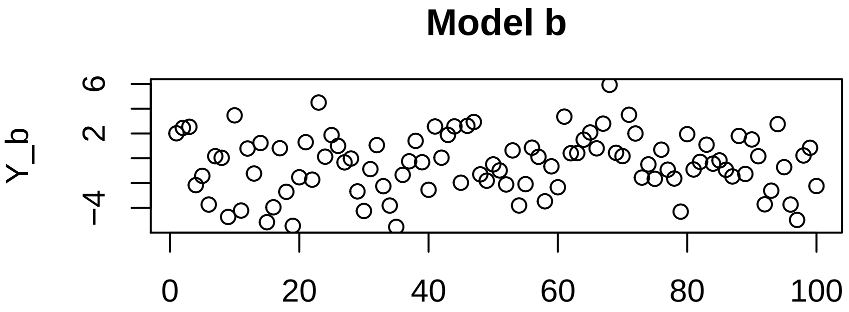
3.1 Part 1

We will generate samples from above model in two cases:

a) X_1 and X_2 are independent and $X \sim N(0, I_2)$,

b) X_1 and X_2 are dependent and $X \sim N(0, \Sigma)$.





3.2 Part 2 and 3

We have the following models:

$$y_n = \alpha_1 X_{1n} + \alpha_2 X_{2n} + \varepsilon_n$$

$$y_n = \beta X_{1n} + u_n$$

We will use the formula from subsection 1.3 to obtain α_1 and β in those models. Below we compare the results for different sample sizes.

	N	theoretical_alpha_1	alpha_1_est	beta_est
1	10.00000	1.00000	1.01685	1.00558
2	100.00000	1.00000	0.99905	1.00060
3	1000.00000	1.00000	1.00147	1.00059

Table 4: Estimator of alpha1 and beta for model a

	N	theoretical_alpha_1	alpha_1_est	beta_est
1	10.00000	1.00000	1.00154	1.25233
2	100.00000	1.00000	0.99924	1.25446
3	1000.00000	1.00000	1.00023	1.25047

Table 5: Estimator of alpha1 and beta for model a