Partial Differential Equations with Applications in Industry

Problem Set 2: Classification of PDEs

1. (*Classification of PDEs: A step-by-step tutorial*) Consider a general second-order PDE which is linear in its highest derivatives

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + F(x,y,u,u_x,u_y) = 0,$$

where a, b, c are smooth coefficients while F is a smooth and possibly nonlinear function.

(a) Start by introducing an arbitrary set of new variables: $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$. Assume that its Jacobian is nonvanishing, i.e. $\xi_x\eta_y - \xi_y\eta_x \neq 0$. Explain why the latter condition is sufficient for an existence of the inverse transformation $x = x(\xi,\eta)$ and $y = y(\xi,\eta)$.

(b) Express the solution of above equation in new variables and find what the considered PDE transforms into. More specifically, define $v(\xi,\eta) := u(x(\xi,\eta), y(\xi,\eta))$, which gives $u(x,y) = v(\xi(x,y), \eta(x,y)$. Show that v is a solution of

$$A(\xi,\eta)v_{\xi\xi} + 2B(\xi,\eta)v_{\xi\eta} + C(\xi,\eta)v_{\eta\eta} + G = 0,$$

where G contains independent variables, function v and its derivatives, while

$$\begin{aligned}
A &= a\xi_x^2 + 2b\xi_x\xi^y + c\xi_y^2,\\
B &= a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y,\\
C &= a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2.
\end{aligned}$$

Notice that we do not write functions' arguments explicitly.

(c) It is time to choose a specific form of ξ and η. To this end, observe that A and C in the above formulas has exactly the same form. Therefore, if we define $\phi = \phi(x,y)$ to be a solution of

$$a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2 = 0,$$

then, choosing $\xi(x,y) = \phi(x,y)$ or $\eta(x,y) = \phi(x,y)$ lets us take $A = 0$ or $C = 0$ which simplifies our PDE considerably. Assume that $a \neq 0$ and $\phi_y \neq 0$ in some region. Show that the ratio ϕ_x/ϕ_y is then a solution of the quadratic which has a determinant

$$\Delta := b^2 - ac.$$

(d) (*Hyperbolic case*) Prove that if $\Delta > 0$ then there are two independent solutions of the quadratic which satisfy a set of equations

$$\phi_x + \frac{-b \pm \sqrt{b^2 - ac}}{a}\phi_y = 0,$$

Choosing $\xi = \phi_1$ and $\eta = \phi_2$, where $\phi_{1,2}$ are solutions of the above equations, yields $A = C = 0$. Show that this implies that v can be written as

$$v_{\xi\eta} = H(\xi,\eta,v,v_\xi,v_\eta),$$

for some function H. Further, demonstrate that by choosing another set of variables

$$\alpha = \frac{\xi+\eta}{2}, \quad \beta = \frac{\xi-\eta}{2},$$

helps us to conclude that

$$w_{\alpha\alpha} - w_{\beta\beta} = K(\alpha,\beta,w,w_\alpha,w_\beta),$$

where $w(\alpha,\beta) = v(\alpha+\beta, \alpha-\beta)$ and K is some function. Both of the above forms of the hyperbolic equation are called *canonical*.

(e) (*Elliptic case*) Mimic the arguments for the hyperbolic case to derive a canonical form for the elliptic equation (do not be afraid of the complex numbers).

(f) (*Parabolic case*) If $\Delta = 0$ the function ϕ has to satisfy

$$\phi_x + \frac{b}{a}\phi_y = 0,$$

which gives us only one solution. Put $\xi = \phi$ which gives $A = 0$. As for the second variable choose η to be *any* function independent of ξ (vanishing Jacobian). Show that then $B = 0$. Therefore, the canonical form of the parabolic equation has the form

$$v_{\eta\eta} = H(\xi, \eta, v, v_\xi, v_\eta),$$

for some function H.

2. Find all functions $u = u(x, y)$ satisfying the following PDE

$$\frac{\partial u}{\partial x} = 0.$$

3. (*Classification*) Classify a given PDE (parabolic, hyperbolic or elliptic) and transform it to its canonical form in the appropriate region.

a) $u_{xx} + xu_{yy} = 0$; b) $u_{xx} + yu_{yy} + \frac{1}{2}u_y = 0$; c) $4y^2u_{xx} - e^{2x}u_{yy} - 4y^2u_x = 0$;
d) $y^2u_{xx} - x^2u_{yy} = 0$; e) $xu_{xx} + yu_{yy} = 0$; f) $(\sin x)^2u_{xx} - 2y(\sin x)u_{xy} + y^3u_{yy} = 0$.

4. (*Transformation*) Let $u = u(x, y)$ be a solution of the following equation

$$u_{xx} + au_x + bu_y + cu + f(x, y) = 0,$$

where a, b and c are constants. Introduce a function $v = v(x, y)$ as follows

$$u(x, y) = e^{\alpha x + \beta y}v(x, y),$$

and transform the PDE into a simpler form. Here, α and β are constants yet to be chosen.

5. Reduce the following equation to its canonical form and further simplify it

$$au_{xx} + 2au_{xy} + au_{yy} + bu_x + cu_y + u = 0.$$

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