# Estimation theory – Report 4

Marta Frankowska, 208581
Agnieszka Szkutek, 208619

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# 1 Exercise 1

We generate a time series of 20 observations according to the model $y_t = \alpha + \varepsilon_t$, where $\varepsilon_t \sim N(0, \sigma^2)$ and *iid*.

## 1.1 Part 1

The density function of a single observation is

$$f(y_t, \theta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_t-\alpha)^2}{2\sigma^2}},$$

the likelihood function is

$$L(\theta; y_1, \ldots, y_N) = f(y_1, \ldots, y_N; \theta) = \prod_{i=1}^{N} f(y_i; \theta) =$$

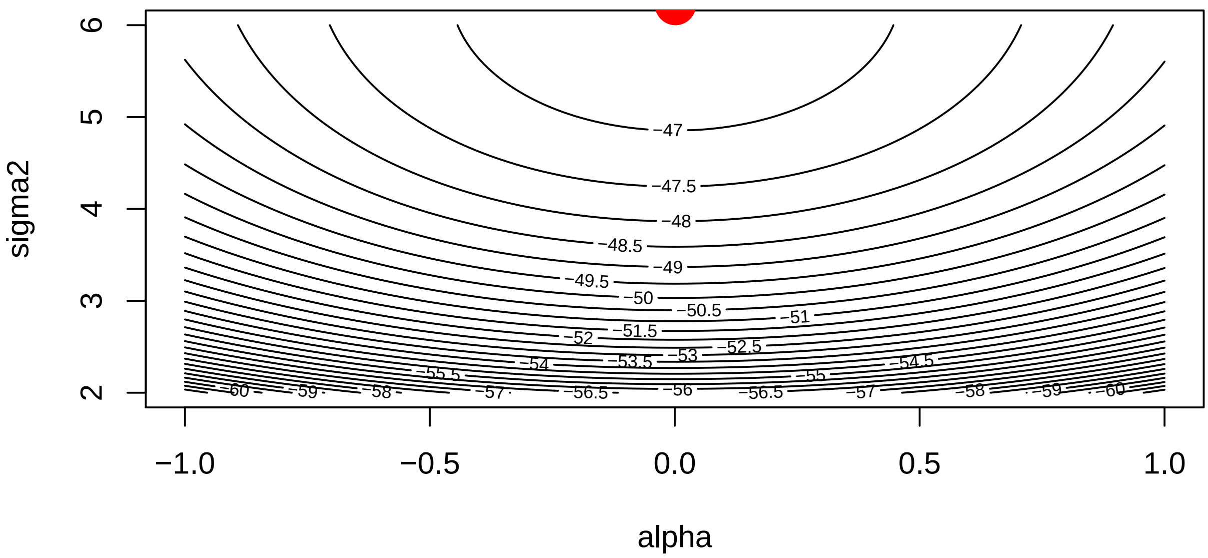
$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_1-\alpha)^2}{2\sigma^2}} \ldots \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_N-\alpha)^2}{2\sigma^2}} = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(y_i-\alpha)^2},$$

and the log-likelihood is as follows

$$l(\theta; y_1, \ldots, y_N) = \ln L(\theta; y_1, \ldots, y_N) = -\frac{N}{2}\ln 2\pi - \frac{N}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{N}(y_i - \alpha)^2.$$

## 1.2 Part 2

The contour plot of log-likelihood function is displayed below.




## 1.3 Part 3

The First Order Condition is as follows $\frac{\partial \ln L}{\partial \theta}$, where vector $\theta$ is equal to $\theta = \begin{bmatrix} \alpha \\ \sigma^2 \end{bmatrix}$. It gives the following set of equations

$$\begin{cases} \frac{\partial \ln L}{\partial \alpha} = 0 \\ \frac{\partial \ln L}{\partial \sigma^2} = 0 \end{cases} \Rightarrow \begin{cases} \frac{1}{\sigma^2} \sum_{i=1}^{N} (y_i - \alpha) = 0 \\ -\frac{N}{2\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum_{i=1}^{N} (y_i - \alpha)^2 = 0 \end{cases} \Rightarrow \begin{cases} \frac{N}{N} \sum_{i=1}^{N} y_i - N\alpha = 0 \\ \frac{1}{\sigma^2} \sum_{i=1}^{N} (y_i - \alpha)^2 = N \end{cases}$$

$$\Rightarrow \begin{cases} N(\bar{y} - \alpha) = 0 \\ \sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (y_i - \alpha)^2 \end{cases} \Rightarrow \begin{cases} \alpha = \bar{y} \\ \sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (y_i - \bar{y})^2 \end{cases}$$

So the ML estimators of the model parameters are

$$\begin{cases} \hat{\alpha} = \bar{y} \\ \hat{\sigma}^2 = \frac{1}{N} \sum_{i=1}^{N} (y_i - \bar{y})^2 \end{cases}$$

## 1.4 Part 4

### 1.4.1 Variance-covariance matrix

The variance-covariance matrix of parameters $\hat{\alpha}$ and $\hat{\sigma}^2$:

$$\Sigma_{(\hat{\alpha}, \hat{\sigma}^2)} = \begin{bmatrix} Var\hat{\alpha} & Cov(\hat{\alpha}, \hat{\sigma}^2) \\ Cov(\hat{\alpha}, \hat{\sigma}^2) & Var\hat{\sigma}^2 \end{bmatrix},$$

where $Var\hat{\alpha} = Var\bar{y} = Var\left(\frac{1}{N} \sum_{i=1}^{N} y_i\right) = \frac{\sigma^2}{N}$.

We know that $\frac{N\hat{\sigma}^2}{\sigma^2} \sim \chi^2(N-1)$ and $Var[\chi^2(N-1)] = 2(N-1)$ so

$$Var\left(\frac{N\hat{\sigma}^2}{\sigma^2}\right) = \frac{N^2}{\sigma^4}Var(\hat{\sigma}^2) = 2(N-1) \quad \Rightarrow \quad Var(\hat{\sigma}^2) = \frac{2(N-1)\sigma^4}{N^2}$$

To calculate $Cov(\hat{\alpha}, \hat{\sigma}^2)$ we will use the following fact.

**Lemma 1** *Let $X_1, ..., X_N \sim N(\mu, \sigma^2)$. Then $\bar{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$ and $\hat{\sigma}^2 = \frac{1}{N}\sum_{i=1}^{N}(X_i - \bar{X})^2$ are independent. That means $Cov(\bar{X}, \hat{\sigma}^2) = 0$.*

Thus the variance-covariance matrix of parameters $\hat{\alpha}$ and $\hat{\sigma}^2$ is

$$\Sigma_{(\hat{\alpha},\hat{\sigma}^2)} = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \frac{2(N-1)\sigma^4}{N^4}. \end{bmatrix}$$

### 1.4.2 The asymptotic distribution of ML estimator

The asymptotic distribution of ML estimator is

$$\hat{\theta} \xrightarrow{d} N(\theta_0, I(\theta_0)^{-1}),$$

where $I(\theta_0) = -E_0(H(\theta_0)) = -E_0\left(\frac{\partial^2 \ln L}{\partial\theta_0 \partial\theta_0'}\right)$.

We calculate partial derivatives:

$$\frac{\partial \ln L}{\partial \alpha} = \frac{1}{\sigma^2}N(\bar{y} - \alpha), \quad \frac{\partial^2 \ln L}{\partial \alpha^2} = -\frac{N}{\sigma^2}$$

$$\frac{\partial \ln L}{\partial \sigma^2} = -\frac{N}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{N}(y_i - \alpha)^2, \quad \frac{\partial^2 \ln L}{\partial(\sigma^2)^2} = \frac{N}{2\sigma^4} - \frac{1}{(\sigma^2)^3}\sum_{i=1}^{N}(y_i - \alpha)^2$$

$$\frac{\partial^2 \ln L}{\partial\alpha\partial\sigma^2} = \frac{\partial^2 \ln L}{\partial\sigma^2\partial\alpha} = -\frac{1}{\sigma^4}\sum_{i=1}^{N}(y_i - \alpha)$$

Now we calculate expected value of the above derivatives:

$$E\frac{\partial^2 \ln L}{\partial \alpha^2} = -\frac{N}{\sigma^2}$$

$$E\frac{\partial^2 \ln L}{\partial(\sigma^2)^2} = E\left(\frac{N}{2\sigma^4}\right) - E\left(\frac{1}{\sigma^6}\sum_{i=1}^{N}(y_i - \alpha)^2\right) = \frac{N}{2\sigma^4} - \frac{N}{\sigma^4} = -\frac{N}{2\sigma^4}$$

$$E\frac{\partial^2 \ln L}{\partial\alpha\partial\sigma^2} = -\frac{1}{\sigma^4}E\sum_{i=1}^{N}(y_i - \alpha) = -\frac{1}{\sigma^4}\sum_{i=1}^{N}E(y_i - \alpha) = 0$$

So

$$I^{-1}(\theta_0) = \begin{bmatrix} \frac{N}{\sigma^2} & 0 \\ 0 & \frac{N}{2\sigma^4} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{\sigma^2}{N} & 0 \\ 0 & \frac{2\sigma^4}{N} \end{bmatrix}$$

We see that asymptotic covariance of $\hat{\alpha}$ and $\hat{\sigma}^2$ is equal to 0.

### 1.5 Part 5

To calculate ML estimator of $1 + \alpha + \alpha^2$ we use below property.

**Lemma 2 (Invariance property)** *$\hat{g}(\theta) = g(\hat{\theta})$, where $\hat{\theta}$ is MLE of $\theta$ and $g$ is continous and continously differentaible funcion.*

ML estimator of $\beta = 1 + \alpha + \alpha^2$ is

$$\hat{\beta} = \hat{g}(\alpha) = g(\hat{\alpha}) = 1 + \hat{\alpha} + \hat{\alpha}^2 = 1 + \bar{y} + \bar{y}^2.$$

Now we calculate $Var\hat{\beta}$ using fact that $\hat{\alpha} \sim N(\alpha, \frac{\sigma^2}{N})$.

$$\begin{aligned} Var(\hat{\beta}) = Var(1 + \hat{\alpha} + \hat{\alpha}^2) = Var(\hat{\alpha} + \hat{\alpha}^2) = E(\hat{\alpha} + \hat{\alpha}^2)^2 - (E(\hat{\alpha} + \hat{\alpha}^2))^2 = \\ = E\hat{\alpha}^2 + 2E\hat{\alpha}^3 + E\hat{\alpha}^4 - (E\hat{\alpha})^2 - 2E\hat{\alpha}E\hat{\alpha}^2 - (E\hat{\alpha}^2)^2 \end{aligned}$$

Using formula for normal distribution moments we get:

$$E\hat{\alpha} = \alpha, \quad E\hat{\alpha}^2 = \alpha^2 + \frac{\sigma^2}{N}, \quad E\hat{\alpha}^3 = \alpha^3 + 3\alpha\frac{\sigma^2}{N}, \quad \text{and} \quad E\hat{\alpha}^4 = \alpha^4 + 6\alpha^2\frac{\sigma^2}{N} + 3\frac{\sigma^4}{N^2}$$

$$\begin{aligned} Var(\hat{\beta}) = \alpha^2 + \frac{\sigma^2}{N} + 2\alpha^3 + 6\alpha\frac{\sigma^2}{N} + \alpha^4 + 6\alpha^2\frac{\sigma^2}{N} + 3\frac{\sigma^4}{N^2} - \alpha^2 - 2\alpha\left(\alpha^2 - \frac{\sigma^2}{N}\right) - \left(\alpha^2 + \frac{\sigma^2}{N}\right)^2 = \\ = \frac{\sigma^2}{N} + 4\alpha\frac{\sigma^2}{N} + 4\alpha^2\frac{\sigma^2}{N} + 2\frac{\sigma^4}{N^2} \end{aligned}$$

## 2 Exercise 2

In this exercise we will be using data set from file *datalab4-1.xlsx*. We will assume that $y$ has a mixed normal distribution,

$$y_n \sim \begin{cases} N(0,1), & \text{for } p \\ N(\mu, \sigma^2), & \text{for } 1-p, \end{cases} \qquad \text{depending on } \theta = \begin{bmatrix} \mu \\ \sigma^2 \\ p \end{bmatrix}.$$

### 2.1 Part 1

The density function of $y$ is equal to

$$f(y;\theta) = f(y;\mu,\sigma^2,p) = p \cdot f(y;0,1) + (1-p) \cdot f(y;\mu,\sigma^2),$$

where

$$f(y;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

is the density function for normal distribution with parameters $\mu$ and $\sigma^2$.

The likelihood function of $y$ is obtained in the following way

$$\begin{aligned} L(\theta; y_1, \ldots, y_N) = \prod_{i=1}^{N} f(y_i;\theta) = \prod_{i=1}^{N} \left(p \cdot f(y_i;0,1) + (1-p) \cdot f(y_i;\mu,\sigma^2)\right) = \\ = \prod_{i=1}^{N} \left(p \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y_i^2}{2}} + (1-p)\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_i-\mu)^2}{2\sigma^2}}\right). \end{aligned}$$

The log-likelihood function of $y$ is as follows

$$l(\theta; y_1, \dots, y_N) = \log L(\theta; y_1, \dots, y_N) = \log \prod_{i=1}^{N} f(y_i; \theta) = \sum_{i=1}^{N} \log f(y_i; \theta) =$$

$$\sum_{i=1}^{N} \log \left( p \cdot f(y_i; 0, 1) + (1 - p) \cdot f(y_i; \mu, \sigma^2) \right).$$

## 2.2 Part 2