Optimization Theory

Applied Mathematics

Problem set 4

1. Using some additional binary variables (used as logical values) write the following problems as integer programs:

   (a)

$$
\begin{array}{lll}
\text{Maximize} & 3x_1 + 4x_2 - x_3 & \\
\text{subject to:} & \text{at least 2 of constraints} & x_1 + 3x_2 \le 2 \\
 & & 2x_1 + x_3 \le 3 \\
 & & x_1 + x_2 + 2x_3 \le 4 \\
 & \text{and at least one of} & x_1 + x_3 \ge 1 \\
 & & x_2 + x_3 \ge 1 \\
 & & x_1 + x_3 \ge 1 \\
 & \text{are satisfied,} & \\
 & x_1, x_2, x_3 \ge 0 &
\end{array}
$$

   (b)

$$
\begin{array}{ll}
\text{Maximize} & 2x_1 - 6x_2 \\
\text{subject to} & x_1 + 2x_2 \le 6 \Longrightarrow \left\{ \begin{array}{l} x_1 + x_2 \ge 3 \\ 2x_1 + x_2 \ge 4 \end{array} \right. \\
 & x_1, x_2 \ge 0
\end{array}
$$

   (c) Suppose constraint $x_1 + x_2 \ge 3$ is replaced with $x_1 - x_2 \ge 3$. We cannot we use the same technique (introduced on the lecture) to write the problem as an integer program? Why?

2. n tasks are to be assigned to 1 of 3 identical processors P_1, P_2 and P_3. Performing task i on any processor takes $t_i > 0$ nanoseconds. A schedule for the processors is given by three disjoint subsets S_1, S_2 and S_3 of $\{1, \dots, n\}$ such that $S_1 \cup S_2 \cup S_3 = \{1, \dots, n\}$. Tasks from set S_i are performed on processor P_i in an arbitrary order. The time when all the tasks are completed can be computed as

$$
T_{\max} = \max \left\{ \sum_{i \in S_1} t_i, \sum_{i \in S_2} t_i, \sum_{i \in S_3} t_i \right\}.
$$

   Write an integer program finding a schedule minimizing $T_{\max}$.

3. ([BHM77]) Graph the following integer program:

$$
\begin{array}{ll}
\text{Maximize} & x_1 + 5x_2, \\
\text{subject to:} & -4x_1 + 3x_2 \le 6, \\
 & 3x_1 + 2x_2 \le 18, \\
 & x_1, x_2 \ge 0 \text{ and integer.}
\end{array}
$$

   Solve it using the branch and bound procedure, graphically solving each linear-programming problem encountered.

4. ([Tr13]) Find the distance between the line $y = -x - 1$ and the parabola $y = x^2 + 1$ using the method of Lagrange multipliers.

5. Using the method of Lagrange multipliers find the minimum of $f(x, y, z, t) = x^2 + y^2 + z^2 + t^2$ subject to $x - y - z + t = 5$ and $x - 3y + z - t = 6$.

6. Show that the minimum of $f(x, y) = x + y$ subject to $(x - 1)^2 + y^2 = 1$ and $(x - 2)^2 + y^2 = 4$ cannot be found using Lagrange multipliers even though it exists. What assumptions of the theorem we are using are not satisfied for this problem?

7. ([Tr13]) Let $p_1, \dots, p_n$ and s be positive numbers. Maximize

$$
f(x_1, \dots, x_n) = (1 - x_1)^{p_1} (1 - x_2)^{p_2} \dots (1 - x_n)^{p_n}
$$

   subject to $x_1 + x_2 + \dots + x_n = s$.

8. Using the method introduced during the lecture find the minimum and the maximum of $f(x,y,z) = x^2 + xy + xz + yz$ subject to $x^2 + y^2 + z^2 = 1$.

9. Find the minimum and the maximum of $f(x,y,z) = 9x^2 + 4y^2 + 2z^2 - 12xy + 12xz + 8yz$ subject to $9x^2 + 4y^2 + z^2 = 1$.
   Hint: Change the variables in such a way that the problem can be solved using the technique from the previous problem.

10. Generalizing the method used to solve two previous problems to the case with inequality constraints find the minimum and the maximum of $f(x,y) = 2x^2 + 2y^2 + 3z^2 - 2xz + 2yz$ subject to $1 \leq x^2 + y^2 + z^2 \leq 4$.

11. ([Be99]) Using Karush-Kuhn-Tucker conditions find (the matrix form of) the maximum value v and the optimal solution x of the problem
$$\begin{array}{ll} \text{Maximize} & c^T x \\ \text{subject to} & x^T Q x \leq 1, \end{array}$$
   where Q is a symmetric positive definite matrix and c is a nonzero vector.

12. Consider the problem
$$\begin{array}{ll} \text{Maximize} & (x_1 + 4)x_2 \\ \text{subject to} & x_1 + x_2 \leq 10, \\ & \frac{4}{3}x_1 + \frac{2}{3}x_2 \leq 12, \\ & \frac{1}{2}x_1 + \frac{3}{2}x_2 \leq 12, \\ & x_1, x_2, x_3 \geq 0. \end{array}$$
   Find the optimum of this problem using Karush-Kuhn-Tucker conditions. Suppose we can increase the RHS of one of the constraints by 1. Which one should we increase (give an answer based on the interpretation of Lagrange multipliers as shadow prices).

References:

[Be99] D.P. Bertsekas, Nonlinear Programming, Athena Scientific, Belmont, MA: 1999.
[BHM77] S.P. Bradley, A.C. Hax, T.L. Magnanti, Applied Mathematical Programming, Addison-Wesley Publishing Company, 1977
[Tr13] W.F. Trench, The Method of Lagrange Multipliers, 2013, available at http://works.bepress.com/william_trench/130/