

Estimation theory – Report 1

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1 Exercise 1

1.1 Part 1

We generate a vector of $Y \sim N(\mu = 2, \sigma^2 = 4)$ with length $N = 1000$ and transform it into variable Y_1 :

```
Y <- rnorm(n = 1000, mean = 2, sd = 2)
# transform Y to Y1
Y1 <- 3 * (Y - 1)
```

Y_1 has normal distribution, because it is a linear combination of Y . We can calculate analytical mean μ_1 and variance σ_1^2 as follows:

$$\mu_1 = E(Y_1) = E(3(Y - 1)) = 3E(Y - 1) = 3E(Y) - 3 = 3 \cdot 2 - 3 = 3,$$

$$\sigma_1^2 = Var(Y_1) = Var(3(Y - 1)) = 9Var(Y) = 9 \cdot 4 = 36$$

and compute them numerically:

```

Y <- rnorm(n = 1000, mean = 2, sd = 2)
Y1 <- 3 * (Y - 1)
mean(Y1)

## [1] 3.290679

var(Y1)

## [1] 36.98737

```

As expected, numerical and analytical results are quite similar. Now we can plot the frequency histogram of Y_1 and analytical normal density with $\mu = 3$ and $\sigma^2 = 36$.

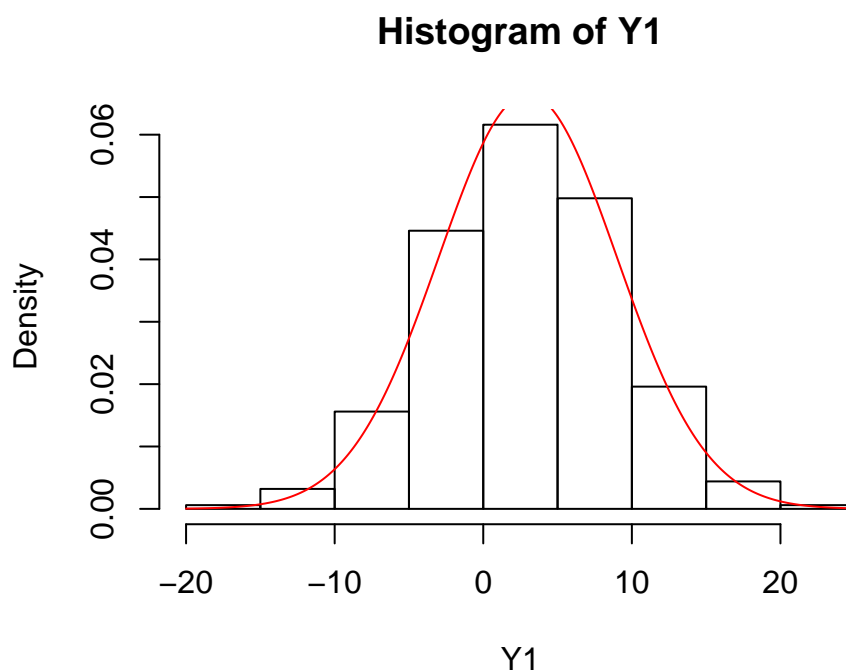


Figure 1: Frequency histogram of Y_1 and analytical density of $N(\mu = 3, \sigma^2 = 36)$.

1.2 Part 2

Next, we create variable $Y_2 = \left(\frac{Y-2}{2}\right)^2$, which is a quadratic function of Y , so we know that distribution of Y_2 is not normal. What is more,

$$\frac{Y-2}{2} \sim N(0, 1).$$

It means that Y_2 , as a sum of squared standard normally distributed variables, has χ^2 distribution with 1 degree of freedom.

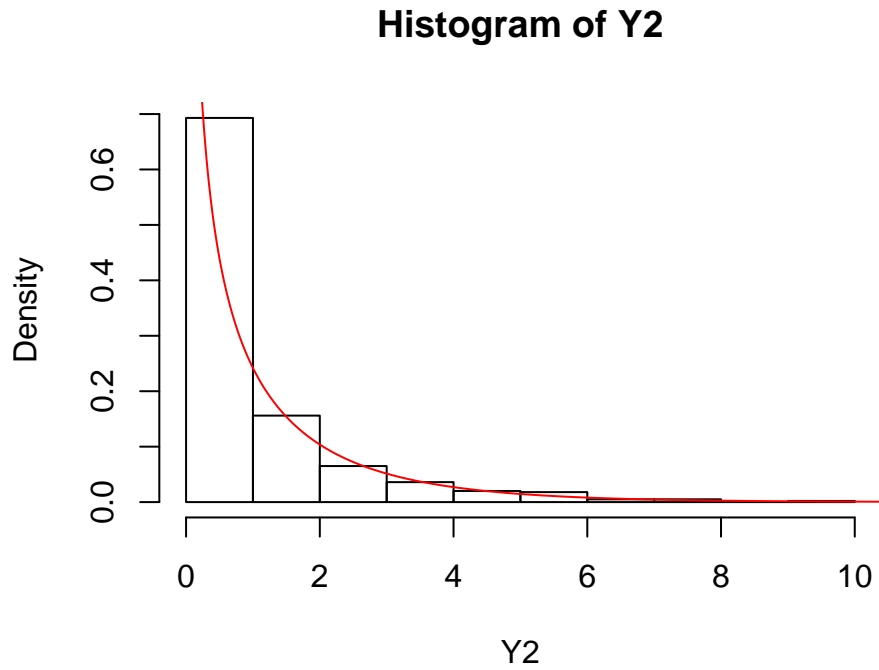


Figure 2: Histogram of Y_2 and theoretical probability density function of $\chi^2(1)$ distribution.

1.3 Part 3 and 4

Next we will compute a sequence of means m_n and a sequence of variances σ_n^2 for the variable Y , where

$$m_n = \frac{1}{n} \sum_{i=1}^n Y_i,$$

$$v_n = \frac{1}{n} \sum_{i=1}^n (Y_i - m_n)^2$$

and plot the results.

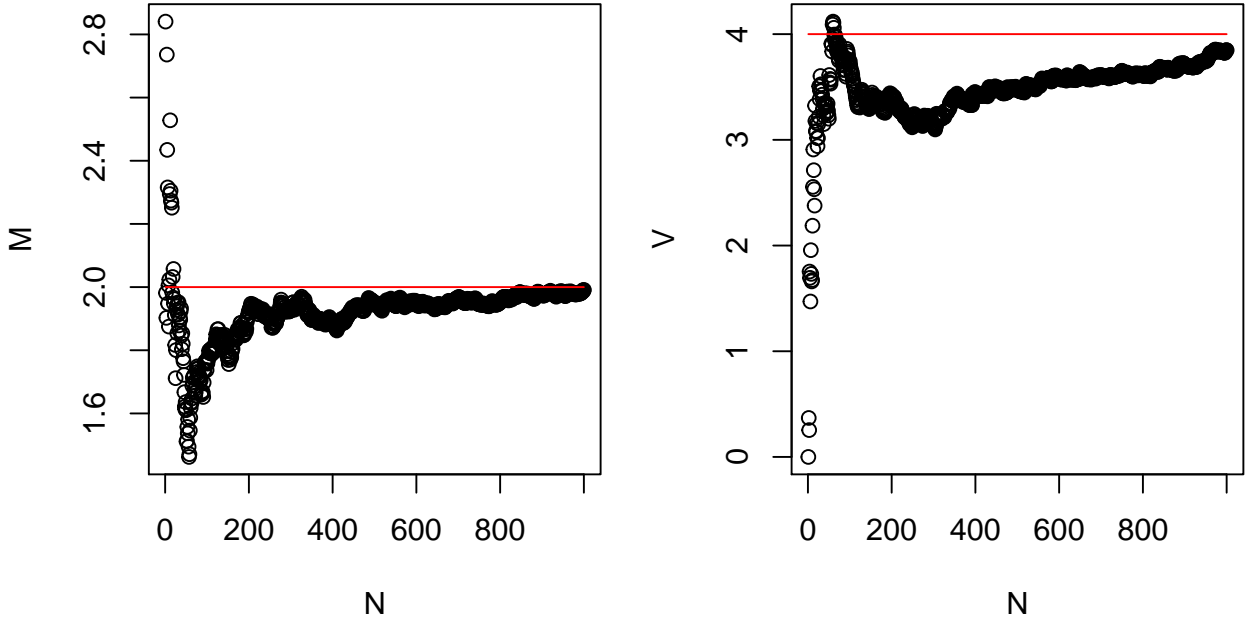


Figure 3: Sequence of means and variances and their respective analytical values.

The sequences m_n and v_n^2 converge to theoretical mean and variance, respectively. To examine the variability of the sequences we can calculate relative errors for both values.

$$err_{m_n} = \left| \frac{m_n - \mu_n}{\mu_n} \right|, \quad err_{v_n} = \left| \frac{v_n - \sigma_n^2}{\sigma_n^2} \right|$$

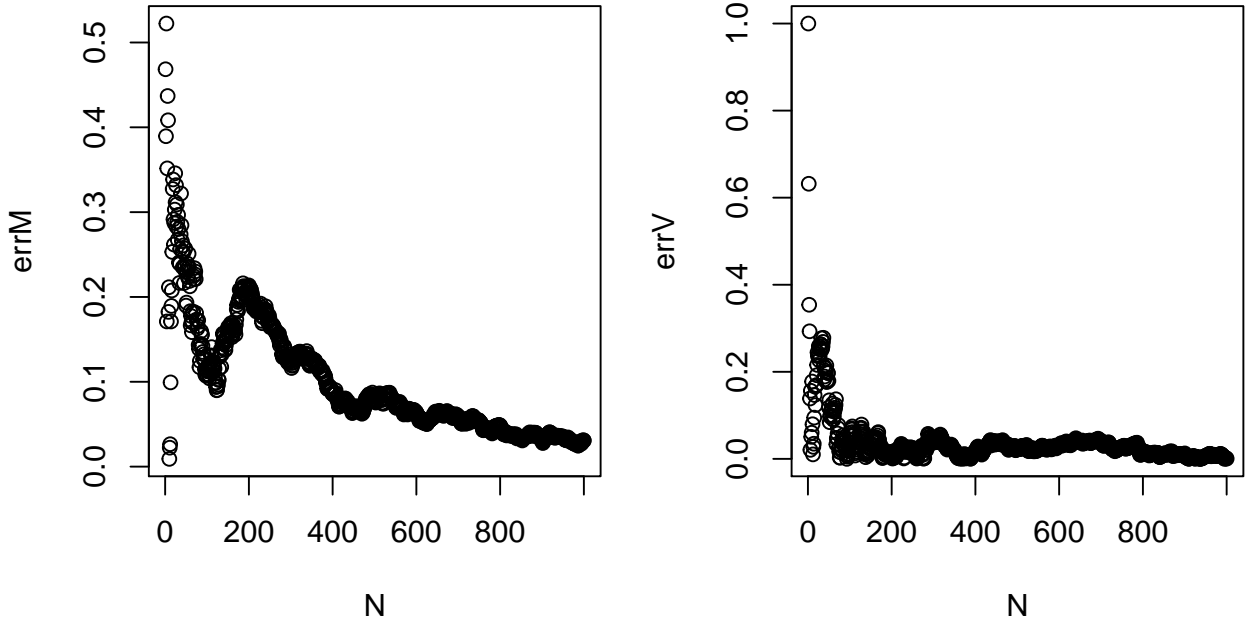


Figure 4: Relative errors for sequences of means and variances.

We can observe that the values of both err_{m_n} and err_{σ^2} are bounded by 0.1.

2 Exercise 2

2.1 Part 1

We simulate 10000 times and then plot 2-dimensional random variable $X \sim N(0, I_2)$.

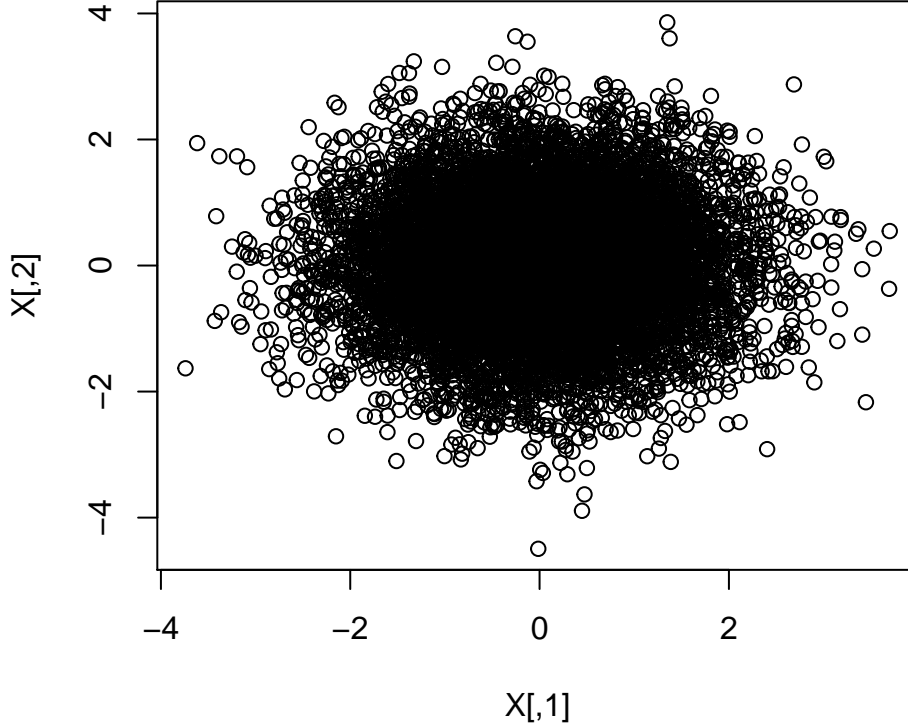


Figure 5: 2-dimensional random variable $X \sim N(0, I_2)$.

2.2 Part 2

To transform variable X into variable $Y \sim N(\mu, \Sigma)$, where

$$\mu = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 2 \end{bmatrix},$$

we should find vector a and matrix A , such that $Y = AX + a$. Expected value of Y is equal to

$$E(Y) = E(AX + a) = AE(X) + a = a,$$

and variance

$$\text{Var}(Y) = \text{Var}(AX + a) = \text{Var}(AX) = A\text{Var}(X)A' = AIA' = AA' = \Sigma.$$

We know that Σ is a symmetric and positive definite matrix, so we can use Cholesky decomposition (`chol()` function in R) to obtain the value of A :

```
Sigma <- matrix(c(2, 0.5, 0.5, 2), 2, 2)
chol(Sigma)
```

```
##           [,1]      [,2]
## [1,] 1.414214 0.3535534
## [2,] 0.000000 1.3693064
```

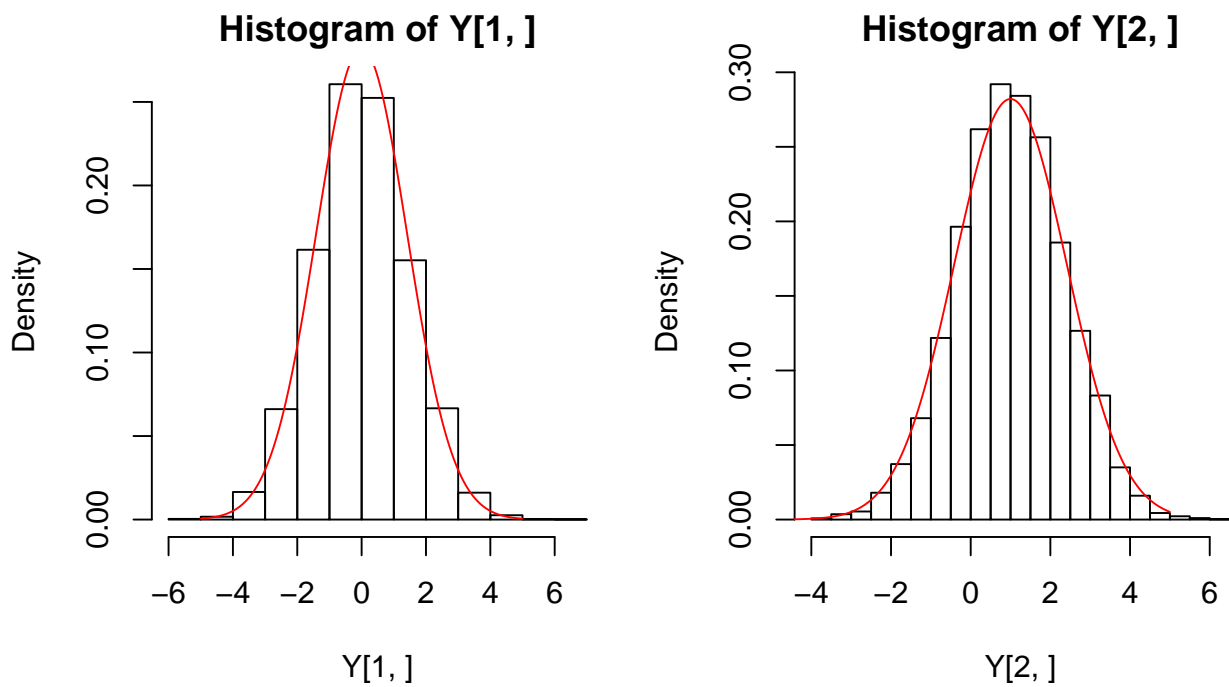


Figure 6: Histograms and probability density functions of Y

2.3 Part 3

Now we can plot the 3D histogram of the random variable $Y = \begin{bmatrix} 1.414214 & 0.3535534 \\ 0.0 & 1.3693064 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

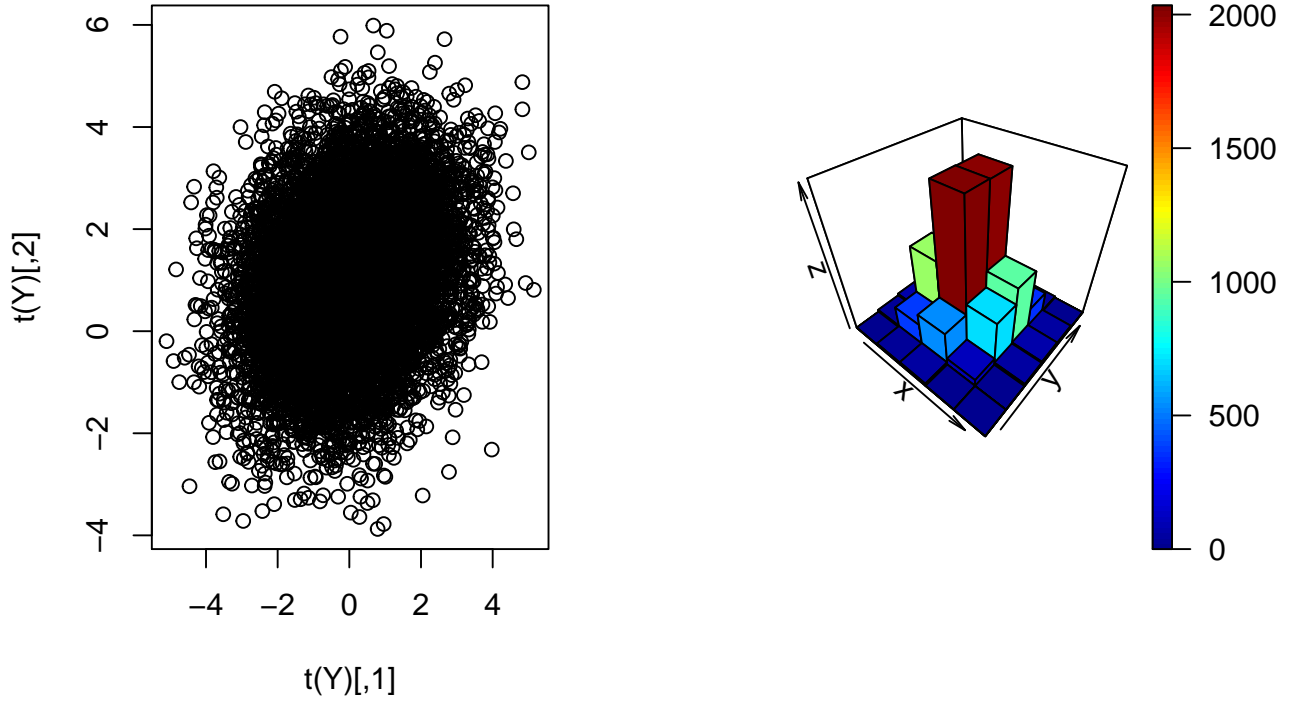


Figure 7: Random variable Y and its 3D histogram with 30 bins and bars colored according to height.

2.4 Part 4

We are going to transform the variable Y into the variable $Z = (Y - \mu)' \Sigma^{-1} (Y - \mu)$, and Σ is a non-singular matrix.

$$Z = (Y - \mu)' \Sigma^{-1} (Y - \mu) = (Y - \mu)' \Sigma^{-0.5} \Sigma^{-0.5} (Y - \mu) = (\Sigma^{-0.5} (Y - \mu))' (\Sigma^{-0.5} (Y - \mu))$$

Let's take $\Sigma^{-0.5} (Y - \mu) = B$. We know that $B \sim N(0, I)$, so $B' B \sim \chi^2(k)$, where $k = 2$.

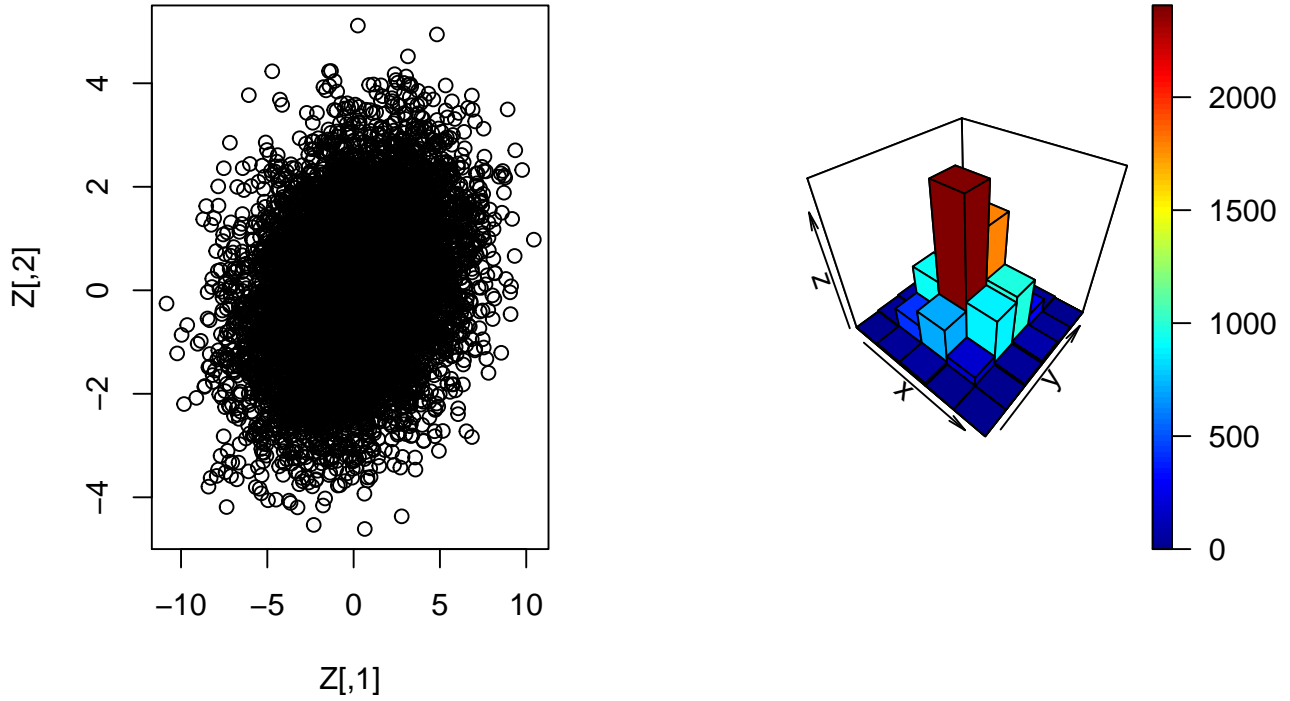


Figure 8: Random variable Z and its 3D histogram with 30 bins and bars colored according to height.

3 Exercise 3

Let $\hat{\beta}$ be a sequence of estimators of a $(K \times 1)$ vector β , which is asymptotically normal with

$$\sqrt{N}(\hat{\beta} - \beta) \rightarrow^d N(0, \Sigma).$$

3.1 Part 1

If $R \neq 0$ is an $(M \times K)$ matrix, then $\sqrt{N}(R\hat{\beta} - R\beta) \rightarrow^d N(\mu, \sigma^2)$. We will compute the mean and the variance of $\sqrt{N}(R\hat{\beta} - R\beta)$:

$$\mu = E\left(\sqrt{N}(R\hat{\beta} - R\beta)\right) = E\left(\sqrt{N}R(\hat{\beta} - \beta)\right) = R E\left(\sqrt{N}(\hat{\beta} - \beta)\right) = 0$$

and

$$\sigma^2 = Var\left(\sqrt{N}(R\hat{\beta} - R\beta)\right) = Var\left(\sqrt{N}R(\hat{\beta} - \beta)\right) = {}^1 R Var\left(\sqrt{N}(\hat{\beta} - \beta)\right) R' = R\Sigma R',$$

so

$$\sqrt{N}(R\hat{\beta} - R\beta) \rightarrow^d N(0, R\Sigma R'), \text{ for } R \neq 0.$$

¹From task 2 point 2.

3.2 Part 2

If $p\lim \hat{A} = A$, then what does $\sqrt{N}\hat{A}(\hat{\beta} - \beta)$ converge to? We know that if $x_n \rightarrow^d x$ and $y_n \rightarrow^p c$, then $x_n y_n \rightarrow^d cx$, so if

$$\hat{A} \rightarrow^p A \quad \text{and} \quad \sqrt{N}R(\hat{\beta} - \beta) \rightarrow^d N(0, R\Sigma R') \text{ for } R \neq 0,$$

then

$$\sqrt{N}\hat{A}(\hat{\beta} - \beta) \rightarrow^d N(0, A\Sigma A').$$

3.3 Part 3

We will prove that $N(\hat{\beta} - \beta)' \hat{\Sigma}^{-1} (\hat{\beta} - \beta) \rightarrow^d \chi^2(K)$ if Σ is a non-singular matrix and $p\lim \hat{\Sigma} = \Sigma$.

$$\begin{aligned} & N(\hat{\beta} - \beta)' \hat{\Sigma}^{-1} (\hat{\beta} - \beta) \\ &= \\ & \sqrt{N}(\hat{\beta} - \beta)' \hat{\Sigma}^{-0.5} \hat{\Sigma}^{-0.5} (\hat{\beta} - \beta) \sqrt{N} \\ & \quad \downarrow^d \\ & \sqrt{N}(\hat{\beta} - \beta)' \Sigma^{-0.5} \Sigma^{-0.5} (\hat{\beta} - \beta) \sqrt{N} \\ &= \\ & \left(\sqrt{N} \Sigma^{-0.5} (\hat{\beta} - \beta) \right)' \left(\sqrt{N} \Sigma^{-0.5} (\hat{\beta} - \beta) \right). \end{aligned}$$

From part 1 of task 3 we know that

$$\left(\sqrt{N} \Sigma^{-0.5} (\hat{\beta} - \beta) \right) \rightarrow^d N(0, \underbrace{\Sigma^{-0.5} \Sigma \Sigma^{-0.5}}_{I_K}).$$

Let $C = \left(\sqrt{N} \Sigma^{-0.5} (\hat{\beta} - \beta) \right)$. We have $C'C$, where $C \rightarrow^d N(0, I_K)$, so

$$C'C \rightarrow^d \chi^2(K).$$