Partial Differential Equations

Problem Set 1: First order equations and shock waves

1. (*Conservation*) Let $u = u(x,t)$ denote the density per unit length of some substance, which fulfils the conservation law without any sources. Show that, provided the flux q vanishes at infinity and u is x-integrable, the total amount of u in space is constant in time.

2. Solve the following problems. Draw their characteristics on the $x-t$ plane and sketch the solution $u(x,t)$ for several times.

$$a)\begin{cases} u_t + 10u_x = 0, & t>0, \quad x\in\mathbb{R};\\ u(x,0) = \frac{1}{1+x^2} & x\in\mathbb{R}, \end{cases} \qquad b)\begin{cases} u_t + 2tu_x = -u, & t>0, \quad x\in\mathbb{R};\\ u(x,0) = e^{-x^2}, & x\in\mathbb{R}. \end{cases}$$

$$c)\begin{cases} u_t - x^2u_x = \sin u, & t>0, \quad x>0;\\ u(x,0) = 2\arctan x, & x\geq 0, \end{cases} \qquad d)\begin{cases} u_t + 2tu_x = xtu, & t>0, \quad x\in\mathbb{R};\\ u(x,0) = x, & x\in\mathbb{R}. \end{cases}$$

3. (*Signaling problem*) Suppose that a lighthouse is located at $x = 0$ and constantly sends its signals with the intensity $\psi(t)$.

   (a) As you know, light travels through any medium with a constant speed c. Let $u = u(x,t)$ denote the light intensity at (x,t). Find $u(x,t)$ for $x > 0$ and $t \in \mathbb{R}$ (lighthouse has been sending its signals for ever).

   (b) Solve the above problem but now assume that the light is absorbed by clouds and other aerosols with a rate proportional to its intensity.

4. (*Initial-boundary value problems*) Sometimes we have to solve a problem where both the initial and boundary data is given. Consider the following equations.

$$a)\begin{cases} u_t + c\,u_x = 0, & t>0, \quad x>0, \quad c>0;\\ u(x,0) = \phi(x), & x>0;\\ u(0,t) = \psi(t), & t>0, \quad \phi(0)=\psi(0)=0, \end{cases} \qquad b)\begin{cases} u_t + tu_x = -u^2, & t>0, \quad x>0;\\ u(x,0) = x, & x>0;\\ u(0,t) = \sin t, & t>0, \end{cases}$$

   What conditions have to be fulfilled for the $a)$ case to have a solution with $c < 0$?

5. (*Quasi-linear equations*) Solve the following problems and sketch the characteristics.

$$a)\begin{cases} u_t + \ln u\, u_x = 0, & t>0, \quad x\in\mathbb{R};\\ u(x,0) = e^x, & x\in\mathbb{R}, \end{cases} \qquad b)\begin{cases} u_t + u_x = u^2, & t\in\mathbb{R}, \quad x>0;\\ u(0,t) = \psi(t), & t\in\mathbb{R}. \end{cases}$$

$$c)\begin{cases} u_t + u^2u_x = 0, & t>0, \quad x\in\mathbb{R};\\ u(x,0) = x, & x\geq 0, \end{cases} \qquad d)\begin{cases} u_t + u^{-1}u_x = 0, & t>0, \quad x\in\mathbb{R};\\ u(x,0) = \frac{1}{1+x}, & x\in\mathbb{R}. \end{cases}$$

6. Solve the following full quasi-nonlinear problem.

$$a)\begin{cases} u_t + uu_x = -u, & t>0, \quad x\in\mathbb{R};\\ u(x,0) = -x, & x\in\mathbb{R}. \end{cases} \qquad b)\begin{cases} u_t + uu_x = 2t, & t>0, \quad x\in\mathbb{R};\\ u(x,0) = x, & x\in\mathbb{R}. \end{cases}$$

7. (*Shock waves*) Find the solutions of given problems. Draw the characteristics and a shock wave trajectory.

$$a)\begin{cases} u_t + u^2u_x = 0, & t>0, \quad x\in\mathbb{R};\\ u(x,0) = \begin{cases} 2, & x<0;\\ 1, & 0>0, \end{cases} \end{cases} \qquad b)\begin{cases} u_t + 2uu_x = 0, & t>0, \quad x\in\mathbb{R};\\ u(x,0) = \begin{cases} 3, & x<0;\\ 2, & x>0, \end{cases} \end{cases}$$

8. (*Shock fitting*) Analyse the following shock-fitting problems. Draw the solution $u(x,t)$ for several times.

$$\text{a)} \begin{cases} u_t + uu_x = 0, \quad t > 0, \quad x \in \mathbb{R}; \\ u(x,0) = \begin{cases} 1, & x < 0; \\ -1, & 0 < x < 1; \\ 0, & x > 1, \end{cases} \end{cases} \qquad \text{b)} \begin{cases} u_t + uu_x = 0, \quad t > 0, \quad x \in \mathbb{R}; \\ u(x,0) = \begin{cases} 1, & x < 0; \\ 1 - x, & 0 < x < 1; \\ 0, & x > 1. \end{cases} \end{cases}$$

9. (*Paint*) The paint flows down the wall and has a thickness $u(x,t)$. As we derived, the governing equation has the form

$$u_t + u^2 u_x = 0, \quad t > 0, \quad x \in \mathbb{R}.$$

Solve the paint flow problem with the given initial profile

$$u(x,0) = \begin{cases} 0, & x < 0 \text{ or } x > 1; \\ 1, & 0 < x < 1. \end{cases}$$

10. (*Car traffic*) Let $u = u(x,t)$ be a density of cars per unit length on a road at time t. Define the traffic flux $q(x,t)$ as the number of cars per unit time passing through a fixed x at time t.

(a) Argue that the conservation of cars leads to the equation

$$u_t + q_x = 0.$$

(b) Define the car velocity as $v = q/u$ (as in the example with convection). A simple model of velocity distribution assumes that there exists a critical density at which cars stop (traffic jam)

$$v(u) = v_m \left(1 - \frac{u}{u_c}\right).$$

From this, find q and plug it in the conversation of cars to obtain the governing equation.

(c) (*Red light turns green*) Solve the car flow problem for a case where the red light at $x = 0$ turns green at $t = 0$, i.e.

$$u(x,0) = \begin{cases} u_c, & x < 0; \\ 0, & x > 0. \end{cases}$$

(d) (*Green light turns red*) Now, a red light suddenly turns on and the traffic with a density u_0 suddenly have to stop at $x = 0$.

$$u(x,0) = u_0 < u_c, \quad x \in \mathbb{R}, \quad u(0,t) = u_c, \quad t > 0.$$

11. (*Continuation of the discontinuity*) Solve the following equation

$$u_t + uu_x = 0, \quad t > 0, \quad x \in \mathbb{R},$$

for each of the two initial conditions

$$\text{a) } u(x,0) = \Phi_1(x) := \begin{cases} 1, & x \le 0; \\ 1 - \frac{x}{\epsilon}, & 0 < x < \epsilon; \\ 0, & x \ge \epsilon, \end{cases} \qquad \text{b) } u(x,0) = \Phi_2(x) := \begin{cases} 0, & x \le 0; \\ \frac{x}{\epsilon}, & 0 < x < \epsilon; \\ 1, & x \ge \epsilon. \end{cases}$$

In both cases examine the limit $\epsilon \to 0$.

12. (*Flood hydrograph*) Imagine a long and narrow river. We can introduce a curvilinear variable x which denotes the position measured along its bed. Moreover, let $A = A(x)$ be the wetted cross-section at x. Conservation of mass along with Manning's Law states that

$$A_t + A^m A_x = 0, \quad m > 0.$$

Assume also that $A(x, 0) = A_0(x)$.

   (a) Find a solution of the above equation given in an implicit form.
   (b) A flood can arise as a result of a sudden rainfall at some point x. To model this situation assume that $A_0(x) = \delta(x)$, where δ is the Dirac delta[1]. Draw the characteristics.
   (c) Find the shock wave. You do not have to solve the equation in order to do that.
   (d) A flood hydrograph is a graph of the flux at some fixed point x_0. Hydrologists use this tool a lot. Draw the hydrograph by yourself.

13. (*Blow-up time*) For a general initial condition it is not usually straightforward to determine the shock wave development. It is, however, possible to find the first time of a blow-up. Consider

$$\begin{cases} u_t + c(u)u_x = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(x,0) = \phi(x) \end{cases},$$

where $c > 0$ and $\phi > 0$ have an opposite monotonicity (say, $c' > 0$ and $\phi' < 0$).

   (a) Pick any characteristic $X = X(t)$ and define $P(t) := u_x(X(t), t)$. Compute P' in terms of u and its derivatives.
   (b) Use the differential equation for u in order to get rid of the second derivatives in the expression for P'.
   (c) Solve the obtained equation for P and whence, for u_x.
   (d) Define $F(\xi) := c(\phi(\xi))$ and show that the first time t_b for which u_x becomes infinite is

$$t_b = \frac{1}{\max_\xi |F'(\xi)|}.$$

14. Find the first time of a blow-up for the following problems.

$$a) \begin{cases} u_t + uu_x = 0, & t > 0, \quad x \in \mathbb{R}; \\ u(x,0) = e^{-x^2} \end{cases} \qquad b) \begin{cases} u_t + uu_x = 0, & t > 0, \quad x \in \mathbb{R}; \\ u(x,0) = \frac{1}{\cosh x^2}. \end{cases}$$

$$c) \begin{cases} u_t + uu_x = 0, \quad t > 0, \quad x \in \mathbb{R}; \\ u(x,0) = \begin{cases} 2 - x^2, & x < 1; \\ 1, & x > 1. \end{cases} \end{cases} \qquad d) \begin{cases} u_t + uu_x = 0, \quad t > 0, \quad x \in \mathbb{R}; \\ u(x,0) = \begin{cases} 1, & x \le 0; \\ \cos x, & 0 < x \le \frac{\pi}{2}; \\ 0, & x > \frac{\pi}{2}. \end{cases} \end{cases}$$

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[1] If you do not feel comfortable with distributions, think about some narrow and high spike of unit mass such as $\frac{1}{\epsilon}\chi_{(-\epsilon/2, \epsilon/2)}(x)$ or $\frac{1}{\epsilon}\varphi(x/\epsilon)$, where φ is any integrable function with $\int |\varphi| dx = 1$.