Estimation theory – Laboratory 1.

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1 Exercise 1

1.1 Part 1

We generate a vector of $Y \sim N(\mu = 2, \sigma^2 = 4)$ with length $N = 1000$ and transform it into variable Y_1:

```
Y <- rnorm(n = 1000, mean = 2, sd = 2)
# transform Y to Y1
Y1 <- 3 * (Y - 1)
```

Y_1 has normal distribution, because it is a linear combination of Y. We can calculate analytical mean μ_1 and variance σ_1^2 as follows:

$$\mu_1 = E(Y_1) = E(3(Y-1)) = 3E(Y-1) = 3E(Y) - 3 = 3 \cdot 2 - 3 = 3,$$

$$\sigma_1^2 = Var(Y_1) = Var(3(Y-1)) = 9Var(Y) = 9 \cdot 4 = 36$$

and compute them numerically:

```
Y <- rnorm(n = 1000, mean = 2, sd = 2)
Y1 <- 3 * (Y - 1)
mean(Y1)

## [1] 3.055195

var(Y1)

## [1] 35.52233
```

As expected, numerical and analytical results are quite similar. Now we can plot the frequency historam of Y_1 and analytical normal density with $\mu = 3$ and $\sigma^2 = 36$.

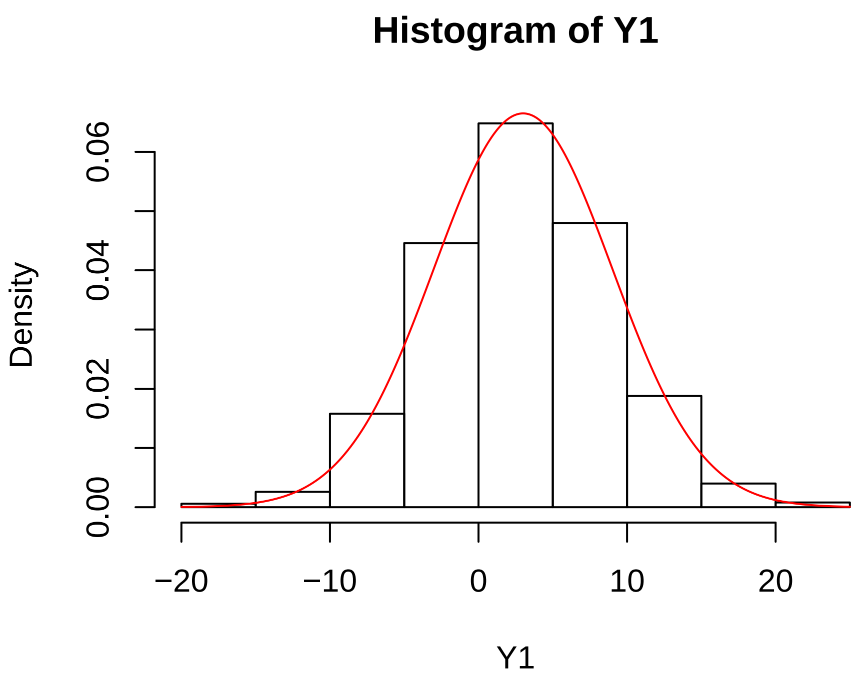

Figure 1: Frequency historam of Y_1 and analytical density of $N(\mu = 3,\ \sigma^2 = 36)$.

1.2 Part 2

Next, we create variable $Y_2 = \left(\frac{Y-2}{2}\right)^2$, which is a quadratic function of Y, so we know that distribution of Y_2 is not normal. What is more,

$$\frac{Y-2}{2} \sim N(0,1).$$

It means that Y_2, as a sum of squared standard normally distributed variables, has χ^2 distribution with 1 degree of freedom.

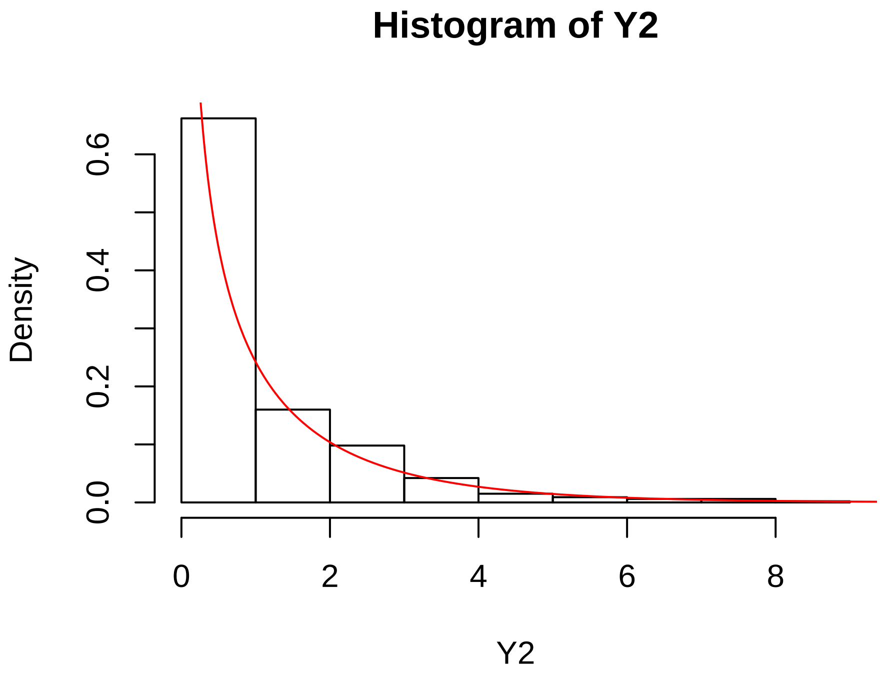

Figure 2: Histogram of Y_2 and theoretical probability density function of $\chi^2(1)$ distribution.

1.3 Part 3 and 4

Next we will compute a sequence of means m_n and a sequence of variances σ_n^2 for the variable Y, where

$$m_n = \frac{1}{n}\sum_{i=1}^{n} Y_i,$$

$$v_n = \frac{1}{n}\sum_{i=1}^{n} (Y_i - m_n)^2$$

and plot the results.

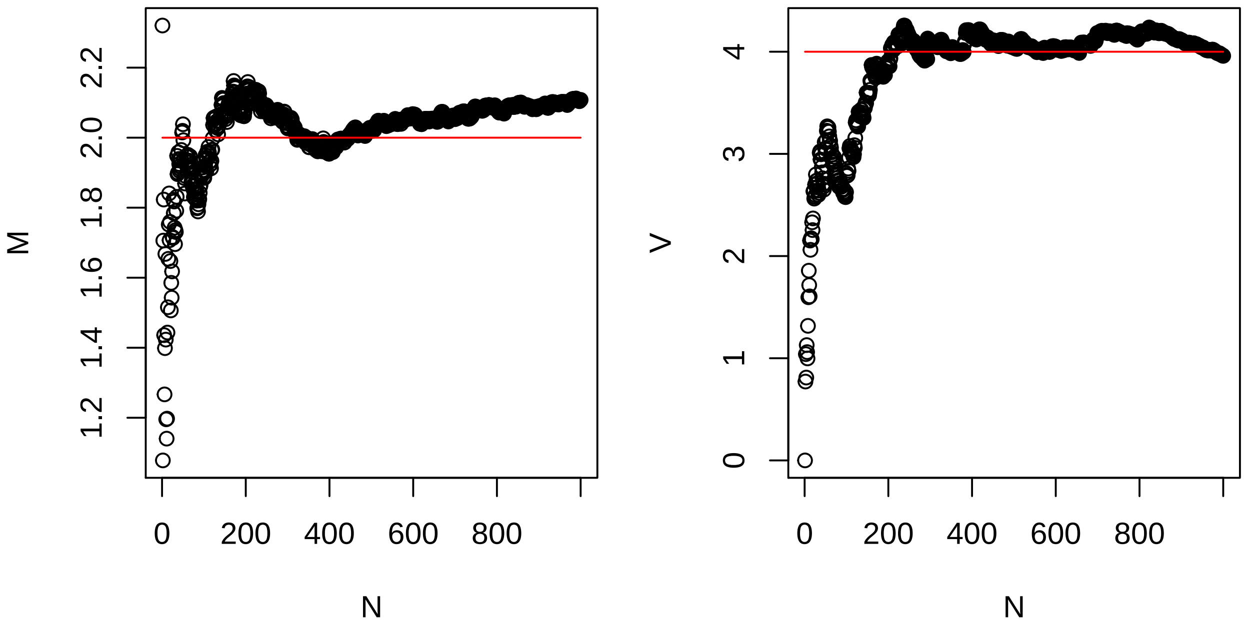

Figure 3: Sequence of means and variances and their respective analytical values.

The sequences m_n and v_n^2 converge to theoretical mean and variance, respectively. To examine the variability of the sequences we can calculate relative errors for both values.

$$err_{m_n} = \left| \frac{m_n - \mu_n}{\mu_n} \right|, \qquad err_{v_n} = \left| \frac{v_n - \sigma_n^2}{\sigma_n^2} \right|$$

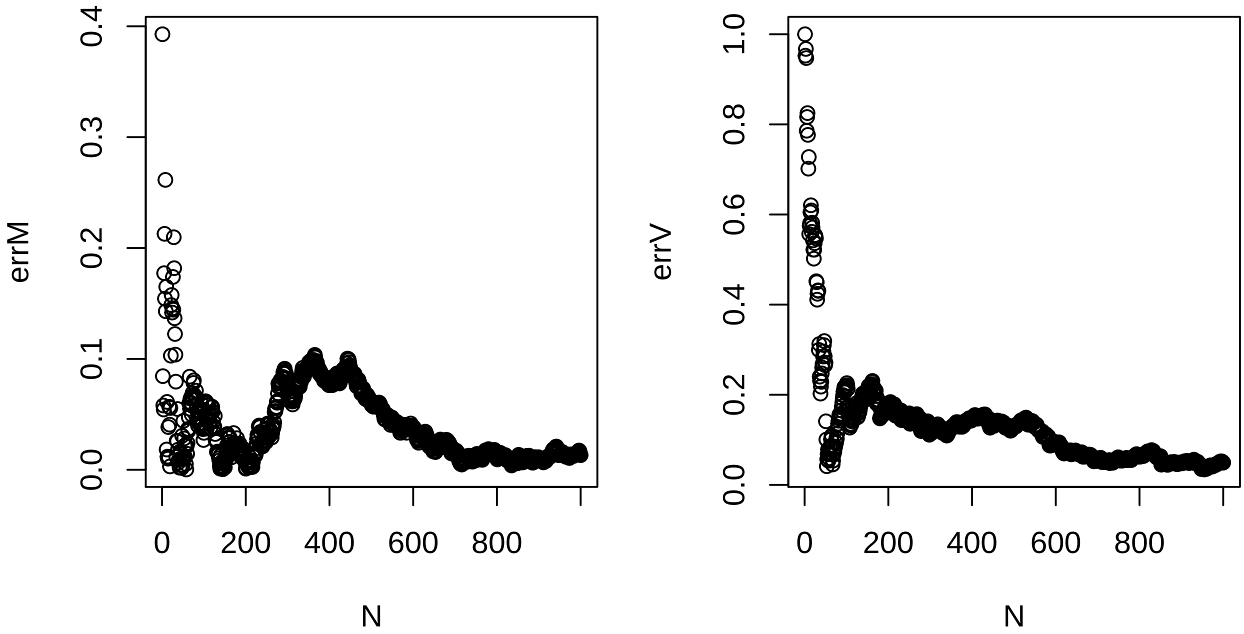

Figure 4: Relative errors for sequences of means and variances.

We can observe that the values of both err_{m_n} and err_{σ^2} are bounded by 0.1.

2 Exercise 2

2.1 Part 1

We simulate 10000 times and then plot 2-dimensional random variable $X \sim N(0, I_2)$.

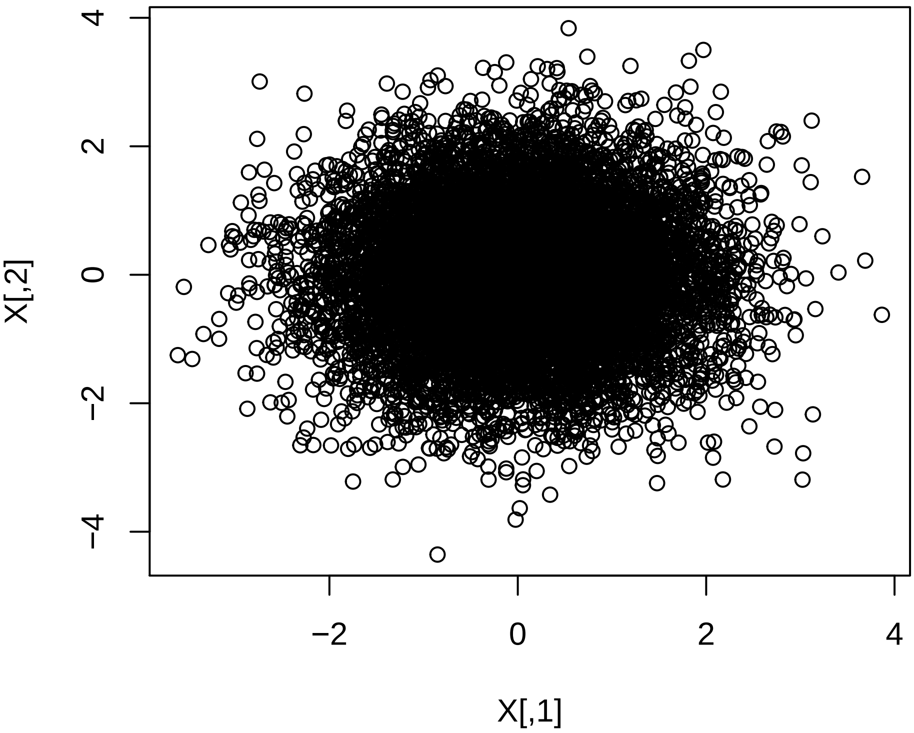

Figure 5: 2-dimensional random variable $X \sim N(0, I_2)$.

2.2 Part 2

To transform variable X into variable $Y \sim N(\mu, \Sigma)$, where

$$\mu = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 2 \end{bmatrix},$$

we should find vector a and matrix A, such that $Y = AX + a$. Expected value of Y is equal to

$$E(Y) = E(AX + a) = AE(X) + a = a,$$

and variance

$$Var(Y) = Var(AX + a) = Var(AX) = AVar(X)A' = AIA' = AA' = \Sigma.$$

We know that Σ is a symmetric and positive definite matrix, so we can use Cholesky decomposition (chol() function in R) to obtain the value of A:

```
Sigma <- matrix(c(2, 0.5, 0.5, 2), 2, 2)
chol(Sigma)

##          [,1]      [,2]
## [1,] 1.414214 0.3535534
## [2,] 0.000000 1.3693064
```

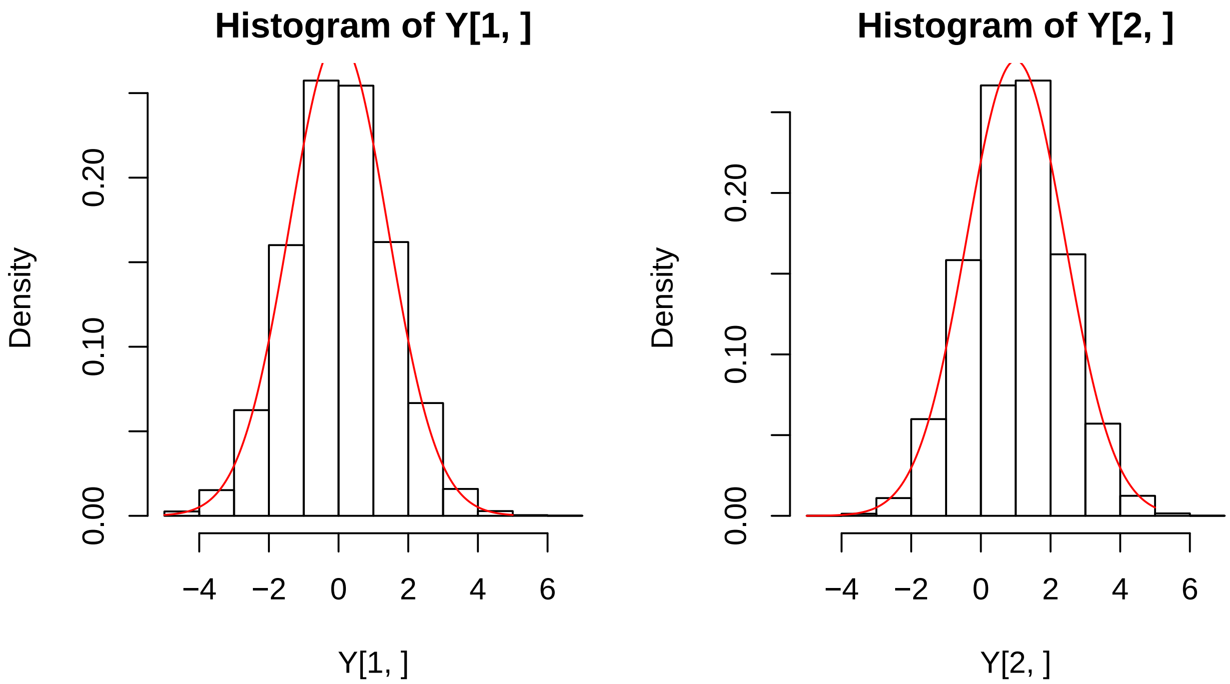

Figure 6: Histograms and probability density functions of Y

2.3 Part 3

Now we can plot the 3D histogram of the random variable $Y = \begin{bmatrix} 1.414214 & 0.3535534 \\ 0.0 & 1.3693064 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

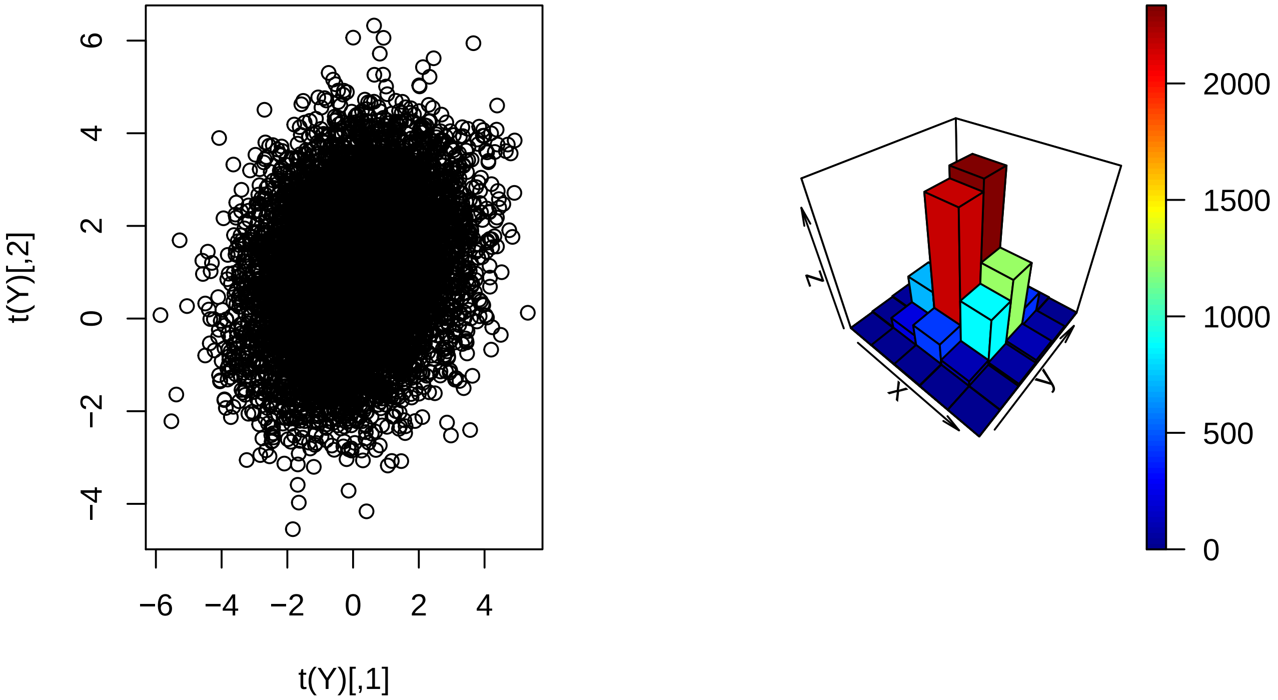

Figure 7: Random variable Y and its 3D histogram with 30 bins and bars colored according to height.

2.4 Part 4

We are going to transform the variable Y into the variable $Z = (Y-\mu)'\Sigma^{-1}(Y-\mu)$, and Σ is a non-singular matrix.

$$Z = (Y-\mu)'\Sigma^{-1}(Y-\mu) = (Y-\mu)'\Sigma^{-0.5}\Sigma^{-0.5}(Y-\mu) = \left(\Sigma^{-0.5}(Y-\mu)\right)'\left(\Sigma^{-0.5}(Y-\mu)\right)$$

Let's take $\Sigma^{-0.5}(Y-\mu) = B$. We know that $B \sim N(0, I)$, so $B'B \sim \chi^2(k)$, where $k = 2$.

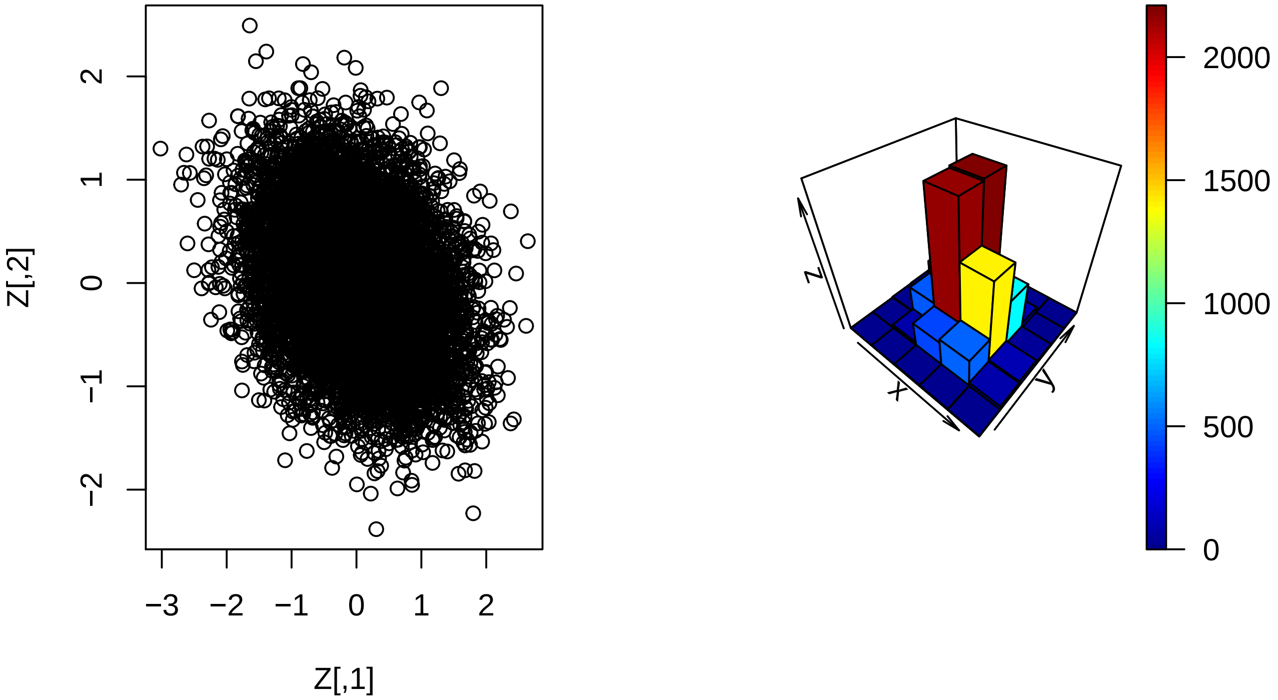

Figure 8: Random variable Z and its 3D histogram with 30 bins and bars colored according to height.

3 Exercise 3

Let $\hat{\beta}$ be a sequence of estimators of a $(K \times 1)$ vector β, which is asymptotically normal with

$$\sqrt{N}(\hat{\beta} - \beta) \to^d N(0, \Sigma).$$

3.1 Part 1

If $R \neq 0$ is an $(M \times K)$ matrix, then $\sqrt{N}(R\hat{\beta} - R\beta) \to^d N(\mu, \sigma^2)$. We will compute the mean and the variance of $\sqrt{N}(R\hat{\beta} - R\beta)$:

$$\mu = E\left(\sqrt{N}(R\hat{\beta} - R\beta)\right) = E\left(\sqrt{N}R(\hat{\beta} - \beta)\right) = R\ E\left(\sqrt{N}(\hat{\beta} - \beta)\right) = 0$$

and

$$\sigma^2 = Var\left(\sqrt{N}(R\hat{\beta} - R\beta)\right) = Var\left(\sqrt{N}R(\hat{\beta} - \beta)\right) =^{1} R\ Var\left(\sqrt{N}(\hat{\beta} - \beta)\right) R' = R\Sigma R',$$

so

$$\sqrt{N}(R\hat{\beta} - R\beta) \to^d N(0, R\Sigma R'), \text{ for } R \neq 0.$$

[1] From task 2 point 2.

3.2 Part 2

If $p\lim \hat{A} = A$, then what does $\sqrt{N}\hat{A}(\hat{\beta} - \beta)$ converge to? We know that if $x_n \to^d x$ and $y_n \to^p c$, then $x_n y_n \to^d cx$, so if

$$\hat{A} \to^p A \quad \text{and} \quad \sqrt{N}R(\hat{\beta} - \beta) \to^d N(0, R\Sigma R') \text{ for } R \neq 0,$$

then

$$\sqrt{N}\hat{A}(\hat{\beta} - \beta) \to^d N(0, A\Sigma A').$$

3.3 Part 3

We will prove that $N\left(\hat{\beta} - \beta\right)' \hat{\Sigma}^{-1}\left(\hat{\beta} - \beta\right) \to^d \chi^2(K)$ if Σ is a non-singular matrix and $p\lim \hat{\Sigma} = \Sigma$.

$$N\left(\hat{\beta} - \beta\right)' \hat{\Sigma}^{-1}\left(\hat{\beta} - \beta\right)$$
$$=$$
$$\sqrt{N}\left(\hat{\beta} - \beta\right)' \hat{\Sigma}^{-0.5}\hat{\Sigma}^{-0.5}\left(\hat{\beta} - \beta\right)\sqrt{N}$$
$$\downarrow^d$$
$$\sqrt{N}\left(\hat{\beta} - \beta\right)' \Sigma^{-0.5}\Sigma^{-0.5}\left(\hat{\beta} - \beta\right)\sqrt{N}$$
$$=$$
$$\left(\sqrt{N}\Sigma^{-0.5}\left(\hat{\beta} - \beta\right)\right)'\left(\sqrt{N}\Sigma^{-0.5}\left(\hat{\beta} - \beta\right)\right).$$

From part 1 of task 3 we know that

$$\left(\sqrt{N}\Sigma^{-0.5}\left(\hat{\beta} - \beta\right)\right) \to^d N(0, \underbrace{\Sigma^{-0.5}\Sigma\Sigma^{-0.5}}_{I_K}).$$

Let $C = \left(\sqrt{N}\Sigma^{-0.5}\left(\hat{\beta} - \beta\right)\right)$. We have $C'C$, where $C \to^d N(0,\ I_K)$, so

$$C'C \to^d \chi^2(K).$$