Optimization Theory

Applied Mathematics

Problem set 3

1. Rewrite the following linear programming problems using the form of linear programs presented during the lecture:

$$(LP1)\quad \begin{array}{ll} \text{Minimize} & c^T x \\ \text{subject to} & Ax = b, x \geq 0, \end{array}$$

$$(LP2)\quad \begin{array}{ll} \text{Minimize} & c^T x \\ \text{subject to} & Ax \leq b, x \leq 0. \end{array}$$

2. Consider a linear programming problem with n variables $x_1, x_2, \ldots, x_n$ unbounded with respect to their sign. How can it be written equivalently using only $n+1$ nonnegative variables?

3. Consider the linear program:

$$\begin{array}{ll} \text{Maximize} & \alpha x_1 + 2x_2 + x_3 - 4x_4 \\ \text{subject to} & x_1 + x_2 - x_4 = 4 + 2\Delta, \\ & 2x_1 - x_2 + 3x_3 - 2x_4 = 5 + 7\Delta, \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, \end{array}$$

where $\alpha, \Delta \in \mathbb{R}$ are viewed as parameters.

(a) Using the fact that the two first constraints are equalities, rewrite the problem using only 2 variables.

(b) Using graphical interpretation of the problem, identify when the problem is unbounded (has no optimal solution) and when it has an optimal solution. What is it then (there will be several cases depending on the value of Δ)?

4. ([BHM77]) Consider the linear program:

$$\begin{array}{ll} \text{Maximize} & -3x_1 + 6x_2 \\ \text{subject to} & 5x_1 + 7x_2 \leq 35, \\ & -x_1 + 2x_2 \leq 2, \\ & x_1 \geq 0, x_2 \geq 0. \end{array}$$

(a) Solve this problem graphically. Are there multiple optimal solutions?

(b) Solve it using the simplex method. How the fact that there are multiple optimal solutions can be determined using the final simplex tableau?

5. Solve the linear programming problem

$$\begin{array}{ll} \text{Maximize} & x_1 + 2x_2 \\ \text{subject to} & x_1 \geq 1, \\ & x_2 \leq 5, \\ & x_1 + x_2 \geq 4, \\ & x_1 - x_2 \leq 2, \\ & 2x_1 + x_2 \leq 6, \\ & x_1 \geq 0, x_2 \geq 0 \end{array}$$

using the simplex algorithm (you will need to use both phases). Draw the graphical interpretation of the problem. At each step of the simplex procedure, draw the movements that you have made on the picture.

6. Formulate the following problem as a linear program: A bus company must provide drivers for buses. It may employ full-time drivers, who work for 8 consecutive hours, or part-time drivers, working for 4 consecutive hours. The demand for drivers varies during the day. In the period between 6 and 10 it needs 3 drivers, in 10–14 – 8 drivers, in 14–18 – 6 drivers, while from 18 to 22 – 4 drivers. In addition it assumes that the work for each driver begins at 6, 10, 14 or 18. Full-time drivers are paid \$10 per hour while part-time drivers are given \$12 per hour. Write a linear program to minimize the wages paid daily by the company subject to all the demand constraints.

7. ([BHM77]) The processing division of the Sunrise Breakfast Company must produce one ton (2000 pounds) of breakfast flakes per day to meet the demand for its Sugar Sweets cereal. Cost per pound of the three ingredients is: ingredient A – \$4 per pound, ingredient B – \$3 per pound, ingredient C – \$2 per pound. Government regulations require that the mix contain at least 10% ingredient A and 20% ingredient B. Use of more than 800 pounds per ton of ingredient C produces an unacceptable taste. Write the linear program to determine the minimum-cost mixture that satisfies the daily demand for Sugar Sweets. Solve it using the simplex method. You can simplify the procedure by introducing new variables which will differ from original ones by constants.

8. Consider the following problem: We have a set of n-dimensional observations $x^i = (x_1^i, x_2^i, \dots, x_n^i)$, $i = 1, \dots, m$, with $m >> n$. Each observation should satisfy the equations $a_j^T x = b_j$ where the vectors a_j and the numbers b_j, $j = 1, \dots, k$ are unknown. As each observation has an error, we try to find A and b which minimize the L_1 error of the approximation $\sup_{i \le n} \sum_{l=1}^{k} |a_l^T x - b_l|$. Formulate the problem of finding A and b (together with the optimal value of the error) as a linear program.

9. Consider the (non-linear) problem

$$\begin{array}{ll} \text{Maximize} & \frac{2x_1 - 2x_2 - 2}{x_1 + 3x_2 + 4} \\ \text{subject to} & -x_1 + x_2 \le 4, \\ & 2x_1 + x_2 \le 14, \\ & x_2 \le 6, \\ & x_1 \ge 0, x_2 \ge 0 \end{array}$$

Introduce 3 new variables which will enable you to transform it into a linear program. Solve it using the simplex method. Compute the values of the original variables corresponding to the solution obtained.

10. ([BHM77]) Find the dual problem associated with each of the following problems:

$$\begin{array}{ll} \text{Maximize} & 3x_1 + 2x_2 \\ \text{subject to} & x_1 + 3x_2 \le 3, \\ & 6x_1 - x_2 = 4, \\ & x_1 + 2x_2 \le 2, \\ & x_1 \ge 0, x_2 \ge 0. \end{array}$$

$$\begin{array}{ll} \text{Maximize} & 3x_1 + 2x_2 - 3x_3 + 4x_4 \\ \text{subject to} & x_1 - 2x_2 + 3x^3 + 4x_4 \le 3, \\ & x_2 + 3x_3 + 4x_4 \ge -5, \\ & 2x_1 - 3x_2 - 7x_3 - 4x_4 = 2, \\ & x_1 \ge 0, x_4 \le 0. \end{array}$$

11. ([Mu76]) Show that the problem's

$$\begin{array}{ll} \text{Maximize} & -x - y - z \\ \text{subject to} & y - z \le 1, \\ & -x + z \le 1, \\ & x - y \le 1, \\ & x \ge 0, y \ge 0, z \ge 0 \end{array}$$

dual is the problem itself.

12. Write the dual problem associated with the problem from excercise 5. Read the solution to this problem from the final simplex tableau. Check that indeed it satisfies all the constraints of the dual problem and that it gives the value equal to the optimal value of the primary problem from excercise 5.

13. Formulate the dual problem associated with the glasses factory production problem from the lecture. Read the optimal solution of the dual from the final simplex tableau. Given the interpretation of the optimal dual variables as shadow prices try to answer the question: Which of the following three possibilities is more profitable for the glass factory owner:

    (a) Invest in extra storage space, given that expanding it by each 1% costs approximately \$10 a week.

    (b) Invest in advertisment of juice glasses, given that an investment of \$15 per week will increase the demand by 1%.

    (c) Pay for extra hours of the employees, given that paying extra \$50 per week will increase their working time by an hour.

References:

[BHM77] S.P. Bradley, A.C. Hax, T.L. Magnanti, Applied Mathematical Programming, Addison-Wesley Publishing Company, 1977

[Mu76] K.G. Murty, Linear and Combinatorial Programming, John Wiley & Sons, Inc., New York, N.Y. 1976.