Differentiation

Derivative of a scalar function with respect to a vector

Let $f(\beta)$ be a scalar function that depends on a $(K \times 1)$ vector $\beta = (\beta_1, ..., \beta_K)'$. Then

$$\frac{\partial f(\beta)}{\partial \beta} = \begin{bmatrix} \frac{\partial f(\beta)}{\partial \beta_1} \\ \dots \\ \frac{\partial f(\beta)}{\partial \beta_K} \end{bmatrix}_{(K\times 1)}$$

$$\frac{\partial f(\beta)}{\partial \beta'} = \begin{bmatrix} \frac{\partial f(\beta)}{\partial \beta_1} & \dots & \frac{\partial f(\beta)}{\partial \beta_K} \end{bmatrix}_{(1\times K)}$$

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Basic rules of vector differentiation

Let $f(\beta)$ and $g(\beta)$ be real valued functions, $c_1, c_2 \in R$. Then

- linearity

$$\frac{\partial c_1 f(\beta) + c_2 g(\beta)}{\partial \beta'} = c_1 \frac{\partial f(\beta)}{\partial \beta'} + c_2 \frac{\partial g(\beta)}{\partial \beta'}$$

- product rule

$$\frac{\partial f(\beta) g(\beta)}{\partial \beta'} = f(\beta) \frac{\partial g(\beta)}{\partial \beta'} + g(\beta) \frac{\partial f(\beta)}{\partial \beta'}$$

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Example

Suppose, $f(\beta) = \beta_1 + 3\beta_2 - 2\beta_3$ and $g(\beta) = 2\beta_1^2$. Compute

- $\partial f(\beta)/\partial\beta$
- $\partial g(\beta)/\partial\beta$
- $\partial(f(\beta)g(\beta))/\partial\beta$
- $\partial(f(\beta)^2)/\partial\beta$

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Example

Suppose, a and β are $(K \times 1)$ vectors. Compute

$$\frac{\partial a'\beta}{\partial \beta'} = ?$$

Derivative of a vector function with respect to a vector

Let $f(\beta)$ be a vector function $f(\beta) = (f_1(\beta), ..., f_N(\beta))_{(1\times N)}$ that depends on a $(K \times 1)$ vector β.

$$\frac{\partial f(\beta)}{\partial \beta} = \left[\begin{array}{ccc} \frac{\partial f_1(\beta)}{\partial \beta} & \dots & \frac{\partial f_N(\beta)}{\partial \beta} \end{array} \right]_{(n\times K)}$$

where $\partial f_n(\beta)/\partial \beta$ are columns of a dimension $(K \times 1)$

Example

Suppose, β a $(K \times 1)$ vector and $g(\beta)$ is a $(N \times 1)$ vector function and a a $(N \times 1)$ vector. What it is

$$\frac{\partial(a' g(\beta))}{\partial \beta'}$$

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$$\frac{\partial(a' g(\beta))}{\partial \beta'}$$

Example

Let $g(\beta)$ be a $(N \times 1)$ vector function. What it is

$$\frac{\partial(g(\beta)'g(\beta))}{\partial\beta'}$$

Example

Suppose, $f(\beta) = A\beta$, where A is a $(m \times n)$ matrix and β a $(n \times 1)$ vector.

- What are the dimensions of $f(\beta)$
- Should we compute $\partial f(\beta)/\partial\beta$ or $\partial f(\beta)/\partial\beta'$?
- Compute $\partial f(\beta)/\partial\beta'$

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Some rules - linear function

Let A be a $(m \times n)$ matrix and β a $(n \times 1)$ vector. Then

- $$\frac{\partial A\beta}{\partial \beta'} = A$$
- $$\frac{\partial \beta' A'}{\partial \beta'} = A'$$

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Some rules - quadratic form

Let A be a $(n \times n)$ matrix and β a $(n \times 1)$ vector. Then

- $$\frac{\partial \beta' A \beta}{\partial \beta} = (A + A')\beta$$
- $$\frac{\partial \beta' A \beta}{\partial \beta'} = \beta'(A + A')$$

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Example

Suppose $\beta = [1,1]'$, $A = \begin{pmatrix} 1 & -1 \end{pmatrix}$ and

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Compute

- $\frac{\partial A\beta}{\partial \beta'} = ?$
- $\frac{\partial \beta' B\beta}{\partial \beta} = ?$

Example

Show that

$$\frac{\partial (a - A\beta)' B (a - A\beta)}{\partial \beta'} = -2(a - A\beta)' BA$$

Example

Let $f(\beta) = \sum_{n=1}^{N} (y_n - x_n\beta)'(y_n - x_n\beta)$, where $x_n \in R^K$ and $\beta \in R^K$.

- Express the $f(\beta)$ as a function of matrices $y = [y_1, ..., y_N]'$ and $X = [x_1, ..., x_N]'$.
- Compute $\partial f(\beta)/\partial\beta'$

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Example

Let $f(\beta) = \sum_{n=1}^{N}(y_n - G(\beta; x_n))'(y_n - G(\beta; x_n))$, where $x_n \in R^K$ and $\beta \in R^K$.

- Express the $f(\beta)$ as a function of matrices $y = [y_1, ..., y_N]'$ and $G(\beta; X) = [G(\beta; x_1), ..., G(\beta; x_N)]'$.
- Compute $\partial f(\beta)/\partial\beta'$

Example

Let $f(\beta) = \sum_{n=1}^{N} (y_n - G(\beta; x_n))'(y_n - G(\beta; x_n))$, where $x_n \in R^K$ and $\beta \in R^K$.

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- Compute $\partial f(\beta)/\partial \beta'$