Optimization Theory

Applied Mathematics

Problem set 5

1. (Be99]) Consider the problem

$$\text{minimize } \frac{1}{2}(-x_1^2 + x_2^2) \text{ subject to } x_1 = 1.$$

   (a) Find its solution x^* and the value of the optimal Lagrange multiplier λ^*.

   (b) Show that for any $c < 1$ the augmented Lagrangian function L_c has no local minima for any $\lambda \neq \lambda^*$.

   (c) Calculate the successive iterations of the method of multipliers for $c^0 > 1$, $\lambda^0 = 0$ and $x^0 = (0, 0)$, assuming that the computations of all unconstrained minima are done using some method giving their exact value (since the objective function is quadratic, Newton's method would be enough to acheive that).

   (d) Show that for any bounded nondecreasing sequence $\{c^k\}$ with $c^0 > 1$ which exceeds 2 for k big enough, $\lambda^k \to \lambda^*$ and $x^k \to x^*$.

   (e) Show (using the definition of superlinear/linear convergence) that the rate of convergence of sequences $\{\lambda^k\}$ and $\{x^k\}$ is superlinear if $c^k \to \infty$ and linear if $\{\lambda^k\}$ is a bounded sequence.

2. (Be99]) Consider the problem

$$\text{minimize } \frac{1}{2}(x_1^2 - x_2^2) - 3x_2 \text{ subject to } x_2 = 0.$$

   (a) Calculate the optimal solution and the Lagrange multiplier.

   (b) For $k = 0, 1, 2$ and $c^k = 10^{l+1}$ calculate and compare the iterates of the quadratic penalty method with $\lambda^k \equiv 0$ and the method of multipliers with $\lambda^0 = 0$. How far from the correct optimal solution are they? As before, assume the gradient method used to compute the unconstrained minima gives exact values of minima for quadratic functions.

3. Consider the problem

$$\text{minimize } -x_1^4 + x_2^2 \text{ subject to } x_1 = 0.$$

   (a) Show that it has a unique global minimum x^*, but its augmented Lagrangian L_c does not have a local minimum for any value of λ and any value of c.

   (b) Show that if we replace the penalty function $\frac{c}{2}\|h(x)\|^2$ with $\frac{c}{p}\|h(x)\|^p$ with $p > 4$, the augmented Lagrangian has a local minimum for every values of λ and c.

   (c) How is it possible that the Lagrangian function (if we take $c = 0$ in the augmented Lagrangian, we obtain the Lagrangian function) does not have a local minimum even for the optimal value of the Lagrange multiplier. Why it does not violate the necessary second-order condition for the existence of the local constrained minimum?

4. By introducing additional nonnegative variables z_i convert the standard minimization problem with inequality constraints given on the lecture into a problem with equality constraints. Finding minimum of the augmented Lagrangian with repect to variables z_i show that the way that the formula for the augmented Lagrangian for inequality case given on the lecture is correct.

5. Design a method to find a projection of a point $x \in \mathbb{R}^n$ on

   (a) a closed ball with center x_0 and radius r;

   (b) a hyperrectangle $\{x \in \mathbb{R}^n : a_i \leq x_i \leq b_i\}$ for some given values a_i, b_i, $i = 1, \ldots, n$;

   (c) (only in case $n = 2$) a compact polytope $\{x \in \mathbb{R}^2 : Ax \leq b\}$.

6. Consider the problem

$$\begin{array}{ll} \text{minimize} & f(x_1, x_2) = \frac{1}{2}(x_1 - 3)^2 + \frac{1}{2}(x_2 - 4)^2 \\ \text{subject to} & -x_1 + x_2 \leq 2 \\ & x_1 + 2x_2 \leq 7 \\ & 2x_1 - x_2 \leq 4 \\ & x_1 \geq 0, x_2 \geq 0 \end{array}$$

   (a) Solve it using the Franck-Wolfe's method.

   (b) Solve it again using the gradient projection method.

   In both cases assume you are able to compute an exact minimum of a quadratic function of one variable when necessary.

References:

[Be99] D.P. Bertsekas, Nonlinear Programming, Athena Scientific, Belmont, MA: 1999.