Normal and related distributions

Expectations

Useful measures

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- Variance: $Var(x) = E((x - Ex)^2) = \int_(x - Ex)^2 f(x)dx$
- Skewness: $E((x - Ex)^3)$
- Kurtosis: $E((x - Ex)^4)$

Suppose $E(x) = \mu$ and $Var(x) = \sigma^2$. Let $z = ax + b$. Find $E(z)$ and $Var(z)$.

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Normal distribution

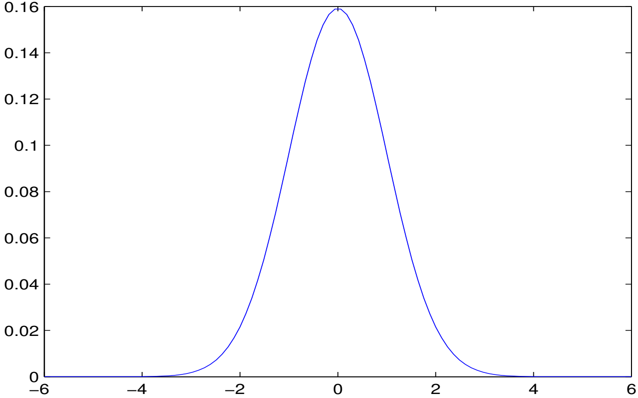
For a 1-dimensional random variable y with a mean μ and a variance σ^2 we will say that y has a univariate normal distribution

$$y \sim N(\mu, \sigma^2)$$

if its density is

$$f(y) = \frac{1}{(2\pi)^{0.5}\sigma} \exp\left[-0.5\frac{(y-\mu)^2}{\sigma^2}\right]$$

Normal distribution $N(0, 1)$



Linear transformation

Normal distribution preserves under linear transformation.

$$a + bx \sim N(a + b\mu, b^2\sigma^2)$$

Find a and b such that $z = a + bx \sim N(0, 1)$.

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Chi-squared distribution

Suppose, $z \sim N(0,1)$, then $x = z^2 \sim \chi^2(1)$

- For independent $z_1 \sim \chi^2(n_1)$ and $z_2 \sim \chi^2(n_2)$ the sum $z_1 + z_2 \sim \chi^2(n_1 + n_2)$.
- For independent $z_i \sim \chi^2(1)$, what is the distribution of $\sum_{i=1}^{n} z_i$?
- For independent $x_i \sim N(0,1)$, what is the distribution of $\sum_{i=1}^{n} z_i$?
- Suppose, $x_i \sim N(\mu, \sigma^2)$ are independent. Transform it into $z\chi^2(n)$.

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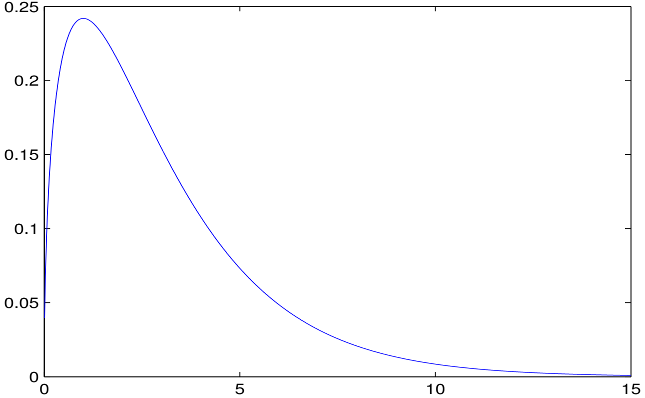

Moments - expected value

Let $x \in R^n$ be a multivariate random variable. The expected value is a $(n \times 1)$ vector

$$E(x) = \begin{bmatrix} E(x_1) \\ E(x_2) \\ \vdots \\ E(x_n) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix} = \mu$$

Moments - covariance matrix

The covariance matrix of the rando vector x is a $(n \times n)$ matrix

$$Var(x) = E((x-\mu)(x-\mu)') = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \cdots & & & \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix} = \Sigma$$

Moments - some rules

Suppose, $x \sim (\mu, \Sigma)$ and $z = Ax + b$. Then

- $E(z) = A\mu + b$
- $Var(z) = A\Sigma A'$
- Find A and b for which $z \sim (0, I)$.

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Multidimensional Normal distribution

A K-dimensional vector of random variables $y = (y_1, y_2, \ldots, y_K)'$ has a multivariate normal distribution with a mean vector μ and a covariance matrix Σ

$$y \sim N(\mu, \Sigma)$$

if its density is

$$f(y) = \frac{1}{(2\pi)^{K/2}} \det(\Sigma)^{-0.5} \exp[-0.5(y-\mu)'\Sigma^{-1}(y-\mu)]$$

Example: Multidimensional Normal

Normal distribution can be defined by marginal distributions and a covariance structure.

$$y \sim N\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & 0.5 \\ 0.5 & 3 \end{bmatrix}\right)$$

What is the marginal distribution of y_1, y_2 and what is their covariance?

Linear transformation of a Normal random vector

Suppose $y \sim N(\mu, \Sigma)$, A is a $(M \times K)$ matrix and b is a $(M \times 1)$ vector. What is the distribution of $x = Ay + b$?

$$x = Ay + b \sim N(A\mu + b, A\Sigma A')$$

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Example: Linear transformation

What is the distribution of $x = y_1 + 0.5y_2 + 1$ from the previous example?

- $x = 1 + [1, 0.5]y$
- $E(x) =$3
- $Var(x) =$2.25

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Knowing that $x =\sim N(\mu, \Sigma)$, find a distribution of

- $C'x$, when $x \sim N(0, I_K)$ and C be a square matrix, such that $C'C = I_K$.
- $\Sigma^{-1/2}(x - \mu)$.

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Quadratic form of the normal variable

Show that for $x \sim N(0, I_K)$

- $x'x \sim \chi^2(K)$
- if all characteristic roots of A are either 1 or 0 (with J non-zero roots) then $x'Ax \sim \chi^2(J)$
- if A is idempotent matrix then $x'Ax \sim \chi^2(tr(A))$

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Problem 1

Suppose $y \sim N(0_{K\times 1}, \Sigma)$, with a quadratic, non-singular variance-covariance matrix Σ. Show that

$$y'\Sigma^{-1}y \sim \chi^2(K)$$

Problem 2

Let $y = [y_n]$ be a $(N \times 1)$ vector of i.i.d $y_n \sim N(\mu, \sigma^2)$. What is the distribution of

- $\hat{\mu} = \frac{1}{N}\sum_{n=1}^{N} y_n$
- $\frac{\hat{\sigma}^2}{\sigma^2} = \frac{1}{N}\sum_{n=1}^{N}(y_n - \bar{y}_N)^2$

Hint: find matrices A and M (idempotent) such that: $\hat{\mu} = Ay$ and $\hat{\sigma}^2 = y'My$

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Normal vector and a quadratic form

Suppose

$$y \sim N(\mu, \sigma^2 I_K)$$

and

- M is $(K \times K)$, symmetric and idempotent matrix.
- A is a $(N \times K)$ matrix.
- $AM = 0$

Show that Ay and $y'My$ are independent.

Example

Let $y = [y_n]$ be a $(N \times 1)$ vector of $y_n \sim N(\mu, \sigma^2)$. Show that the estimators of a mean and a variance are independent.

- $\hat{\mu} = \frac{1}{N} \sum_{n=1}^{N} y_n$
- $\hat{\sigma}^2 = \frac{1}{N} \sum_{n=1}^{N} (y_n - \bar{y}_N)^2$

Hint: use matrices A and M from previous example.

t-distribution

Suppose $z \sim N(0,1)$ and $u \sim \chi^2(m)$ are independent. Then

$$T = \frac{z}{\sqrt{u/m}}$$

has a t-distribution with m degrees of freedom.

$$T \sim t(m)$$

Example

Let $y = [y_n]$ be a $(N \times 1)$ vector of $y_n \sim N(\mu, \sigma^2)$ and

- $\hat{\mu} = \frac{1}{N}\sum_{n=1}^{N} y_n$
- $\hat{\sigma}^2 = \frac{1}{N}\sum_{n=1}^{N}(y_n - \bar{y}_N)^2$

Show that

$$\sqrt{N-1}\frac{\hat{\mu}-\mu}{\sqrt{\hat{\sigma}^2}} \sim t(N-1)$$

F-distribution

Suppose $u \sim \chi^2(m)$ and $v \sim \chi^2(n)$ are independent. Then

$$F = \frac{u/m}{v/n}$$

has a *F-distribution* with m and n degrees of freedom.

$$F \sim F(m, n)$$