Simulation techniques

Introduction

Simulations

When we use simulations-based method?

- infer the characteristics of random variables
  - estimators
  - functions of estimators
  - test statistics
  - ...
- construct estimators that involve complicated integrals that do no exists in a closed form
  - forecasts, impulse responses
  - critical values of tests
  - ...

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MC simulations

MC studies

Simulated data have various uses in econometrics:

- properties of estimators
- comparison of different estimators
- evaluation of tests properties
- computation of critical values

Estimator properties - problems

Typically, only asymptotic results are available. Problems:

- small sample,
- violation of regularity conditions,
- non-linear transformation.

Example 1

For a normally distributed variables

$$x_n \sim N(\mu, \sigma^2)$$

which is a better estimator of μ:

- mean,
- median?

Example 1

Theoretical, asymptotic results:

$$\sqrt{N}\bar{x}_n \rightarrow N(\mu, \sigma^2)$$

$$\sqrt{N}M_n \rightarrow N(\mu, \pi/2\sigma^2)$$

So mean is more efficient.

Question

- Is mean more efficient in small samples?
- Does the result depend on normality assumption?

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- Does the result depend on normality assumption?

Example 1

Design of MC experiment:

1. Set a sample size ($T = 10, 25, 100, 1000$) and the number of MC iterations ($N_{MC} = 1000$).
2. Generate a sample from a given distribution ($N(0, 1)$ or t-Student with $3, 6$ or 10 degrees of freedom).
3. Estimate the mean and the median
4. Asses the results by mean square error around the true value ($\mu = 0$)

$$\frac{1}{K}\sum_{k=1}^{N_{MC}}(\hat{\mu}_k - 0)^2$$

Results - relative MSE

T	$N(0,1)$	$t(3)$	$t(6)$	$t(10)$
10	1.339	0.556	1.029	1.202
25	1.558	0.669	1.106	1.210
100	1.539	0.544	1.178	1.269
1000	1.494	0.644	1.161	1.277

Test properties

In the statistical test, two hypothesis are considered

$$H_0 : h(\theta) = 0 \tag{1}$$

Against the alternative

$$H_1 : h(\theta) \neq 0 \tag{2}$$

Errors:

- rejection the null, when it is true
- not rejection of null when it is false .

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- Type I error The null is rejected when it is true
  - with probability α
- Type II error Test fails to reject the null when it is false
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A size and a power of a test

Size of a test is the probability of a type I error (α) - called also the significance level.

Power of a test is the probability that the null is rejected when it is false:

$$\textit{power} = 1 - \beta$$

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The size and the power

Typically,

- The higher the α, the higher the power and hence the lower the β.
- For a given α we want β to be as small as possible and power as large as possible.

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The test is called unbiased if its power $1 - \beta$ is larger than or equal to its size α for all values of the parameters.

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Empirical size and power

Empirical size

- Simulate the process under the null
- Compute the test statistics
- Compute the frequency of rejections

Empirical power

- Simulate the process under the alternative
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Example 2

Consider a linear regression model

$$y_i = \alpha + \beta x_i + \gamma z_i + e_i$$

where $e_i \sim N(0, \sigma^2)$ for $i = 1, ..., 50$. Suppose wa want to test if

$$H_0 : \gamma = 0.$$

The LM test statistic is

$$LM = e_0 X(X'X)^{-1} X' e_0 / (e_0' e_0 / N),$$

where e_0 are residuals under the null.

Question: how does the violation of normality assumption affects the test size and power?

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Example 2

In each MC iteration (sample size $T = 20$, number of iterations $N_{MC} = 1000$):

1. generate x and z - coming from $N(0,1)$ and residuals e from $N(0,1)$, $t(6)$ or $t(3)$,
2. choose parameter values: $\alpha = 0$, $\beta = 1$ and $\gamma \in [-1, -0.9, ..., 0.9, 1]$ (remark: under the null $\gamma = 0$),
3. generate $y = \alpha + \beta x + \gamma y + e$,
4. estimate the model under the null: $y = \alpha + \beta x + e$ and compute the LM statistic,
5. when $LM > 3.84$ reject the null.

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Based on the experiment, get a frequency of rejecting the null

- empirical size: frequency of rejection for $\gamma = 0$,

γ	N(0,1)	t(6)	t(3)
0	0.0520	0.0510	0.0590

- empirical power: frequency of rejection for $\gamma \neq 0$.

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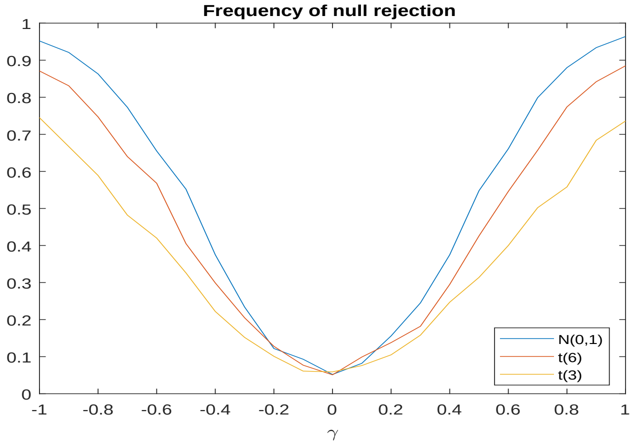
Frequency of null rejection



Example 2

When the normality assumption is violated and the true errors come from $t(3)$ or $t(6)$ then

- the size of the test remain correct,
- the power of the test is reduced: we may not reject the null even if it is false.

Example 3

Lets consider the DF unit root test

$$\triangle y_t = \beta y_{t-1} + e_t$$

Hypothesis

$$H_0 : \beta = 0$$

$$H_1 : \beta < 0$$

Propose a MC experiment, which allows to check, if the t-student distribution is correct when testing for $\beta = 0$.

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Example 3

The frequency of rejecting the null

$$H_0 : \beta = 0$$

for the significance level $\alpha = 0.05$ and t-student asymptotic distribution

T/β	-2	-1	-0.5	-0.1	-0.05	0
100	1.000	1.000	1.000	0.947	0.576	0.133
200	1.000	1.000	1.000	1.000	0.942	0.128
1000	1.000	1.000	1.000	1.000	1.000	0.124

Question: what is the size of the test? Is it correct?

Monte Carlo - results

The results are subjected to the error. How to calculate the error of estimation of the test size?

- Suppose, that the nominal significance level is 5% then the rejection variable comes from a distribution

$$h_i \sim 0.05^{h_i} 0.95^{1-h_i}$$

- What is the expected value and the variance of the frequency estimator?

$$\bar{h} = \frac{1}{N_{MC}} \sum_{m=1}^{N_{MC}} h_i$$

Limitations of MC simulations

The main problem of MC simulations is specificity:

- distribution of variables needs to be predefined,
- there are many parameters combinations which needs to be considered,
- results are related to the design of the experiment and are difficult to generalize.

Simulation of critical values and confidence intervals

Simulation of critical values or confidence intervals

- a small sample,
- unknown distribution,
- confidence intervals of nonlinear function of parameters

Example 4

Suppose,

$$y_i = \alpha + \beta x_i + \gamma z_i + e_i$$

and we want to test if

$$H_0 : \gamma = 0$$

$$H_1 : \gamma \neq 0$$

The t-Student statistic is

$$t = \frac{\hat{\gamma}}{\sqrt{var(\hat{\gamma})}}$$

For small sample (for example $N = 10$) may differ from asymptotic distribution.

Example 4

For a given values of x and z, the empirical size of the test (for asymptotic critical values $t^* = 2.262$) and significance level $\alpha = 0.05$

γ	N(0,1)	t(6)	t(3)
0	0.0989	0.1008	0.0974

Notice: the empirical size is much higher than the significance level. What are the potential consequences of it?

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Example 4 - design of the experiment

Lets simulate the model under the null. How to choose the parameters for the particular application?

- the variables x and z are known (does not need to be simulated) and $X = [1, x, z]$
- the true parameters are equal to the estimates under the null ($\hat{\theta}_R = [\hat{\alpha}_R, \hat{\beta}_R, 0]'$)
- the residuals' variance is equal to its estimate ($\hat{\sigma}^2_R$)

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Example 4 - design of the experiment

Set the number of MC iterations ($N_{MC} = 10000$) and for each iteration

1. generate residuals e from $N(0, 1)$, $t(6)$ or $t(3)$ and multiply by the $\sqrt{\hat{\sigma}_R^2}$,
2. generate $y = X\hat{\theta}_R + e$,
3. estimate the model and compute the t statistic for the parameter γ,
4. estimate the critical value as the 975% of the empirical distribution of the statistics.

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Example 4 - results

Estimated critical values

t^*	N(0,1)	t(6)	t(3)
2.262	2.842	2.773	2.926

Bootstrap

Bootstrap

Bootstrap is used to obtain a description of sampling properties of empirical estimators using the sample data rather then broad theoretical results.

Applications

- for particular empirical problem,
- approximation of the distribution of parameters: confidence intervals (complicated, nonlinear problems),
- bias correction.

Bootstrap

Suppose, $\hat{\theta}_N$ is

- an estimator of θ,
- based on a sample $Z = [(y_1, x_1), ..., (y_N, x_N)]$

What is the variance of the estimator $Var(\hat{\theta}_N)$?

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Bootstrap

The distribution of $\hat{\theta}$ can be obtain by sampling m observations with replacement from Z.

For each bootstrap iteration, $b = 1, ..., B$

- a new sample, $Z(b)$ is randomly chosen from Z,
- a new estimator $\hat{\theta}(b)_m$ is computed.

Desired characteristics are computed from

$$\hat{\Theta} = [\hat{\theta}(1)_m, \hat{\theta}(2)_m, ..., \hat{\theta}(B)_m].$$

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Bootstrap

Based on the bootstraped estimates of the parameters

$$\hat{\Theta} = [\hat{\theta}(1)_m, \hat{\theta}(2)_m, ..., \hat{\theta}(B)_m]$$

we can compute:

- the estimator expected value

$$\bar{\hat{\theta}}_B = \frac{1}{B}\sum_{b=1}^{B} \hat{\theta}(b)_m$$

- the estimator variance

$$Var(\hat{\theta}_N) = \frac{1}{B-1}\sum_{b=1}^{B} (\hat{\theta}(b)_m - \bar{\hat{\theta}}_B)(\hat{\theta}(b)_m - \bar{\hat{\theta}}_B)'$$

- the estimator median ....

Example 5

Suppose, we want to compute a standard error of a median for a *t*-student distribution:

- no exact formula for the variance of median,
- unknown the number of degrees of freedom and the mean (if non-zero)

Example 5

For each bootstrap iteration ($B = 100000$) and the sample size $N = 500$

- randomly draw with replacement N observations
- compute median $\hat{M}_b$

Compare the estimates with the median of the sample and compute the RMSE.

median	mean
0.0570	0.0515

Example 6

Lets use bootstrap to get the confidence intervals of the parameter γ from the previous example.

$$y_i = \alpha + \beta x_i + \gamma z_i + e_i,$$

for $i = 1, 2, ..., 10$. The estimator is $\hat{\gamma} = 0.109$

Question: What is the sample?

$$Z = [y, 1, x, z]$$

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$$Z = \left(\begin{array}{c|ccc} -0.6790 & 1.0000 & -1.1082 & -1.0121 \\ -0.4171 & 1.0000 & -0.1438 & 0.6886 \\ -2.1285 & 1.0000 & -0.5226 & 0.4817 \\ 1.4352 & 1.0000 & 0.9272 & -0.2933 \\ -1.1544 & 1.0000 & -0.3550 & -0.2965 \\ -0.5688 & 1.0000 & -0.2709 & 0.3545 \\ -3.3412 & 1.0000 & -0.4944 & -1.5773 \\ -1.3398 & 1.0000 & -1.5380 & -1.3614 \\ -0.2702 & 1.0000 & -0.0391 & -0.6914 \\ 0.8011 & 1.0000 & 0.9009 & -0.7236 \end{array}\right)$$

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For each bootstrap iteration

1. randomly draw T raw from Z with replacement,
2. for the new sample, estimate $\hat{\gamma}$

Collect the results and approximate the distribution of $\hat{\gamma}$

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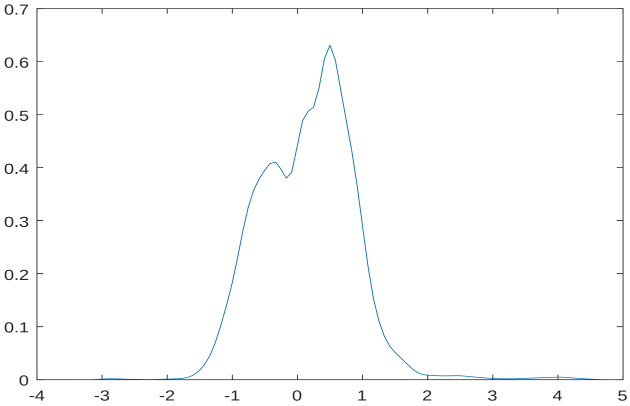

Bootstrap

How to bootstrap a sample?

- paired bootstrap - joint sampling of y_i and x_i,
- parametric bootstrap

$$y_i(b) = x_i(b)\hat{\theta}_n + \hat{e}_i(b)$$

Bootstrap - parametric bootstrap

How to bootstrap is an AR model?

$$y_t = \alpha + \beta_1 y_{t-1} + ... + \beta_p y_{t-1} + e_t$$

parametric bootstrap:

- Estimate the model parameters: $\hat{\theta} = [\hat{\alpha}, \hat{\beta}_1, ..., \hat{\beta}_p]'$
- estimate the model residuals: $\hat{e}_t = y_t - \hat{\alpha} - \hat{\beta}_1 y_{t-1} - ... - \hat{\beta}_p y_{t-1}$,
- sample from e: $e(b)$
- simulate $y_t = \hat{\alpha} + \hat{\beta}_1 y_{t-1} + ... + \hat{\beta}_p y_{t-1} + \hat{e}(b)_t$

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- sample from e: $e(b)$
- simulate $y_t = \hat{\alpha} + \hat{\beta}_1 y_{t-1} + ... + \hat{\beta}_p y_{t-1} + \hat{e}(b)_t$

Example 7

Suppose

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + e_t$$

and we need to compute forecast intervals for one and two days ahead forecasts.

Future values:

$$y_{t+1} = \alpha_0 + \alpha_1 y_t + e_{t+1}$$

$$y_{t+2} = \alpha_0(1 + \alpha_1) + \alpha_1^2 y_t + e_{t+2} + \alpha_1 e_{t+1}$$

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Problems:

- unknown distribution of residuals e_{t+1} and e_{t+2} (what if not normal?),
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- unknown distribution of residuals e_{t+1} and e_{t+2} (what if not normal?),
- 2-days ahead forecast nonlinear in parameters,
- estimation uncertainty (unknown parameters),
- ...

Example 7

Parametric bootstrap (can we use the paired bootstrap?): does not take into account parameter uncertainty:

1. Estimate the model parameters: $\hat{\alpha}_0$ and $\hat{\alpha}_1$ and point forecasts
$$\hat{y}_{t+1|t} = \hat{\alpha_0} + \hat{\alpha}_1 y_t$$
$$\hat{y}_{t+2|t} = \hat{\alpha_0}(1 + \hat{\alpha}_1) + \hat{\alpha}_1^2 y_t$$
2. Compute the residuals $e_t = y_t - \hat{\alpha_0} + \hat{\alpha_1} y_{t-1}$.
3. Draw randomly $e_{t+1}(b)$ and $e_{t+2}(b)$ from residuals, with replacement and simulate
$$y_{t+1}(b) = \hat{y}_{t+1|t} + e_{t+1}(b)$$
$$y_{t+2}(b) = \hat{y}_{t+2|t} + e_{t+2}(b) + \hat{\alpha}_1 e_{t+1}(b)$$
4. Using $y_{t+1}(b)$ and $y_{t+2}(b)$ construct confidence intervals.

Example 7

How to include parameter estimation uncertainty in the simulation of forecast intervals?