Partial Differential Equations with Applications in Industry

Problem Set 3: Heat equation

1. (*Diffusion equation*) Consider a conserved substance described by the concentration $u(\mathbf{x}, t)$. By assuming Fick's Law stating that the diffusive flux (what is that?) is *proportional to the gradient of the concentration* (give an interpretation) derive the governing equation.

2. (*Non-insulating surface*) The iron bar looses its heat through the lateral surface according to the Newton's Cooling Law. Find equation for its temperature $u = u(x, t)$ at time t and point x assuming that any other sources (or sinks) are absent.

3. (*Existence and regularity*) Prove that the function

$$u(x,t) := \sum_{n=1}^{\infty} a_n e^{-\left(\frac{\alpha n \pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right), \quad a_n = \frac{2}{L}\int_0^L \phi(x) \sin\left(\frac{n\pi}{L}x\right) dx,$$

is a solution of the following problem

$$\begin{cases} u_t = \alpha^2 u_{xx}, & (x,t) \in (0,L) \times (0,T), \\ u(x,0) = \phi(x), & x \in [0,L] \\ u(0,t) = 0, & u(L,t) = 0, \end{cases}$$

where ϕ is continuous. Moreover, show that u defined by the above formula is an infinitely differentiable function. *Hint.* Follow the sketch given during the lecture.

4. (*Some heat conduction problems*) Solve each of the following heat equation with given boundary conditions and for initial temperature distribution $u(x, 0) = f(x)$. In each case give a physical interpretation of the boundary conditions and compute the exact Fourier coefficients for $f(x) = U_0$=const. Finally, find the temperature for $t \to \infty$.

   a) $u(0,t) = 0; \quad u_x(L,t) = 0,$
   b) $u_x(0,t) = 0; \quad u_x(L,t) = 0,$
   c) $u_x(0,t) = 0; \quad u(L,t) = 0,$
   d) $u_x(0,t) = -hu(0,t); \quad u(L,t) = 0.$
   d) $u_x(0,t) = -hu(0,t); \quad u_x(L,t) = 0,$
   d) $u_x(0,t) = -hu(0,t); \quad u_x(L,t) = -ku(L,t).$

5. (*Periodic boundary conditions*) Suppose that a homogeneous thin wire is bent into a circular ring of length $2L$ (for simplicity). Assume that its surface is insulated and formulate a boundary value problem modelling temperature distribution in the ring. Finally, solve the stated problem.

6. (*Almost steady-state*)

   (a) Find a temperature distribution of a bar $0 \leq x \leq L$ with isolated surface and initial temperature equal to $f(x)$. Both ends of the bar are kept in a zero temperature.

   (b) Explicitly solve the case where $f(x) = U_0$ = const., i.e. determine the series which constitute the solution. Next, estimate the error of approximating the whole series by its $n-$th partial sum .

   (c) Determine the time T after which the whole series summed from the *second* term and then divided by the *first* term will be smaller than ϵ.

7. (*Constant boundary conditions*) Find an explicit solution (compute the Fourier series) of the heat conduction problem with constant boundary conditions

$$\begin{cases} u_t = \alpha^2 u_{xx}, & (x,t) \in (0,L) \times (0,T), \\ u(x,0) = 0, & x \in [0,L] \\ u(0,t) = u_0, & u(L,t) = u_1. \end{cases}$$

Compare the function U with solution of the steady-state equation $u_{xx} = 0$.

8. (*von Neumann boundary conditions*) Devise a solution to the problem with von Neumann BCs

$$\begin{cases} u_t = \alpha^2 u_{xx}, & (x,t) \in (0,L) \times (0,T), \\ u(x,0) = 0, & x \in [0,L] \\ u_x(0,t) = \mu(t), & u_x(L,t) = \nu(t). \end{cases}$$

Next, write an explicit solution for μ and ν constant.

9. (*Robin boundary conditions*) Solve the following heat conduction problem with Robin boundary condition by reducing it to Dirichlet BC for a nonhomogeneous equation. How to proceed when additionally $u(L,t) = -\kappa(u(L,t) - \sigma(t))$?

$$\begin{cases} u_t = \alpha^2 u_{xx}, & (x,t) \in (0,L) \times (0,T), \\ u(x,0) = 0, & x \in [0,L] \\ u_x(0,t) = -\lambda(u(0,t) - \theta(t)), & u(L,t) = 0, \qquad t \geq 0. \end{cases}$$

Hint. You can introduce a new function $v(x,t) = e^{\lambda x}(\theta(t) + u(x,t))$.

10. (*Nonhomogeneous problems*) Reduce given nonhomogeneous heat conduction problem into a collection of homogeneous ones. Then, write its solution in terms of the Green function. Pay particular attention to the choice of the appropriate Green function.

$$\begin{cases} u_t = \alpha^2 u_{xx} + f, & (x,t) \in (0,L) \times (0,T), \\ u(x,0) = \phi(x), & x \in [0,L], \end{cases}$$

a) $u_x(0,t) = \mu(t)$; $u(L,t) = \nu(t)$, b) $u(0,t) = \mu(t)$; $u_x(L,t) = -\lambda(u(L,t) - \theta(t))$,
c) $u_x(0,t) = \mu(t)$; $u_x(L,t) = -\lambda(u(L,t) - \theta(t))$.

11. Solve the equation derived in Prob. 2.

12. (*Practical way of finding self-similar solutions*) Usually self-similar solution of many PDEs are found by introducing the following transformation

$$u(x,t) = t^a U(z), \quad z = xt^b,$$

and choosing a and b for the equation to be satisfied. Do so for the heat equation.

13. (*Self-similarity and the half-line*) Use the self-similar solution technique to solve the heat equation on the half-line with constant u_0

$$\begin{cases} u_t = \alpha^2 u_{xx}, & (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ u(x,0) = 0, & x > 0, \\ u(0,t) = u_0, & x > 0. \end{cases}$$

14. Solve the following problem on the real-line

$$\begin{cases} u_t = \alpha^2 u_{xx}, & (x,t) \in \mathbb{R} \times (0,T), \\ u(x,0) = \begin{cases} u_1, & x < 0; \\ u_2, & x > 0 \end{cases}, \end{cases}$$

What is the value of $u(0,t)$?

15. (*Heat kernel*) Show that

$$\frac{1}{\sqrt{4\alpha^2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4\alpha^2}} dz = 1.$$

16. (*von Neumann BC on half-line*) Utilize the method of reflections to find a solution of the heat equation on the half-line with Neumann boundary condition

$$\begin{cases} u_t = \alpha^2 u_{xx}, & (x,t) \in \mathbb{R}_+ \times (0,T), \\ u(x,0) = 0, \\ u_x(0,t) = \mu(t). \end{cases}$$

Hint. Odd extension will not necessarily work.

17. Using method of reflections solve the following problem on the half-line for constant u_0

$$\begin{cases} u_t = \alpha^2 u_{xx}, & (x,t) \in \mathbb{R}_+ \times (0,T), \\ u(x,0) = u_0, \\ u(0,t) = 0. \end{cases}$$

18. (*Nonhomogeneous problems on the half-line*) Using similar techniques as for the real-line, devise a solution of the following nonhomogeneous problem on $\mathbb{R}_+$

$$\begin{cases} u_t = \alpha^2 u_{xx} + f, & (x,t) \in \mathbb{R}_+ \times (0,T), \\ u(x,0) = \phi(x), \\ u(0,t) = \mu(t). \end{cases}$$

19. (*Wine cellar*) We want to build a wine cellar under our garden. The main assumption is to found it on an appropriate depth in order to make the temperature best for our wine. Let $u = u(x,t)$ be the temperature of Earth on depth x and time t.

    (a) Justify that the following problem is a sensible model for our case (why the boundary condition is as so?)

    $$\begin{cases} u_t = c^2 u_{xx}, & x > 0, \quad t > 0, \\ u(0,t) = T_0 + A\sin(\omega t), & t > 0. \end{cases}$$

    (b) Why the solution of the above problem can be sough in the given ansatz?

    $$u(x,t) = T_0 + f(x)\sin\left(\omega t - \delta(x)\right),$$

    Where f and δ are unknown functions.

    (c) Find the bounded solution of our winecellar equation. Why we do not need the initial condition? Describe what you got.

    (d) The best depth to build a wine cellar is to have the smallest variations of temperature around the whole year. This means that in the Summer we would like to have colder conditions beneath the surface than above it while in the Winder the opposite should hold. Hence, we look for x_0 such that $\delta(x_0) = \pi$. Find $u(x_0,t)$ and compare with $u(x,t)$.

20. (*Porous medium equation*) Diffusion in many porous media such as soil or minerals is described by the following nonlinear PDE (it also arises in many other contexts; For instance hydrology, semiconductors or gasdynamics)

$$u_t = \left(u^m u_x\right)_x, \quad m > 0.$$

Use the techniques from Problem 12 to find the self-similar solution of the above equation with

$$u_x(0,t) = 0, \quad \int_{-\infty}^{\infty} u(x,t)\,dx = 1, \quad x \in \mathbb{R}.$$

This is the celebrated *Barenblatt's solution* and is associated with modelling a sudden release of energy at $x = 0$ (such as in a-bomb).

Łukasz Płociniczak