Valuation of Games: A Selected Survey

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- What is a game? A game is a competitive situation where two or more persons pursue their own interests and no one person can dictate the outcome.
- The goal of each participant is to maximize his own payoff, while taking into account that the other participants are doing the same.
- Game theory is generally divided into two approaches: the cooperative approach and the non-cooperative approach. These approaches should not be regarded as analyzing different kinds of games; they are different ways of looking at the same game.

- The non-cooperative approach is strategy oriented. It studies what we expect the players to do in the game. The cooperative approach, on the other hand, studies the outcomes of the game we expect.
- The non-cooperative approach is a kind of micro theory; it involves precise descriptions of what happens.
- The cooperative approach studies games from a macro point of view, focusing on the feasible outcomes that can be obtained through cooperation. That is, we look at the outcomes directly, not how one gets there. We ask questions how coalitions can form, what coalitions will form, and how coalitions that do form divide what they achieve.

- For two-person zero-sum games, the maximin value yields a sensible and unique evaluation of the game for each of the players.
- Shapley applies the idea of evaluating the prospect of playing a game to n-person coalitional games.
- Why studying valuations of games? Help an individual assess his prospects from participation in the game; determine an equitable distribution of the wealth to the players through their participation in the game.

- The cornerstone of the theory of cooperative *n*-person games is the characteristic function, a concept first formulated by John von Neumann in 1928. The idea is to capture in a single numerical index the potential worth of each coalition of players. Mathematically, the characteristic function is a function ν from the subsets of players to the real numbers. With the characteristic function in hand, all questions of strategies, information, physical transactions, etc. are left behind.
- A *n*-person game in coalitional form is a pair (N, ν) where $N = \{1, 2, \cdots, n\}$ is the set of players and $\nu : 2^N \longrightarrow \Re$ is the characteristic function. For each coalition $S \subseteq N$, $\nu(S)$ is the worth (transferable utility) that members of coalition S can achieve, regardless of what the outsiders (members of $N \setminus S$) do. Let Γ^N denote the set of all characteristic functions for player set N.

- Consider a market with 8 individuals each having 1 horse for sale and 10 other individuals each wishing to buy 1 horse. Horses are homogenous, but traders have different subjective valuations of the worth of horse ownership. There are no restrictions on communication, on transfers of money, or on transfers of horses. All players have a linear utility for money.

Seller	Valuation of a Horse	Buyer	Valuation of a Horse
A_1	\$10	B_1	\$30
A_2	\$11	B_2	\$28
A_3	\$15	B_3	\$26
A_4	\$17	B_4	\$24
A_5	\$20	B_5	\$22
A_6	\$21.50	B_6	\$21
A_7	\$25	B_7	\$20
A_8	\$26	B_8	\$18
		B_9	\$17
		B_{10}	\$15

Characteristic Function

- In calculating the characteristic function of a game we need no a priori assumptions about the schedule of bids and offers, or other procedures that serve to bring the traders to the agreement on prices and terms. The reason for this is clear: whenever a horse is transferred from A_i to B_j, the joint profit to A_i and B_j, which is all that matters, is independent of the amount of money that changes hands. Thus,

$$v(\{A_i, B_j\}) = \max\{0, b_j - a_i\}, \tag{1}$$

where a_i, b_j are the subjective valuations of seller A_i and buyer B_j. For any coalition $S = \{A_{i_1}, \cdots, A_{i_k}, B_{j_1}, \cdots, B_{j_l}\}$,

$$v(S) = \sum_{h=1}^{\min\{k,l\}} v(\{A_{i_h}, B_{j_h}\}), \tag{2}$$

where $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_l$.

A Basis for n-Person Coalitional Games

- For each $T \subseteq N$, define $v_T : 2^N \longrightarrow \Re$ by $v(\emptyset) = 0$ and

$$v_T(S) = \begin{cases} 1, & \text{is } S \supseteq T; \\ 0, & \text{otherwise.} \end{cases}$$

  for all $S \subseteq N$.

- For any $v : 2^N \longrightarrow \Re$ with $v(\emptyset) = 0$, there exists a unique linear representation:

$$v = \sum_{T \subseteq N} \lambda_T v_T$$

  where

$$\lambda_T = \sum_{R \subseteq T} (-1)^{|T|-|S|} v(S).$$

- Value of Games: $\phi : \Gamma^N \longrightarrow \Re^N$.
- Efficiency: $\sum_{i \in N} \phi_i(v) = v(N)$, $v \in \Gamma^N$.
- Symmetry: $\phi_i(v) = \phi_j(v)$ for all $i, j \in N$ such that $v(S \cup i) = v(S \cup j)$ for $S \subseteq N \setminus \{i, j\}$.
- Dummy: $\phi_i(v) = 0$ for all $i \in N$ such that $v(S) = v(S \cup i)$ for all $S \subseteq N \setminus \{i\}$.
- Additivity: $\phi(v + v') = \phi(v) + \phi(v')$ for all $v, v' \in \Gamma$.

Theorem

(Shapley 1953) There exists a unique value on Γ^N satisfying the Shapley axioms. Furthermore, it is given by

$$\phi_i(v_T) = \begin{cases} \frac{1}{|T|}, & \text{if } i \in T; \\ 0, & \text{otherwise.} \end{cases}$$

for all $T \subseteq N$ and

$$\phi_i(v) \sum_{S \subseteq N: S \ni i} \frac{(|S|-1)!(n-|S|)!}{n!}[v(S) - v(S \setminus i)]$$

for all $v \in \Gamma^N$. Equivalent representation:

$$\begin{aligned} \phi_i(v) &= \textstyle\sum_{S \subseteq N: i \notin S} \frac{|S|!(n-|S|-1)!}{n!}[v(S \cup i) - v(S)] \\ &= \textstyle\frac{1}{n!} \sum_{\pi \in \Pi}[v(P_i(\pi) \cup i) - v(P_i(\pi))] \end{aligned}$$

where $P_i(\pi) = \{j \in N : \pi_i(j) < \pi(i)\}$.

- Suppose we choose a random order of the players with all $n!$ orders *equally* likely. Then we enter the players according to this order. If coalition $S \ni i$ is formed after i enters, he receives $v(S) - v(S \setminus i)$. Given $S \ni i$, the number of permutations in which the $|S| - 1$ members of $S \setminus i$ enter before i and those of $N \setminus S$ enter after i is $(|S - 1|)!(n - |S|)!$. Thus, the probability that the $|S| - 1$ members of coalition S come first and then player i is

$$\frac{(|S - 1)!(n - |S|)!}{n!}.$$

- Expected payoff to player i.
- Fair entry fee for player i.
- Utility to player i for the prospect of playing the game.
- What i deserves to get (as a judge's problem of fair distribution).

Application: The Shapley-Shubik Power Index

- A coalitional game (N, v) is *simple* if for every coalition $S \subseteq N$, either $v(S) = 1$ (a winning coalition) or $v(S) = 0$ (a losing coalition). Player i is pivotal for coalition $S \ni i$ if S is winning and $S \setminus i$ is losing.
- Special Simple Games
  - The Majority Rule Game: $v(S) = 1$ if and only if $|S| > n/2$; $v(S) = 0$ otherwise.
  - The Unanimity Game: $v(N) = 1$ and $v(S) = 0$ for all $S \subsetneqq N$.
  - The Dictator Game: For some player $i \in N$ (the dictator), $v(S) = 1$ if and only if $i \in S$ and $v(S) = 0$ otherwise.
  - Weighted Voting Games: $v(S) = 1$ if $\sum_{i \in S} w_i > q$ and $v(S) = 0$ if $\sum_{i \in S} w_i \leq q$ for some non-negative vector $(w_i)_{i \in N}$ and positive number q.
- If (N, v) is simple, then

$$\phi_i(v) = \sum_{S \subseteq N : i \text{ is pivotal}} \frac{(|S - 1)!(n - |S|)!}{n!}.$$

- **Example:** Consider a game with players 1, 2, 3, 4 having 10, 20 , 30, 40 shares of stocks, respectively in a corporation. Decisions require approval by a majority of the shares. Then, for player 1, $v(S) - v(S \setminus 1) = 0$ unless $S = \{1,2,3\}$; for player 2, $v(S) - v(S \setminus 2) = 0$ unless $S = \{2,4\}$, $\{1,2,3\}$, $\{1,2,4\}$; for player 3, $v(S) - v(S \setminus 2) = 0$ unless $S = \{2,4\}$, $\{1,2,3\}$, $\{1,2,4\}$; and for player 4, $v(S) - v(S \setminus 2) = 0$ unless $S = \{2,4\}$, $\{3,4\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$. Thus,

$$\phi_1 = 1/12, \phi_2 = \phi_3 = 1/4, \phi_4 = 5/12.$$

- Change the stockholdings to 10, 30, 30, 30. Then, player 1 is not essential. He is not pivotal for any coalition. This is reflected in his Shapley-Shubik power Index: $\phi_1 = 0$.

Valuations of Strategic-Form Games

- Games in Strategic Form: $\Gamma = \{C_i, u_i\}_{i \in N}$ where C_i is the set of pure strategies of player i.
- Max-Min Worth (von Neumann 1928):

$$v(S) = \max_{x_S \in \Delta(C_S)} \min_{x_{N \setminus S} \in \Delta(C_{N \setminus S})} \sum_{i \in S} u_i(x_S, x_{N \setminus S}).$$

- Equilibrium Representation:

$$v(S) = \sum_{i \in S} u_i(x_S^*, x_{N \setminus S}^*), \; v(N \setminus S) = \sum_{i \in N \setminus S} u_i(x_S^*, x_{N \setminus S}^*)$$

where $(x_S^*, x_{N \setminus S}^*)$ is a Nash equilibrium for the 2-player game with (i) players S and $N \setminus S$; (ii) pure strategy sets C_S for player S and $C_{N \setminus S}$ for player $N \setminus S$; and (iii) utility functions $\sum_{i \in S} u_i(c_S, c_{N \setminus S})$ for player S and $\sum_{i \in N \setminus S} u_i(c_S, c_{N \setminus S})$ for player $N \setminus S$.

- Rational-Threat Representation (Harsanyi 1963): Given $S \subseteq N$. define $A, B : C_S \times C_{N\setminus S} \longrightarrow \Re$ by

$$A(c_S, c_{N\setminus S}) = \sum_{i\in S} u_i(c_S, c_{N\setminus S}) + \sum_{i\in N\setminus S} u_i(c_S, c_{N\setminus S})$$

and

$$B(c_S, c_{N\setminus S}) = \sum_{i\in S} u_i(c_S, c_{N\setminus S}) - \sum_{i\in N\setminus S} u_i(c_S, c_{N\setminus S}).$$

Payoff function A represents the common interests of the two groups of players; denote its "max-max" value by $a(S, N \setminus S)$. The second represents their opposing interests; denote the minimax value of the zero-sum game between the two groups by $b(S, N \setminus S)$. Then, the bargaining worth of coalition S is:

$$v(S) = \frac{1}{2}a(S, N \setminus S) + \frac{1}{2}b(S, N \setminus S).$$

- If a player wishes to sell his right to play the game, how much should he sell it for?

Cooperation Structures

- Players may cooperate in a game by forming a series of bilateral agreements among themselves. Denote the bilateral agreement by $i:j$. Then, a cooperation structure is a collection of bilateral agreements. Mathematically, a cooperation structure is a (undirected) graph with players as the nodes and bilateral agreements as the links.
- $G = \{g | g \subseteq g^N\}$ where $g^N = \{i:j | i,j \in N : i \neq j\}$.
- Given $g \in G$ and given $S \subseteq N$, S/g denotes the partition of S into connected components.
- How should the should the outcome of a game depend on which players cooperate with whom?
- For $(g, v) \in G \times \Gamma^N$, Define $v^g : 2^N \longrightarrow \Re$ by

$$v^g(S) = \sum_{R \in S/g} v(R).$$

- Myerson Axioms: A value is a mapping $\psi : G \times \Gamma^N \longrightarrow \Re$.
    - Feasibility: $\sum_{i \in S} \psi_i(g, v) = v(S)$, $\forall S \in N/g$.
    - Fairness: $\psi_i(g) - \psi_i(g \setminus i:j) = \psi_j(g) - \psi_j(g \setminus i:j)$,

Theorem

(Myerson 1977) There exists a unique value on $G \times \Gamma^N$ satisfying Myerson axioms. Furthermore, it is given by

$$\phi(g, v) = \phi(v^g).$$

Example: Let $N = \{1, 2, 3\}$, $v(\{1\}) = 0$, $v(\{i\}) = v(\{1, i\}) = 1$ for $i = 2, 3$, and $v(\{2, 3\}) = v(N) = 3$. Then, player 1 is dummy. Now consider $Ng = \{1 : 2, 1 : 3\}$. Then, $v^g(\{1\}) = 0$, $v^g(i) = v^g(\{1, i\}) = 1$, $v^g(\{2, 3\}) = 2$, and $v^g(\{1, 2, 3\}) = 3$. Then, player 1 is not dummy and $\psi_1(g, v) = 1/6$.

A Cooperation-Formation Game

- Given a game (N, v), what cooperation structures will the players form?
- $C_i = \{S \subset N : S \ni i\}$, $i \in N$. Strategy in C_i is a set of other players with whom player i proposes to form bilateral links. Given $c \in C$, $g(c) = \{i : j | i \in c_i, j \in c_j\}$.
- Consider a game in strategic form $\Gamma = \{C_i, u_i\}_{i \in N}$. How might players' payoff functions u_i be determined?
  - $u(\cdot) = (u_i(\cdot))_{i \in N}$ is feasible:

$$\sum_{i \in S} u_i(c) = v(S), \ \forall S \in N/g(c), \forall c \in C.$$

  - $\Gamma = \{C_i, u_i\}_{i \in N}$ has a potential: $\exists Q : C \longrightarrow \Re$ such that

$$u_i(c_i', c_{-i}) - u_i(c_i, c_{-i}) = Q(c_i', c_{-i}) - Q(c_i, c_{-i})$$

for all $c_i, c_i' \in C_i$, $c_{-i} \in C_{-i} = \times_{j \neq i} C_j$, and for all $i \in N$.

Theorem

(Qin 1996) Given a coalitional game (N, v)*, there exists a unique payoff function* $u(\cdot) = (u_i(\cdot))_{i \in N}$ *such that* $u(\cdot)$ *is feasible and* $\Gamma = \{C_i, u_i\}_{i \in N}$ *has potential. Furthermore,* $u(c) = \psi(g(c), v), i \in N$ *and* $\psi(g(c^*), v) = \phi(v)$ *whenever* $Q(c^*) = \max_{c \in C} Q(c)$ *and* v *is superadditive.*

- Fully endogenous cooperation-formation?

- A value function is a mapping $v : G \longrightarrow \Re$. Set $V = \{v | v : G \longrightarrow \Re\}$. A network game is a pair (N, v) with $v \in V$.
- Given $g \in G$ define $v_g \in V$ by

$$v_g(g') = \begin{cases} 1, & \text{if } g' \supseteq g; \\ 0, & \text{otherwise.} \end{cases}$$

For any $v \in V$, there is a unique linear representation,

$$v = \sum_{g \in G} \lambda_g v_g$$

where

$$\lambda_g = \sum_{g' \subseteq g} (-1)^{l(g) - l(g')} v(g').$$

Axioms on Value Functions

- Allocation Rules: $\psi : G \times V \longrightarrow \Re^N$.
- What would be appropriate axioms on allocations rules?
- Given $g \in G$, $N(g) = \{i \in N | \exists j \in N : i : j \in g\}$, $n(g) = |N(g)|$. Is

$$\psi_i(g, v_g) = \begin{cases} \frac{1}{n(g)}, & \text{if } i \in N(g); \\ 0, & \text{otherwise.} \end{cases}$$

  reasonable?

- Jackson (2005).

Conclusion

- In cooperative theory we are interested in feasible outcomes. Anything that players could get is taken into consideration, even if it is not incentive compatible for them. This is done by assuming that there is a mechanism, such as a court, that enforces agreements, so that all feasible outcomes should be considered.
- The aim of the valuation approach to games is to associate with every game a reasonable outcome. The criteria of reasonableness are expressed axiomatically.
- The reasonable outcome can be taken as determining an equitable distribution of the wealth (cost, risk, etc) to the players through their participation in the game.