Normalisation for Ordinary Agents: A Solution to the Problem of Logical Omniscience

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I. Introduction

Bayesianism, hereafter the Bayesian model, is the interpretation of probability as a quantification of degrees of belief. Because probabilities must adhere to the Kolmogorov axioms, it follows from the axiom of normalisation that a rational agent must believe all logical truths, even those unknown. This impossible demand on rationality is the problem of logical omniscience.

This paper will discuss the problem and demonstrate that its conclusion is problematic because it demands us to abandon a valuable model of rationality despite the lack of alternatives. I will then argue for a pragmatic solution to the problem by relaxing the axiom of normalisation, insofar as a rational agent must assign a maximum degree of belief to logical truths only if such truths can be justified to be known. This is the most ideal solution because it directly addresses the impossible demand posed by the problem, while also preserving the normative value of the model and avoiding complications that render the model impractical.

II. The Problem of Logical Omniscience

In the Bayesian model, an agent is rational only if their degrees of belief in propositions, hereafter credences, satisfy the mathematical conditions of a probability function.[1] These conditions are the Kolmogorov axioms which state that, for any set of propositions S, the credence function $Cr : S \to \mathbb{R}$ must satisfy the following:

Axiom 1. *(Non-negativity) For any proposition $A \in S$, one has $Cr(A) \geq 0$.*

Axiom 2. *(Normalisation) For any logical truth $T \in S$, one has $Cr(T) = 1$.*

Axiom 3. *(Finite additivity) For any propositions $A_1, A_2, ..., A_n \in S$ such that for integers $j, k \in [1, n]$ where $j \neq k$ and $Cr(A_j \wedge A_k) = 0$, one has $Cr(\bigvee_{i=1}^{n} A_i) = \sum_{i=1}^{n} Cr(A_i)$.*

1. Wolfgang Schwarz, *Belief, Desire, and Rational Choice*, https://github.com/wo/bdrc/blob/master/bdrc.pdf, December 20, 2022, 13.

What if S includes propositions whose truth we have yet to ascertain? For such propositions, the axiom of normalisation demands an ordinary agent to assign maximum credence to its truth, otherwise they would be considered irrational. However, it is unreasonable to qualify rationality on knowledge of truths that ordinary agents cannot prove conclusively. Consider the case of Fermat's Last Theorem, which took over three hundred years to prove, and only successfully so with cutting-edge mathematics that would be inaccessible to an ordinary agent.[2] Is it fair to consider someone in the past to be irrational if they did not assign maximum credence in the truth of the theorem?[3] This clearly impossible demand is the problem of logical omniscience, which may be formalised as:[4]

P1: No ordinary agent can have credences in an infinite number of propositions.

P2: There are an infinite number of logical truths.

P3: If the Bayesian model is applicable to ordinary agents, then any ordinary agent must have maximum credence in any logical truth.

P4: No ordinary agent can have credences in all logical truths.

C: The Bayesian model is inapplicable to ordinary agents.

I will demonstrate that the argument is sound. Suppose there exists an ordinary agent who knows an infinite number of propositions, so they must have successfully enumerated through an infinite set of propositions and determined each individual credence. However, it is nigh impossible for an ordinary agent to even begin constructing this set, much less to enumerate through it. This contradiction proves **P1**.

2. For background, the theorem states that: For any $n \in \mathbb{Z}_{\geq 2}$, the equation $a^n + b^n = c^n$ has no positive integer solutions. The theorem seems innocuous, but it was a notorious unsolved problem that required mathematics developed after Fermat's time to prove conclusively.

3. Sinan Dogramaci, "Solving the Problem of Logical Non-Omniscience," *Philosophical Issues* 28, no. 1 (2018): 108, https://doi.org/10.1111/phis.12118.

4. Carneades.org, "The Problem of Logical Omniscience (Bayesian Epistemology)," YouTube video, 6:24, February 22, 2015, youtu.be/pkGFSkGzg4M.

Additionally, let the greatest number of logical truths be a finite number n, so the set of logical truths is $T = \{T_1, T_2, ..., T_n\}$. For any i with $1 \leq i \leq n$, we know $T_i \in T$ is true, so the tautology $(T_i \vee \neg T_i)$ is also a logical truth. This additional logical truth means that there are at least $n+1$ logical truths, so this contradicts the definition of n as the greatest number of logical truths. Since there is no upper bound on the number of logical truths, there must be an infinite number of logical truths, so **P2** is true.

Consequently, **P3** restates the axiom of normalisation. **P4** also follows naturally: if there are an infinite number of logical truths (**P1**) and no ordinary agent can have credences in an infinite number of propositions (**P2**), then **P4** must be true. Hence, the conclusion is obtained by *modus tollens* of **P4** on **P3**.

The problem of this conclusion is that we are compelled to discard a valuable normative model of rationality on a basis of a technicality, even when we have no alternatives. The conclusion that the model is inapplicable to any ordinary agent implies that it is pointless to evaluate rationality on the Bayesian model. This cannot be the case, because the model is clearly valuable and widely-applicable on account of the balance between rigour and intuition it provides. It literally pays to abide by the model, since the a simple Dutch Book argument demonstrates that if an agent has credences that violate the axioms, then there exist a set of bets that lead to a guaranteed loss if bought.[5] Hence, by abandoning the model just because ordinary agents cannot fulfil the impossible demand of logical omniscience, we forgo a definition of rationality grounded in easily-tested principles in favour of either a subjective, laissez-faire approach to defining rationality or the sweeping generalisation that all ordinary agents are irrational. The former cannot be the right approach, since it is reasonable to expect normative claims of rationality to be grounded in some objectively measurable standards. Therefore, it is preferable to preserve the utility of the model by attempting to address the problem of logical omniscience and reconcile it with the model, rather than completely give the model up.

5. Schwarz, *Belief, Desire, and Rational Choice*, 44.

III. Reformulating Normalisation

How might we account for the shortcomings of the Bayesian model? The most obvious approach would be to accept that the model only prescribes how a logically omniscient agent must assign their credences. The axioms merely provide a starting point for further amendments to make them more reflective of how ordinary agents behave. Thus, given that the current formulation of the axiom of normalisation is the critical flaw of the model, we could reformulate it to:

Axiom 4. *(Weak Normalisation, WN) For any* ***known*** *logical truth $T \in S$, one has $Cr(T) = 1$.*

This reformulation implies that an ordinary agent ought to have maximum credence in a logical truth, only if they know that it is one.[6] To know a logical truth would mean that an agent is able to justify its truth or non-falsity. Consequently, they must assign maximum credence to the truth of this proposition to qualify as rational. This reformulation is most desirable because it renders truths that are unknown to the agent irrelevant to considerations of rationality. Furthermore, it upholds the normative standards of the Bayesian model, as it is a reasonable demand for an agent to justify their credences. Therefore, applying WN to the example of Fermat's Last Theorem, the ordinary agent in the past remains rational for having a non-maximum credence in its truth, because prior to its proof it would have been impossible for anyone to produce a valid reason for their credence in it. However, since the theorem has now been proved, even if the agent does not understand the proof, they must assign maximum credence in its truth because they can ground their credence in this truth on the authority and rigorous review that the proof should have underwent. Hence, this is a more pragmatic and reflective approach to modelling the credences of ordinary agents.

A potential challenge to this approach is that invoking an arbitrary criteria to "know" a logical truth would taint an otherwise objective Bayesian model with subjectivity. Agents may utilise different criteria to determine what counts as knowledge, so privileging the definition of

6. Hanti Lin, "Bayesian Epistemology," in *The Stanford Encyclopedia of Philosophy*, Fall 2022, ed. Edward N. Zalta and Uri Nodelman (Metaphysics Research Lab, Stanford University, 2022), https://plato.stanford.edu/archives/fall2022/entries/epistemology-bayesian/.

knowledge as justified true belief may be seen as arbitrary. Since the model is grounded in this potentially contentious perspective, the objection would yet again undermine its normative value. However, I contend that the aim of the reformulation is not to demand universal agreement on a definition of knowledge. It is beyond the scope of the Bayesian model to account for this underlying epistemological challenge, since the model only aims to provide a principled framework for rational credence assignment. By limiting the demands to conditions that rational agents ought to fulfil, the reformulation of the axiom accommodates for the diversity of perspectives in defining knowledge and it remains agnostic on a specific definition. Thus, subjectivity is not introduced by the reformulation and this solution to preserve the value of the Bayesian model is still valid.

IV. An Extension?

By reformulating the axioms, the natural question that follows would be: how far should we go to make the model more reflective of any ordinary agent's credences? I argue that this limit must be WN, if we wish to prevent complications from arising. Any reformulation destabilises the mathematical foundations of the whole system of probability. When we relax the axioms to compensate for logical non-omniscience, we lose some applications to probability that make the Bayesian model appealing in the first place.[7] In particular, results that hinge upon the precision that the axioms offer would become either contradictory or nonsensical if the reformulations become excessive.

To illustrate this, suppose we take the view that credences need not be single real values, but are a real interval. This is not implausible, because ordinary agents may account for a range of uncertainty or indifference that is more faithful to their epistemic position. Therefore, if one knows a logical truth, their credence ought to be an interval that includes the number one.[8] Formally, we represent this as:

7. Michael G. Titelbaum, "Old Evidence and Logical Omniscience," in *Fundamentals of Bayesian Epistemology 2: Arguments, Challenges, Alternatives* (Oxford University Press, 2022), 436.
8. Lin, "Bayesian Epistemology."

Axiom 5. *(Very Weak Normalisation, VWN) For any known logical truth $T \in S$, there exists c with $0 \leq c \leq 1$ such that $c \leq Cr(T) \leq 1$.*

However, this causes some useful results in probability to become undone. Consider the case of conditional credences, defined as:

Definition 1. *For propositions $A, B \in S$ with $Cr(B) > 0$, one has $Cr(A|B) = \frac{Cr(A \wedge B)}{Cr(B)}$.*

Intuitively, this represents an agent's credence that a proposition A is true, supposing that B is true.[9] If we take B to be a known logical truth, by WN we have that $Cr(B) = 1$, so $Cr(A|B) = Cr(A \wedge B)$. Also, because the truth of the proposition $(A \wedge B)$ depends solely on A, we know that $Cr(A \wedge B) = Cr(A)$. Thus we obtain a well-defined conditional credence. However, if we allow VWN to stand, then $Cr(B)$ may be a real interval, so considering conditional credences becomes mathematically absurd because it is impossible to divide $Cr(A \wedge B)$ over an interval.

Perhaps we accept that conditional credence as an interval is meaningful because it is still a credence, so one might pursue the logic in WN and let $Cr(A \wedge B) = Cr(A) = a$ for some $0 \leq a \leq 1$. Dividing $Cr(A)$ over the bounds of $Cr(B)$, we get $Cr(A|B) = [a, \frac{a}{c}]$. However, if we let $a = 1$ and $c \leq 1$, we get a conditional credence that may be greater than the maximum credence in a logical truth. This strange result violates our logical intuitions, because it suggests that in certain situations, there exist propositions that we may believe in more than logical truths. It is obviously inconceivable, if not outright impossible, to have "more" belief in a proposition than *'A triangle has three angles'*. Therefore, VWN is too naïve, thus it cannot be the right approach to account for logical non-omniscience.

One could contend that the above examples are contrived and unfair. There could be methods that reconcile VWN with a consistent and improved Bayesian model. However, I argue that introducing further mathematical complexity causes the model to lose its intuitive appeal. As stated previously, the appeal of the model comes from the balance between rigour in its

9. Schwarz, *Belief, Desire, and Rational Choice*, 32.

mathematical foundations and its simplicity.[10] Adding layers of sophistication to the model infringes upon its simplicity and imposes additional implicit conditions for rationality. This results in a situation not much different from the impossible demand for rationality posed by the problem of logical omniscience, since a mathematically-refined Bayesian model could demand so much calculation from an ordinary agent that it may be impossible to ever evaluate if they are truly rational. Those advocating this solution will find themselves back where they started because of the mathematical complexity that they have introduced. Hence, reformulating the axiom excessively is not a pragmatic solution to the problem of logical omniscience, so the best solution is to simply relax the axioms to the WN formulation.

V. Conclusion

This paper has demonstrated that the best solution to the problem of logical omniscience so is to adopt the WN formulation of normalisation and limit the assignment of maximum credence to logical truths that ordinary agents can justify. This is the only possible compromise to preserve the normative value of the Bayesian model, because adopting weaker formulations like WVN would destabilise the system of probability, and also cause impossible demands to present themselves again in the form of a model that is too convoluted to be useful for an ordinary agent.

10. See pg. 3

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