Normalisation for Ordinary Agents: A Solution to the Problem of Logical Omniscience

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I. Introduction

Bayesianism, hereafter the Bayesian model, is the interpretation of probability as a quantification of degrees of belief. Because probabilities must adhere to the Kolmogorov axioms, it follows from the axiom of normalisation that a rational agent must believe all logical truths, even those unknown. This impossible demand on rationality is the problem of logical omniscience.

This paper will discuss the problem and demonstrate that its conclusion is problematic because it compels us to abandon the Bayesian model, despite the lack of equally valuable alternatives. I will then advance a pragmatic solution to the problem by relaxing the axiom of normalisation, insofar as a rational agent must assign a maximum degree of belief to logical truths only if they can be known. This solution is most ideal because it preserves the utility of the model and avoids complications that render it impractical.

II. The Problem of Logical Omniscience

In the Bayesian model, an agent is rational only if their degrees of belief in propositions, hereafter credences, satisfy the mathematical conditions of a probability function.[1] These conditions are the Kolmogorov axioms which state that, for any set of propositions S, the credence function[2] $Cr : S \to \mathbb{R}$ must satisfy the following:

Axiom 1. *(Non-negativity) For any proposition $A \in S$, one has $Cr(A) \geq 0$.*

Axiom 2. *(Normalisation) For any logical truth $T \in S$, one has $Cr(T) = 1$.*

Axiom 3. *(Finite Additivity) For any propositions $A_1, A_2, \ldots, A_n \in S$ such that for integers $j, k \in [1, n]$ where $j \neq k$ and $Cr(A_j \wedge A_k) = 0$,[3] one has $Cr(\bigvee_{i=1}^{n} A_i) = \sum_{i=1}^{n} Cr(A_i)$.*

1. Wolfgang Schwarz, *Belief, Desire, and Rational Choice*, https://github.com/wo/bdrc/blob/master/bdrc.pdf, December 20, 2022, 13.
2. The credence function Cr assigns propositions in a set S to some real number.
3. In terms of the content taught in the module, this is a partition of propositions on a set S.

What if S includes propositions whose truth we have yet to ascertain? Unfortunately, if we are to fully abide by the axioms, the axiom of normalisation demands that an ordinary agent must assign maximum credence to the truth of such unknown propositions to be deemed rational. But it is clearly unreasonable to qualify rationality on knowledge of truths that ordinary agents cannot prove conclusively. Consider the case of Fermat's Last Theorem, which went unsolved for over three hundred years.[4] Is it fair to consider someone in the past to be irrational if they did not assign maximum credence in the truth of the theorem?[5] This clearly impossible demand is the problem of logical omniscience, which may be formalised as:[6]

P1: No ordinary agent can have credences in an infinite number of propositions.

P2: There are an infinite number of logical truths.

P3: If the Bayesian model is applicable to ordinary agents, then any ordinary agent must have maximum credence in any logical truth.

P4: No ordinary agent can have credences in all logical truths.

C: The Bayesian model is inapplicable to ordinary agents.

I will demonstrate that the above argument is sound. Suppose there exists an ordinary agent who knows an infinite number of propositions, which means that they have enumerated through an infinite set of propositions and assigned a credence to each proposition within the set. However, it is nigh impossible for an ordinary agent to begin constructing this set, much less to enumerate through it, because time is finite for an ordinary agent. Hence **P1** must be true.

Next, suppose we let the greatest number of logical truths be a finite number n, so the set of logical truths is $T = \{T_1, T_2, ..., T_n\}$. For any i with $1 \leq i \leq n$, we know $T_i \in T$ is true, so the

4. The theorem states that for any $n \in \mathbb{Z}_{\geq 2}$, the equation $a^n + b^n = c^n$ has no positive integer solutions. This theorem seems innocuous, but its proof required extensive use of mathematics developed after Fermat's time, which would be inaccessible to an ordinary agent.

5. Sinan Dogramaci, "Solving the Problem of Logical Non-Omniscience," *Philosophical Issues* 28, no. 1 (2018): 108, https://doi.org/10.1111/phis.12118.

6. Carneades.org, "The Problem of Logical Omniscience (Bayesian Epistemology)," YouTube video, 6:24, February 22, 2015, youtu.be/pkGFSkGzg4M.

tautology $(T_i \vee \neg T_i)$ is also a logical truth. This additional logical truth means that there are at least $n+1$ logical truths, which contradicts the definition of n as the greatest number of logical truths. Thus **P2** is true since there is no upper bound on the number of logical truths.

Consequently, **P3** restates the axiom of normalisation. **P4** also follows naturally: if there are an infinite number of logical truths (**P1**) and no ordinary agent can have credences in an infinite number of propositions (**P2**), then **P4** must be true. Hence, the conclusion is obtained by *modus tollens* of **P4** on **P3**.

The problem of this conclusion is that we are compelled to discard a valuable model of rationality on the basis of a technicality, even though we lack viable alternatives. The conclusion that the model is inapplicable to any ordinary agent implies that it is pointless to evaluate rationality on the criteria stipulated by the Bayesian model. However, this cannot be the case, because even with its shortcomings, the Bayesian model is generally useful, widely-applicable and valuable for its balance between rigour and simplicity. It literally pays to abide by the model, since we can show by way of a simple Dutch Book argument that if an agent has credences that violate the axioms, then they may be duped into buying bets which lead to a guaranteed loss.[7] Hence, abandoning the model just because ordinary agents never attain the tall order of logical omniscience forgoes well-grounded criteria for rationality in favour of either a more laissez-faire approach in defining rationality, or the sweeping generalisation that any ordinary agent is irrational. The latter is intuitively repulsive, and the former cannot be the right approach, since a subjective criteria of rationality is not generalisable to all agents and hence holds little purpose as universal standard of rationality. Therefore, it is preferable to preserve the utility of the model by attempting to reconcile it with the problem of logical omniscience, rather than give it up while we lack a better substitute.

7. Schwarz, *Belief, Desire, and Rational Choice*, 44.

III. Reformulating Normalisation

To account for the shortcomings of the Bayesian model, we could accept that the model merely prescribes how a logically omniscient agent must assign their credences. In order to apply the model to ordinary agents, the axioms must be duly amended to reflect how ordinary agents actually behave. Since the axiom of normalisation is the critical flaw that leads to the problem, we could reformulate it to:

Axiom 4. *(Weak Normalisation, WN) For any* ***known*** *logical truth* $T \in S$*, one has* $Cr(T) = 1$.

This reformulation implies that an ordinary agent ought to have maximum credence in T, only if they know that it is a logical truth.[8] An ordinary agent knows T, if they can justify its truth or non-falsity. Consequently, to qualify as rational, this agent must assign maximum credence to the truth of T.

This reformulation is most desirable because it renders truths that are unknown to an ordinary agent as irrelevant to considerations of rationality, so it eliminates the demand of logical omniscience. Furthermore, it is coherent with the demands on rationality prescribed by the Bayesian model, as it is reasonable to expect that a rational agent can justify their credences. Therefore, applying WN to the example of Fermat's Last Theorem, an ordinary agent in the past remains rational for having a non-maximum credence in its truth, because prior to its proof it would have been impossible for anyone to produce a valid reason for their credence in it. However, with the proof of the theorem, even if a rational agent cannot reproduce the proof, they may ground their credence in it on the rigorous review that the proof underwent, so they must assign maximum credence in its truth. Hence, this constitutes a more pragmatic and reflective approach to modelling the credences of ordinary agents.

8. Hanti Lin, "Bayesian Epistemology," in *The Stanford Encyclopedia of Philosophy*, Fall 2022, ed. Edward N. Zalta and Uri Nodelman (Metaphysics Research Lab, Stanford University, 2022), https://plato.stanford.edu/archives/fall2022/entries/epistemology-bayesian/.

A potential challenge to this approach is that invoking an arbitrary criteria to "know" a logical truth would taint the otherwise objective Bayesian model with subjectivity. Agents may utilise different criteria to determine what counts as knowledge, so privileging the definition of knowledge as justified true belief may be seen as arbitrary. Since the model is grounded in a potentially contentious perspective of knowledge, this objection rejects the view that the model is widely applicable and hence implicitly undermines its utility. However, I contend that reformulating the axioms in this manner does not demand universal agreement on a definition of knowledge. It is beyond the scope of the Bayesian model to account for this underlying epistemological challenge, as the model only aims to provide a principled framework for rational credence assignment. The WM formulation of the axiom accommodates for the diversity of perspectives in defining knowledge by limiting its demands to conditions that rational agents ought to fulfil, so it remains agnostic on a specific definition of knowledge. Thus, subjectivity is not introduced by the reformulation and we preserve the value of the model in a sound manner.

In reformulating the axioms, the natural consideration that follows would be to determine the limits of further reformulations. I propose that if we wish to prevent complications from arising, this limit must be the WN formulation, because going further would completely destabilise the mathematical foundations of the conventional system of probability based on the axioms. By relaxing the axioms to compensate for logical non-omniscience, we already lose some applications to probability that make the Bayesian model appealing.[9] Excessive reformulations are undesirable as they yield contradictory or nonsensical results, especially for applications which hinge upon the precision of the axioms.

To illustrate this view, suppose that credences need not be single real values, but are a real interval. This is plausible, because ordinary agents may account for a range of uncertainty or indifference that is more faithful to their epistemic position. Therefore, if a rational agent knows a

9. Michael G. Titelbaum, "Old Evidence and Logical Omniscience," in *Fundamentals of Bayesian Epistemology 2: Arguments, Challenges, Alternatives* (Oxford University Press, 2022), 436.

logical truth, their credence ought to be an interval that includes the number one.[10] We may thus reformulate the axiom of normalisation to:

Axiom 5. *(Very Weak Normalisation, VWN) For any known logical truth $T \in S$, there exists c with $0 \leq c \leq 1$ such that $c \leq Cr(T) \leq 1$.*

However, this causes some useful results of probability to become undone. Consider the case of conditional credences. Intuitively, this represents an agent's credence that a proposition A is true, supposing that B is true.[11] This is defined as:

Definition 1. *For propositions $A, B \in S$ with $Cr(B) > 0$, one has $Cr(A|B) = \frac{Cr(A \wedge B)}{Cr(B)}$.*

If we take B to be a known logical truth, by WN we have that $Cr(B) = 1$, so $Cr(A|B) = Cr(A \wedge B)$. Also, the truth of the proposition $(A \wedge B)$ depends solely on A as B is logically true, so $Cr(A \wedge B) = Cr(A)$. Thus we obtain a well-defined conditional credence which is $Cr(A|B) = Cr(A)$. However, in VWN, $Cr(B)$ may be a real interval, so considering conditional credences becomes mathematically absurd, for it is impossible to divide $Cr(A \wedge B)$ over an interval. This already undermines the case for weakening the axioms in this manner.

Perhaps we still accept that conditional credences as intervals are meaningful, so we compromise on mathematical rigour. We may pursue the logic of WN to let $Cr(A \wedge B) = a$ for some $0 \leq a \leq 1$, and divide $Cr(A)$ over the bounds of $Cr(B)$ to obtain $Cr(A|B) = [a, \frac{a}{c}]$ for some $0 \leq c \leq 1$. However, if we let $a = 1$ and $c \leq 1$, we get a conditional credence that may be greater than the maximum credence in a logical truth. This result violates our logical intuitions, because it suggests that in certain situations, there exist propositions that we may believe in more than logical truths. It is inconceivable, if not outright impossible, to have "more" belief in a proposition than *'A triangle has three angles'*. Therefore, VWN is too naïve, thus it cannot be the right approach to account for logical non-omniscience.

10. Lin, "Bayesian Epistemology."
11. Schwarz, *Belief, Desire, and Rational Choice*, 32.

One could contend that the above examples are unfair because there could be methods that reconcile VWN with a consistent and improved Bayesian model. However, introducing further mathematical complexity causes the model to become convoluted and hence lose its intuitive appeal. This appeal comes from the balance between rigour in its mathematical foundations and its simplicity.[12] Therefore, adding layers of sophistication to the model infringes upon its simplicity by implicitly imposing more conditions on rationality. This result is not much different from the impossible demands on rationality posed by the problem of logical omniscience, since a mathematically-refined Bayesian model could demand so much calculation from an ordinary agent that it may be impossible to ever determine if they are truly rational. Those advocating this solution will find themselves back where they started as a consequence of the unnecessary mathematical complexity that they have introduced. Thus, extreme reformulations are not a pragmatic solution to the problem of logical omniscience, so the best solution is to adopt a moderate approach espoused by the WN formulation.

IV. Conclusion

This paper has demonstrated that the best solution to the problem of logical omniscience is to adopt the WN formulation of normalisation, which limits the assignment of maximum credence only to logical truths that ordinary agents can know. This is the only possible compromise to preserve the utility of the Bayesian model, because adopting weaker formulations like WVN destabilises the conventional system of probability, while also causing new impossible demands on rationality to reappear as a model that is too convoluted to be useful for an ordinary agent.

12. See page 3.

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