Study of a stochastic differential equation

Equation

$$\begin{cases} dX(t) = (\frac{3}{2}X(t) - X^2(t))dt + X(t)dB(t), \\ X(0) = 1. \end{cases}$$

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Stochastic differential equations

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Contents

Chapter 1

Analytical part

1.1 Testing the assumptions of the existence theorem and the unambiguity of SRR solutions

In general, we consider a stochastic differential equation (SDE) of the form

$$\begin{cases} dX(t) = a(t, X(t))dt + b(t, X(t))dB(t), & t \in [0, T], \\ X(0) = x_0, \end{cases} \tag{1.1}$$

so we are able to isolate the functions

$$\begin{aligned} &a : [0, T] \times \mathbb{R} \to \mathbb{R}, \quad \forall_{t \in [0,T]} \forall_{x \in \mathbb{R}} \quad a(t, x) = \frac{3}{2}x - x^2, \\ &b : [0, T] \times \mathbb{R} \to \mathbb{R}, \quad \forall_{t \in [0,T]} \forall_{x \in \mathbb{R}} \quad b(t, x) = x. \end{aligned}$$

In general, this equation is a nonlinear stochastic differential equation of the form

$$dX(t) = rX(t)(K - X(t))dt + \beta X(t)dB(t), \quad X(0) = x_0 > 0, \tag{1.2}$$

where $r = 1$, $K = \frac{3}{2}$ and $\beta = 1$. This form of the equation is often used to model population growth of size $X(t)$ in a crowded environment of a stochastic nature. The constant K is called the carrying capacity of the environment, the constant r a measure of the quality of the environment, and the constant *beta* a measure of the amount of noise in the system. [2]

Moreover, there is a general form of SDE for expressions with a polynomial form of the deterministic part, i.e.:

$$dX(t) = (aX^n(t) + bX(t))dt + cX(t)dB(t), \tag{1.3}$$

where for $n = 2$ it's called a Verhulst equation.

We will first examine the assumptions of the existence theorem and the unambiguity of SDE solutions

Theorem 1 *[1] Let $(B_t, \mathcal{F}_t)_{t \geq 0}$ be the specified Brownian motion process and let the stochastic differential equation as above be given. In addition, we assume that* *the components $a(t,x)$, $b(t,x)$ of this stochastic differential equation are measurable functions and will satisfy the Lipschitz condition*

$$\exists_{L\in\mathbb{R}}\,\forall_{x,y\in\mathbb{R}}\,\forall_{t\in[0,T]}\quad |f(t,x)-f(t,y)|+|g(t,x)-g(t,y)|\le L|x-y| \tag{1.4}$$

for a certain constant L and that these components satisfy the condition of linear growth, i.e.:

$$\exists K\in\mathbb{R}\,\forall_{x,y\in\mathbb{R}}\,\forall_{t\in[0,T]}\quad |f(t,x)+g(t,x)|\le K(1+|x|). \tag{1.5}$$

Then, for any initial condition $\mathcal{F}_{\prime}$-measurable x_0, there is an unambiguous solution $(X(t))_{t\ge0}$ of the equation (1.1).

Indeed, the functions a and b are smooth and L^2-measurable functions on $[0,T]\times\mathbb{R}$. Checking the Lipschitz condition, we have:

$$\begin{aligned}&|a(t,x)-a(t,y)|+|b(t,x)-b(t,y)|=|\frac{3}{2}x-x^2-\frac{3}{2}y-y^2|+|x-y|=\\&=|\frac{3}{2}(x-y)-(x^2-y^2)|+|x-y|=|x-y||\frac{3}{2}-(x+y)|+|x-y|=\\&=|x-y||\frac{5}{2}-(x+y)|\end{aligned}$$

Since the inequality is to be satisfied for any pair $x,y\in\mathbb{R}$ then we will not find such a constant L. We can transform this inequality so that it actually depends on the factor $|x^2-y^2|$ ie.

$$\begin{aligned}&|\frac{3}{2}x-x^2-\frac{3}{2}y-y^2|+|x-y|=\\&=|\frac{3}{2}(x-y)-(x^2-y^2)|+|x-y|\le\frac{3}{2}|x-y|+|x^2-y^2|+|x-y|=\\&=\frac{5}{2}|x-y|+|x^2-y^2|.\end{aligned}$$

Note that this factor is actually a check for the Lipschitz condition of the function $f(x)=x^2$, which is not globally Lipschitzian, but is locally Lipschitzian if you take some subset of the $f:[-\hat{L},\hat{L}]\to\mathbb{R}$ then

$$|x^2-y^2|\le 2\hat{L}|x-y|.$$

There is a theorem (18.16 Theorem, Schilling [1]) which proves that for the existence of an unambiguous solution it is sufficient to satisfy the Lipschitz condition locally, but still the linear increment condition is required, so let's check it:

$$|a(t,x)+b(t,x)|=|\frac{3}{2}x-x^2|+|x|=|x||\frac{3}{2}-x|+|x|=|x|(|\frac{3}{2}-x|+1)\le|x|(\frac{5}{2}+|x|).$$

So we see that this condition is also not satisfied globally (here we do not allow for the possibility of local) which indicates the explosion of the solution in the deterministic part.

So we will try to make the appropriate substitution and bring our equation to linear form. The linear form of the SDE is as follows

$$dX(t) = (\alpha(t) + \beta(t)X(t))dt + (\gamma(t) + \delta(t)X(t))dB(t), \tag{1.6}$$

where $\alpha, \beta, \gamma, \delta : [0, \infty) \to \mathbb{R}$ Are non-random coefficients. Suppose $(X(t))_{t\geq 0}$ is the solution of some SDE and let us denote by $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ A function that is monotonic, has continuous derivatives $f_t(t, x), f_x(t, x), f_{xx}(t, x)$ and there is its inverse function $g(t, \cdot) := f^{-1}(t, \cdot)$ such as $f(t, g(t, x)) = g(t, f(t, x)) = x$.

We then define the process $Z(t) := f(t, X(t))$ so that the transformed stochastic differential equation

$$dZ(t) = \hat{a}(t, Z(t))dt + \hat{b}(t, Z(t))dB(t) \tag{1.7}$$

had an unambiguous solution, from which we will then determine the primary solution$X(t) = g(t, Z(t))$. In this situation, the new coefficients $\hat{a}$, $\hat{b}$ are respectively:

$$\hat{a}(t, Z(t)) = f_t(t, X(t)) + f_x(t, X(t))a(t, X(t)) + \frac{1}{2}f_{xx}(t, X(t))b^2(t, X(t)), \tag{1.8}$$
$$\hat{b}(t, Z(t)) = f_x(t, X(t))b(t, X(t)).$$

The following theorem occurs

Theorem 2 *[1] Let $(B(t))_{t\geq 0}$ will be a Brownian motion process. The autonomous stochastic differential equation (form (1.1) for $a(t, X(t)) = a(X(t))$ and $b(t, X(t)) = b(X(t))$) can be converted to the linear form of SDE if and only if.*

$$\frac{d}{dx}\left(\frac{\frac{d}{dx}(\kappa'(x)\sigma(x))}{\kappa'(x)}\right) = 0, \tag{1.9}$$

where $\kappa(x) = \frac{b(x)}{\sigma(x)} - \frac{1}{2}\sigma'(x)$.

In our case $b \equiv a$ and $\sigma \equiv b$ so

$$\kappa(x) = \frac{a(x)}{b(x)} - \frac{1}{2}b = 1 - x,$$
$$\frac{d}{dx}\left(\frac{\frac{d}{dx}(\kappa'(x)b(x))}{\kappa'(x)}\right) = \frac{d}{dx}\left(\frac{\frac{d}{dx}(-1 \cdot x)}{-1}\right) = \frac{d}{dx}\left(\frac{-1}{-1}\right) = 0,$$

so we conclude that our equation is reducible to linear form. Going further, as stated in [1], let

$$d(x) = \int_{x_0}^{x} \frac{dy}{b(y)} = \int_{1}^{x} \frac{dy}{y} = \ln x$$

and

$$\delta = -\frac{[\frac{d}{dx}(\kappa'(x)b(x))]}{\kappa'(x)} = -1.$$

Then the transformation $f(X(t)) = Z(t)$ (we consider the autonomous case, independent of time t) is given as

$$f(x) = \begin{cases} e^{\delta d(x)} : & \delta \neq 0, \\ \gamma d(x) : & \delta = 0 \end{cases} = e^{-\ln x} = \frac{1}{x}. \tag{1.10}$$

Note that the function f is monotonic, having continuous derivatives

$$f_t(t,x) = 0,$$
$$f_x(t,x) = -\frac{1}{x^2},$$
$$f_{xx}(t,x) = \frac{2}{x^3}$$

and the inverse function $g(t,y) = \frac{1}{y}$ such as $g(t, f(x)) = x$ and $f(t, g(y)) = y$ That is, it meets the assumptions made about it. Returning to the new coefficients $\hat{a}$, $\hat{b}$ (1.8) we get

$$\hat{a}(t,Z(t)) = 0 + (-\frac{1}{X^2(t)} \cdot (\frac{3}{2}X(t) - X^2(t))) + \frac{1}{2}\frac{2}{X^3(t)}X^2(t) = 1 - \frac{1}{2}\frac{1}{X(t)} = 1 - \frac{1}{2}Z(t),$$
$$\hat{b}(t,Z(t)) = -\frac{1}{X^2(t)}X(t) = -\frac{1}{X(t)} = -Z(t).$$

Ultimately, we get an initial SDE of the form

$$\begin{cases} dX(t) = (\frac{3}{2}X(t) - X^2(t))dt + X(t)dB(t), \\ X(0) = 1 \end{cases}$$

transformed to a linear SDE of the form

$$\begin{cases} dZ(t) = (1 - \frac{1}{2}Z(t))dt - Z(t)dB(t), \\ Z(0) = 1. \end{cases} \tag{1.11}$$

Checking again the conditions of the theorem of existence and unambiguity of solutions of SRR of linear SDE (1.11) we get that

- functions $\hat{a}, \hat{b}$ specified above are smooth and L^2-measurable at $[0,T] \times \mathbb{R}$,
- satisfy the condition of linear growth i.e..:
$$|1 - \frac{1}{2}x + (-x)| = |1 - \frac{3}{2}x| \leq 1 + |-\frac{3}{2}x| =$$
$$= 1 + \frac{3}{2}|x| \leq 2 \cdot (1 + |x|)$$
so there exists a constant $K = 2$ that the inequality (1.5) is satisfied for all $x \in \mathbb{R}$ and $t \in [0,T]$,
- globally satisfy the Lipschitz condition i.e..:
$$|1-\frac{1}{2}x-1+\frac{1}{2}y|+|-x+y| = |-\frac{1}{2}||(x-y)|+|-1||x-y| = \frac{1}{2}|x-y|+|x-y| \leq 2|x-y|$$
so there exists a constant $L = 2$ that the inequality (1.4) is satisfied for all $x, y \in \mathbb{R}$ and $t \in [0,T]$,
- Z(0) = 1 is measurable and independent of $(B(s) : 0 \leq s \leq T)$, $\mathbb{E}Z^2(0) = 1 < \infty$.

We conclude that there is an unambiguous solution of the linear SDE based on which we obtain the solution of the initially considered stochastic differential equation.

1.2 Determination of functions of consecutive moments $\mathbb{E}X(t)$, $\mathbb{E}X^2(t)$, $VarX(t)$, $EX^3(t)$

Since we are considering the case of a nonlinear equation reduced to a linear equation, we will determine the functions of consecutive moments for $f(X(t)) = Z(t)$. We have shown that there is an unambiguous solution so we will impose an expectation value on it. We have:

$$\begin{aligned}
&dZ(t) = (1 - \frac{1}{2}Z(t))dt - Z(t)dB(t),\\
&Z(t) = Z(0) + \int_0^t (1 - \frac{1}{2}Z(s))ds + \int_0^t -Z(s)dB(s),\\
&\mathbb{E}Z(t) = \mathbb{E}[Z(0) + \int_0^t (1 - \frac{1}{2}Z(s))ds + \int_0^t -Z(s)dB(s)].
\end{aligned}$$

Then, from Fubini's theorem and the property of the integral of Itô and applying the differential, we get the following

$$\begin{aligned}
&\mathbb{E}Z(t) = 1 + \int_0^t (1 - \frac{1}{2}\mathbb{E}Z(s))ds + 0 = 1 + \int_0^t (1 - \frac{1}{2}\mathbb{E}Z(s))ds,\\
&d(\mathbb{E}Z(t)) = (1 - \frac{1}{2}\mathbb{E}Z(t))dt,\\
&dm_1(t) = (1 - \frac{1}{2}m_1(t))dt,\\
&m_1'(t) + \frac{1}{2}m_1(t) = 1.
\end{aligned}$$

So we have a first-order inhomogeneous linear equation, which we will solve by the integral factor $e^{\int \frac{1}{2}dt} = e^{\frac{1}{2}t}$:

$$\begin{aligned}
&m_1'(t) + \frac{1}{2}m_1(t) = 1,\\
&(e^{\frac{1}{2}t}m_1(t))' = e^{\frac{1}{2}t},\\
&e^{\frac{1}{2}t}m_1(t) = 2e^{\frac{1}{2}t} + C,\\
&m_1(t) = Ce^{-\frac{1}{2}t} + 2.
\end{aligned}$$

Using the initial condition

$$\begin{aligned}
&m_1(0) = \mathbb{E}Z(0) = 1 \ \wedge \ m_1(0) = C + 2 \Rightarrow C = -1,\\
&m_1(t) = -e^{-\frac{1}{2}t} + 2
\end{aligned}$$

we get that the function of the first moment is given by the formula

$$\mathbb{E}Z(t) = -e^{-\frac{1}{2}t} + 2. \tag{1.12}$$

We proceed to determine the second moment. From [[5]] we know that

$$Z^2(t) = Z^2(0) + \int_0^t -Z^2(s) + 2Z(s) + (-Z(s))^2 ds + 2\int_0^t (-Z^2(s))dB(s)$$

where again from Fubini's theorem and the properties of the integral Itô we have

$$\begin{aligned}
\mathbb{E}Z^2(t) &= Z^2(0) + \int_0^t \mathbb{E}[-Z^2(s) + 2Z(s) + (-Z(s))^2]ds + 0,\\
m_2(t) &= Z^2(0) + \int_0^t \mathbb{E}[2Z(s)]ds,\\
m_2(t) &= Z^2(0) + \int_0^t 2m_1(s)ds,\\
m_2(t) &= 1 + \int_0^t 2(-e^{-\frac{1}{2}t} + 2)ds,\\
m_2(t) &= 1 + \int_0^t (4 - 2e^{-\frac{1}{2}t})ds,\\
m_2(t) &= 4(t + e^{-\frac{1}{2}t}) - 3.
\end{aligned}$$

Thus, the second moment function is expressed as

$$\mathbb{E}Z^2(t) = 4(t + e^{-\frac{1}{2}t}) - 3. \tag{1.13}$$

Going forward, the variance is given by the formula $VarX(t) = \mathbb{E}X^2(t) - (\mathbb{E}X(t))^2$ so using (1.12) and (1.14) in our case it is

$$\begin{aligned}
VarZ(t) &= 4(t + e^{-\frac{1}{2}t}) - 3 - (-e^{-\frac{1}{2}t} + 2)^2 =\\
&= 4t + 4e^{-\frac{1}{2}t} - 3 - (e^{-t} - 4e^{-\frac{1}{2}t} + 4) =\\
&= 4t - e^{-t} + 8e^{-\frac{1}{2}t} - 7
\end{aligned}$$

To calculate the third moment, we will introduce the function $F(t,x) = x^3$ with the partial derivatives

$$\begin{aligned}
F_t(t,x) &= 0,\\
F_x(t,x) &= 3x^2,\\
F_{xx}(t,x) &= 6x.
\end{aligned}$$

We use the Itô formula for this function so

$$\begin{aligned}
d(F(t,Z(t)) &= 3Z^2(t)dZ(t) + \frac{1}{2}6Z(t)d[Z,Z](t) =\\
&= 3Z^2(t)[(1 - \frac{1}{2}Z(t))dt - Z(t)dB(t)] + 3Z(t)(-Z(t))^2dt =\\
&= (3Z^2(t) + \frac{3}{2}Z^3(t))dt - 3Z^3(t)dB(t).
\end{aligned}$$

Superimposing the expected value on this, we get the following

$$\begin{aligned}
\mathbb{E}(dZ^3(t)) &= [3\mathbb{E}Z^2(t) + \frac{3}{2}\mathbb{E}Z^3(t)]dt,\\
m_3'(t) &= 3m_2(t) + \frac{3}{2}m_3(t),\\
m_3'(t) - \frac{3}{2}m_3(t) &= 3(4(t + e^{-\frac{1}{2}t}) - 3).
\end{aligned}$$

Once again using the integral factor $e^{\int -\frac{3}{2}dt} = e^{-\frac{3}{2}t}$ we have

$$\begin{aligned}
m_3'(t) - \frac{3}{2}m_3(t) &= 12t + 12e^{-\frac{1}{2}t} - 9,\\
(e^{-\frac{3}{2}t}m_3(t))' &= 12te^{-\frac{3}{2}t} + 12e^{-2t} - 9e^{-\frac{3}{2}t},\\
e^{-\frac{3}{2}t}m_3(t) &= -\frac{2}{3}e^{-\frac{3}{2}t}(12t-1) - 6e^{-2t} + E,\\
m_3(t) &= Ee^{\frac{3}{2}t} - \frac{2}{3}(12t-1) - 6e^{-\frac{1}{2}t}.
\end{aligned}$$

Using the initial condition

$$\begin{aligned}
m_3(0) &= EZ^3(0) = 1 \ \wedge \ m_3(0) = E + \frac{2}{3} - 6 \Rightarrow E = \frac{19}{3},\\
m_3(t) &= \frac{19}{3}e^{\frac{3}{2}t} - \frac{2}{3}(12t-1) - 6e^{-\frac{1}{2}t}.
\end{aligned}$$

Thus, the third moment function is expressed by

$$\mathbb{E}Z^3(t) = \frac{19}{3}e^{\frac{3}{2}t} - \frac{2}{3}(12t-1) - 6e^{-\frac{1}{2}t}. \tag{1.14}$$

1.3 Determining the solution analytically

In order to determine the solution of our SDE, we want to find a stochastic process $Y(t)$ such that.

$$d(Z(t)Y(t)) = Y(t)dZ(t) - Y(t)Z(t)(-\frac{1}{2}dt - dB(t)) = Y(t)dt. \tag{1.15}$$

Suppose that $Y(t)$ is an Itô process such that.

$$dY(t) = f_1(t, Y(t))dt + f_2(t, Y(t))dB(t), \quad Y(0) = 1.$$

Let's apply the Itô formual for $d(Z(t)Y(t))$:

$$\begin{aligned}
d(Z(t)Y(t)) &= Y(t)dZ(t) + Z(t)dY(t) + dZ(t)dY(t) = \qquad (1.16)\\
&= Y(t)dZ(t) + Z(t)(f_1(t, Y(t))dt + f_2(t, Y(t))dB(t)) + (-Z(t))f_2(t, Y(t))dt.
\end{aligned}$$

We get

$$\begin{aligned}
&-Y(t)Z(t)(-\frac{1}{2}dt - dB(t)) = Z(t)(f_1(t, Y(t))dt + f_2(t, Y(t))dB(t)) + (-Z(t))f_2(t, Y(t))dt,\\
&\begin{cases} Y(t)Z(t)dB(t) = Z(t)f_2(t, Y(t))dB(t),\\ \frac{1}{2}Y(t)Z(t)dt = (Z(t)f_1(t, Y(t)) - Z(t)f_2(t, Y(t)))dt \end{cases}
\end{aligned}$$

and infer that $f_2(t, Y(t)) = Y(t)$ which results in

$$\begin{aligned}
\frac{1}{2}Y(t)Z(t)dt &= (Z(t)f_1(t, Y(t)) - Z(t)Y(t))dt,\\
Y(t)(\frac{1}{2}Z(t) + Z(t))dt &= Z(t)f_1(t, Y(t))dt,\\
f_1(t, Y(t)) &= \frac{3}{2}Y(t).
\end{aligned}$$

so the process $Y(t)$ satisfies

$$dY(t) = \frac{3}{2}Y(t)dt + Y(t)dB(t), \quad Y(0) = 1$$

Multiplying by the factor $Y^{-1}(t)$ we get

$$Y^{-1}(t)dY(t) = \frac{3}{2}dt + dB(t)$$

from where

$$d\ln Y(t) = \frac{3}{2}dt + dB(t).$$

Let's put $f(Y(t)) = \ln(Y(t))$ and use the Itô formula

$$\begin{aligned}
df &= f'(Y(t))dY(t) + \frac{1}{2}f''(Y(t))(dY(t))^2 = \\
&= \frac{1}{Y(t)}(\frac{3}{2}Y(t)dt + Y(t)dB(t)) - \frac{1}{2}\frac{1}{Y^2(t)}(Y^2(t)dt) = \\
&= \frac{3}{2}dt + dB(t) - \frac{1}{2}dt = dt + dB(t)
\end{aligned}$$

so

$$\begin{aligned}
d\ln Y(t) &= dt + dB(t), \\
\ln Y(t) &= \int_0^t dt + \int_0^t dB(t), \\
Y(t) &= \exp(t + B(t)).
\end{aligned}$$

Going next

$$\begin{aligned}
&d(Y(t)X(t)) = Y(t)dt, \\
&Y(t)Z(t) = Z(0) + \int_0^t Y(s)ds, \\
&Z(t) = Y^{-1}(t) + Y^{-1}(t)\int_0^t Y(s)ds, \\
&Z(t) = \exp(-t - B(t)) \cdot (1 + \int_0^t \exp(s + B(s))ds).
\end{aligned}$$

Thus, the analytical solution of our linear SDE is of the form

$$Z(t) = \exp(-t - B(t)) \cdot (1 + \int_0^t \exp(s + B(s))ds). \tag{1.17}$$

Returning to the nonlinear form of the SDE, recall that we can obtain the primary solution by $X(t) = g(t, Z(t)) = \frac{1}{Z(t)} = Z^{-1}(t)$. We have

$$X(t) = [\exp(t + B(t)) \cdot (1 + \int_0^t \exp(s + B(s))ds)]^{-1} =$$
$$= \frac{\exp(t + B(t))}{(1 + \int_0^t \exp(s + B(s))ds)}$$

Note that our original solution has a form similar to the distributant of a logistic distribution shifted by an exponent depending on time with the addition of noise in the form of a Brownian motion process. The solution of the transformed linear system, on the other hand, is close to the distributions of the exponential family of distributions also with some exponential time-dependent shift with the addition of random noise.

Moreover, both solutions (the original and the linear form) are always positive for any $tgq0$ due to the exponential function present.

Reviewing the literature [[2]], we find a reference to the form of the equation mentioned at the beginning with parameters r, K and β. One of the recommended tasks is to show that the solution of this equation is

$$X_t = \frac{\exp((rK - \frac{1}{2}\beta^2)t + \beta B_t)}{x_0^{-1} + r\int_0^t \exp((rK - \frac{1}{2}\beta^2)s + \beta B_s)ds}.$$

Note that if we substitute the parameters from our original equation $r = 1$, $K = Frac32$ and $\beta = 1$ then we get

$$X_t = \frac{\exp((1 \cdot \frac{3}{2} - \frac{1}{2} \cdot 1^2)t + 1 \cdot B_t)}{1 + 1 \cdot \int_0^t \exp((1 \cdot \frac{3}{2} - \frac{1}{2} \cdot 1^2)s + \cdot B_s)ds} =$$
$$= \frac{\exp(t + B_t)}{1 + \int_0^t \exp(s + B_s)ds}$$

Which is exactly the primary solution we derived. This proves the validity of the solution, the selection of appropriate methods and the correct calculations.

Chapter 2

Simulation part

2.1 Analyzed methods

Again, we consider a stochastic differential equation of the form Itô.

$$dX(t) = a(t, X(t))dt + b(t, X(t))dB(t).$$

As we showed in the analytical section, the form of the linear transformation from the original form of the SDE is of the form

$$dZ(t) = (1 - \frac{1}{2}Z(t))dt - Z(t)dB(t)$$

with initial condition $Z(0) = 1$. The analytical solution, on the other hand, is given by the equation

$$Z(t) = \exp(-t - B(t)) \cdot (1 + \int_0^t \exp(s + B(s))ds).$$

Thus, in the simulation part, we will consider the functions

$$a(t, x) = 1 - \frac{1}{2}x,$$

$$b(t, x) = -1.$$

We will use two approximation methods, the first of which is the Euler-Maruyama method, in which we assume that there is a stochastic differential equation Itô

$$dX(t) = a(t, X(t))dt + b(t, X(t))dB(t)$$

and satisfies the linear growth condition, the Lipschitz condition, and the

$$\forall_{t_1,t_2 \in [0,T]} \; |a(t_1, x) - a(t_2, x)| + (b(t_1, x) - b(t_2, x)| \leq K|t_1 - t_2|^{\frac{1}{2}}.$$

The SDE we are considering, as shown in the analytical section, satisfies these conditions therefore the Euler-Maruyama method converges and approximates the trajectory $X(t)$ at fixed points t_i, $X(t_i) \approx X_i$ on the interval $[0, T]$ based on the formula

$$X_{i+1} = X_i + a(t_i, X_i)dt + b(t_i, X_i)dB_i$$

for $i \in \{0, ..., N - 1\}$ where

1. $0 = t_0 < t_1 < ... < t_{k-1} < t_k = T$,
2. $dt = t_{i+1} - t_i = T/N$,
3. $dB_i = B(t_{i+1}) - B(t_i) = \sqrt{dt}\eta_i$,
4. $\eta_i \sim \mathcal{N}(0, 1)$ i.i.d.,
5. $X_0 = X(0)$.

Another method is the Milstein method, in which for determinations as above we have approximations through:

$$X_{i+1} = X_i + a(t_i, X_i)dt + b(t_i, X_i)dB_i + \frac{1}{2}b(t_i, X_i)\frac{\partial b}{\partial x}(t_i, X_i)((dB_i)^2 - dt).$$

2.2 Presentation of sample process trajectories

We will first present some example process trajectories for a certain set of selected ω by selecting the appropriate grain. These realizations are as follows:

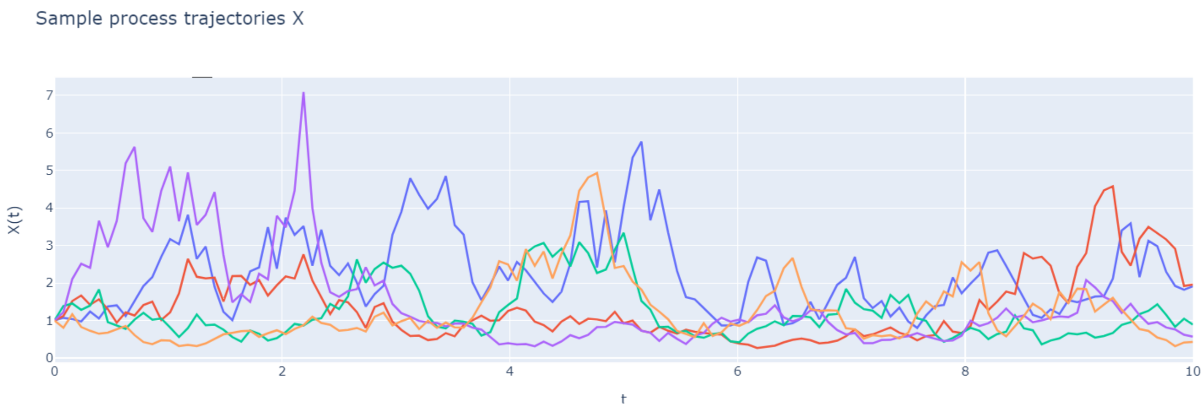

Figure 2.1: Examples of process trajectories $X(t)$.

In the graph above, we can see the stochastic nature of the process and how diverse values it can take. Interestingly, the process does not take non-negative values due to the exponential function present. Certainly, it would be incredibly difficult to find a suitable model of deterministic nature to appropriately approximate it. Moreover, even on the basis of historical data from a generated sample of this process using econometric or statistical methods, approximation would be a non-trivial task.

2.3 Comparison of process implementation for the fixed ω analytical form and methods (M) and (E-M)

We will first select one particular realization of a stochastic process (some fixed seed) and, based on it, generate the values of this process along with their approximations obtained by the Euler-Maruyama (E-M) and Milstein (M) methods. The results can be found in the chart below.

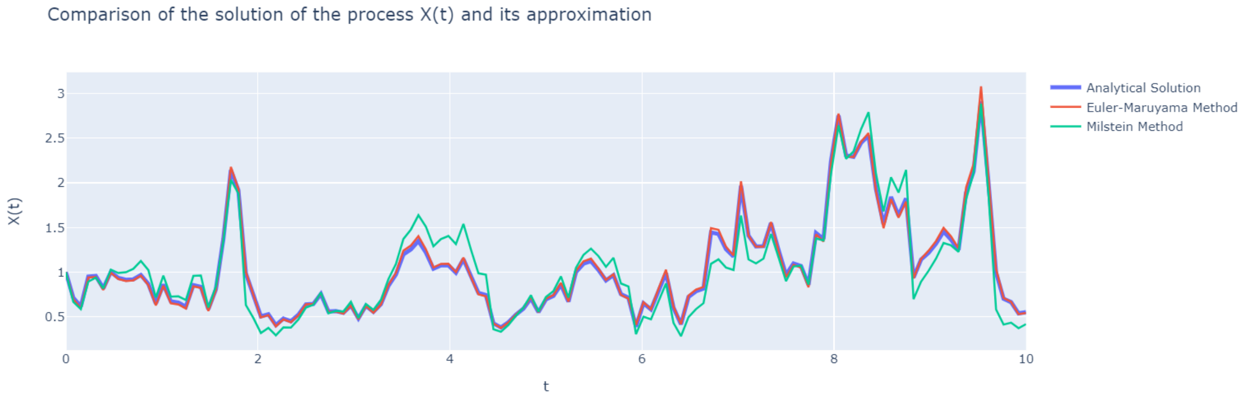

Figure 2.2: Comparison of trajectory realizations.

We can see how, for the selected realization of the process, its approximation is close to the value determined analytically. As mentioned, doing this by other methods would be a difficult task to say the least, so we can see here the enormous potential of stochastic modeling and the use of stochastic differential equations.

2.4 Examine the function of consecutive moments $\mathbb{E}X(t)$, $\mathbb{E}X^2(t)$, $VarX(t)$, $\mathbb{E}X^3(t)$

We will compare the values of the moments of the considered process for different lengths of intervals $N \in \{2^5, 2^7\}$ and for different numbers of simulations $L \in \{100, 1000\}$ along with their approximation. We will first analyze the first moment of $\mathbb{E}(X(t))$.

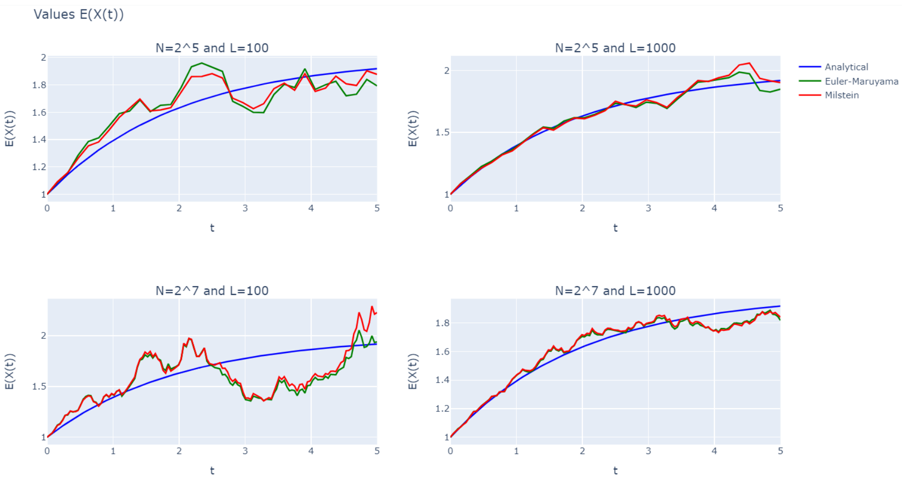

Figure 2.3: Study of the first moment function $\mathbb{E}(X(t))$.

While the values of the process itself approximate satisfactorily well, the values of the features of the process would already require a better method. The first moment, that is, the average value in the approximations has large fluctuations with respect to the analytical solution. However, you can see how changing the size of the division and the number of simulations improves the situation. For this realization, the higher number of iterations introduces larger fluctuations, while the increased number of simulations certainly smooths our approximation.

Now we will examine the second moment function $\mathbb{E}(X^2(t))$.

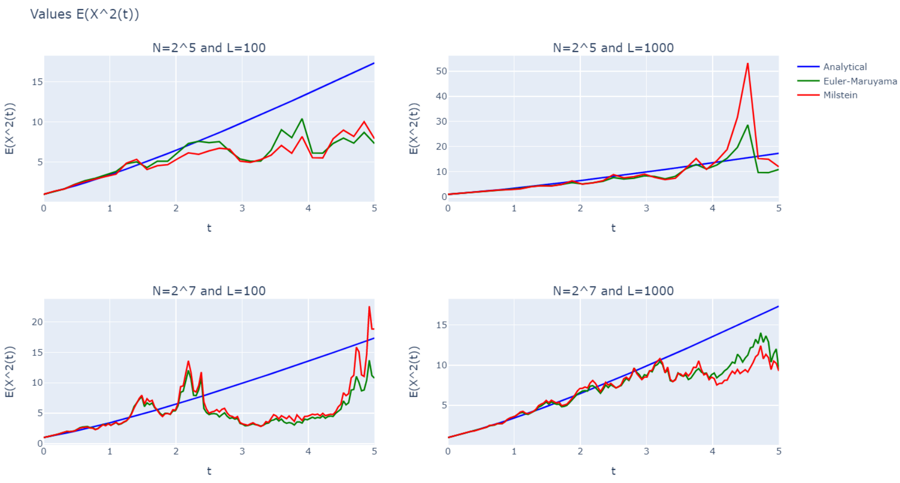

Figure 2.4: Study of the second moment function $\mathbb{E}(X^2(t))$.

The approximation of the value of the second moment already looks a little worse. At the beginning of the interval with small values, the results are relatively good while with time they start to deviate significantly. Again, increasing the number of simulations improves the situation. Let's see the situation for the third moment function $\mathbb{E}(X^3(t))$.

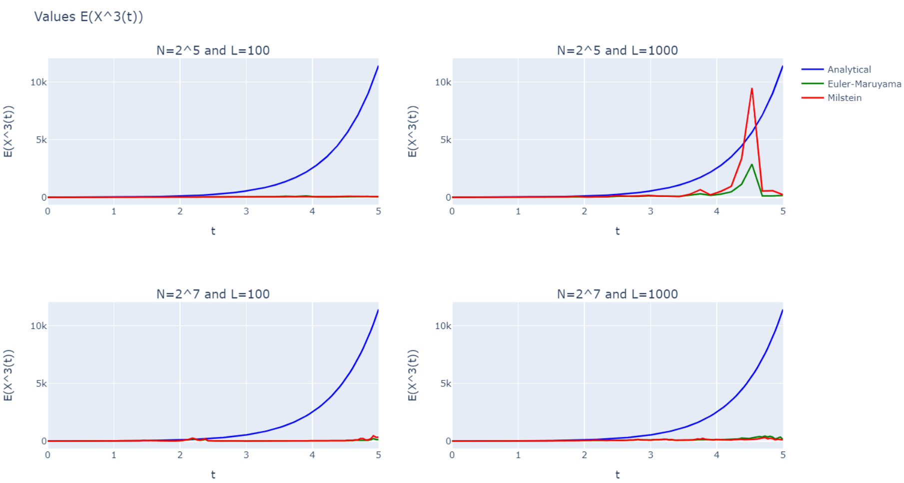

Figure 2.5: Study of the third moment function $\mathbb{E}(X^3(t))$.

The value of the third moment is approximated in a reasonably approximate way only for the beginning of the interval. Later, in the analytical solution evidently the role of the leader is taken over by the exponential function $e^{frac32t}$ and causes an increase in the value of the moment. Obviously, this increase exceeds the simple raising of the simulated values to the third power hence such a discrepancy. The remaining variance value to be examined $VarX(t)$.

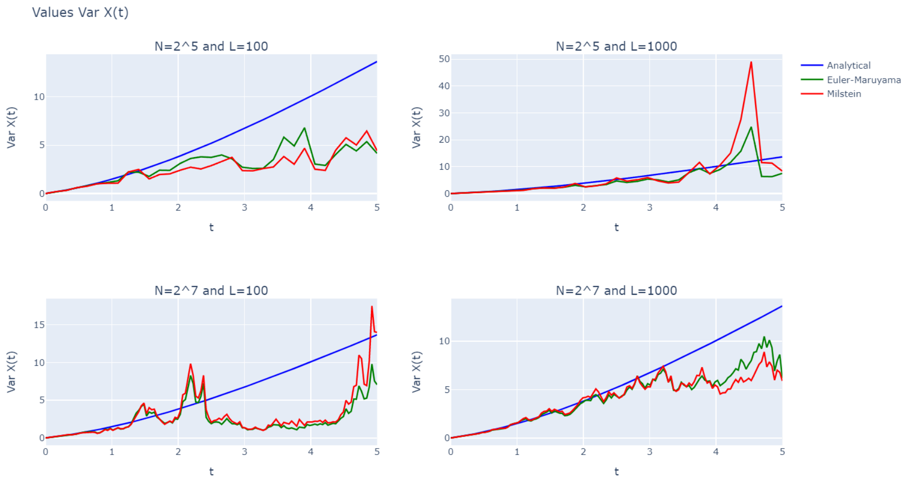

Figure 2.6: Testing the variance function $VarX(t)$.

The approximations of the variance values are similar in terms of accuracy to the approximations of the second moment function. An increase in the simulation value results in better approximations, but it is still possible to notice a certain discrepancy in the results from the simulation and the analytical solution.

In summary, the tested methods do an excellent job of approximating the execution of the process itself, but when examining its attributes, an oversimplification relative to analytical methods appears. Nevertheless, the appropriate choice of parameters, including the number of simulations and the length of the interval has the potential to improve the results, but the time to generate them could make it uneconomical.

2.5 Examine the distribution of the process $X(t)$

We will now examine the distribution of the process under consideration based on different time points and the approximation methods adopted. So, we will consider a number of moments in time $t \in \{0.5, 1, 5\}$ to study the distribution and compare in doing so the analytical method, (M) and (E-M) for $L = 1000$. Let's start with $t = 0.5$.

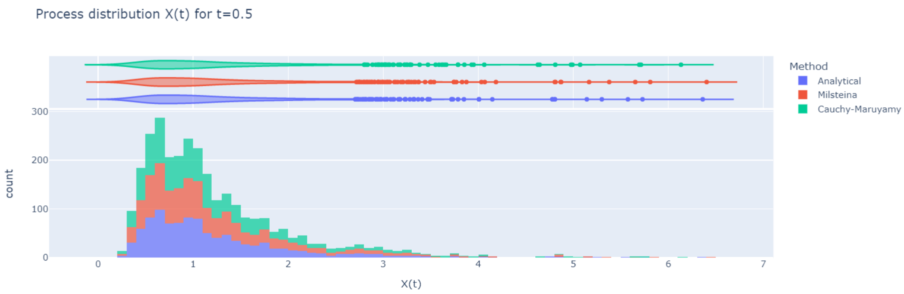

Figure 2.7: Histogram of the stochastic process $X(0.5)$.

For the point $t = 0.5$, we can observe a high density at 1 mainly for approximation methods, where the analytical solution is more "flatly" distributed. As conjectured, these distributions somewhat resemble those of the family of exponential distributions with appropriate parameters. Let's move on to $t = 1$.

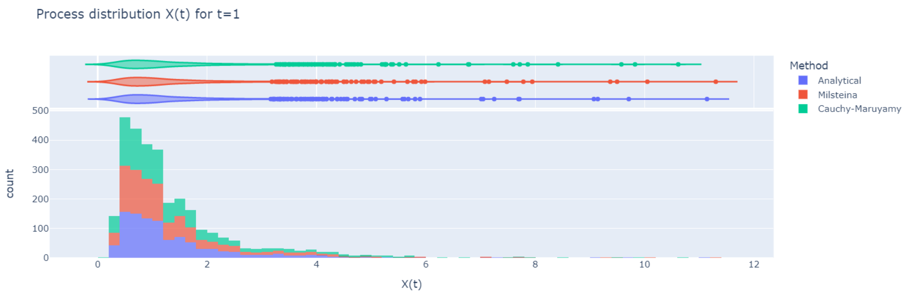

Figure 2.8: Histogram of the stochastic process $X(1)$.

For $t = 1$ the situation changes slightly, the analytical solution begins to be denser at 1, and the distribution actually begins to resemble more of an exponential distribution. In contrast, the solution obtained by the algorithms behave similarly to the previous point and also oscillate strongly at one point. In addition, they also already indicate more of an exponential distribution than a normal distribution with long tails. The last point is $t = 5$.

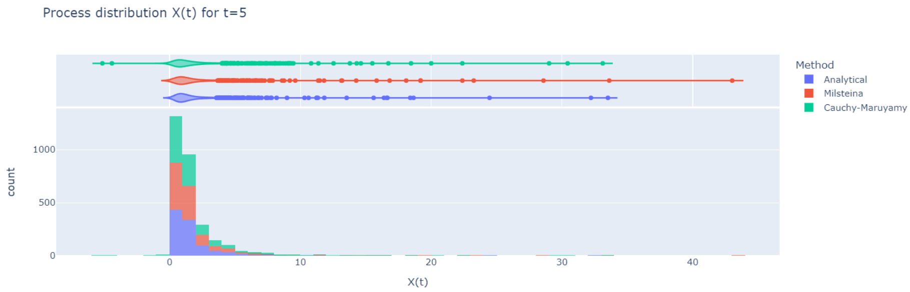

Figure 2.9: Histogram of the stochastic process $X(5)$.

For $t = 5$, the main cluster occurs in the interval $[0, 1]$. Moreover, the exponential nature of the distribution of the process becomes much more pronounced. Note also that based on this point, as well as the previous ones, it can be concluded that the process does not take negative values due to exponential functions. Of course, one could try to fit many other distributions with appropriate parameters here, but as mentioned, the analytical solution resembles the distribution of a distribution from the family of exponential distributions.

2.6 Testing the rate of convergence of the (EM) and (M) methods

For a fixed division dt at time T, we define the approximation error as

$$e_{dt} = \mathbb{E}(|X(T) - X_N|)$$

Approximation converges strongly of order $\gamma > 0$ if $e_{dt} = O((dt)^\gamma)$.

We will study the absolute error for $L \in \{10, 50\}$ simulation at the point $T = 1$ and the value of the $dt \in \{2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}, 2^{-10}\}$.

We will compare them to the theoretical values. The Euler-Maruyama method is of strong order $\gamma = \frac{1}{2}$ so we will compare it to the simple $y = \frac{1}{2}x + b$, $b \in \mathbb{R}$. Milstein's method, on the other hand, is of strong order $\gamma = 1$ so we will compare it to the simple $y = x + b$, $b \in \mathbb{R}$. We will show the comparative results in log-log plot and tabular form. Let's start by assuming $L = 10$.

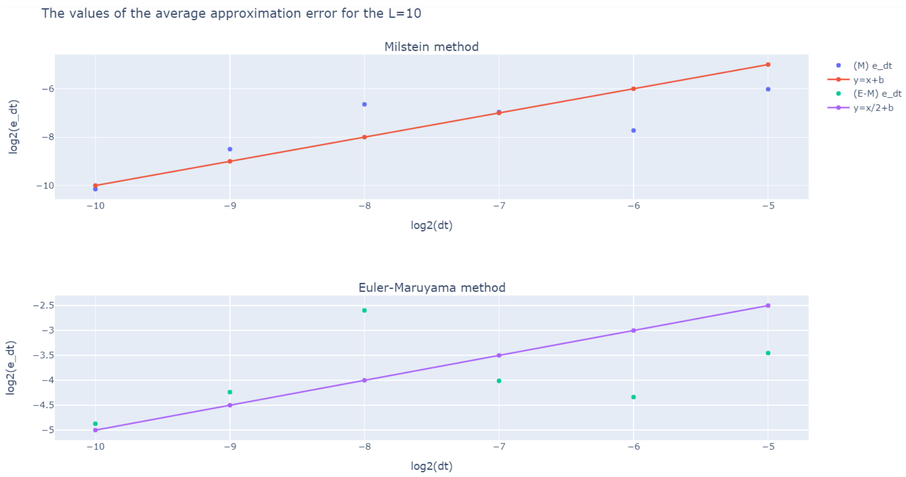

Figure 2.10: The values of the average approximation error for the two methods for the $L = 10$.

We can observe that both methods deviate slightly from the theoretical values, but they are characterized by the same trend, i.e. the error values increase as the time interval increases. Let's move on to compare the two methods.

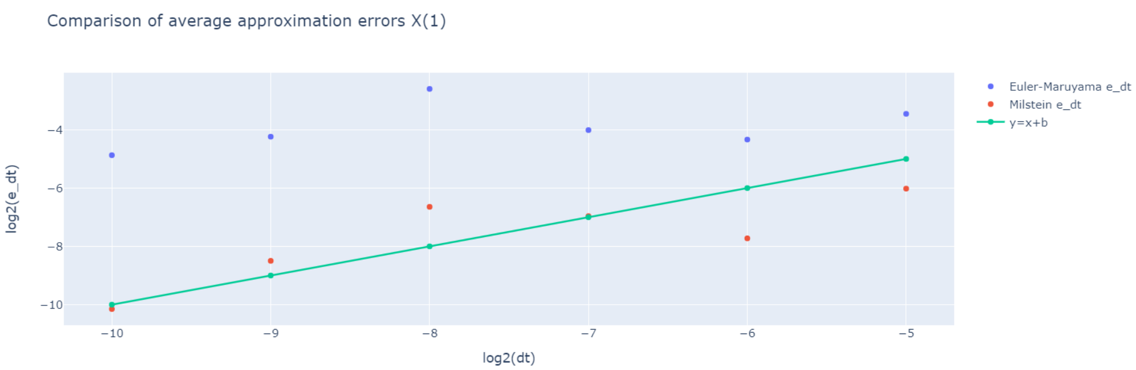

Figure 2.11: Comparison of approximation error of two methods for $L = 50$.

Comparing the two methods, one can see differences in the value of the approximation error, and it can be assumed that Milstein's method handles the approximation task better, but let's put them together in a table without scaling.

Split dt	Error e_{dt} Milstein	Error e_{dt} Euler-Maruyama
2^{-5}	0.015418	0.100519
2^{-6}	0.004729	0.057749
2^{-7}	0.008017	0.041709
2^{-8}	0.010001	0.052885
2^{-9}	0.002770	0.032787
2^{-10}	0.000881	0.020148

Table 2.1: Approximation error table for $L = 10$.

Indeed, the error of the Milstein method has lower values in general compared to the error of the Euler-Maruyama method. The value of the error of Milstein's method generally decreases as the value of the pitch decreases, as do the values of the error of the Euler-Maruyama method. It is worth noting that the values of the approximation error of the Milstein method according to the theoretical order of convergence decrease more rapidly with an increase in the value of the pitch (thus reducing the values of the intervals). Let's check the situation for $L = 50$.

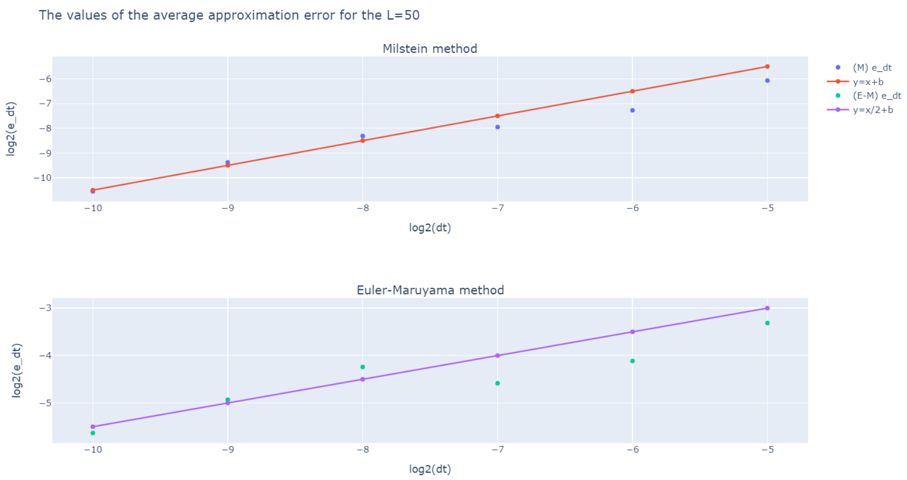

Figure 2.12: The values of the average approximation error for the two methods for the $L = 10$.

We can see that the increased number of simulations changes the situations in the context of approximation to the theoretical values. Milstein's method is almost completely covered by the simple $y = x + b$, which indicates that the order of convergence of the theoretical to the real is consistent. For the Euler-Maruyama method, there is also an apparent improvement, but with slightly larger deviations. Again, let's compare the methods to each other.

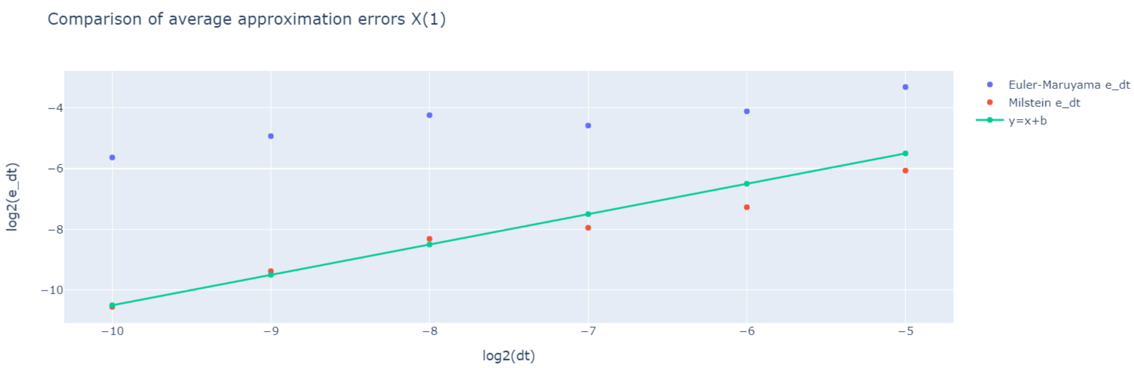

Figure 2.13: Comparison of approximation error of two methods for $L = 50$.

Once again, we are able to observe higher error values for the Euler-Maruyama method than for the Milstein method. We can also see the juxtaposition of different orders of convergence by the faster decrease in the approximation error of Milstein's method. Let's see the results in tabular form.

Split dt	Error e_{dt} Milstein	Error e_{dt} Euler-Maruyama
2^{-5}	0.014918	0.100519
2^{-6}	0.006474	0.057749
2^{-7}	0.004051	0.041709
2^{-8}	0.003145	0.052885
2^{-9}	0.001503	0.032787
2^{-10}	0.000666	0.020148

Table 2.2: Approximation error table for $L = 50$.

The table shows confirmation of the superiority of the Milstein method over the Euler-Maruyama method in terms of approximation error. Milstein's method achieves relatively lower values and converges more quickly to zero error as the time differences decrease. Already for the number of $L \in \{10, 20\}$ simulations, however, the differences are negligible, so we can assume that with an increased number of simulations the averaged values would be similar and the difference will be mainly affected by the value of the division.

Moreover, the Euler-Maruyama method is simpler to obtain because of the consideration of fewer factors. Nevertheless, the above analysis also suggests that when using the Milstein method, a small number of simulations is sufficient to obtain approximations close to analytical solutions, which is a huge advantage in terms of the required computational time. In both cases, the theoretical order of convergence can also be observed. Ultimately, the choice of method depends on many factors and it is worth considering both options.

Chapter 3

Summary

The whole project concerned the study of a certain stochastic differential equation of the form

$$\begin{cases} dX(t) = (\frac{3}{2}X(t) - X^2(t))dt + X(t)dB(t), \\ X(0) = 1. \end{cases}$$

The first part dealt with the analytical solution. First, the assumptions of the theorem on the existence and unambiguity of solutions were examined, from which it followed that this form of the SDE does not satisfy them. Thus, it was proved on the basis of another theorem that this form is reducible to a linear form. After an appropriate transformation, the linear form was further investigated. As part of this, the functions of successive moments were determined, including the function of the first moment of $\mathbb{E}(X(t))$, the second moment of $\mathbb{E}(X^2(t))$, the third moment of $\mathbb{E}(X^3(t))$ and the variance function of $VarX(t)$. Finally, the solution of the linear form was determined analytically.

The second part dealt with the simulation part. The first part presented the approximation methods considered, namely the Euler-Maruyama method and the Milstein method. Example trajectories of the stochastic process were also presented. Then, for a certain fixed realization of the ω process, the results of the approximation algorithms were compared. Going further, the functions of successive moments were examined and compared to the forms determined analytically. An analysis of the distribution of the process at selected time points was also performed. Finally, the rate of convergence of the two approximation methods was examined and conclusions were drawn.

The project captures the complexity and difficulty of the subject of stochastic differential equations, and at the same time their potential in modeling various real-world processes that would be heavy with other methods. It also demonstrates the effectiveness of numerical methods in approximating solutions of these equations, which entails a whole spectrum of possibilities in applications.

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