Bootcamp Assessment 2022

Name: ______________________________

This assessment is to help you gauge your understanding of basic undergraduate level math and statistics materials. We would strongly encourage you to try to complete this assessment without reference to any materials.

Please show your work for all problems, even if you do not arrive at a solution!

1. For which values of $p > 0$ does $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge? ________________________
   Series of the form $\sum \frac{1}{n^p}$, $p > 0$, are called p-series. Using the integral test, we know that p-series converges if and only if $p > 1$. Note that when p=1, it is the harmonic series which diverges.

2. If $X = \begin{bmatrix} -2 & 2 \\ -1 & -1 \end{bmatrix}$ and $Y = \begin{bmatrix} 3 & -8 \\ 4 & -1 \end{bmatrix}$, does $XY = YX$? ________________________
   Yes.

3. Male verbal GRE scores are normally distributed, with a mean of 149 and a standard deviation of 9. Female verbal GRE scores are also normally distributed, with a mean of 149 and a standard deviation of 8. 55% of the students who take the GRE are female.

   What is the probability that a randomly chosen student is female, given that their verbal GRE score is 170? ____________________

   Let G denote gender, M and F denote male and female, S denote the verbal GRE scores. The joint PDF of G and S is

$$f_{G,S}(g,s) = f_{S|G}(s \mid G=g) f_G(G=g),$$

   where $p_{S|G}$ is the conditional PDF of GRE scores given gender, and p_G is the PMF of gender. The marginal PDF of S is

$$\begin{aligned} f_S(s) &= f_{G,S}(M,s) + f_{G,S}(F,s) \\ &= f_{S|G}(s \mid G=M) f_G(G=M) + f_{S|G}(s \mid G=F) f_G(G=F) \\ &= dnorm(s \mid \mu=149, \sigma=9)(1-0.55) + dnorm(s \mid \mu=149, \sigma=8)0.55, \end{aligned}$$

   where dnorm denote the normal PDF. Therefore the conditional PMF of $G \mid S$ is

$$f_{G|S}(g \mid S=s) = \frac{f_{G,S}(g,s)}{f_S(s)}.$$

   And the probability that a randomly chosen student is female given a verbal GRE score of 170 is

$$\begin{aligned} f_{G|S}(F \mid S=170) &= \frac{f_{G,S}(F,170)}{f_S(170)} \\ &\approx 0.4. \end{aligned}$$

4. Let $U \sim Unif(0,1)$ and define $X = -\ln U$.
   What is the the distribution of X? ____________________
   Applying change of variable, we can see that $X \sim Exponential(1)$.

5. Let $X_1, \ldots, X_n$ be iid samples from $Pois(\lambda)$. Show that both $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$ are unbiased estimators for λ.
Note that

$$\mathbb{E}[X_i] = \mathbb{V}[X_i] = \lambda, \quad \mathbb{E}[X_i^2] = \mathbb{V}[X_i] + \mathbb{E}[X_i]^2 = \lambda + \lambda^2$$

Therefore $\mathbb{E}[\bar{X}] = \lambda$, *hence it is an unbiased estimator for* λ. *Also*

$$\begin{aligned} \mathbb{V}[\bar{X}] &= \frac{1}{n^2}\sum_{i=1}^n \mathbb{V}[X_i] = \frac{\lambda}{n} \\ \mathbb{E}[\bar{X}^2] &= \mathbb{V}[\bar{X}] + \mathbb{E}[\bar{X}]^2 = \frac{\lambda}{n} + \lambda^2. \end{aligned}$$

If we expand S^2, *we have*

$$\begin{aligned} S^2 &= \frac{1}{n-1}\left(\sum_{i=1}^n X_i^2 + n\bar{X}^2 - 2\sum_{i=1}^n X_i\bar{X}\right) \\ &= \frac{1}{n-1}\left(\sum_{i=1}^n X_i^2 - n\bar{X}^2\right). \end{aligned}$$

Therefore

$$\mathbb{E}[S^2] = \frac{1}{n-1}(n(\lambda + \lambda^2) - n\frac{\lambda}{n} - n\lambda^2) = \lambda.$$

This shows that S^2 *is also an unbiased estimator of* λ.

6. Skewness is a measure of the asymmetry of a probability distribution of a random variable about its mean. We define the skewness of a r.v. X as $\mathsf{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]$. A random variable that is right-skewed will have positive skewness, and a variable that is left-skewed will have negative skewness.

If $X \sim Exp(\lambda)$, what is the skewness of X? ______________________

$$\begin{aligned}\mathsf{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] &= \frac{E[X^3]-3\mu E[X^2]+3\mu^2 E[X]-\mu^3}{\sigma^3}\\ &= \frac{E[X^3]-3\mu(\sigma^2+\mu^2)+3\mu^3-\mu^3}{\sigma^3}\\ &= \frac{E[X^3]-3\mu\sigma^2-\mu^3}{\sigma^3}\end{aligned}$$

For $X \sim Exp(\lambda)$, $\mu = \frac{1}{\lambda}$ and $\sigma^2 = \frac{1}{\lambda^2}$, and

$$E[X^3] = \int_0^\infty x^3 \lambda e^{-\lambda x} dx = \ldots = 6(1/\lambda)^3$$

So the skewness is $\left(\frac{6}{\lambda^3} - \frac{3}{\lambda^3} - \frac{1}{\lambda^3}\right)\lambda^3 = 2.$

7. Let $X_1, \ldots, X_n$ be iid $Beta(\theta, 1)$ random variables, where $\theta > 0$. Find the MLE of $\frac{1}{\theta}$.

MLE = ______________________

$$L(\theta; y) = \prod_{i=1}^n \frac{\Gamma(\theta+1)}{\Gamma(\theta)\Gamma(1)} x_i^{\theta-1}(1-x_i)^{1-1} = \theta^n (\prod x_i)^{\theta-1}$$

$$l(\theta; y) = n\log(\theta) + (\theta-1)\sum log(x_i)$$

$$\frac{d}{d\theta} l(\theta; y) = \frac{n}{\theta} + \sum \log(x_i) \Rightarrow \hat{\theta} = -\frac{n}{\sum \log(x_i)}$$

So by invariance property of MLEs, MLE of $1/\theta$ is $-\frac{\sum \log(x_i)}{n}$

8. The joint pdf of X and Y is

$$f(x,y) = \frac{e^{-yx^2/2}}{\sqrt{2\pi/y}} \cdot ye^{-y}, \quad x \in \mathbb{R}, y > 0$$

(a) Find the conditional density $f_{X|Y}(x|y)$ of X given $Y = y$. ____________________
Hint: consider decomposing the joint into the product of conditional and marginal densities.
By inspection, $X|Y \sim N(0, 1/y)$

(b) What is $\mathsf{E}[X|Y]$? ____________________
$E[X|Y=y] = 0$

(c) What is $\mathsf{Var}(X|Y)$? ____________________
$Var(X|Y) = 1/Y$

(d) What is $\mathsf{Var}(X)$? ____________________
$Var(X) = E[Var(X|Y)] + Var(E[X|Y]) = E[1/Y] + 0 = \int_0^\infty \frac{1}{y} ye^{-y} dy = 1$

9. Let $\mathbf{Z} = (Z_1, \dots, Z_m)^T$ be a vector of m i.i.d. $N(0,1)$ random variables. Then $\mathsf{E}[\boldsymbol{Z}] = \mathbf{0}$ and $Cov(\boldsymbol{Z}) = \boldsymbol{I}_m$. Recall that we say a random n-dimensional vector $\boldsymbol{Y}$ is distributed multivariate normal if it has the same distribution as $\boldsymbol{AZ} + \boldsymbol{b}$ where $\boldsymbol{A}$ is some fixed $n \times m$ matrix and $\boldsymbol{b}$ is some fixed $n \times 1$ vector. That is, $\boldsymbol{Y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{V})$, where $\boldsymbol{\mu} = \boldsymbol{b}$ and $\boldsymbol{V} = \boldsymbol{AA}^T$.

   However, it is not clear that if we know $\boldsymbol{b}$ and $\boldsymbol{AA}^T$ then $\boldsymbol{Y}$ is uniquely or well defined. Notice that the covariance matrix of $\boldsymbol{Y}$ is $\boldsymbol{V} = \boldsymbol{AA}^T$. But because matrix square roots are not unique, we could have some matrix $\boldsymbol{B}$ such that $\boldsymbol{V} = \boldsymbol{BB}^T$ but $\boldsymbol{B} \neq \boldsymbol{A}$.

   With this information, prove that if $\boldsymbol{Y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{V})$, and $\boldsymbol{W} \sim N_n(\boldsymbol{\mu}, \boldsymbol{V})$, then $\boldsymbol{Y}$ and $\boldsymbol{W}$ follow the same distribution.

   Suppose $\boldsymbol{Y} = \boldsymbol{AZ} + \boldsymbol{\mu}$, $\boldsymbol{W} = \boldsymbol{BZ} + \boldsymbol{\mu}$, $\boldsymbol{V} = \boldsymbol{AA}^T = \boldsymbol{BB}^T$ and $\boldsymbol{A}$ and $\boldsymbol{B}$ are not necessarily equal. We want to show that $\boldsymbol{Y}$ and $\boldsymbol{W}$ follow the same distribution.

   Recall that MGFs uniquely characterize a distribution. The MGF for the random vector $\boldsymbol{Z}$ is

$$M_{\boldsymbol{Z}}(\boldsymbol{t}) = \mathbb{E}[e^{\boldsymbol{t}^T\boldsymbol{Z}}] = \mathbb{E}[e^{\sum_{i=1}^n t_i Z_i}] = \prod_{i=1}^n \mathbb{E}[e^{t_i Z_i}]$$
$$= \prod_{i=1}^n e^{\frac{1}{2}t_i^2} = e^{\frac{1}{2}\boldsymbol{t}^T\boldsymbol{t}}.$$

   Note that this made use of the fact that Z_i's are i.i.d. $N(0,1)$.
   The MGF for $\boldsymbol{Y}$ is

$$M_{\boldsymbol{Y}}(\boldsymbol{t}) = \mathbb{E}[e^{\boldsymbol{t}^T\boldsymbol{Y}}] = e^{\boldsymbol{t}^T\boldsymbol{\mu}}\mathbb{E}[e^{(\boldsymbol{A}^T\boldsymbol{t})^T\boldsymbol{Z}}] = e^{\boldsymbol{t}^T\boldsymbol{\mu}} M_{\boldsymbol{Z}}(\boldsymbol{A}^T\boldsymbol{t})$$
$$= e^{\boldsymbol{t}^T\boldsymbol{\mu}} e^{\frac{1}{2}(\boldsymbol{A}^T\boldsymbol{t})^T(\boldsymbol{A}^T\boldsymbol{t})} = e^{\boldsymbol{t}^T\boldsymbol{\mu}} e^{\frac{1}{2}\boldsymbol{t}^T\boldsymbol{AA}^T\boldsymbol{t}} = e^{\boldsymbol{t}^T\boldsymbol{\mu} + \frac{1}{2}\boldsymbol{t}^T\boldsymbol{Vt}}.$$

   Similarly the MGF for $\boldsymbol{W}$ is also $e^{\boldsymbol{t}^T\boldsymbol{\mu} + \frac{1}{2}\boldsymbol{t}^T\boldsymbol{Vt}}$. This shows that $\boldsymbol{Y}$ and $\boldsymbol{W}$ necessarily follow the same distribution, and that mean and covariance matrix uniquely characterize a multivariate normal distribution.