Mathematics/Statistics Bootcamp

Part III: Distributions

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Outline

Random Variables

Univariate Distributions

Multivariate Distributions

Building New Variables

Random Variables

Random Variables

A **random variable** is a (measurable) function from a sample space to an outcome space ($\mathbb{R}$, $\mathbb{Z}$, sentences, brain scans, etc).

Common notation:

- Ω is the set of all possible outcomes of an experiment.
- $\omega \in \Omega$ is a particular outcome.
- $Y = Y(\omega)$ is a function of ω (the random variable).

Imagine our experiment is rolling two dice. What is Ω? Give an example of ω and a few random variables.

Cumulative Distribution Functions

The **cumulative distribution function** (CDF) is

$$F_X(x) = P_X(X \leq x) \qquad \forall x \in \mathbb{R}$$

A function $F : \mathbb{R} \to [0, 1]$ is a CDF if and only if the following are true:

- $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$.
- $F(x)$ is non-decreasing.
- $F(x)$ is right continuous.

A random variable is **continuous** if $F_X(x)$ is continuous. A random variable is **discrete** if $F_X(x)$ is a step function.

Example

Consider flipping a biased coin. Let p denote the probability of getting a head. If we define

$$X = \text{number of tosses until a head,}$$

then for $x \in \mathbb{N}$,

$$\begin{aligned} P(X \leq x) &= p + (1-p)p + (1-p)^2 p + \cdots + \cdots (1-p)^{x-1} p \\ &= \sum_{i=1}^{x} (1-p)^{i-1} p \\ &= 1 - (1-p)^x. \end{aligned}$$

Probability Density/Mass Functions

The **probability mass function** (PMF) for a discrete random variable is

$$f_X(x) = P(X = x); \quad \forall x \in \Omega.$$

The **probability density function** (pdf) for an (absolutely) continuous random variable is a function f_X such that

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\mathrm{d}t; \quad \forall x \in \Omega.$$

Recall PMFs/PDFs sum/integrate to 1. Can help you avoid calculus.

Univariate Distributions

Bernoulli Distribution

Represents a single coin flip with success ($X = 1$) probability p.

$$
\begin{aligned}
X &\sim \text{Bernoulli}(p) \\
P(X = x) &= p^x (1-p)^{1-x}, \quad x \in \{0, 1\} \\
\mathbb{E}[X] &= p \\
\mathbb{V}[X] &= p(1-p)
\end{aligned}
$$

Binomial Distribution

Counts the number of successes in n independent trials all with the same success probability p.

$$X \sim \text{Binomial}(n, p)$$

$$P(X = x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}, \quad x \in \{0, 1, \ldots, n\}$$

$$\mathbb{E}[X] = np$$

$$\mathbb{V}[X] = np(1-p)$$

Can decompose $X = \sum_{i=1}^{n} Y_i$ where the Y_i are iid $\text{Bernoulli}(p)$.

Poisson Distribution

Counts events occurring with rate λ:

$$X \sim \text{Poisson}(\lambda)$$

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x \in \{0, 1, 2, \ldots\}$$

$$\mathbb{E}[X] = \mathbb{V}[X] = \lambda$$

Not always a great fit: real data are often zero-inflated and over-dispersed.

If $X \sim Po(\lambda)$ and $Y \sim Po(\eta)$ are independent, then $X + Y \sim Po(\lambda + \eta)$.

Geometric Distribution

Counts the number of failures until the first success in sequential independent Bernoulli trials with success probability p.

$$X \sim \text{Geom}(p)$$

$$P(X = k) = (1-p)^k p, \quad k \in \{0, 1, 2, \ldots\}$$

$$\mathbb{E}[X] = \frac{1-p}{p}$$

$$\mathbb{V}[X] = \frac{1-p}{p^2}$$

Caution: some parameterizations start at 1.

Normal Distributions

A random variable $X \sim N(\mu, \sigma^2)$ has PDF:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} \right)$$

Sometimes write in terms of **precision** (inverse variance): $X \sim N(\mu, \phi^{-1})$.

If $X \sim N(\mu, \sigma^2)$, then

$$a + bX \sim N(a + b\mu, b^2\sigma^2).$$

If $Z \sim N(0, 1)$ then the distribution of Z is **standard normal.**

Normal Distributions Properties

Can always decompose $X \sim N(\mu, \sigma^2)$ as $X = \mu + \sigma Z$.

If $X \sim N(\mu, \sigma^2)$ and $Y \sim N(\eta, \tau^2)$ are jointly normal, then

$$X + Y \sim N(\mu + \eta, \sigma^2 + \tau^2 + 2\text{Cov}[X, Y]).$$

Jointly normal random variables are independent if and only if they are uncorrelated.

Chi-Squared Distribution

If $Z_1, Z_2, \ldots, Z_k$ are independent, standard normal random variables, then

$$\sum_{j=1}^{k} Z_j^2 \sim \chi_k^2$$

follows a Chi-Squared distribution with k degrees of freedom. Special case of the Gamma distribution.

You will see statements like $X \sim c\chi_k^2$ in linear models. This means $X/c \sim \chi_k^2$.

Gamma Distribution

A random variable $X \sim \text{Gamma}(\alpha, \beta)$ has PDF:

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp\{-x\beta\}$$

$$\mathbb{E}[X] = \frac{\alpha}{\beta}$$

$$\mathbb{V}[X] = \frac{\alpha}{\beta^2}$$

$$\alpha, \beta > 0$$

$$x \in (0, \infty)$$

This is the rate parameterization. You will also see the scale parameterization with $\theta = 1/\beta$.

If $X \sim \text{Gamma}(1, \beta)$, then $X \sim \text{Exponential}(\lambda = \beta)$.

Gamma Distribution Properties (Optional)

Useful facts:

- If $X \sim \text{Gamma}(\alpha_1, \beta)$ and $Y \sim \text{Gamma}(\alpha_2, \beta)$ are independent, then $X + Y \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$. Consequence: sum of independent exponentials is Gamma.
- If $\alpha = \frac{\nu}{2}$ and $\beta = \frac{1}{2}$ then $X \sim \chi^2_\nu$.
- If $X \sim \text{Gamma}(\alpha_1, \beta)$ and $Y \sim \text{Gamma}(\alpha_2, \beta)$ are independent, then $X/(X + Y) \sim \text{Beta}(\alpha_1, \alpha_2)$.
- Let $X_i \sim \text{Gamma}(\alpha_i, \beta)$, $i = 1, ..., n$ be independent. Set $T = \sum_i X_i$. Then $(X_1/T, ..., X_n/T) \sim \text{Dirichlet}(\alpha_1, ...\alpha_n)$.
- If $X \sim \text{Gamma}(\alpha, \beta)$, then $\frac{1}{X} \sim \text{Inverse} - \text{Gamma}(\alpha, \beta)$.

Student's-t Distribution

A random variable T follows a Student's-t distribution with ν degrees of freedom if

$$T = \frac{Z}{\sqrt{V/\nu}},$$
$$Z \sim N(0, 1),$$
$$V \sim \chi^2_\nu$$

and Z and V are independent.

Like a normal, but with heavier tails. The $\nu = 1$ case is a Cauchy distribution with undefined mean and variance.

As $\nu \to \infty$, $T \to N(0, 1)$.

Useful for confidence intervals with unknown variance.

F Distribution

A random variable X follows a F-distribution with numerator degrees of freedom ν_1 and denominator degrees of freedom ν_2 if

$$X = \frac{V_1/\nu_1}{V_2/\nu_2}$$

where V_1 and V_2 are independent χ^2 random variables with degrees of freedom equal to ν_1 and ν_2 respectively.

Useful for model selection, e.g. comparing variances in ANOVA.

Beta Distribution

A random variable $X \sim \mathrm{Beta}(\alpha, \beta)$ has PDF:

$$f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad x \in (0,1), \quad \alpha, \beta > 0$$

$$\mathbb{E}[X] = \frac{\alpha}{\alpha+\beta}$$

$$\mathbb{V}[X] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Useful for eliciting probability distributions for proportions.

Exercises

1. Give an example of a random variable which is not discrete or continuous.
2. Let $X_1, ..., X_n$ be iid $N(\mu, \sigma^2)$. Find the distributions of
   a) $\bar{X} = \sum_{i=1}^{n} X_i/n$.
   b) $(X_i - \bar{X})^2$. Recall $Cov[\cdot, \cdot]$ is linear in both components.
   c) $(X_i - \mu)/(X_j - \mu)$
3. A $\mathrm{Weibull}(\alpha, \beta)$ distribution has density

$$f(x) = \alpha\beta(\beta x)^{\alpha-1} \exp(-(\beta x)^{\alpha}).$$

   where $\alpha, \beta, x > 0$. Find the CDF.

Multivariate Distributions

Random Vectors

A d-dimensional **random vector** is a collection of d random variables:

$$\mathbf{X} = (X_1, \cdots, X_d)^T$$

The joint distribution function of the random vector $\mathbf{X}$ is

$$\begin{aligned} F_X(\mathbf{x}) &= F_X(x_1, \ldots, x_d) \\ &= P(X_1 \leq x_1, \ldots, X_d \leq x_d) \\ &= P(\mathbf{X} \leq \mathbf{x}) \end{aligned}$$

If F_X is absolutely continuous, then the joint density function f_X of $\mathbf{X}$ is

$$f_X(\mathbf{x}) = f_X(x_1, \ldots, x_d) = \frac{\partial^d F_X(x_1, \ldots, x_d)}{\partial x_1 \cdots \partial x_d}$$

Independence

We can find the **marginal density** of a random variable by integrating/summing out the others. For example, if we have a joint bivariate density $f_{X,Y}(x, y)$, then

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dy \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dx$$

The components of a random vector $\mathbf{X}$ are **independent** if the joint CDF (equivalently PDF) factors as a product of marginals:

$$F_X(\mathbf{x}) = \prod_{i=1}^{d} F_i(x_i), \quad f_X(\mathbf{x}) = \prod_{i=1}^{d} f_i(x_i)$$

Multivariate Moments

The expected value of a random vector $\mathbf{X}$ is

$$\mu_X = E(\mathbf{X}) = (E(X_1), \cdots, E(X_d)) = (\mu_1, \cdots, \mu_d)^T$$

and the $d \times d$ **covariance matrix** is

$$\mathbf{\Sigma}_X = \mathrm{Cov}(\mathbf{X}, \mathbf{X}) = E[(\mathbf{X} - \mu_X)(\mathbf{X} - \mu_X)^T]$$

If $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$, then

$$\mu_Y = \mathbf{A}\mu_x + \mathbf{b},$$
$$\mathbf{\Sigma}_Y = \mathbf{A}\mathbf{\Sigma}_\mathbf{X}\mathbf{A}^T.$$

Multivariate Normal Distribution

Let $\mathbf{Z}$ be a standard normal vector - i.e., $\mathbf{Z} = (Z_1, ..., Z_n)$ where the Z_i are iid standard normal.

A random vector $\mathbf{X}$ has a multivariate normal distribution with mean vector μ and positive definite symmetric covariance matrix $\mathbf{\Sigma}$ if and only if

$$\mathbf{X} = \mu + L\mathbf{Z}$$

where $\mathbf{L}\mathbf{L}^T = \mathbf{\Sigma}$.

We write $\mathbf{X} \sim N_n(\mu, \mathbf{\Sigma})$. The density is

$$f(\mathbf{x}) = (2\pi)^{-n/2}|\mathbf{\Sigma}|^{-1/2}e^{-\frac{1}{2}(\mathbf{x}-\mu)^T\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)}.$$

Multivariate Normal Properties

If $\mathbf{X} \sim N(\mu_{\mathbf{X}}, \mathbf{\Sigma}_{\mathbf{X}})$, then

$$\mathbf{A}\mathbf{X} + \mathbf{b} \sim N(\mathbf{A}\mu_{\mathbf{X}} + \mathbf{b}, \mathbf{A}\mathbf{\Sigma}_{\mathbf{X}}\mathbf{A}^T)$$

Note: can always rotate $\mathbf{X}$ to make coordinates independent.

If $\mathbf{Y} \sim N(\mu_{\mathbf{Y}}, \mathbf{\Sigma}_{\mathbf{Y}})$ is independent of $\mathbf{X}$, then

$$\mathbf{X} + \mathbf{Y} \sim N(\mu_{\mathbf{X}} + \mu_{\mathbf{Y}}, \mathbf{\Sigma}_{\mathbf{X}} + \mathbf{\Sigma}_{\mathbf{Y}})$$

Sum of correlated MVN is still MVN, provided $\mathbf{X}$ and $\mathbf{Y}$ are jointly normal.

If $\mathbf{X}$ and $\mathbf{Y}$ are jointly normal, then $\mathbf{X}|\mathbf{Y}$ and $\mathbf{Y}|\mathbf{X}$ are also normal.[1]

[1] Hard exercise: derive the conditional distributions.

High Dimensional Vectors (Optional)

Let $\mathbf{Z} \sim N(\mathbf{0}, \mathbf{I}_n)$ by an n-dimensional normal vector. Samples of each component are clustered around $\mathbf{0}$.

Intuitively, we expect samples of $\mathbf{Z}$ to be clustered around $\mathbf{0}_n$. This is wrong!

In high dimensions, $\mathbf{Z}$ *concentrates around an* $n-1$ *dimensional sphere* of radius $\sqrt{n}$:

$$N(\mathbf{0}, \mathbf{I}_n) \approx Unif(\sqrt{n}\mathbb{S}^{n-1})$$

Figure from "High-Dimensional Probability" by Roman Vershynin.

Exercises

Consider the linear model $\mathbf{Y} \sim N(\mathbf{X}\beta, \sigma^2\mathbf{I})$ where σ^2 is known, $\mathbf{Y} \in \mathbb{R}^n$ and $\mathbf{X} \in \mathbb{R}^{n \times p}$ (rank $p \leq n$). We found

$$\hat{\beta} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}$$

minimized $||\mathbf{Y} - \mathbf{X}\beta||^2$, giving fitted values

$$\hat{\mathbf{Y}} = \mathbf{X}\hat{\beta} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} = \mathbf{P}\mathbf{Y}.$$

Find distributions for the following:

1. $\hat{\beta}$, $\hat{\beta}_1$, and $(\hat{\beta}_1, \hat{\beta}_7)^T$ (if $p \geq 7$).
2. $\hat{\mathbf{Y}}$ and $\mathbf{Y} - \hat{\mathbf{Y}} = (\mathbf{I} - \mathbf{P})\mathbf{Y}$. Guess the distribution of $\mathbf{Y}^T(\mathbf{I} - \mathbf{P})\mathbf{Y}$.
3. $(\mathbf{X}^T\mathbf{X})^{1/2}(\hat{\beta} - \beta)$ and $||\mathbf{X}(\hat{\beta} - \beta)||^2$.

What do these quantities mean?

Building New Variables

Univariate Change of Variables

If $Y = g(X)$ for some monotone g, then

$$\begin{aligned} F_Y(y) &= P_Y[Y \leq y] = P_X[g(X) \leq y] = P_X[X \leq g^{-1}(y)] \\ &= F_X(g^{-1}(y)). \end{aligned}$$

Taking derivatives,

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}(g^{-1}(y)) \right|.$$

E.g. if $Y = g(X) = 2X$ then $f_Y(y) = f_X(y/2)/2$.

If g is not monotone, then

$$f_Y(y) = \sum_{k=1}^{n(y)} f_X(g_k^{-1}(y)) \left| \frac{d}{dy}(g_k^{-1}(y)) \right|.$$

where $g_1^{-1}(y), ..., g_{n(y)}^{-1}(y)$ are the $n(y)$ solutions to $g(x) = y$.

Example

Let $X \sim \mathrm{Normal}(0,1)$ and $Y = g(X) = X^2$.

If $y = g(x)$, then $x = g^{-1}(y) = \pm\sqrt{y}$. Let $g_1^{-1}(y) = -\sqrt{y}$ and $g_2^{-1}(y) = \sqrt{y}$. Almost always $n(y) = 2$. Therefore

$$
\begin{aligned}
f_Y(y) &= f_X(g_1^{-1}(y))\left|\frac{d}{dx}(g_1^{-1}(y))\right| + f_X(g_2^{-1}(y))\left|\frac{d}{dx}(g_2^{-1}(y))\right| \\
&= \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-(-\sqrt{y})^2}{2}\right)\left|-\frac{1}{2\sqrt{y}}\right| + \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-(\sqrt{y})^2}{2}\right)\left|\frac{1}{2\sqrt{y}}\right| \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{y}}\exp\left(\frac{-y}{2}\right)
\end{aligned}
$$

This is a $\mathrm{Gamma}(1/2, 1/2)$ (equivalently χ^2_1) density.

Multivariate Change of Variables

If $\mathbf{Y} = g(\mathbf{X})$ with differentiable inverse. Then

$$f_Y(\mathbf{y}) = f_X(g^{-1}(\mathbf{y}))\left|J_{g^{-1}}(\mathbf{y})\right|$$

where $J_{g^{-1}}$ is the Jacobian of g^{-1}.

E.g. let X be Y independent and set $(U, V) = (2X, 2X + Y)$. Then $g^{-1}(u, v) = (u/2, v - u)$, so

$$|\mathbf{J}_{g^{-1}}| = \begin{vmatrix} 1/2 & 0 \\ -1 & 1 \end{vmatrix} = \frac{1}{2}$$

and $f_{U,V}(u, v) = f_X(u/2)f_V(v - u)/2$.

Finite Mixtures (Optional)

Let $(F_1, ..., F_n)$ be a collection of CDFs with PDFs $(f_1, ..., f_n)$. Let $\mathbf{w} = (w_1, ..., w_n)$ a weight vector summing to 1.

A **finite mixture** is a random variable with CDF/PMF given by

$$F(x) = \sum_{i=1}^{n} w_i F_i(x), \quad f(x) = \sum_{i=1}^{n} w_i f_i(x).$$

Implementation: sample an index $i \in \{1, ..., n\}$ with probability w_i, then sample from f_i.

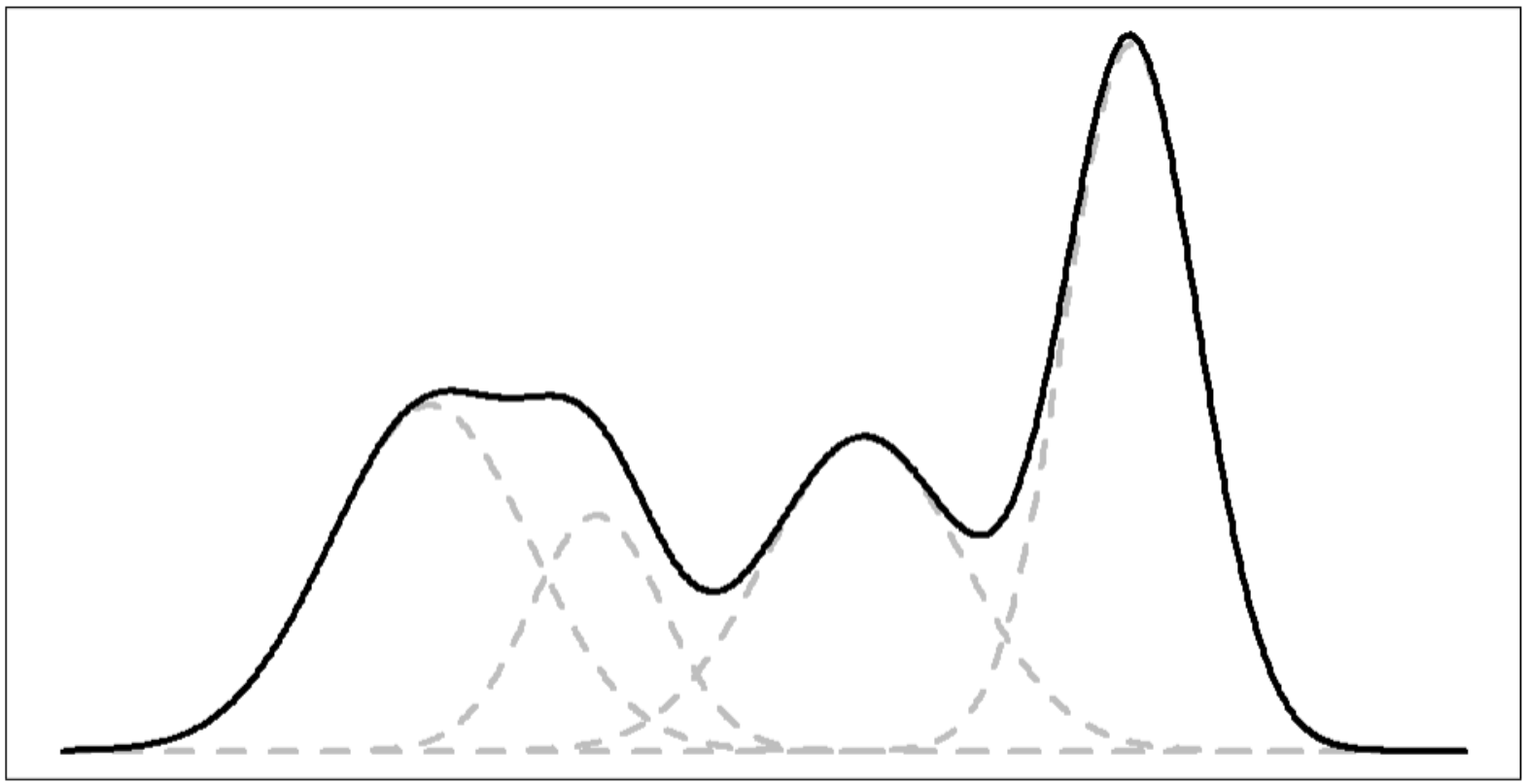
Extremely important: clustering, density estimation, image segmentation, zero inflated Poisson (discuss?), etc.

Example (Optional)

Key example is a location-scale mixture of univariate normals:

$$f(x) = \sum_{i=1}^{n} w_i N(x; \mu_i, \sigma_i^2).$$

Note we could relabel the groups (not identifiable).



Other Techniques (Optional)

May need a more flexible distribution. Leads to **infinite mixture** models:

$$f(x) = \sum_{i=1}^{\infty} w_i f_i(x) \quad \text{or even} \quad f(x) = \int f(x|w)p(w)dw.$$

May need complicated multivariate distributions (e.g., correlated Poissons) with asymmetric correlation structures (e.g., stock crashes). Leads to **copulas.**

Exercises

1. Let $X_1, ..., X_n$ be iid with CDF F and PDF f. Order the variables $X_{(1)} \leq \cdots \leq X_{(n)}$.
   a) Find the CDF and PDF for $X_{(1)}$.
   b) Find the CDF and PDF for $X_{(n)}$.
   c) Evaluate these quantities when $X_i \sim Exp(\lambda)$.
2. Let $X \sim Gamma(\alpha, 1)$ and $Y \sim Gamma(\beta, 1)$ be independent.
   a) Write down the joint density, $f_{X,Y}(x, y)$.
   b) Find the joint density of $(U, V) = (X/(X + Y), X + Y)$.
   c) Identify the marginal distributions of U and V.

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