Improved blow-up conditions for a dyadic model of the Navier-Stokes equations

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Abstract

Spatial discretizations of the 3D Navier Stokes equations have provided valuable intuition for understanding plausible conditions under which solutions to the Navier-Stokes equations might i) exist and ii) behave well. We consider a set of discretizations indexed by a family of hyper-parameters $(\alpha, \beta, \nu) \in [0, 1]^3$. Our contribution is an improvement upon the work by Jeong and Li 2015; in particular, we extend the class of known solutions exhibiting finite-time blow-up to hyperparameter values further from one of the corners.

1. Introduction

Understanding the regularity and existence of solutions to the 3D Navier-Stokes equations is a central problem in modern fluid dynamics. Simpler "toy models" of the Navier-Stokes equations provide an tractable opportunity to develop reliable intuition for how this complicated system ought to behave. This article considers a particular toy model consisting of an infinite set of ordinary differential equations (ODEs) motivated by the projection of the Navier Stokes equations onto a sequence of dyadic cubes [Friedlander and Pavlović 2004]. In full generality, these equations are

$$u_j' = \alpha(\lambda^j u_{j-1}^2 - \lambda^{j+1} u_j uj + 1) + \beta(\lambda^j u_{j-1} u_j - \lambda^{j+1} u_{j+1}) - \nu \lambda^{2\gamma j} u_j \tag{1}$$

where the u_j are real valued functions of time, $j \geq 0$, and $u_{-1} = 0$ by convention. Here $(\alpha, \beta, \nu) \in [0, 1]^3$, $\gamma > 0$, and $\lambda > 1$. We fix $\lambda = 2$, but all results generalize to arbitrary λ by re-scaling. Valid solutions must remain in the Sobolev space of sequences H^s $(s > 0)$ for all $t > 0$; that is they must have finite norm

$$||u(t)||_s^2 = \sum_{j=0}^{\infty} \lambda^{2js} u_j^2(t) \tag{2}$$

regardless of the time. The energy $E(t)$ of a solution is the square of the H_0-norm.

Two edge cases are of interest. The $(\alpha, \beta, \nu) = (1, 0, 0)$ case is known as the KP equations after the work of Katz and Pavlovic [Katz and Pavlović 2005, Friedlander and Pavlović 2004]. This system behaves poorly: when $s > 3/2$ any set of initial conditions $u(0) \in H^s$ can be extended to a solution of (1) which blows up in finite time. Such blow-up occurs because the KP equations are *positively monotone*: when any u_j becomes positive at some t_0 it must remain positive for all $t > t_0$. Energy is violently accelerated towards higher modes, guaranteeing catastrophe. Intuitively, α controls the propensity of solutions to (1) to behave poorly.

The other interesting edge case, $(\alpha, \beta, \nu) = (0, 1, 0)$, is known as the Obuhkov equations [Waleffe 2006]. Here solutions are *negatively monotone* and extremely well behaved. When $s > 1$ any set of initial conditions $u(0) \in H^1$ can be extended to a solution that remains in H^s for eternity. Consequently we view β as a parameter which forces the energy backwards, at direct odds with α.

One term of the equation remains undiscussed. Parameters ν and γ are related to the rate at which energy disperses. If $\nu = 1$ is fixed, then increasing γ increases the rate at which energy spreads out and reduces the likelihood of the energy collapsing to a single mode.

We view (1) as an interpolation between the KP and Obuhkov equations; when α is large relative to β we should observe KP-like behaviour, resulting in frequent blow-up. Justification of this intuition due to a lack of monotonicity. At the time of writing, the following appears to be the strongest result.

Theorem 1 (Jeong and Li)**.** *Fix $s_0 > 1/3$, $\gamma \leq 1/3$, $\alpha = 1$, and $\nu = 1$ in (1). Then there exists $\varepsilon_{s_0,\gamma_0} \in (0, 1]$ such that for each $\beta \in [0, \varepsilon_{s_0,\gamma_0}]$ one can find a set of nontrivial solutions to (1) which start in H^{s_0} and blow-up in finite time.*

We obtain a larger class of nontrivial solutions. The proof is purely technical; along the way the authors are forced to place a lower bound on the norm of an assumed solution to (1). Our contribution is an improvement of this bound.

2. Results

Jeong and Li's original proof follows - more details on the computations can be found in the original paper.

Proof. (Jeong and Li's proof of Theorem 1) Fix $\alpha = 1$, $\nu = 1$, $s > 1/3$. Define $b_j =^j u_j$ to transform (1) into

$$b_j' = (\lambda^2 b_{j-1}^2 - b_j b_{j+1}) + \beta(\lambda b_{j-1} b_j - \lambda^{-1} b_{j+1}^2) - \lambda^{2\gamma j} b_j. \tag{3}$$

The body of the proof rests upon studying a weighting of the energy of b_j. Fix $\omega > 1$ and define the time-dependent function

$$A = \sum_{j=0}^{\infty} b_j^2 \omega^{-j}. \tag{4}$$

Assume for contradiction that $||u(t)||_s$ is locally integrable on $[0, \infty)$. Further imposing $\omega^{-1} \leq \lambda^{2(s-1)}$ will ensure $A \leq ||u(t)||_s$, hence A is also locally integrable on $[0, \infty)$. It will be enough to reduce to a system of ODEs in A, and then show that this system exhibits blow-up.

Motivated by this, Jeong and Li multiply (3) by ω^{-j} and sum over j. They prove the resulting sum is bounded by a polynomial in A when $\omega > \lambda^4\gamma$, allowing one to freely rearrange sums and manipulate derivatives. Liberal application of Cauchy-Schwartz on each resulting summand yields

$$\left(\sum_{j=0}^{\infty} b_j \omega^{-j}\right)' \geq (\lambda\omega - 1 - \omega^{-1/2} - \beta\lambda\omega^{-1/2} - \beta\lambda^{-1}\omega)A - \left(\frac{1}{1-\lambda^{4\gamma}\omega^{-1}}\right)\sqrt{A}. \tag{5}$$

Call the sum on the left X, the first constant on the right C_1, and the second constant $-C_2$. Notice by Cauchy-Schwartz we have

$$A \geq \sqrt{1-\omega^{-1}}X^2. \tag{6}$$

If the last constant is denoted C_3, then local integrability of $||u(t)||_s$ is equivalent to local integrability of solutions to

$$\begin{cases} X' \geq C_1 A - C_2\sqrt{A} \\ A \geq C_3 X^2. \end{cases} \tag{7}$$

Jeong and Li further simplify this system by picking some $\eta > 0$ such that

$$\sqrt{A} \leq \eta A + \frac{1}{4\eta}, \tag{8}$$

resulting in the simpler equality

$$X' > D_1 A - D_2. \tag{9}$$

In terms of the old constants, $D_1 = C_3^2(C_1 - \eta C_2)$ and $D_2 = C_2/(4\eta)$. It is easy to see that a solution to (9) will blow-up in finite time if $D_1 > 0$ and the initial conditions $X(0) > \sqrt{D_1/D_2}$. At the expense of taking η veyr small it is enough to jointly solve

$$\begin{aligned} \omega &\geq \lambda^{2(1-s)} \\ \omega &\geq \lambda^{4\gamma} \\ \lambda^2\omega^{-1} - \omega^{-1/2} &\geq \beta(\lambda\omega^{-1/2} + \lambda^{-1}\omega). \end{aligned}$$

If $\beta = 0$ then the first two equations give the restrictions $s > 1/3$ and $\gamma < 1/3$. Once we have fixed these two variables we can then pick ω so that the inequalities are still satisfied for some appropriately small β. □

The first half of our proof is identical to Jeong and Li's - equation (7) is where we diverge.

Proof. (Adapted proof of Theorem 1) Local integrability of $||u(t)||_s$ was seen to be equivalent to local integrability of solutions to (7). Rewriting the first row of (7) establishes

$$X' \geq C_2\sqrt{A}\left(\frac{C_1}{C_2}\sqrt{A} - 1\right). \tag{10}$$

Our desire is that X is always increasing; to this end we need $A(0) > (C_2/C_1)^2$. Further assume we have equality in the second line of (7) at $t = 0$, so $A(0) = C_3 X(0)^2$ and hence $X(0) > C_2/(\sqrt{C_3}C_1)$.

Combining the pieces,

$$X' \geq C_1C_3X^2 - C_2\sqrt{C_3}X. \tag{11}$$

Dividing and integrating further reduces the problem to analysing the inequality

$$\int_0^t \frac{X'}{C_1C_3X^2 - C_2\sqrt{C_3}X}dt \geq \int_0^t 1dt. \tag{12}$$

Partial fractions cracks this integral, resulting in

$$\frac{1}{C_2\sqrt{C_3}} \log\left(C_1C_3 - \frac{C_2\sqrt{C_3}}{X}\right) \geq t - C_4. \tag{13}$$

where C_4 is a constant depending on the initial conditions. Equivalently,

$$C_1C_3 - \exp(C_2\sqrt{C_3}(t - C_4)) \geq \frac{C_2\sqrt{C_3}}{X}. \tag{14}$$

The left hand side becomes negative at some $t_0 > 0$; the right hand side is always positive. This gives us a contradiction on the local integrability of A (and thus $||u||_s$) at $t = t_0$. □

We now establish the (slight) superiority of our new bound.

The smallest lower bound on $X(0)$ from Theorem 1 is $C_2/(C_1C_3)$.

Proof. In terms of C_1, C_2, and C_3, the initial condition in Theorem 1 was

$$\sqrt{\frac{D_1}{D_2}} = \sqrt{\frac{C_2/(4\eta)}{C_3^2(C_1 - \eta C_2)}}. \tag{15}$$

As a function of η, this is minimized at $\eta_0 = C_1/(2C_2)$ and takes value $C_2/(C_1C_3)$. □

Our proof allows a accommodates a lower bound on $X(0)$ of $C_2/(C_1\sqrt{C_3})$. As $C_3 < 1$, this is strictly smaller than the previous bound.

3. Closing Remarks

Our proof inherits many of the drawbacks of the original; in particular, it does not give us a heuristic understanding of why the equations blow up. Without a description of the energy of the system we are unable to verify our intuition that the energy should move forward at some rapid pace. Crude analytic methods such as ours can probably not be expected to provide much more insight into the behaviour of (1), and new methods will have to be contrived.

At the time of writing, nobody appears to have studied the other extreme (when α is small relative to β). Intuitively one would expect these equations to stay regular, but this intuition has proven difficult to verify due to difficulties establishing monotonicity of the energy.

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