Zad. 1

a)

x_k	0	2	4
y_k	-8	8	-8

$$s(x) = \begin{cases} s_1(x) = Ax^3 + Bx^2 + Cx + D & : x \in [0,2] \\ s_2(x) = Ex^3 + Fx^2 + Gx + H & : x \in [2,4] \end{cases}$$

$s_1(0) = -8 \implies D = -8$

$s_2(2) = 8 = s_1(2) \implies 8A + 4B + 2C + D = 8$
$8E + 4F + 2G + H = 8$

$s_2(4) = -8 \implies 64E + 16F + 4G + H = -8$

$$s'(x) = \begin{cases} s_1'(x) = 3Ax^2 + 2Bx + C & : x \in [0,2] \\ s_2'(x) = 3Ex^2 + 2Fx + G & : x \in [2,4] \end{cases}$$

$s_1'(2) = s_2'(2) \implies 12A + 4B + C = 12E + 4F + G$

$$s''(x) = \begin{cases} s_1''(x) = 6Ax + 2B & : x \in [0,2] \\ s_2''(x) = 6Ex + 2F & : x \in [2,4] \end{cases}$$

$s_1''(2) = s_2''(2) \implies 12A + 2B = 12E + 2F$

$s_1''(0) = s_2''(4) = 0 \implies 2B = 24E + 2F = 0 \implies F = -12E$

$D = -8,\ B = 0$

$8A + 4B + 2C = 16$

$8E + 4F + 2G - 64E - 16F - 4G - 16 = 0$

$-56E - 12F - 2G - 16 = 0$

$-56E + 144E - 2G - 16 = 0$

$88E - 2G = 16 \quad /:2$

$44E - G = 8 \implies G = 44E - 8$

$12A + 2B = 12E - 24E = -12E$

$12A = -12E$

$A = -E$

$12A + 4B + C = 12E + 4F + G$

$-12E + C = 12E - 48E + 44E - 8$

$24E - 4E - C = 8$

$20E - 8 = C$

$8A + 4B + 2C + D = 8$

$-8E + 40E - 16 - 8 = 8$

$32E = 32 \implies E = 1,\ A = -1$

$G = 36,\ C = 12,\ F = -12$

$8E + 4F + 2G + H = 8$

$8-48+72+H=8$

$H=8-8+48-72=-24$

$$S(x)=\begin{cases}-x^3+12x-8 : x\in[0,2]\\ x^3-12x^2+36x-24 : x\in[2,4]\end{cases}$$

b)

x_k	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1
y_k	4	2	-6	-24

$h_k=x_k-x_{k-1}$

$M_0=M_3=0$

$$\lambda_k=\frac{h_k}{h_k+h_{k+1}}$$

$h_1=-\frac{1}{2}+1=\frac{1}{2}$

$h_2=\frac{1}{2}+\frac{1}{2}=1$

$h_3=1-\frac{1}{2}=\frac{1}{2}$

$$S_k(x)=h_k^{-1}\left[\frac{1}{6}M_{k-1}(x_k-x)^3+\frac{1}{6}M_k(x-x_{k-1})^3+\left(y_{k-1}-\frac{1}{6}M_{k-1}h_k^2\right)(x_k-x)+\right.$$
$$\left.+\left(y_k-\frac{1}{6}M_kh_k^2\right)(x-x_{k-1})\right]$$ – k-ty segment NIFS3

Obliczamy M_1, M_2

$$\lambda_k M_{k-1}+2M_k+(1-\lambda_k)M_{k+1}=6f[x_{k-1},x_k,x_{k+1}]$$

$$\lambda_1 \overset{0}{M_0}+2M_1+(1-\lambda_1)M_2=6\cdot f[x_0,x_1,x_2]=-16$$

$$\lambda_2M_1+2M_2+(1-\lambda_2)\underset{0}{M_3}=6\cdot f[x_1,x_2,x_3]=112$$

x_k	y_k		
-1	4		
$-\frac{1}{2}$	2	-4	
$\frac{1}{2}$	-6	-8	$-\frac{8}{3}$
1	-24	-36	$-\frac{56}{3}$

Obliczamy λ_1 i λ_2

$$\lambda_1=\frac{\frac{1}{2}}{\frac{3}{2}}=\frac{1}{3} \qquad \lambda_2=\frac{1}{\frac{3}{2}}=\frac{2}{3}$$

$\lambda_1 \underset{0}{M_0}+2M_1+(1-\lambda_1)M_2=6\cdot\left(-\frac{8}{3}\right)$

$2M_1+\frac{2}{3}M_2=-16 \quad /\cdot 3$

$6M_1+2M_2=-48$

$2M_2=-48-6M_1$

$\frac{2}{3}M_1-48-6M_1=-112$

$\frac{16}{3}M_1=64 \Rightarrow M_1=12,\ M_2=-60$

$\lambda_2M_1+2M_2+(1-\lambda_2)\overset{0}{M_3}=6\cdot\left(-\frac{56}{3}\right)$

$\frac{2}{3}M_1+2M_2=-112$

$$\underset{x\in[-1;\frac{1}{2}]}{S_1(x)}=2\left[\frac{1}{6}\cdot12\cdot(x+1)^3+4\left(-\frac{1}{2}-x\right)+\left(2-\frac{1}{6}\cdot\frac{1}{4}\cdot12\right)(x+1)\right]=$$

$$=4(x+1)^3-4-8x+3(x+1)=4x^3+12x^2+12x+4-4-8x+3x+3=$$

$$=4x^3+12x^2+7x+3$$

$$\underset{x\in[-\frac12,\frac12]}{S_2(x)} = \frac{1}{6}\cdot 12\left(\frac{1}{2}-x\right)^3 + \frac{1}{6}\cdot(-60)\left(x+\frac{1}{2}\right)^3 + \left(2-\frac{1}{6}\cdot 12\right)\left(\frac{1}{2}-x\right) +$$
$$+\left(-6-\frac{1}{6}\cdot(-60)\right)\left(x+\frac{1}{2}\right) = 2\left(\frac{1}{2}-x\right)^3 - 10\left(x+\frac{1}{2}\right)^3 + 4\left(x+\frac{1}{2}\right) =$$
$$= \frac{1}{4} - \frac{3}{2}x + 3x^2 - 2x^3 - 10x^3 - 15x^2 - \frac{15}{2}x - \frac{5}{4} + 4x + 2 =$$
$$= -12x^3 - 12x^2 - 5x + 1$$

$$\underset{x\in[\frac12,1]}{S_3(x)} = 2\left[\frac{1}{6}\cdot(-60)(1-x)^3 + \left(-6-\frac{1}{6}\cdot(-60)\cdot\frac{1}{4}\right)(1-x) +\right.$$
$$\left.+(-24)\left(x-\frac{1}{2}\right)\right] = -20(1-x)^3 + 2\left(-\frac{7}{2}\right)(1-x) +$$
$$-48\left(x-\frac{1}{2}\right) = -20 + 60x - 60x^2 + 20x^3 - 7 + 7x - 48x + 24 =$$
$$= 20x^3 - 60x^2 + 19x - 3$$

Zad. 2

1° $f(x_k) = y_k$

2° ciągłość f: $(-1)^3 + 6(-1)^2 + 18(-1) + 13 = -5(-1)^3 - 12(-1)^2 + 7$

$-1 + 6 - 18 + 13 = 5 - 12 + 7$

$0 = 0$ ✓ } ciągła w $x=-1$

$7 = 7$ ✓ } ciągła w $x=0$

$5 - 12 + 7 = -1 + 6 - 18 + 13$

$0 = 0$ ✓ } ciągła w $x=1$

3° ciągłość f': $f_1'(x) = 3x^2 + 12x + 18$

$f_2'(x) = -15x^2 - 24x$

$f_3'(x) = 15x^2 - 24x$

$f_4'(x) = -3x^2 + 12x - 18$

$3 - 12 + 18 = -15 + 24$

$9 = 9$ ✓ } ciągła w $x=-1$

$0 = 0$ } ciągła w $x=0$

$15 - 24 = -3 + 12 - 18$

$-9 = -9$ ✓ } ciągła w $x=1$

4° ciągłość f'': $f_1''(x) = 6x + 12$

$f_2''(x) = -30x - 24$

$f_3''(x) = 30x - 24$

$f_4''(x) = -6x + 12$

$-6 + 12 = 30 - 24$

$6 = 6$ ✓ } ciągła w $x=-1$

$$-24 = -24 \checkmark \; \} \text{ ciągła w } x=0$$

$$\left.\begin{array}{l} 30-24 = -6+12 \\ 6 = 6 \checkmark \end{array}\right\} \text{ ciągła w } x=1$$

5° $f_{1,2,3,4}(x) \in \Pi_3$ ✓

6° $f''(x_0) = f''(x_n)$

$f''(-2) = 0 = f''(2)$ ✓

funkcja $f(x)$ jest NIFS3

Zad. 3

$$f(x) = \begin{cases} f_1(x) = 2020x & : x \in [-2,-1] \\ f_2(x) = ax^3+bx^2+cx+d & : x \in [-1,1] \\ f_3(x) = -2020x & : x \in [1,2] \end{cases}$$

1) ciągłość f

$f_1(-1) = -2020$

$f_2(-1) = -a+b-c+d = -2020$

$f_3(1) = -2020$

$f_2(1) = a+b+c+d = -2020$

2) ciągłość f'

$f_1'(-1) = 2020$

$f_2'(-1) = 3a-2b+c = 2020$

$f_3'(1) = -2020$

$f_2'(1) = 3a+2b+c = -2020$

3) ciągłość f''

$f_1''(-1) = 0$

$f_2''(-1) = -6a+2b = 0$

$f_3''(1) = 0$

$f_2''(1) = 6a+2b = 0$

Mamy układ równań

$$\left\{\begin{array}{l} -a+b-c+d = -2020 \\ a+b+c+d = -2020 \\ \left.\begin{array}{l} 3a-2b+c = 2020 \\ 3a+2b+c = -2020 \end{array}\right\} -4b = 4040 \Rightarrow b = -1010 \\ \left.\begin{array}{l} -6a+2b = 0 \\ 6a+2b = 0 \end{array}\right\} 4b = 0 \Rightarrow b = 0 \end{array}\right\} \text{sprzeczność}$$

Zatem nie istnieją takie a, b, c, d

Zad. 4

$$(*)\ \lambda_k M_{k-1} + 2M_k + (1-\lambda_k)M_{k+1} = d_k \quad (k = 1, 2, \dots, n-1)$$

gdzie $M_0 = M_n = 0$, $d_k = 6f[x_{k-1}, x_k, x_{k+1}]$, $\lambda_k = \frac{h_k}{h_k + h_{k+1}}$,

$h_k = x_k - x_{k-1}$

Algorytm:

$$\left.\begin{aligned} &q_0 := 0 \\ &u_0 := 0 \\ &p_k := \lambda_k q_{k-1} + 2 \\ &q_k := (\lambda_k - 1)/p_k \\ &u_k := (d_k - \lambda_k u_{k-1})/p_k \end{aligned}\right\} k = 1, 2, \dots, n-1$$

$$(**)\begin{cases} M_{n-1} = u_{n-1}, \\ M_k = u_k + q_k M_{k+1} \\ (k = n-2, n-3, \dots, 1) \end{cases}$$

Dowód indukcyjny $(**)$:

dla $k = 1$

$p_1 = \lambda_1 \overset{0}{q_0} + 2 = 2$, $u_1 = (d_1 - \lambda_1 \overset{0}{u_0})/p_1 = \frac{d_1}{p_1}$,

$q_1 = \frac{\lambda_1 - 1}{p_1}$

$(*)\quad \lambda_1 \overset{0}{M_0} + 2M_1 + (1-\lambda_1)M_2 = d_1 \quad /: p_1 = 2$

$$M_1 + \left(\frac{1-\lambda_1}{p_1}\right)M_2 = \frac{d_1}{p_1}$$

$$M_1 - \left(\frac{\lambda_1 - 1}{p_1}\right)M_2 = \frac{d_1}{p_1}$$

$$M_1 = \frac{d_1}{p_1} + \left(\frac{\lambda_1 - 1}{p_1}\right)M_2 = u_1 + q_1 M_2 \ ✓$$

Załóżmy, że $(**)$ zachodzi dla k, pokażemy dla $k+1$:

$(*)$ dla $k+1$: $\lambda_{k+1}M_k + 2M_{k+1} + (1-\lambda_{k+1})M_{k+2} = d_{k+1}$

$(**)$ dla k: $M_k = u_k + q_k M_{k+1} \quad / \cdot \lambda_{k+1}$

$$M_k \lambda_{k+1} = u_k \lambda_{k+1} + q_k M_{k+1} \lambda_{k+1} \quad / - (*) \text{ dla } k+1$$

$$\overbrace{M_k \lambda_{k+1} - \lambda_{k+1} M_k}^{=0} - 2M_{k+1} - (1-\lambda_{k+1}) M_{k+2} = u_k \lambda_{k+1} + q_k M_{k+1} \lambda_{k+1} - d_{k+1}$$

$$-2M_{k+1} - q_k M_{k+1} \lambda_{k+1} - (1-\lambda_{k+1}) M_{k+2} = u_k \lambda_{k+1} - d_{k+1} \quad / \cdot (-1)$$

$$\underbrace{(2 + q_k \lambda_{k+1})}_{= p_{k+1}} M_{k+1} - (\lambda_{k+1} - 1) M_{k+2} = d_{k+1} - u_k \lambda_{k+1} \quad / : p_{k+1}$$

$$M_{k+1} - \underbrace{\left(\frac{\lambda_{k+1} - 1}{p_{k+1}}\right)}_{q_{k+1}} M_{k+2} = \underbrace{\frac{d_{k+1} - u_k \lambda_{k+1}}{p_{k+1}}}_{u_{k+1}}$$

$$M_{k+1} - q_{k+1} M_{k+2} = u_{k+1}$$

$$M_{k+1} = u_{k+1} + q_{k+1} M_{k+2} \quad \blacksquare$$

Momenty znajdujemy w czasie liniowym.