Zad. 5

Z wykładu wiadomo, że:

$$a = \frac{(N+1)s_4 - s_1 s_3}{(N+1)s_2 - s_1^2}, \quad b = \frac{s_2 s_3 - s_1 s_4}{(N+1)s_2 - s_1^2}$$

$N+1 = 8$
$N = 8$
$s_i = \sum_{k=0}^{N} x_k^i \ (i = 1,2)$
$s_3 = \sum_{k=0}^{N} f(x_k)$
$s_4 = \sum_{k=0}^{N} x_k f(x_k)$

$$s_1 = \sum_{k=0}^{N=7} x_k = 365 \qquad s_2 = \sum_{k=0}^{N=7} x_k^2 = 26525$$

$$s_3 = \sum_{k=0}^{N=7} f(x_k) = 514,5$$

$$s_4 = \sum_{k=0}^{N=7} x_k f(x_k) = 671 + 1328 + 1968 + 2584 + 4944 + 5700 = 22685$$

$$a = \frac{8 \cdot 22685 - 365 \cdot 514,5}{8 \cdot 22685 - 365^2} = \frac{-6312,5}{78975} = -0,0799$$

$$b = \frac{26525 \cdot 514,5 - 365 \cdot 22685}{8 \cdot 26525 - 365^2} = \frac{5367087,5}{78975} = 67,9593$$

Zad. 2

$$y(x) = ax(2021x - 2020) + 1977$$

$$\|f - y\|_2 = \sqrt{\sum_{k=0}^{N} [f(x_k) - y(x_k)]^2} = \sqrt{\underbrace{\sum_{k=0}^{N} [f(x_k) - a x_k (2021 x_k - 2020) + 1977]^2}_{E(a)}}$$

Szukamy minimum $E(a)$, więc policzmy pochodną tej funkcji i sprawdźmy, kiedy jest równa 0:

$$E'(a) = 2\sum_{k=0}^{N} \left[\left(f(x_k) - a x_k (2021 x_k - 2020) + 1977 \right) \left(-2021 x_k^2 + 2020 x_k \right) \right] = 0 \ /:2$$

$$\sum_{k=0}^{N} \left[\left(f(x_k) - a x_k (2021 x_k - 2020) + 1977 \right) (2020 - 2021 x_k) x_k \right] = 0$$

$$\sum_{k=0}^{N} \left[(f(x_k) + 1977)(2020 - 2021 x_k) x_k - a x_k (2020 - 2021 x_k)^2 x_k \right] = 0$$

$$\sum_{k=0}^{N} (f(x_k) + 1377)(2020 - 2021 x_k) x_k - \sum_{k=0}^{N} a x_k (2020 - 2021 x_k)^2 x_k = 0$$

$$a = \frac{\sum_{k=0}^{N} (f(x_k) + 1377)(2020 - 2021 x_k) x_k}{\sum_{k=0}^{N} x_k^2 (2020 - 2021 x_k)}$$

Zad. 3

$$E(a) = \sum_{k=0}^{N} \frac{e^{x_k} - 2020}{1 + \ln(x_k^2 + 1)} \left[y_k - a(\cos(2x_k + 2020) + x_k^3) \right]^2$$

Szukamy a, dla którego $E(a)$ ma najmniejszą możliwą wartość

Niech $b_k = \frac{e^{x_k} - 2020}{1 + \ln(x_k^2 + 1)}$, $c_k = \cos(2x_k + 2020) + x_k^3$

$$E(a) = \sum_{k=0}^{N} b_k (y_k - a c_k)^2$$

Policzmy pochodną i sprawdźmy, kiedy jest równa 0:

$$E'(a) = 2\sum_{k=0}^{N} b_k (y_k - a c_k) \cdot (-c_k) = 0 \ /:(-2)$$

Policzmy pochodną i sprawdźmy, kiedy jest równa 0:

$$E'(a) = 2\sum_{k=0}^{N} b_k(y_k - ac_k)\cdot(-c_k) = 0 \quad /:(-2)$$

$$\sum_{k=0}^{N} b_k(y_k - ac_k)c_k = 0$$

$$\sum_{k=0}^{N} b_k y_k c_k - ac_k^2 b_k = 0$$

$$\sum_{k=0}^{N} b_k y_k c_k - a\sum_{k=0}^{N} c^2 b_k = 0$$

$$a = \frac{\sum_{k=0}^{N} b_k y_k c_k}{\sum_{k=0}^{N} c_k^2 b_k}$$

$$a = \frac{\sum_{k=0}^{N} \frac{e^{x_k}-2020}{1+\ln(x_k^2+1)}\cdot\left(\cos(2x_k+2020)+x_k^3\right)\cdot y_k}{\sum_{k=0}^{N} \frac{e^{x_k}-2020}{1+\ln(x_k^2+1)}\cdot\left(\cos(2x_k+2020)+x_k^3\right)^2}$$

Zad. 6

$$y = e^{ax+b} \quad /\ln$$

$$\ln y = ax+b$$

$$g = ax+b$$

$$y = e^g$$

$$\|f-g\|_2 = \sqrt{\sum_{k=0}^{N}\left[f(x_k)-g(x_k)\right]^2} = \sqrt{\sum_{k=0}^{N}\left[f(x_k)-ax_k-b\right]^2}$$

Z wykładu wiadomo, że:

$$a = \frac{(N+1)s_4 - s_1 s_3}{(N+1)s_2 - s_1^2}, \qquad b = \frac{s_2 s_3 - s_1 s_4}{(N+1)s_2 - s_1^2}$$

$$s_1 = \sum_{k=0}^{N} x_k \qquad s_2 = \sum_{k=0}^{N} x_k^2 \qquad s_2 = \sum_{k=0}^{N} x_k^i \ (i=1,2)$$

$$s_3 = \sum_{k=0}^{N} f(x_k) \qquad s_4 = \sum_{k=0}^{N} x_k f(x_k) \qquad s_3 = \sum_{k=0}^{N} f(x_k) \qquad s_4 = \sum_{k=0}^{N} x_k f(x_k)$$

$$y = e^{\frac{(N+1)s_4 - s_1 s_3}{(N+1)s_2 - s_1^2}\cdot x + \frac{s_2 s_3 - s_1 s_4}{(N+1)s_2 - s_1^2}}$$

Zad. 7

Pokażemy, że są spełnione następujące warunki:

1° $\|f\| \geqslant 0$

$$\sum_{k=0}^{N} \underset{\uparrow}{p(x_k)}\, \underbrace{f(x_k)^2}_{>0} \geqslant 0 \implies \|f\| \geqslant 0$$

$p(x_k) > 0$ dla $x \in \mathfrak{X}$

$2^\circ \quad \|\alpha f\| = |\alpha| \cdot \|f\|$

$$\|\alpha f\| = \sqrt{\sum_{k=0}^{N} p(x_k)(\alpha f(x_k))^2} = |\alpha| \sqrt{\sum_{k=0}^{N} p(x_k) f(x_k)^2} = |\alpha| \cdot \|f\|$$

$3^\circ \quad \|f+g\| \leq \|f\| + \|g\|$

$$\|f+g\| = \sqrt{\sum_{k=0}^{N} p_k \cdot \underbrace{(f_k+g_k)^2}_{= f_k^2 + 2f_kg_k + g_k^2}} \leq \sqrt{\sum_{k=0}^{N} p_k \cdot f_k^2} + \sqrt{\sum_{k=0}^{N} p_k \cdot g_k^2} \quad /()^2$$

$$\sum_{k=0}^{N} p_k f_k^2 + 2\sum_{k=0}^{N} p_k f_k g_k + \sum_{k=0}^{N} p_k g_k^2 \leq \sum_{k=0}^{N} p_k f_k^2 + 2 \cdot \sqrt{\sum_{k=0}^{N} p_k \cdot f_k^2} \cdot \sqrt{\sum_{k=0}^{N} p_k \cdot g_k^2} + \sum_{k=0}^{N} p_k g_k^2$$

$$\sum_{k=0}^{N} p_k f_k g_k \leq \sqrt{\sum_{k=0}^{N} p_k \cdot f_k^2} \cdot \sqrt{\sum_{k=0}^{N} p_k \cdot g_k^2}$$

$$\sum_{k=0}^{N} (\sqrt{p_k} \cdot f_k)(\sqrt{p_k} \cdot g_k) \leq \sqrt{\sum_{k=0}^{N} (\sqrt{p_k} \cdot f_k)^2} \cdot \sqrt{\sum_{k=0}^{N} (\sqrt{p_k} \cdot g_k)^2} \quad /()^2$$

$$\underbrace{\left[\sum_{k=0}^{N} (\sqrt{p_k} \cdot f_k)(\sqrt{p_k} \cdot g_k)\right]^2 \leq \left(\sum_{k=0}^{N} (\sqrt{p_k} \cdot f_k)^2\right)\left(\sum_{k=0}^{N} (\sqrt{p_k} \cdot g_k)^2\right)}_{(*) \text{ NIERÓWNOŚĆ CAUCHY'EGO - SCHWARZA}}$$

(*) - dowód

$$\left(\sum_{i=0}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=0}^{n} x_i^2\right)\left(\sum_{i=0}^{n} y_i^2\right)$$

$$\sum_{i=0}^{n}\sum_{j=0}^{n} (x_i y_j - x_j y_i)^2 = \sum_{i=0}^{n}\sum_{j=0}^{n} (x_i^2 y_j^2 - 2x_i y_j x_j y_i + x_j^2 y_i^2) =$$

$$= \sum_{i=0}^{n}\sum_{j=0}^{n} x_i^2 y_j^2 - 2\sum_{i=0}^{n}\sum_{j=0}^{n} x_i y_j x_j y_i + \sum_{i=0}^{n}\sum_{j=0}^{n} x_j^2 y_i^2 =$$

$$= \sum_{i=0}^{n} x_i^2 \sum_{j=0}^{n} y_j^2 + \sum_{j=0}^{n} x_j^2 \sum_{i=0}^{n} y_i^2 - 2\sum_{i=0}^{n} x_i y_i \sum_{j=0}^{n} x_j y_j =$$

$$= 2\left(\sum_{i=0}^{n} x_i^2\right)\left(\sum_{i=0}^{n} y_i^2\right) - 2\left(\sum_{i=0}^{n} x_i y_i\right)^2 \geq 0 \quad \left(\text{bo } \sum_{i=0}^{n}\sum_{j=0}^{n} \overbrace{(x_i y_j - x_j y_i)^2}^{\geq 0} \geq 0\right)$$

czyli $\left(\sum_{i=0}^{n} x_i^2\right)\left(\sum_{i=0}^{n} y_i^2\right) - \left(\sum_{i=0}^{n} x_i y_i\right)^2 \geq 0$,

zatem $\left(\sum_{i=0}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=0}^{n} x_i^2\right)\left(\sum_{i=0}^{n} y_i^2\right)$ $\blacksquare$