

Zadanie 2

$$A = \begin{bmatrix} 2 & 0 & 2 & 1 \\ -4 & 2 & -4 & 0 \\ -6 & 8 & -4 & 5 \\ -10 & 12 & -24 & 9 \end{bmatrix}$$

$$① \quad u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}$$

$$② \quad l_{ji} = \frac{1}{u_{ii}} \left(a_{ji} - \sum_{k=1}^{i-1} l_{jk} u_{ki} \right)$$

Dla $j = 1, 2, \dots, n$: wykonujemy kolejno dwie poniższe operacje:

Dla $i = 1, 2, \dots, j$: wykorzystujemy wzór z punktu 1) do wyznaczenia elementów u_{ij} .

Dla $i = j+1, j+2, \dots, n$: wykorzystujemy wzór z punktu 2) do wyznaczenia elementów l_{ij} .

$j=1$

$$u_{11} = a_{11} = 2$$

$$l_{21} = \frac{1}{u_{11}} \cdot a_{21} = -2$$

$$l_{31} = \frac{1}{u_{11}} \cdot a_{31} = -3$$

$$l_{41} = \frac{1}{u_{11}} \cdot a_{41} = -5$$

$j=2$

$$u_{12} = a_{12} = 0$$

$$u_{22} = a_{22} - l_{21} u_{12} = 2 + 2 \cdot 0 = 2$$

$$l_{32} = \frac{a_{32} - l_{31} u_{12}}{u_{22}} = \frac{8 + 3 \cdot 0}{2} = 4$$

$$l_{42} = \frac{a_{42} - l_{41} u_{12}}{u_{22}} = \frac{12 + 5 \cdot 0}{2} = 6$$

$j=3$

$$u_{13} = a_{13} = 2$$

$$u_{23} = a_{23} - l_{21} u_{13} = -4 + 2 \cdot 2 = 0$$

$$u_{33} = a_{33} - l_{31} u_{13} - l_{32} u_{23} = -4 + 3 \cdot 2 - 4 \cdot 0 = 2$$

$$l_{43} = \frac{a_{43} - l_{41} u_{13} - l_{42} u_{23}}{u_{33}} = \frac{-24 + 5 \cdot 2 - 6 \cdot 0}{2} = -7$$

$j=4$

$$u_{14} = a_{14} = 1$$

$$u_{24} = a_{24} - l_{21} u_{14} = 0 + 2 \cdot 1 = 2$$

$$u_{34} = a_{34} - l_{31} u_{14} - l_{32} u_{24} = 5 + 3 \cdot 1 - 4 \cdot 2 = 0$$

$$u_{44} = a_{44} - l_{41} u_{14} - l_{42} u_{24} - l_{43} u_{34} = 9 + 5 \cdot 1 - 6 \cdot 2 + 7 \cdot 0 = 2$$

$$U = \begin{bmatrix} 2 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -3 & 4 & 1 & 0 \\ -5 & 6 & -7 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 4 & 1 & 0 \\ -5 & 6 & -7 & 1 \end{bmatrix}$$

$$\det(U) = 2 \cdot 2 \cdot 2 \cdot 2 = 16 \quad \det(L) = 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

$$\det(A) = \det(U) \cdot \det(L) = 16$$

$$A^{-1} = U^{-1}L^{-1}$$

Obliczamy L^{-1}

$$[L|I] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -3 & 4 & 1 & 0 & 0 & 0 & 1 & 0 \\ -5 & 6 & -7 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\substack{U_2 + 2U_1 \\ U_4 + 5U_1 \\ U_3 + 3U_1}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 6 & -7 & 1 & 5 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\substack{U_3 - 4U_2 \\ U_4 - 6U_2}}$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -5 & -4 & 1 & 0 \\ 0 & 0 & -7 & 1 & -7 & -6 & 0 & 1 \end{array} \right] \xrightarrow{U_4 + 7U_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -5 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & -42 & -34 & 7 & 1 \end{array} \right] \underset{L^{-1}}{\parallel}$$

Obliczamy U^{-1}

$$[U|I] = \left[\begin{array}{cccc|cccc} 2 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\substack{U_1/2 \\ U_2/2 \\ U_3/2 \\ U_4/2}} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \end{array} \right] \xrightarrow{\substack{U_1 - U_3 \\ U_2 - U_4}}$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \end{array} \right] \xrightarrow{U_1 - \frac{1}{2}U_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{4} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \end{array} \right] \underset{U^{-1}}{\parallel}$$

$$A^{-1} = U^{-1} \cdot L^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{4} \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -5 & -4 & 1 & 0 \\ -42 & -34 & 7 & 1 \end{bmatrix} = \begin{bmatrix} \frac{27}{2} & \frac{21}{2} & -\frac{9}{4} & -\frac{1}{4} \\ 22 & \frac{35}{2} & -\frac{7}{2} & -\frac{1}{2} \\ -\frac{5}{2} & -2 & \frac{1}{2} & 0 \\ -21 & -17 & \frac{7}{2} & \frac{1}{2} \end{bmatrix}$$