ANL - lista 3

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Zad. 1

a) $4\cos^2 x - 3$

Nie działa, gdy

$4\cos^2 x \approx 3$

$\cos^2 x = \frac{3}{4}$

$\cos x = \frac{\sqrt{3}}{2} \vee \cos x = -\frac{\sqrt{3}}{2}$

$x \approx \frac{\pi}{6} + 2k\pi \vee x \approx -\frac{\pi}{6} + 2k\pi \quad k \in \mathbb{Z}$

z 1 trygonometrycznej

$4\cos^2 x - 3 = 4\cos^2 x - 3\sin^2 x - 3\cos^2 x =$

$= \cos^2 x - \sin^2 x - 2\sin^2 x =$ $\cos 2x$

$= (\cos 2x - 2\sin^2 x) \cdot \frac{\cos x}{\cos x} =$ $\sin 2x$

$= \frac{\cos 2x \cos x - 2\sin x \cos x \sin x}{\cos x} =$

$\cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha+\beta)$

$= \frac{\cos 2x \cos x - \sin 2x \sin x}{\cos x} =$

$= \frac{\cos 3x}{\cos x}$

b) $x^{-3}\left(\frac{\pi}{2} - x - \operatorname{arctg}(x)\right) =$

$= x^{-3}\left(\frac{\pi}{2} - x - \left(\frac{\pi}{2} - \operatorname{arctg}(x)\right)\right) =$

$= x^{-3}(\operatorname{arctg}(x) - x)$

Nie działa, gdy $x = 0$

$\operatorname{arctg}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

$x^{-3}(\operatorname{arctg}(x) - x) = -\frac{1}{3} + \frac{x^2}{5} - \frac{x^4}{7} + \frac{x^6}{9} - \dots$

Zad. 2

$x_{1,2} = \left(-b \pm \sqrt{b^2 - 4ac}\right)$

Zad. 2

$$x_{1,2} = \frac{\left(-b \pm \sqrt{b^2-4ac}\right)}{2a}$$

Nie działa dla dużych b i małych a i c.

Korzystamy z wzorów Viete'a:

$$x_1 x_2 = \frac{a}{c} \qquad x_1 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$x_2 = \frac{c}{a} \cdot x_1 = \frac{2c}{-b-\sqrt{b^2-4ac}}$$

(działa dla $b>0$, dla $b<0$ musimy policzyć x_1 korzystając z wzoru z +)

Zad. 3

$$x = \left(v + \sqrt{q^3+v^2}\right)^{\frac{1}{3}} + \left(v - \sqrt{q^3+v^2}\right)^{\frac{1}{3}}$$

$a^3+b^3 = (a^2-ab+b^2)(a+b)$

Problem jest, gdy v jest duże a q małe

$$x = \overset{a=}{\sqrt[3]{v+\sqrt{q^3+v^2}}} + \overset{b=}{\sqrt[3]{v-\sqrt{q^3+v}}} = \frac{(a+b)(a^2-ab+b^2)}{a^2-ab+b^2}$$

Gdy v jest małe a q duże to też się psuje!!!

$$= \left(\sqrt[3]{v+\sqrt{q^3+v^2}} + \sqrt[3]{v-\sqrt{q^3+v}}\right) \cdot \frac{\left(\sqrt[3]{v+\sqrt{q^3+v^2}}\right)^2 - \left(\sqrt[3]{v+\sqrt{q^3+v^2}}\,\sqrt[3]{v-\sqrt{q^3+v^2}}\right) + \left(\sqrt[3]{v-\sqrt{q^3+v^2}}\right)^2}{\left(\sqrt[3]{v+\sqrt{q^3+v^2}}\right)^2 - \left(\sqrt[3]{v+\sqrt{q^3+v^2}}\,\sqrt[3]{v-\sqrt{q^3+v^2}}\right) + \left(\sqrt[3]{v-\sqrt{q^3+v^2}}\right)^2} =$$

$$= \frac{v+\sqrt{q^3+v^2}+v-\sqrt{q^3+v^2}}{\left(\sqrt[3]{v+\sqrt{q^3+v^2}}\right)^2 - \sqrt[3]{v^2-q^3-v^2} + \left(\sqrt[3]{v-\sqrt{q^3+v^2}}\right)^2} = \frac{2v}{\left(\sqrt[3]{v+\sqrt{q^3+v^2}}\right)^2 + \left(\sqrt[3]{v-\sqrt{q^3+v^2}}\right)^2 + q} =$$

$$= \frac{2v}{q + \left(\sqrt[3]{v+\sqrt{q^3+v^2}}\right)^2 + \left(\frac{\sqrt[3]{v-\sqrt{q^3+v^2}} \cdot \sqrt[3]{v+\sqrt{q^3+v^2}}}{\sqrt[3]{v+\sqrt{q^3+v^2}}}\right)^2} =$$

$$= \frac{2r}{q + \left(\sqrt[3]{r + \sqrt{q^3 + r^2}}\right)^2 + \left(\sqrt[3]{\frac{-q^3}{r + \sqrt{q^3+r^2}}}\right)^2} =$$

$$= \frac{2r}{\left(\sqrt[3]{r + \sqrt{q^3 + r^2}}\right)^2 + \left(\sqrt[3]{\frac{1}{r + \sqrt{q^3+r^2}}}\right)^2 \cdot q^2 + q}$$

Zad. 4

Względna zmiana danych:

$$\left| \frac{(x+h) - x}{x} \right| = \left| \frac{h}{x} \right|$$

Względna zmiana wyniku:

$$\left| \frac{f(x+h) - f(x)}{f(x)} \right|$$

cond(f(x)) - uwarunkowanie zadania

$$\mathrm{cond}(f(x)) = \frac{\left| \frac{f(x+h) - f(x)}{f(x)} \right|}{\left| \frac{h}{x} \right|} = \left| \frac{f(x+h) - f(x)}{f(x)} \right| \cdot \left| \frac{x}{h} \right| =$$

$$= \left| \frac{f(x+h) - f(x)}{h} \right| \cdot \left| \frac{x}{f(x)} \right| = \left| f'(x) \right| \cdot \left| \frac{x}{f(x)} \right| = \left| \frac{x \cdot f'(x)}{f(x)} \right|$$

Zad. 5

a) $f(x) = x^3 + 2020 \qquad f'(x) = 3x^2$

$$\mathrm{cond}(f(x)) = \left| \frac{x \cdot 3x^2}{x^3 + 2020} \right| = \left| \frac{3x^3}{x^3 + 2020} \right| = \left| \frac{3}{1 + \frac{2020}{x^3}} \right|$$

$$\mathrm{cond}(f(x)) = \left| \frac{x}{x^3+2020} \cdot \frac{3x^2}{x^3+2020} \right| = \left| \frac{3x^3}{x^3+2020} \right|$$

$$\lim_{x \to -\sqrt[3]{2020}} \left| \frac{3}{1+\frac{2020}{x^3}} \right| = \left[\left| \frac{3}{0} \right| \right] = \infty$$

Źle uwarunkowane dla $x \approx -\sqrt[3]{2020}$

b) $f(x) = x^{-1} \ln(x)$ $\qquad f'(x) = \frac{1-\ln x}{x^2}$

$$\mathrm{cond}(f(x)) = \left| \frac{x \cdot \left(\frac{1-\ln x}{x^2}\right)}{\frac{\ln x}{x}} \right| = \left| \frac{1-\ln x}{x} \right| \cdot \left| \frac{x}{\ln x} \right| =$$

$$= \left| \frac{1-\ln x}{\ln x} \right| = \left| \frac{1}{\ln x} - 1 \right|$$

$$\lim_{x \to 1} \left| \frac{1}{\ln x} - 1 \right| = \infty$$

Źle uwarunkowane dla $x \approx 1$

c) $f(x) = \cos(5x)$ $\qquad f'(x) = -5 \sin 5x$

$$\mathrm{cond}(f(x)) = \left| \frac{x \cdot (-5 \sin 5x)}{\cos 5x} \right| = \left| -5x \,\mathrm{tg}\, 5x \right|$$

$$5x = \frac{\pi}{2} + k\pi$$

$$x = \frac{\pi}{10} + \frac{k\pi}{5} \quad (k \in \mathbb{Z})$$

$$\lim_{x \to \frac{\pi}{10} + \frac{k\pi}{5}} \left| -5x \,\mathrm{tg}\, 5x \right| = \infty$$

Źle uwarunkowane dla $x \approx \frac{\pi}{10} + \frac{k\pi}{5}$, $k \in \mathbb{Z}$

d) $f(x) = \left(\sqrt{x^4+2020} + x\right)^{-1}$

$$f'(x) = -\frac{2x^3 + \sqrt{x^4+2020}}{\left(\sqrt{x^4+2020}+x\right)^2 \cdot \sqrt{x^4+2020}}$$

$$\mathrm{cond}(f(x)) = \left| \frac{\left(2x^4 + x\sqrt{x^4+2020}\right)\left(\sqrt{x^4+2020}+x\right)}{\left(\sqrt{x^4+2020}+x\right)^2 \cdot \sqrt{x^4+2020}} \right| = \left| \frac{2x^4 + x\sqrt{x^4+2020}}{\left(\sqrt{x^4+2020}+x\right) \cdot \sqrt{x^4+2020}} \right| =$$

$$= \left| \frac{2x^4 + x\sqrt{x^4+2020}}{x^4+2020+x\sqrt{x^4+2020}} \right|$$

$$= \left| \frac{2x^4 + x\sqrt{x^4+2020}}{x^4+2020+x\sqrt{x^4+2020}} \right|$$

$$\lim_{x\to} \left| \frac{2x^4 + x\sqrt{x^4+2020}}{x^4+2020+x\sqrt{x^4+2020}} \right|$$

Zad. 7

$$\omega(x) = x + \frac{1}{x} \qquad x \neq 0$$

$$\tilde{\omega}(x) = \left(x + \frac{1}{x}(1+\varepsilon_1)\right)(1+\varepsilon_2) = x + x\varepsilon_2 + \frac{1+\varepsilon_1+\varepsilon_2+\varepsilon_1\varepsilon_2}{x} =$$

$$= x(1+\varepsilon_2) + \frac{1}{x}(1+\varepsilon_1+\varepsilon_2+\varepsilon_1\varepsilon_2) = x(1+\varepsilon_2) + \frac{1}{x}(1+E_1)$$

?

Zad. 8

dla liczb maszynowych:

$$I = x_1\cdot(1+\varepsilon_1)\cdot x_2\cdot(1+\varepsilon_2)\cdot x_3\cdot(1+\varepsilon_3)\cdot \ldots$$

$$I = \prod_{i=1}^{n} x_i(1+\varepsilon_i), \qquad \varepsilon_1 = 0$$

$$I = \prod_{i=1}^{n} x_i \cdot \prod_{i=1}^{n}(1+\varepsilon_i) = \left(\prod_{i=1}^{n} x_i\right)(1+E)$$

z tw. o kumulacji błędów

$$|\varepsilon_i| \leq 2^{-t} \Rightarrow |E| \leq n\cdot 2^{-t}$$

Zatem algorytm jest numerycznie poprawny

dla liczb niemaszynowych:

$$I = x_1(1+\alpha_1)(1+\varepsilon_1)\cdot x_2(1+\alpha_2)(1+\varepsilon_2)\cdot \ldots$$

$$I = \prod_{i=0}^{n} x_i(1+\alpha_i)(1+\varepsilon_i), \qquad \varepsilon_1 = 0$$

$$I = \prod_{i=0}^{n} x_i \cdot \prod_{i=0}^{n}(1+\alpha_i)\cdot \prod_{i=0}^{n}(1+\varepsilon_i) = \left(\prod_{i=0}^{n} x_i\right)(1+E)$$

$$|\alpha_i| \leq 2^{-t} \quad |\varepsilon_i| \leq 2^{-t} \Rightarrow |E| \leq 2n\cdot 2^{-t} = n\cdot 2^{-t+1}$$

$$1 = \prod_{i=0} x_i \cdot \prod_{i=0} (1+\alpha_i) \cdot \prod_{i=0} (1+\varepsilon_i) \cdot \prod_{i=0} \ldots$$

$$|\alpha_i| \leq 2^{-t} \wedge |\varepsilon_i| \leq 2^{-t} \Rightarrow |E| \leq 2n \cdot 2^{-t} = n \cdot 2^{-t+1}$$

Zatem algorytm jest numerycznie poprawny