Zad. 2

Możemy zapisać w jako kombinację liniową $P_0, P_1, \dots, P_{k-1}$.

$$w = \alpha_0 P_0 + \alpha_1 P_1 + \dots + \alpha_{k-1} P_{k-1}$$

Mamy wtedy:

$$(w, P_k)_N = (\alpha_0 P_0 + \alpha_1 P_1 + \dots + \alpha_{k-1} P_{k-1}, P_k)_N =$$

$$= (\alpha_0 P_0, P_k)_N + (\alpha_1 P_1, P_k)_N + \dots + (\alpha_{k-1} P_{k-1}, P_k)_N =$$

$$= \alpha_0 (P_0, P_k)_N + \alpha_1 (P_1, P_k)_N + \dots + \alpha_{k-1} (P_{k-1}, P_k)_N = 0,$$

ponieważ z ortogonalności wiemy, że dla $i \neq k$ $(P_i, P_k)_N = 0$.

∎

Zad. 5

$$B_0 = a_0 + (x - c_1) B_1 - d_2 B_2 = a_0 \overset{1}{\overset{\parallel}{Q_0}} + Q_1 B_1 - d_2 B_2 =$$

$$= a_0 Q_0 + Q_1 (a_1 + (x - c_2) B_2 - d_3 B_3) - d_2 B_2 =$$

$$= a_0 Q_0 + a_1 Q_1 + (x - c_2) Q_1 B_2 - d_3 Q_1 B_3 - d_2 B_2 Q_0 = a_0 Q_0 + a_1 Q_1 + Q_2 B_2 - d_3 Q_1 B_3 =$$

$$Q_2 B_2$$

$$= a_0 Q_0 + a_1 Q_1 + Q_2 (a_2 + (x - c_3) B_3 - d_4 B_4) - Q_1 d_3 B_3 =$$

$$= a_0 Q_0 + a_1 Q_1 + a_2 Q_2 + Q_3 B_3 - Q_2 d_4 B_4 = \dots = \left(\sum_{k=0}^{i} a_k Q_k \right) + Q_{i+1} B_{i+1} - Q_i d_{i+2} B_{i+2} =$$

$$= \left(\sum_{k=0}^{i} a_k Q_k \right) + Q_{i+1} (a_{i+1} + (x - c_{i+2}) B_{i+2} - d_{i+3} B_{i+3}) - Q_i d_{i+2} B_{i+2} =$$

$$= \left(\sum_{k=0}^{i} a_k Q_k \right) + a_{i+1} Q_{i+1} + (x - c_{i+2}) B_{i+2} Q_{i+1} - d_{i+3} B_{i+3} Q_{i+1} - Q_i d_{i+2} B_{i+2} =$$

$$= \left(\sum_{k=0}^{i+1} a_k Q_k \right) + B_{i+2} \Big(\overbrace{(x - c_{i+2}) Q_{i+1} - Q_i d_{i+2}}^{Q_{i+2}} \Big) - d_{i+3} B_{i+3} Q_{i+1} =$$

$$= \left(\sum_{k=0}^{i+1} a_k Q_k \right) + B_{i+2} Q_{i+2} - d_{i+3} B_{i+3} Q_{i+1} = \sum_{k=0}^{m} a_k Q_k(x),$$

bo $B_{\dots} = B_{\dots} = 0$

bo $B_{m+2} = B_{m+1} = 0$. $\blacksquare$

Aby obliczyć Q_m używając tego algorytmu, należy jako a_m przyjąć 1, a za a_k $(0 \le k < m)$ przyjąć 0.

Zad. 6

$$D_4 = \{x_0, x_1, x_2, x_3, x_4\} \quad x_0 = -10,\ x_1 = -5,\ x_2 = 0,\ x_3 = 5,\ x_4 = 10$$

I sposób (ciąg wielomianów ortogonalnych P_n):

$$\begin{cases} P_0 = 1 \\ P_1 = x - c_1 \\ P_k = (x - c_k)P_{k-1} - d_k P_{k-2} \quad (k = 2, 3, \ldots, m) \end{cases}$$

$$c_k = \frac{(xP_{k-1}, P_{k-1})_N}{(P_{k-1}, P_{k-1})_N} \text{ dla } 1 \le k \le m$$

$$d_k = \frac{(P_{k-1}, P_{k-1})_N}{(P_{k-2}, P_{k-2})_N}$$

$$c_1 = \frac{-10 - 5 + 0 + 5 + 10}{1 + 1 + 1 + 1 + 1} = 0$$

$$c_2 = \frac{(-10)^3 + (-5)^3 + 0^3 + 5^3 + 10^3}{(-10)^2 + (-5)^2 + 0^2 + 5^2 + 10^2} = 0 \qquad d_2 = \frac{(-10)^2 + (-5)^2 + 0^2 + 5^2 + 10^2}{1 + 1 + 1 + 1 + 1} = \frac{250}{5} = 50$$

$$P_0 = 1$$
$$P_1 = x$$
$$P_2 = x \cdot x - 50 = x^2 - 50$$

II sposób (ortogonalizacja Grama-Schmidta):

Weźmy liniowo niezależne funkcje: $f_0(x) = 1$, $f_1(x) = x$, $f_2(x) = x^2$

$$\begin{cases} P_0(x) = f_0(x) \\ P_k(x) = f_k(x) - \sum_{j=0}^{k-1} \frac{(f_k, P_j)_N}{(P_j, P_j)_N} \cdot P_j \end{cases}$$

$$P_0(x) = 1$$

$$P_1(x) = x - \frac{-10 - 5 + 0 + 5 + 10}{1 + 1 + 1 + 1 + 1} = x$$

$$P_1(x) = x - \frac{-10-5+0+5+10}{1+1+1+1+1} = x$$

$$P_2(x) = x^2 - \left(\frac{(-10)^2+(-5)^2+0^2+5^2+10^2}{1+1+1+1+1} + \frac{(-10)^3+(-5)^3+0^3+5^3+10^3}{(-10)^2+(-5)^2+0^2+5^2+10^2}\cdot x\right) =$$

$$= x^2 - \frac{250}{5} = x^2 - 50$$

Zad. 7

$x_0 = -10,\ x_1 = -5,\ x_2 = 0,\ x_3 = 5,\ x_4 = 10$

$h(x_0) = 3,\ h(x_1) = -5,\ h(x_2) = -1,\ h(x_3) = -5,\ h(x_4) = 3$

$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = x^2 - 50$

$$\omega_n^*(x) = \sum_{k=0}^{n} a_k P_k \qquad a_k = \frac{(h, P_k)_N}{(P_k, P_k)_N} \quad k \in [0, 2]$$

$$a_0 = \frac{3-5-1-5+3}{5} = -1$$

$$a_1 = \frac{(3\cdot(-10)) + ((-5)\cdot(-5)) + ((-5)\cdot 5) + 3\cdot 10}{(-10)^2+(-5)^2+5^2+10^2} = 0$$

$$a_2 = \frac{(3\cdot 50) + ((-5)\cdot(-25)) + 50 + ((-5)\cdot(-25)) + 3\cdot 50}{50^2 + (-25)^2 + (-50)^2 + (-25)^2 + 50^2} = \frac{150+125+50+125+150}{2500+625+2500+625+2500} =$$

$$= \frac{600}{8750} = \frac{12}{175}$$

$$\omega_2^*(x) = (-1)\cdot 1 + \frac{12}{175}(x^2-50) = \frac{12}{175}x^2 - \frac{600}{175} - \frac{175}{175} = \frac{12}{175}x^2 - \frac{17}{7}$$