Homological stability of the Cremona groups

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The Cremona groups are the groups of birational equivalences of projective spaces. Their abelianisations, and more generally, the homology of these groups, stabilises as the dimension of the projective spaces increases.

Introduction

The Cremona groups $\mathrm{Cr}_n(K) = \mathrm{Bir}(\mathbb{P}^n_K)$ have been the subject of intensive studies for over a century. Nevertheless, it seems fair to say that they are far from well-understood beyond some low-dimensional, exceptional cases, and very little is known about the higher Cremona groups $\mathrm{Cr}_n(K)$. This paper aims to show that we can expect general phenomena to emerge for large n.

Theorem. *For any field K, the canonical homomorphisms*

$$\mathrm{H}_i(\mathrm{Cr}_n(K);\mathbb{Z}) \longrightarrow \mathrm{H}_i(\mathrm{Cr}_{n+1}(K);\mathbb{Z})$$

are epimorphisms for $i \leqslant (n-1)/2$ and isomorphisms for $i \leqslant (n-2)/2$.

This result is Theorem 5.1 in this paper. In particular, for $i = 1$, we see that the abelianisation $\mathrm{Cr}_3(K)^{\mathrm{ab}}$ surjects onto $\mathrm{Cr}_4(K)^{\mathrm{ab}}$, which assumes the stable value: the higher groups $\mathrm{Cr}_4(K)^{\mathrm{ab}} \cong \mathrm{Cr}_5(K)^{\mathrm{ab}} \cong \mathrm{Cr}_6(K)^{\mathrm{ab}} \cong \dots$ are all isomorphic. The phenomenon underlying the theorem is called *homological stability* (see [19] for a recent survey), and our proof follows a strategy introduced by Quillen in his work on the homology of the general linear groups. It is based on a spectral sequence argument that works whenever a family of groups acts nicely on a family of highly connected spaces.

Different actions of the Cremona groups have been studied in a variety of contexts. For example, Wright [18] used an action of $\mathrm{Cr}_2(K)$ on a 2-dimensional simplicial complex that has as vertices certain models of the function field to deduce a description of $\mathrm{Cr}_2(K)$ as an amalgamated product. Earlier, while studying cubic surfaces, Manin [16] introduced a colimit of Picard groups of surfaces that are blown-up projective planes, resulting in an infinite dimensional hyperbolic space, the Picard–Manin space (see also [7]). Very recent progress was made by Lonjou and Urech [15], who constructed CAT(0) cube complexes on which the Cremona groups act by isometries and used these to deduce group-theoretical and dynamical results about Cremona groups, and in [5]. The main subjects of this paper are the actions of the Cremona groups on spaces that resemble buildings and that have, as we shall prove in Theorem 2.1, very nice combinatorial properties: they are Cohen–Macaulay of dimension $n-1$. In particular, they are highly connected.

After some preliminaries about function field extensions (Section 1), we immediately prove Theorem 2.1 about the buildings, and then deduce the high connectivity of some related split buildings (see Theorem 3.1). The use of split buildings in this context has been pioneered by Charney [8]. We then show (in Theorem 4.1) that these split buildings fit into an abstract framework [17] for automorphism groups on homogenous categories. The final Section 5 contains the statement and proof of the theorem above as well as refinements for abelian coefficient modules (see Theorem 5.2) and consequences for the commutator subgroups (see Corollary 5.3).

1 Preliminary results on partial pure bases

We fix a field K. For any integer $n \geqslant 0$, let us write $K[x_1,\dots,x_n]$ for the polynomial K–algebra in the n variables $x_1,\dots,x_n$, and $K(x_1,\dots,x_n)$ for its fraction field. Sometimes, it will be convenient to write $K(n) = K(x_1,\dots,x_n)$ for short. Recall that such field extensions of K are called *pure* or *rational* over K.

Proposition 1.1. *Let $E = K(x_1,\dots,x_n)$ be a rational extension of the field K, and let $J \leqslant \{1,\dots,n\}$ be a set of indices. For intermediate extensions $E|L_j|K$, with $j \in J$,* *we have $L_j = K(x_1,\ldots,\widehat{x_j},\ldots,x_n)$ if and only if, for all $j \in J$, we have $L_j(x_j) = E$ and $x_k \in L_j$ if and only if $k \neq j$. If these conditions are satisfied, then we also have*

$$\bigcap_{j\in J} L_j = K(x_k \,|\, k \notin J).$$

Proof. If we set $L_j = K(x_1,\ldots,\widehat{x_j},\ldots,x_n)$, then $L_j(x_j) = E$ and $x_k \in L_j$ if and only if $k \neq j$. Conversely, if $x_k \in L_j$ for all $k \neq j$, then $K(x_1,\ldots,\widehat{x_j},\ldots,x_n) \leqslant L_j$. Adjoining x_j leads to $E = K(x_1,\ldots,x_n) \leqslant L_j(x_j) \leqslant E$, from which we deduce the equality $L_j(x_j) = E$. The second statement follows immediately from the description $L_j = K(x_1,\ldots,\widehat{x_j},\ldots,x_n)$. □

Throughout this paper, we only consider finitely generated extensions E of K, i.e., finite extensions of rational extensions $K(x_1,\ldots,x_n)$ of K, with n the transcendence degree of E over K. Recall that a set $\{v_1,\ldots,v_n\}$ of elements in a K–algebra R is *algebraically independent* if the morphism $K[x_1,\ldots,x_n] \to R$ of K–algebras that sends x_j to v_j is injective. If that is the case, and $R = E$ is a field extension of K, then this K–algebra morphism admits a unique extension $K(x_1,\ldots,x_n) \to E$, which is also injective. We say that $\{v_1,\ldots,v_n\}$ is a *transcendence basis* of E over K if the extension E is finite over the image $K(v_1,\ldots,v_n)$. We say that it is a *pure basis* if $E = K(v_1,\ldots,v_n)$. Every pure basis is a transcendence basis of E, but the converse is false. Every finitely generated extension of K has a transcendence basis, but only pure extensions have pure bases. Even pure extensions have transcendence bases that are not pure. All transcendence bases have the same cardinality, the transcendence degree of E over K. This is usually shown using the "théorème d'échange" [6, Ch. 5 §14], and we need the following version of it, which adds a second statement so that it is symmetric.

Proposition 1.2. *Suppose $\{v_1,\ldots,v_n\}$ and $\{w_1,\ldots,w_n\}$ are transcendental bases. For every j there is a k such that $\{v_1,\ldots,v_{j-1},\widehat{v_j},v_{j+1},\ldots,v_n,w_k\}$ is a transcendental basis that contains the intersection $\{v_1,\ldots,v_n\} \cap \{w_1,\ldots,w_n\}$. For every j there is an index k such that $\{w_1,\ldots,w_{k-1},\widehat{w_k},w_{k+1},\ldots,w_n,v_j\}$ is a transcendental basis that contains $\{v_1,\ldots,v_n\} \cap \{w_1,\ldots,w_n\}$.*

Proof. The first statement is obviously true whenever the element v_j appears in the basis $\{w_1,\ldots,w_n\}$: in that case, we can take $w_k = v_j$. So, let us assume that v_j does not appear in $\{w_1,\ldots,w_n\}$. Then, the statement follows from the "théorème d'échange" [6, Ch. 5 §14], applied to the generating set $S = \{w_1, \dots, w_n\}$ and the algebraically independent set $T = \{v_1, \dots, \widehat{v_j}, \dots, v_n\}$: we can find a subset $S' = \{w_k\}$ of S such $S' \cap T = \emptyset$, i.e., we have $w_k \notin T$, and $S' \cup T$ is a basis. The intersection $\{v_1, \dots, \widehat{v_j}, \dots, v_n\} \cap \{w_1, \dots, w_n\}$ is obviously contained in the new basis. And $\{v_j\} \cap \{w_1, \dots, w_n\}$ can only be non-empty if the element v_j lies in $\{w_1, \dots, w_n\}$, which we have excluded.

The second statement also follows from the "théorème d'échange" [6, Ch. 5 §14], applied to the generating set $S = \{w_1, \dots, w_n\}$ and the algebraically independent subset $T = (\{v_1, \dots, v_n\} \cap \{w_1, \dots, w_n\}) \cup \{v_j\}$ of $\{v_1, \dots, v_n\}$: we can find a subset S' of S such that $S' \cap T = \emptyset$ and $S' \cup T$ is a basis. $\square$

Proposition 1.3. *Suppose $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_n\}$ are pure bases. Then, so are the bases constructed in Propositon 1.2.*

Proof. If $E = K(v_1, \dots, v_n) = K(w_1, \dots, w_n)$ is a pure extension of K, then the element w_k is contained in $K(v_1, \dots, v_n)$, so also $K(v_1, \dots, \widehat{v_j}, \dots, v_n, w_k)$ is contained in $K(v_1, \dots, v_n)$, and this extension is algebraic. In fact, if we choose an expression $w_k = f(v_1, \dots, v_n)/g(v_1, \dots, v_n)$ of w_k as a rational function of the v's, then $g(v_1, \dots, v_n)w_k - f(v_1, \dots, v_n) = 0$ is an algebraic equation, and the element v_j must occur in it; otherwise, the set $\{v_1, \dots, \widehat{v_j}, \dots, v_n, w_k\}$ of variables would not be algebraically independent. Then, that algebraic equation shows that v_j is algebraic over $K(v_1, \dots, \widehat{v_j}, \dots, v_n, w_k)$. We can now continue in the same way, removing more and more elements of the set $\{v_1, \dots, v_n\}$ and replacing them with elements in the set $\{w_1, \dots, w_n\}$, not involving elements in the intersection. Thereby, we arrive at a tower $E = K(v_1, \dots, v_n) \geqslant K(v_1, \dots, \widehat{v_j}, \dots, v_n, w_k) \geqslant \dots$ of algebraic extensions that ends in $K(w_1, \dots, w_n) = E$, with all v's replaced with w's. As we have $K(v_1, \dots, v_n) = K(w_1, \dots, w_n)$, this tower shows that all extensions are trivial, and this gives the equality $K(v_1, \dots, \widehat{v_j}, \dots, v_n, w_k) = E$ as well. The statement for the basis $\{w_1, \dots, \widehat{w_k}, \dots, w_n, v_j\}$ follows from symmetry. $\square$

We now choose a well-ordering of the underlying set of E. This ordering induces a well-ordering of the subset of elements v that appear as part of a pure or transcendental basis. We will write $\{v_1 < \dots < v_n\}$ to indicate when we list the basis elements in order. The set of bases is well-ordered lexicographically: we have $\{v_1 < \cdots < v_n\} < \{w_1 < \cdots < w_n\}$ if there is a k such that $v_j = w_j$ when $j < k$ and $v_k < w_k$.

Proposition 1.4. *If $V = \{v_1 < \cdots < v_n\} < \{w_1 < \cdots < w_n\} = W$ are bases, then there is a basis $V^\sharp$ with $V^\sharp < W$ that contains $V \cap W$ and differs from W only in one element. If V and W are pure, so is $V^\sharp$.*

Proof. As $V < W$, we have an index k such that $v_j = w_j$ when $j < k$ and $v_k < w_k$. Then we have $w_{k-1} = v_{k-1} < v_k < w_k$, and we see that $v_k \in V \setminus W$. We can insert v_k into W to get a basis $V^\sharp$. We claim that $V^\sharp$ has the properties we want.

Let w_l be the element of $W \setminus V$ that we remove from W to make space for v_k. By construction, the new basis $V^\sharp$ contains $V \cap W$, as the intersection is unaffected by the manoeuvre. It is clear that $V^\sharp$ and W differ only in one element: the former contains v_k, and the latter contains w_l; other than that, they are the same. Finally, when comparing $V^\sharp$ and W, we find that the first $k-1$ elements are the same, and the k–th elements are v_k and one of w_k or w_{k+1}, and the latter are both larger than v_k.

The second statement is clear. $\square$

A finitely generated extension E of K is *stably rational* over K if a rational extension of E is rational over K. If that is the case, then E is a subextension of a rational extension, hence *unirational*. In other words, rational $\Rightarrow$ stably rational $\Rightarrow$ unirational, but by the solutions to the Lüroth problem (see [1, 9, 14]) and the Zariski problem (see [2]), the reverse implications do not hold. All extensions of the field K in this paper are unirational, i.e., subfields of rational extensions.

2 Buildings for the Cremona groups

In this section, we fix a rational extension $K(n) = K(x_1, \ldots, x_n)$ of our groud field K. The starting point for our argument is the simplicial complex $P(K(n)|K)$ of partial pure bases of $K(n)$. The p–simplices in the simplicial complex $P(K(n)|K)$ are the $(p+1)$–element subsets of pure bases of $K(n)$. The simplicial complex $P(K(n)|K)$ is purely $(n-1)$–dimensional: by construction, every simplex is contained in an $(n-1)$–simplex given by a pure basis. For example, it is easy to see that $P(K|K)$ is empty, the complex $P(K(1)|K)$ is the discrete set $\mathrm{PGL}_2(K)$, and $P(K(2)|K)$ is a graph. We shall prove in a moment that the complex $P(K(n)|K)$ is $(n-2)$–connected, i.e., it is an $(n-1)$–dimensional spherical complex. In fact, more is true, and it can be proven simultaneously: the complex $P(K(n)|K)$ is $(n-1)$–dimensional Cohen–Macaulay. There are various minor variants of this concept in the literature; for us, it means that the complex is $(n-1)$–dimensional spherical, and the link of every p–simplex is $(n-p-2)$–spherical. If one thinks of the empty set as a simplex of dimension -1, then the sphericity is a special case. A simplicial complex is weakly $(n-1)$–dimensional Cohen–Macaulay if only the connectivity requirements are met: the complex is $(n-2)$–connected, and the link of every p–simplex is $(n-p-3)$–connected. Note that a weakly $(n-1)$–dimensional Cohen–Macaulay complex is also weakly Cohen–Macaulay of any smaller dimension.

Theorem 2.1. *For all $n \geqslant 0$, the simplicial complex $P(K(n)|K)$ is a Cohen–Macaulay complex of dimension $n-1$.*

Proof. Recall that the link $\mathrm{Link}_X(\sigma)$ of a p–simplex σ in a simplicial complex X is the subcomplex of all simplices τ of X that do not meet σ and such that the union $\sigma \cup \tau$ is still a simplex. For $X = P(K(n)|K)$, we can assume, without loss of generality, that σ is given by the first $p+1$ variables $x_1, \dots, x_{p+1}$. The link $L = \mathrm{Link}_{P(K(n)|K)}(\{x_1, \dots, x_{p+1}\})$ consists of all partial pure bases $\{v_1, \dots, v_k\}$ that do not intersect $\{x_1, \dots, x_{p+1}\}$ and such that $\{x_1, \dots, x_{p+1}, v_1, \dots, v_k\}$ is still a partial pure basis. We see that the complex L is $(n-p-2)$–dimensional and we have to show that it is $(n-p-3)$–connected for all $p = -1, \dots, n-2$. The case $p = -1$ corresponds to the case $L = P(K(n)|K)$.

If $n-p-3 \geqslant -1$, the link is non-empty. To see this, we note that we fixed the first $p+1$ variables $x_1, \dots, x_{p+1}$. The hypothesis $n-p-3 \geqslant -1$ implies $p+1 < n$, so x_n is not one of them. Therefore, a vertex of the link is given by x_n.

If $n-p-3 \geqslant 0$, the link is connected. In fact, any two vertices can be connected by a path consisting of at most two edges. Given two vertices $\{v_1\}$ and $\{w_1\}$, we can find two pure bases $\{x_1, \dots, x_{p+1}, v_1, \dots, v_q\}$ and $\{x_1, \dots, x_{p+1}, w_1, \dots, w_q\}$, respectively, with $q = n-p-1 \geqslant 2$. There is then a pure basis $\{x_1, \dots, x_{p+1}, v_1, \dots, v_{q-1}, w_j\}$ for some $j \in \{1, \dots, q\}$, and we see that there is an edge between $\{v_1\}$ and $\{w_j\}$ and, if $j \neq 1$, another one onward to $\{w_1\}$.

If $n-p-3 \geqslant 1$, the link is even simply-connected. Any loop that involves at most two edges is nullhomotopic. Therefore, it suffices to show that any loop that involves three or more edges is homotopic to a loop that involves fewer edges. To do so, consider a path consisting of three composable edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, and $\{v_3, v_4\}$ that connect vertices v_1, v_2, v_3, and v_4, with the case $v_4 = v_1$ allowed. We can then find pure bases $\{x_1, \dots, x_{p+1}, v_1, v_2, a_1, \dots, a_q\}$, $\{x_1, \dots, x_{p+1}, v_2, v_3, b_1, \dots, b_q\}$, and $\{x_1, \dots, x_{p+1}, v_3, v_4, c_1, \dots, c_q\}$, with $q = r-p-3 \geqslant 1$. These three pure bases represent three $(n-1)$–simplices in the link, each with an edge that is part of the path. We can remove a_q from the first basis and replace it with a vertex from the second basis. This new vertex cannot be v_2, so it is either v_3 or b_j for some j. If it is v_3, we are done because then $\{v_1, v_2, v_3\}$ is a simplex, and the original path is homotopic to one that directly connects v_1 to v_3, using one edge fewer. If it is b_j for some j, then the new basis is $\{x_1, \dots, x_{p+1}, v_1, v_2, a_1, \dots, a_{q-1}, b_j\}$. We can then remove v_3 from the third basis and replace it with a vertex from the new basis. If, on the one hand, the new vertex is v_1 or v_2, we are done, as we can then connect v_1 or v_2 directly with v_4, without going via v_3, resulting in a shorter path. If, on the other hand, the new vertex is a_k for some k or b_j, then we can go from v_1 to v_4 via this new vertex, also resulting in a shorter path.

In the general case, as above in Section 1, we choose a well-ordering of the underlying set of the field $K(x_1, \dots, x_n)$, resulting in a lexicographic well-ordering of the simplices of L of top-dimension $d = n-p-2$. We call a d–simplex τ *redundant* if it has a vertex v whose opposite face $\tau \setminus \{v\}$ is not contained in a d–simplex $\sigma < \tau$. Otherwise, the d–simplex τ is *essential*. Let $R \subseteq L$ be the subcomplex spanned by the redundant d–simplices.

We show that the subcomplex R is contractible by a transfinite induction on the set of simplices, i.e., by showing that for every simplex τ of R, the subcomplex $R(\leqslant \tau) = \{\sigma \leqslant \tau\}$ is contractible. For that purpose, consider $R(< \tau) = \{\sigma \leqslant \tau\}$ and the homotopy pushout diagram

$$\begin{tikzcd} R(<\tau) \arrow[r] & R(<\tau)\cup\tau \arrow[r, equal] & R(\leqslant\tau) \\ R(<\tau)\cap\tau \arrow[u] \arrow[r] & \tau \arrow[u] & \end{tikzcd}$$

that describes how the simplex τ is attached to the smaller simplices. By induction, the complex $R(< \tau)$ of smaller cells is contractible: if $\tau = \sigma + 1$ is a successor, then we have $R(<\tau) = R(\leqslant \sigma)$ and if τ is a limit, then we have $R(<\tau) = \mathrm{hocolim}_{\sigma<\tau} R(\leqslant \sigma)$. As τ is contractible, it remains to be shown that $R(<\tau) \cap \tau$ is contractible. But this intersection is a subset of the boundary of τ, namely the union

$$R(<\tau) \cap \tau = \bigcup_{\tau \setminus \{v\} \subseteq R(<\tau)} \tau \setminus \{v\}.$$

This union is non-empty by Proposition 1.4, and it is not the entire boundary because the simplex τ is redundant: there is a vertex v whose opposite face $\tau \setminus \{v\}$ is not contained in any d–simplex $\sigma < \tau$. Thus, we see that the intersection $R(<\tau) \cap \tau$ is a cone on any such vertex v, and therefore, contractible.

To conclude the argument, note that R is obtained from the d–dimensional complex L by removing the interiors of the essential d–simplices. Hence, the contractible subcomplex R contains the $(d-1)$–skeleton of L, and $L \simeq L/R$ is a wedge of d–spheres, indexed by the essential simplices of L. $\square$

3 Split buildings for the Cremona groups

We fix a field K and a rational extension $K(n) = K(x_1, \dots, x_n)$ of it. We define a simplicial complex $S(K(n)|K)$ as follows. The vertices are the pairs (L_0, v_0) such that there is a pure basis $\{w_1, \dots, w_n\}$ of $K(n)$ such that $w_1 = v_0$ and $K(w_2, \dots, w_n) = L_0$. More generally, a set $\{(L_0, v_0), \dots, (L_p, v_p)\}$ of $p+1$ vertices forms a p–simplex when there is a pure basis of $K(n)$ that contains each v_j and such that each L_j is the rational extension generated by the other variables. We have

$$\bigcap_{j=0}^{p} L_j = K(w_1, \dots, w_{n-p-1}) \tag{3.1}$$

for some set $\{w_1, \dots, w_{n-p-1}\}$ of variables, namely the complement of the subset $\{v_0, \dots, v_p\}$ in a pure basis of $K(n)$ that contains it. Therefore, we can equivalentlydescribe the p–simplex $\sigma = \{(L_0, v_0), \dots, (L_p, v_p)\}$ by the pair $(L, \{v_0, \dots, v_p\})$, with L as in (3.1): we can recover the extensions L_j as $L_j = L(v_0, \dots, \widehat{v_j}, \dots, v_p)$. In other words, a p–simplex is a rational subfield L of $K(n)$ together with a pure basis $\{v_0, \dots, v_p\}$ of $K(n)$ over L of length $p+1$. This implies that the transcendence degree of L over K is $n-p-1$. A colouring argument as in [13, Sec. 3] allows us now to deduce the high connectivity of $S(K(n)|K)$ from the high connectivity of $P(K(n)|K)$ established in Theorem 2.1.

Theorem 3.1. *The simplicial complex $S(K(n)|K)$ is $(n-3)/2$–connected.*

Proof. There is a forgetful map $f\colon S(K(n)|K) \to P(K(n)|K)$ of simplicial complexes. On vertices, it is given by $(L_0, v_0) \mapsto \{v_0\}$, and more generally, a p–simplex $(L, \{v_0, \dots, v_p\})$ is sent to the p–simplex $\{v_0, \dots, v_p\}$. The map f is surjective. In fact, it displays $S(K(n)|K)$ as a join complex over $P(K(n)|K)$ as in [13, Def. 3.2]. We will see this by showing that the simplicial complex $S(K(n)|K)$ is isomorphic to the labelling complex $P(K(n)|K)^{\Lambda}$ for a labelling system Λ on $P(K(n)|K)$.

Recall that a labelling system assigns a non-empty set $\Lambda_{v_0}(\sigma)$ to each vertex v_0 of each simplex σ in such a way that $\Lambda_{v_0}(\sigma) \supseteq \Lambda_{v_0}(\tau)$ holds for $\sigma \subseteq \tau$. Here, the set $\Lambda_{v_0}(\{v_0, \dots, v_p\})$ will be the set of subfields L_0 such that (L_0, v_0) is a vertex of $S(K(n)|K)$ and L_0 contains $\{v_1, \dots, v_p\}$. The simplicial complex $P(K(n)|K)^{\Lambda}$ has vertices the pairs (L_0, v_0) with v_0 in $P(K(n)|K)$ and $L_0 \in \Lambda_{v_0}(\{v_0\})$. A set of pairs $(L_0, v_0), \dots, (L_p, v_p)$ then forms a p–simplex of $P(K(n)|K)^{\Lambda}$ if and only if $\{v_0, \dots, v_p\}$ is a p–simplex σ of $P(K(n)|K)$ and $L_j \in \Lambda_{v_j}(\sigma)$ for all j. We see, using Proposition 1.1, that $S(K(n)|K) = P(K(n)|K)^{\Lambda}$, and this is, therefore, a join complex over $P(K(n)|K)$.

We have already shown in Theorem 2.1 that the target $P(K(n)|K) = f(S(K(n)|K))$ is a Cohen–Macaulay complex of dimension $n-1$, but we need something in addition. The link of a p–simplex $\sigma = \{(L_0, v_0), \dots, (L_p, v_p)\}$ in the simplicial complex $S(K(n)|K)$ is $S(K(n)|K(v_0, \dots, v_p))$. It follows that the image of this link is $P(K(n)|K(v_0, \dots, v_p))$, hence Cohen–Macaulay of dimension $n-p-2$, again by Theorem 2.1. Then this complex is also weakly Cohen–Macaulay of any smaller dimension, and we can apply [13, Thm. 3.6] to conclude that the connectivity of $S(K(n)|K)$ is $(n-1)/2 - 1 = (n-3)/2$. □

4 Homogenous categories from the Cremona groupoid

Given a simplicial complex S, one can form an associated semi-simplicial set S^{ord} that has a p–simplex for every p–simplex of S and every choice of ordering of its vertices. For $S(K(n)|K)$, we can describe $S(K(n)|K)^{\mathrm{ord}}$ directly as follows. Consider the semi-simplicial set $W(K(n)|K)$ with vertex set $W(K(n)|K)_0$ the same as that of $S(K(n)|K)$. It consists of the pairs (L,v) where L is a rational subextension of $K(n)$ and $v \in K(n) \setminus L$ such that $K(n) = L(v)$. More generally, for any integer $p \geqslant 0$, we let $W(K(n)|K)_p$ denote the set of tuples $(L, v_0, \dots, v_p)$, where L is a rational subextension of $K(n)$ and the v_j are distinct elements of the complement $K(n) \setminus L$ such that we have $L(v_0, \dots, v_p) = K(n)$. The boundary operators d_j are given by removing v_j from the list of elements and adjoining it to L:

$$d_j(L, v_0, \dots, v_p) = (L(v_j), v_0, \dots, v_{j-1}, \widehat{v_j}, v_{j+1}, \dots, v_p).$$

The simplices of the $W(K(n)|K)$ are determined by their vertices, and their vertices are all distinct. This property corresponds to condition (A) in [17, p. 558]. Then, we have $W(K(n)|K) = S(K(n)|K)^{\mathrm{ord}}$. As we will apply the results of [17, p. 558], we adopt the notation $S_n(a,x) = S(K(a+nx)|K)$ and $W_n(a,x) = W(K(a+nx)|K)$ for integers $a,n,x \geqslant 1$. In this notation, Theorem 3.1 says that the simplicial complex $S_n(1,1)$ is $(n-2)/2$–connected.

Theorem 4.1. *For all $n \geqslant 1$, the semi-simplicial set $W_n(1,1)$ is $(n-2)/2$–connected.*

Proof. Let $\mathcal{R}_K$ be the category of finitely generated rational K–extensions. We will restrict to its skeleton consisting of the standard examples $K(n) = K(x_1, \dots, x_n)$ for $n \geqslant 0$ so that we can identify the set of objects with the natural numbers. The category $\mathcal{R}_K$ is symmetric monoidal with respect to $K(m) \otimes K(n) = K(m+n)$, satisfies cancellation, and the unit object $K(0) = K$ is initial in it. The groupoid $\mathcal{R}_K^{\times}$ of isomorphisms in $\mathcal{R}_K$ is the disjoint union of the Cremona groups. However, there are endomorphisms that are not automorphisms, and the category $\mathcal{R}_K$ has other features that are less desirable for us. A standard remedy, going back to Quillen [12] (see also [10, 11, 17]), is to take complements into account. This leads us to introduce the category $\mathcal{Q}_K$, which will be homogenous in the sense of [17, Def. 1.3]. This category has the same objects, but where a morphism $K(m) \to K(n)$ is a pair (R, f), where $R \leqslant K(n)$ is a rational subextension, together with an isomorphism $f \colon R(m) \cong K(n)$ that extends the inclusion. We can think of these morphisms as equivalence classes of automorphisms of $K(n)$ modulo change of co-ordinates in the $m-n$ variables belonging to the complementary R, i.e., we can identify the set of morphisms with the set $\mathrm{Cr}_n(K)/\mathrm{Cr}_{m-n}(K)$ of cosets. Composition with a morphism $K(l) \to K(m)$ is given by noting that f embeds $\mathrm{Cr}_m(K)$

into $\mathrm{Cr}_n(K)$ by its action on the m 'free' co-ordinates. Then, we can compose representatives in $\mathrm{Cr}_n(K)$. The identity morphism of $K(n)$ for this composition is $(K, \mathrm{id}_{K(n)})$. The object $K = K(0)$ is still initial in the category $\mathcal{Q}_K$, and the endomorphism monoid of $K(n)$ is $\mathrm{Cr}_n(K)$. In fact, as a morphism $K(m) \to K(n)$ can only exist when $m \leqslant n$, the groupoid $\mathcal{Q}_K^\times$ of isomorphism in $\mathcal{Q}_K$ is the disjoint union of the Cremona groups. Conversely, we can write the category $\mathcal{Q}_K$ as Quillen's bracket construction $\langle \mathcal{Q}_K^\times, \mathcal{Q}_K^\times \rangle$. Both $\mathcal{Q}_K^\times$ and $\mathcal{Q}_K$ inherit the monoidal structure from $\mathcal{R}_K$. The category $\mathcal{Q}_K$ is homogenous in the sense of [17, Def. 1.3.]. This means that the automorphism group of $K(n)$ acts transitively on the set of morphisms $K(m) \to K(n)$ and is the stabiliser of the canonical map $K(m) \to K(m+n)$. The category $\mathcal{Q}_K$ is also locally standard in the sense of [17, Def. 2.5]; this is equivalent to the fact, established above, that the simplices of all the semi-simplicial sets $W_n(a,x)$ are determined by their vertices and their vertices are all distinct. As mentioned above, the semi-simplicial sets $W_n(a,x)$ satisfy condition (A) for all $n \geqslant 0$ because $\mathcal{Q}_K$ is symmetric monoidal. As the connectivity of $S_n(1,1)$ is $(n-2)/2$ for all n by Theorem 3.1, we can now apply [17, Thm. 2.10] to deduce that the connectivity of the $W_n(1,1)$ is also $(n-2)/2$ for all n. □

5 Homological stability

With the connectivity of sufficiently nice spaces in place, we are now ready to deduce the homological stability of the Cremona groups. Quillen developed this proof strategy in the 1970s and used it in the context of the general linear groups. It was later axiomatised in [13] and [17], and that version applies here to give the following result.

Theorem 5.1. *For any field K, the homomorphisms*

$$\mathrm{H}_i(\mathrm{Cr}_n(K);\mathbb{Z}) \longrightarrow \mathrm{H}_i(\mathrm{Cr}_{n+1}(K);\mathbb{Z})$$

are epimorphisms for $i \leqslant (n-1)/2$ and isomorphisms for $i \leqslant (n-2)/2$.

In particular, for $i = 1$, we see that the abelianisation $\mathrm{Cr}_3(K)^{\mathrm{ab}}$ surjects onto $\mathrm{Cr}_4(K)^{\mathrm{ab}}$, which assumes the stable value: $\mathrm{Cr}_4(K)^{\mathrm{ab}} \cong \mathrm{Cr}_5(K)^{\mathrm{ab}} \cong \mathrm{Cr}_6(K)^{\mathrm{ab}} \cong \dots$.

Proof. As in our proof of our Theorem 4.1, we are in the framework of [17]: The category $\mathcal{Q}_K$ is a symmetric monoidal category that is homogeneous. Theorem 4.1

verifies the connectivity condition (LH3) from [17] at the pair $(A,X) = (K(1),K(1))$ of objects in $\mathcal{Q}_K$. Therefore, we can apply [17, Thm. 3.1] to deduce the result. $\square$

There is an extension of Theorem 5.1 to more general coefficients. Let $\mathrm{Cr}_\infty(K)$ the stable Cremona group, i.e., the colimit of the groups $\mathrm{Cr}_n(K)$ under the maps inducing the homomorphisms in Theorem 5.1. (This colimit should not be mixed up with efforts to provide each group $\mathrm{Cr}_n(K)$ with the structure of an ind-algebraic group or an ind-stack in such a way that it becomes the universal object for families of birational transformations: that's impossible [4].) Any module for the abelianisation $\mathrm{Cr}_\infty(K)^{\mathrm{ab}}$ gives rise to $\mathrm{Cr}_n(K)$–modules by restriction.

Theorem 5.2. *For any field K, and any* $\mathrm{Cr}_\infty(K)^{\mathrm{ab}}$*–module M, the homomorphisms*

$$\mathrm{H}_i(\mathrm{Cr}_n(K);M) \longrightarrow \mathrm{H}_i(\mathrm{Cr}_{n+1}(K);M)$$

are epimorphisms for $i \leqslant (n-2)/3$ *and isomorphisms for* $i \leqslant (n-4)/3$*.*

Proof. The argument is similar to our proof of Theorem 5.1, except that we have to refer to [17, Thm. 3.4] instead. $\square$

In particular, the result applies to $M = \mathbb{Z}[\mathrm{Cr}_\infty(K)^{\mathrm{ab}}]$, the group ring of the abelianisation. Shapiro's Lemma implies that the homology with coefficients in the abelianisation is the homology of the commutator subgroup. As in [17, Cor. 3.9], this leads to the following result.

Corollary 5.3. *For any field K, and any* $\mathrm{Cr}_\infty(K)^{\mathrm{ab}}$*–module M, the homomorphisms*

$$\mathrm{H}_i([\mathrm{Cr}_n(K),\mathrm{Cr}_n(K)];\mathbb{Z}) \longrightarrow \mathrm{H}_i([\mathrm{Cr}_{n+1}(K),\mathrm{Cr}_{n+1}(K)];\mathbb{Z})$$

are epimorphisms for $i \leqslant (n-2)/3$ *and isomorphisms for* $i \leqslant (n-4)/3$*.*

Proof. The statement is clear for $i = 0$. If $i \geqslant 1$, it only applies to $n \geqslant 3i+2 \geqslant 4$, and $\mathrm{Cr}_n(K)^{\mathrm{ab}} \cong \mathrm{Cr}_\infty(K)^{\mathrm{ab}}$ for these n, as we have remarked above. Therefore, we can use Theorem 5.2, and the homology groups are those of the respective commutator subgroups with trivial coefficients. $\square$

As for the stable homology, the context of symmetric monoidal categories makes it clear that it can be identified with the homology of an infinite loop space, the group completion of the Cremona groupoids. We refer to [17, Sec. 3.2] and [3, Sec. 2] for details.

References

[1] M. Artin, D. Mumford. Some elementary examples of unirational varieties which are not rational. Proc. London Math. Soc. 25 (1972) 75–95. ◁ 5

[2] A. Beauville, J.-L. Colliot-Thélène, J.-J. Sansuc, P. Swinnerton-Dyer. Varietes stablement rationnelles non rationnelles. Ann. of Math. 121 (1985) 283–318. ◁ 5

[3] A.M. Bohmann, M. Szymik. Boolean algebras, Morita invariance and the algebraic K-theory of Lawvere theories. Math. Proc. Cambridge Philos. Soc. 175 (2023) 253–270. ◁ 13

[4] J. Blanc, J.-P. Furter. Topologies and structures of the Cremona groups. Ann. Math. 178 (2013) 1173–1198. ◁ 12

[5] J. Blanc, S. Lamy, S. Zimmermann. Quotients of higher-dimensional Cremona groups. Acta Math. 226 (2021) 211–318. ◁ 2

[6] N. Bourbaki. Éléments de mathématique. Algèbre. Chapitres 4 à 7. Masson, Paris, 1981. ◁ 3, 4

[7] S. Cantat. Sur les groupes de transformations birationnelles des surfaces. Ann. of Math. 174 (2011) 299–340. ◁ 2

[8] R.M. Charney. Homology stability for GL_n of a Dedekind domain. Invent. Math. 56 (1980) 1–17. ◁ 2

[9] C.H. Clemens, P. Griffiths. The intermediate Jacobian of the cubic threefold. Ann. of Math. 95 (1972) 281–356. ◁ 5

[10] A. Djament, C. Vespa. Sur l'homologie des groupes orthogonaux et symplectiques à coefficients tordus. Ann. Sci. Éc. Norm. Supér. 43 (2010) 395–459. ◁ 10

[11] A. Djament, C. Vespa. Foncteurs faiblement polynomiaux. Int. Math. Res. Not. IMRN (2019) 321–391. ◁ 10

[12] D. Grayson. Higher algebraic K-theory. II (after Daniel Quillen). Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976) 217–240. Lecture Notes in Math. 551. Springer-Verlag, Berlin-New York, 1976. ◁ 10

[13] A. Hatcher, N. Wahl. Stabilization for mapping class groups of 3-manifolds. Duke Math. J. 155 (2010) 205–269. ◁ 8, 9, 11

[14] V.A. Iskovskikh, Yu.I. Manin. Three-dimensional quartics and counterexamples to the Lüroth problem. Mat. Sb. 86 (1971) 140–166. ◁ 5

[15] A. Lonjou, C. Urech. Actions of Cremona groups on CAT(0) cube complexes. Duke Math. J. 170 (2021) 3703–3743. ◁ 2

[16] Y.I. Manin. Cubic Forms: Algebra, Geometry, Arithmetic. North-Holland Mathematical Library, Vol. 4. North-Holland Publishing Co., 1974. ◁ 2

[17] O. Randal-Williams, N. Wahl. Homological stability for automorphism groups. Adv. Math. 318 (2017) 534–626. ◁ 2, 10, 11, 12, 13

[18] D. Wright. Two-dimensional Cremona groups acting on simplicial complexes, Trans. Amer. Math. Soc. 331 (1992) 281–300. ◁ 2

[19] N. Wahl. Homological stability: a tool for computations. Proc. Int. Cong. Math. 2022, Vol. 4, 2904–2927. Ed. D. Beliaev and S. Smirnov. European Mathematical Society, 2023. ◁ 1

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