Quandles, real algebra, and symmetries

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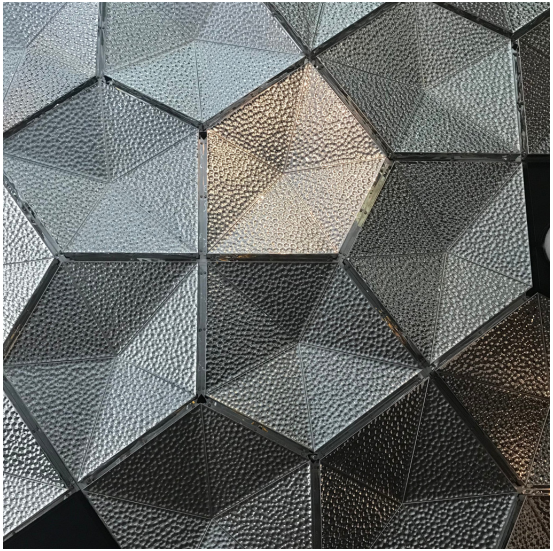
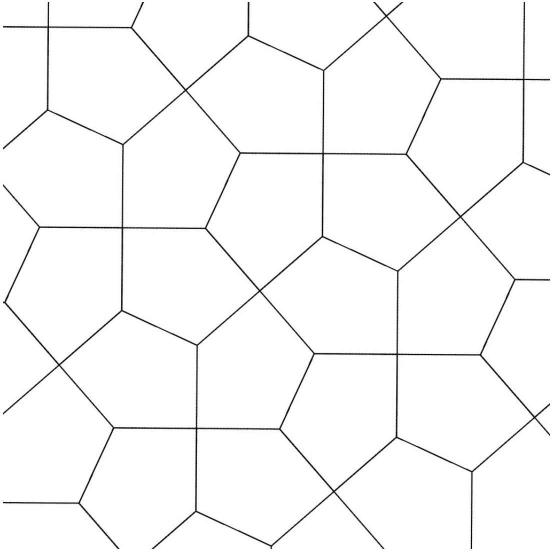
Clarion Hotel The Edge





https://en.wikipedia.org/wiki/Cairo_pentagonal_tiling

Never odd or even

Number theory — Number theory

What is a quandle? — What is a quandle?

Geometry — Geometry

What is a group? — What is a group?

NUMBER THEORY

Galois symmetries

F = a field

$\mathrm{Gal}(F)$ = its absolute Galois group

Example. $\mathrm{Gal}(\mathbb{C}) = \{\mathrm{id}\}$

Example. $\mathrm{Gal}(\mathbb{R}) = \mathbb{Z}/2$

Example. $\mathrm{Gal}(\mathbb{F}_q) = \widehat{\mathbb{Z}}$

Example. $\mathrm{Gal}(\mathbb{Q}) = ?$

It's even difficult to describe elements.

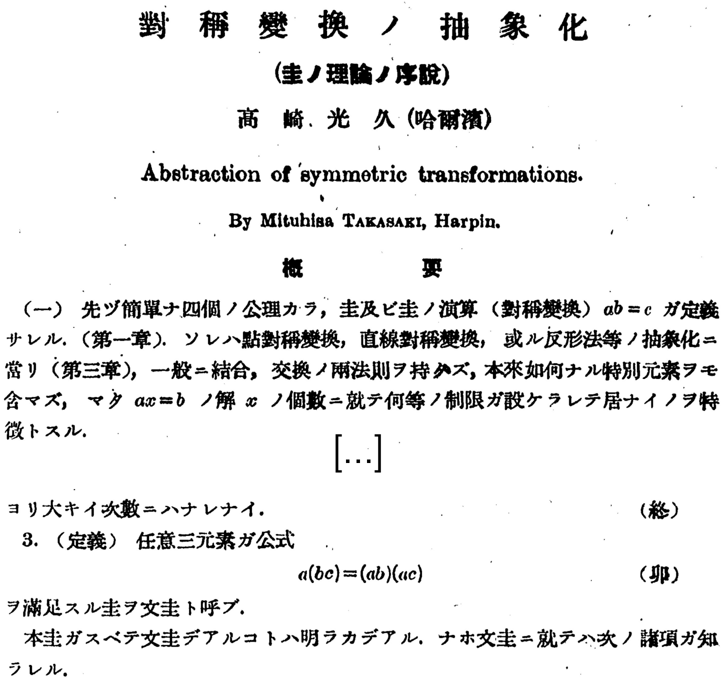
Artin–Schreier quandles

Theorem (Artin–Schreier)
If $g \in \mathrm{Gal}(F)$ has finite order, then $g^2 = \mathrm{id}$.

$$\mathrm{AS}(F) := \{\, g \in \mathrm{Gal}(F) \mid g^2 = \mathrm{id} \neq g \,\}$$

Theorem (S)
This is a pro-finite involutory quandle.
If $F = \mathbb{Q}$, then $\mathrm{AS}(\mathbb{Q})$ is free with a Cantor space basis.

WHAT IS A QUANDLE?


對 稱 變 換 ノ 抽 象 化

(圭ノ理論ノ序說)

高 崎 光 久 (哈爾濱)

Abstraction of symmetric transformations.

By Mituhisa TAKASAKI, Harpin.

概 要

(一) 先ヅ簡單ナ四個ノ公理カラ，圭及ビ圭ノ演算（對稱變換）$ab=c$ ガ定義サレル.（第一章）. ソレハ點對稱變換，直線對稱變換，或ル反形法等ノ抽象化ニ當リ（第三章），一般ニ結合，交換ノ兩法則ヲ持タズ，本來如何ナル特別元素ヲモ含マズ，マタ $ax=b$ ノ解 x ノ個數ニ就テ何等ノ制限ガ設ケラレテ居ナイノヲ特徵トスル.

[...]

ヨリ大キイ次數ニハナレナイ. (総)

3.（定義） 任意三元素ガ公式

$$a(bc)=(ab)(ac) \qquad (\text{卯})$$

ヲ滿足スル圭ヲ文圭ト呼ブ.

本圭ガスベテ文圭デアルコトハ明ラカデアル. ナホ文圭ニ就テハ次ノ諸項ガ知ラレル.

Racks

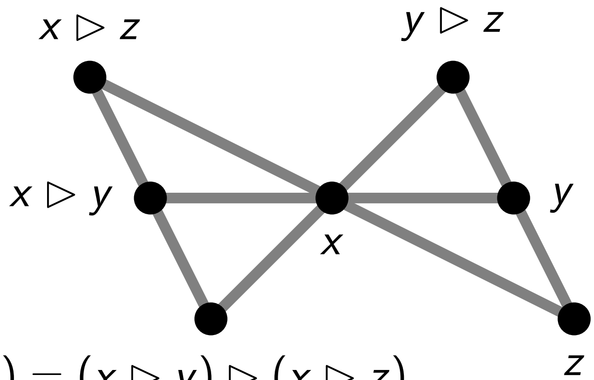
A **rack** is a set together with a binary operation $\rhd$ such that all left-multiplications $y \mapsto x \rhd y$ are automorphisms.

$$x \rhd (y \rhd z) = (x \rhd y) \rhd (x \rhd z)$$

A **pro-finite** rack...

Examples (a reflection on reflections)

$x \triangleright y := y$ reflected in x



$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$$

Note. We have $x \triangleright x = x$ in this example.

Quandles

A rack is a **quandle** if all natural automorphisms are the identity.

$$x \triangleright x = x$$

Examples. When G is a group, and $Q \subseteq G$ is a union of conjugacy classes, then

$$x \triangleright y = xyx^{-1}$$

turns Q into a quandle.

Involutions

A rack/quandle is **involutory** if all left-multiplications square to the identity.

$$x \rhd (x \rhd y) = y$$

Examples. When G is a group, its subset

$$\mathrm{Inv}(G) = \{\, g \in G \mid g^2 = \mathrm{id} \neq g \,\}$$

of involutions gives an involutory quandle.

Note. We have $\mathrm{AS}(F) = \mathrm{Inv}(\mathrm{Gal}(F))$.

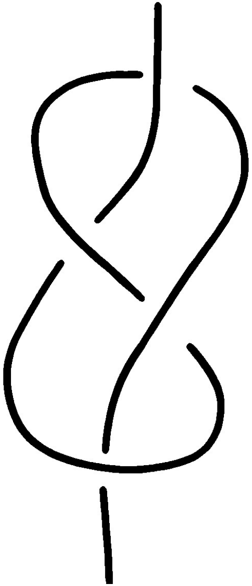
Free models

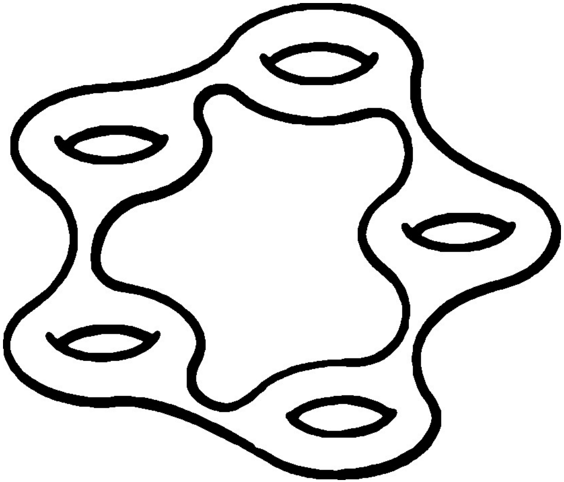
If $G = \mathsf{F}(S) = \mathbb{Z}^{\star|S|}$ is a free (Artin) group on a set S of generators, and $\mathsf{FQ}(S) \subseteq \mathsf{F}(S)$ is the set of conjugates of these generators, then $\mathsf{FQ}(S)$ is a **free quandle** with basis S.

The **free involutory quandle** $\mathsf{FQ}_2(S)$ generated by a set S consists of the conjugates of the basis S inside the free Coxeter group $\mathsf{F}_2(S) = (\mathbb{Z}/2)^{\star|S|}$.

Note. We have $\mathsf{FQ}_2(S) = \mathrm{Inv}(\mathsf{F}_2(S))$ by Kurosh.

GEOMETRY





Riemann surfaces

Riemann surfaces C are classified by their

$$\begin{array}{ll} \mathrm{g}(C) & \text{genus,} \\ \mathrm{H}^\bullet(C) & \text{cohomology,} \\ \pi_1(C) & \text{fundamental group.} \end{array}$$

Which of these is the best invariant?

Moduli spaces of Riemann surfaces

$\mathcal{M}_C^{\mathsf{surf}}$ = component of C in the moduli space of surfaces

Theorem (Dehn, Nielsen, Baer)
If $\mathrm{g}(C) \geqslant 2$, then

$$\mathcal{M}_C^{\mathsf{surf}} \simeq \mathrm{BOut}(\pi_1 C).$$

The fundamental group is a classifying invariant that also knows about the symmetries of surfaces.

Knots

For all knots K, we have

$$\mathrm{J}_K \quad \text{Jones polynomial,}$$

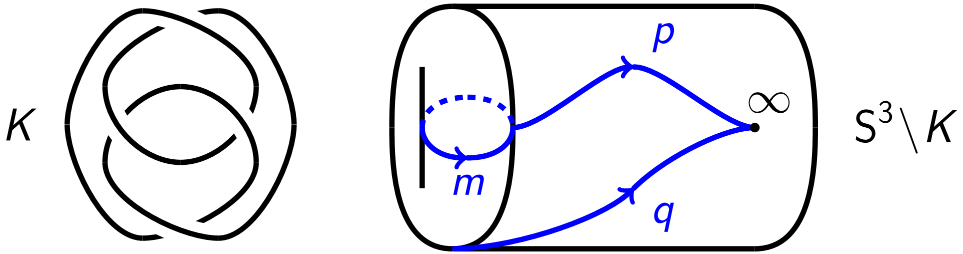
$$\mathrm{Kh}_K \quad \text{Khovanov homology,}$$

$$\pi_K \quad \text{knot group.}$$

Theorem (Joyce, Matveev, Waldhausen)
Knots K are classified by their knot quandles Q_K.

What is the quandle of a knot?

$$\mathrm{Q}_K = \text{homotopy classes of paths boundary}(\,\mathrm{S}^3 \backslash K\,) \to \infty$$
$$= \pi_0\,\text{hofib}(\,\text{boundary}(\,\mathrm{S}^3 \backslash K\,) \to \mathrm{S}^3 \backslash K\,)$$



$p \rhd q = pmp^{-1}q$ for a suitable meridian m

Moduli spaces of knots

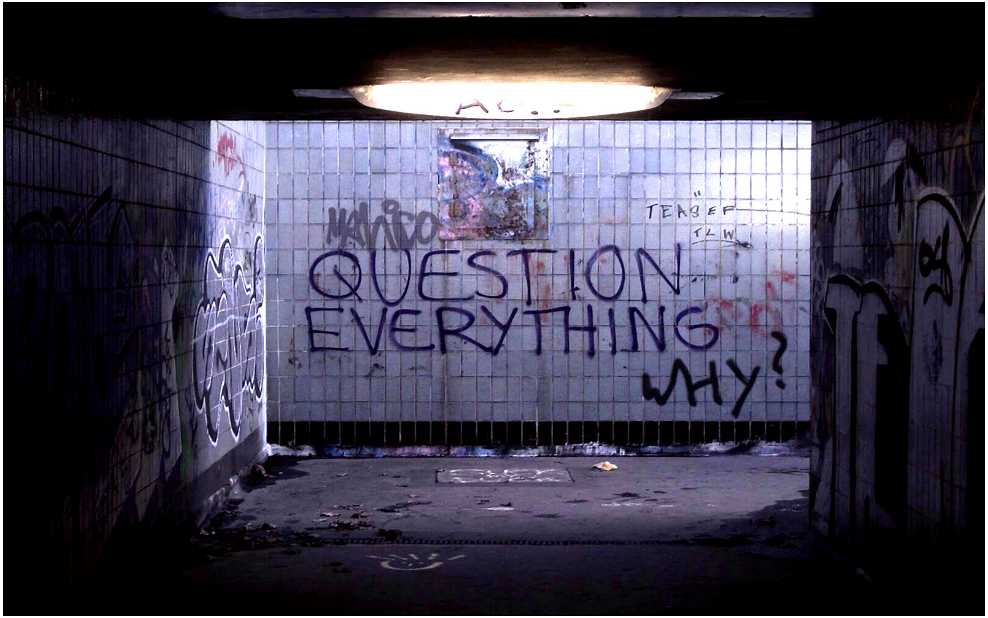
$\mathcal{M}_K^{\mathrm{knot}}$ = component of K in the moduli space of knots

Theorem (S)
For all knots K, we have

$$\mathcal{M}_K^{\mathrm{knot}} \simeq \mathrm{BAut}(\mathrm{Q}_K).$$

The knot quandle is a classifying invariant that also knows about the symmetries of knots.

WHAT IS A GROUP?



Quandles vs groups

joint work with Torstein Vik and James Cranch

$$\begin{tikzcd} \textbf{Quandles} \arrow[rr, bend left, dashed, "\mathsf{L}"] & & \textbf{Groups} \arrow[ll, bend left, "\text{forget}"] \end{tikzcd}$$

$$\mathsf{L}(Q) = \langle x \in Q \mid x \rhd y = xyx^{-1} \rangle$$

Problem. No non-trivial finite group is in the image of L.

Operations in groups

$\{$ natural n–ary operations on groups $\} \cong \mathsf{F}\{1, \ldots, n\}$

Examples

$n = 0$	e	neutral element
$n = 1$	$g \mapsto g^k,\ k \in \mathbb{Z}$	power operations
$n = 2$	$(g, h) \mapsto gh, ghg^{-1} \ldots$	multiplication, conjugation. . .

Power quandles...

...are defined so that they can remember the conjugation, the power operations, and the neutral element from a group.

$$\begin{tikzcd} \mathbf{PowerQuandles} \arrow[rr, bend left, dashed, "\mathsf{Gr}"] & & \mathbf{Groups} \arrow[ll, bend left, "\mathsf{Pq}"] \end{tikzcd}$$

Results

Theorem (S–Vik)

If G and H are finite groups with $\mathrm{Pq}(G) \cong \mathrm{Pq}(H)$, then we have $\mathrm{Z}(G) \cong \mathrm{Z}(H)$ and $G/\mathrm{Z}(G) \cong H/\mathrm{Z}(H)$.

When, in addition, both G and H are in the image of Gr, we have $G \cong H$.

Theorem (Cranch–S)

There are still finite groups not in the image of Gr, and the theorem above cannot be sharpened to infer $G \cong H$ in general.

Reconstructing a group from its power quandle

Theorem (S–Vik)
There is a central extension

$$0 \to \mathrm{A}(G) \to \mathrm{Gr}\,\mathrm{Pq}(G) \to G \to 1$$

that satisfies a universal property.

The abelian group $\mathrm{A}(G)$ can be computed via

$$0 \to \mathrm{H}_2(G;\mathbb{Z}) \to \mathrm{A}(G) \to \mathrm{A}'(G) \to \mathrm{H}_1(G;\mathbb{Z}) \to 0$$

with

$$\mathrm{A}'(G) = \mathbb{Z}\{\text{conjugacy classes in } G\}/k[g] = [g^k].$$

SUMMARY

A quandle is a set with a binary operation such that all left-multiplications are automorphisms, but all natural automorphisms are the identity.

Quandles are useful in algebra and geometry to formulate theorems about more classical objects.

There is some progress in understanding the relation between quandles and groups, but more needs to be done.