Estimation of Low Rank and Sparse Covariance Matrices

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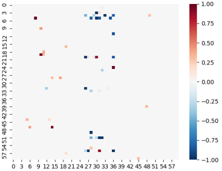
Agenda

What connections do we look for?

We are looking for matrices that are **sparse** and has **low rank**.

- Sparse, full rank
- Low rank not sparse



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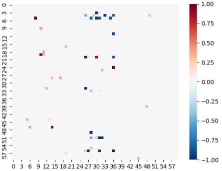

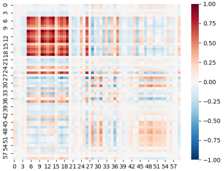

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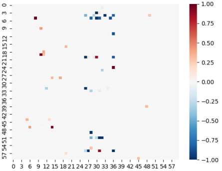

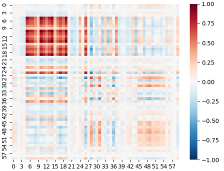

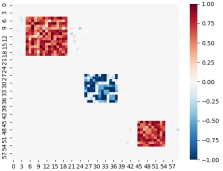

Introducing **Low-Rank Sparse Estimator** (LoRSEr)

$$\hat{B} := \underset{B}{\operatorname{argmin}} \left\{ \frac{1}{2}||Y - B||_F^2 + \lambda_N \|B\|_* + \lambda_L \|\operatorname{vec}(W \circ B)\|_1 \right\}$$

Where:

- Y - empirical covariance matrix.
- $||Y - B||_F$ - Frobenius norm.
- $||B||_*$ - nuclear norm responsible for low rank.
- W - matrix of weights.
- $\|\operatorname{vec}(W \circ B)\|_1$ - l_1 norm responsible for sparsity.
- λ_N, λ_L - scales fine-tuned in the CV process.

Alternating Direction Method of Multipliers (ADMM)

Numerical Solution

$$F(B) := \underbrace{\frac{1}{2}||Y - B||_F^2}_{f(B)} + \underbrace{\lambda_N \|B\|_*}_{g(B)} + \underbrace{\lambda_L \|\operatorname{vec}(W \circ B)\|_1}_{h(B)}.$$

Alternating Direction Method of Multipliers (ADMM)

Low - **R**ank **S**parse **E**stimator (LoRSEr)

$$B^{[k+1]} := \underset{B}{\operatorname{argmin}} \left\{ 2f(B) \ + \ \delta_1^{[k]} \left\| D^{[k]} + \frac{Z_1^{[k]}}{\delta_1^{[k]}} - B \right\|_F^2 \right\}$$

$$C^{[k+1]} := \underset{C}{\operatorname{argmin}} \left\{ 2g(C) \ + \ \delta_2^{[k]} \left\| D^{[k]} + \frac{Z_2^{[k]}}{\delta_2^{[k]}} - C \right\|_F^2 \right\}$$

$$D^{[k+1]} := \underset{D}{\operatorname{argmin}} \left\{ 2h(D) \ + \ \delta_1^{[k]} \left\| D + \frac{Z_1^{[k]}}{\delta_1^{[k]}} - B^{[k+1]} \right\|_F^2 + \delta_2^{[k]} \left\| D + \frac{Z_2^{[k]}}{\delta_2^{[k]}} - C^{[k+1]} \right\|_F^2 \right\}$$

$$\left\{ \begin{array}{l} Z_1^{[k+1]} := \ Z_1^{[k]} + \delta_1^{[k]} \left(D^{[k+1]} - B^{[k+1]} \right) \\ Z_2^{[k+1]} := \ Z_2^{[k]} + \delta_2^{[k]} \left(D^{[k+1]} - C^{[k+1]} \right) \end{array} \right. .$$

Properties of LoRSEr

Properties

Property 1

For any pair of the tuning parameters, there **exists** solution to optimisation problem and this solution is **unique**.

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Solution is symmetric

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Property 2

Solution is symmetric.

Property 3

Solution is invariable under reordering applied to rows and columns of Y matrix.

Experiments (Generating data and loss calculation)

X - experiment matrix
X_i - i-th row of matrix

$$X_i \sim \mathcal{N}(\mu, \Sigma) .$$

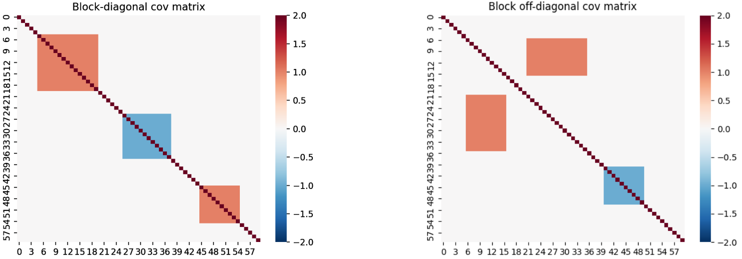
Where μ is a zero vector and Σ is predefined covariance matrix we want to estimate after data generation.

Loss Function

$$\text{MSEr} = ||\hat{B} - \Sigma||_{F^\star}^2 / ||\Sigma||_{F^\star}^2$$

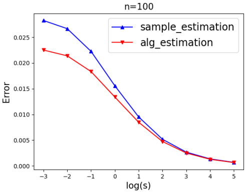
where $||.||_*$ is Frobenius Norm of matrix with diagonal entries excluded.

Experiments (Examples of covariance matrices)

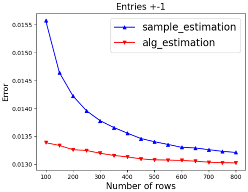


Rysunek 1: Examples of covariance matrices we want to recover from data

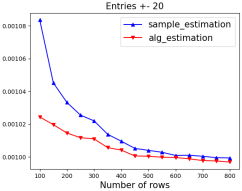
Differences in performance



(a) Different values

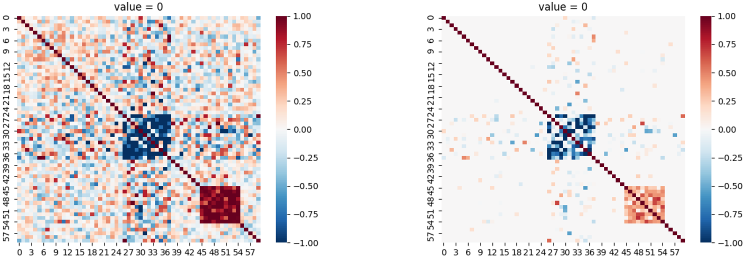


(b) Small values



(c) Big values

Differences in outcome



Rysunek 3: Examples of covariance matrices recovered from data.

End

Thank You for Your Attention

Alternating Direction Method of Multipliers (ADMM)

$$F(B) := \underbrace{\frac{1}{2}||Y - B||_F^2}_{f(B)} + \underbrace{\lambda_N \|B\|_*}_{g(B)} + \underbrace{\lambda_L \|\operatorname{vec}(W \circ B)\|_1}_{h(B)}.$$

Introducing variables C and D

$$\underset{B,C,D}{\operatorname{argmin}} \; \{f(B) + g(C) + h(D)\} \qquad \text{such that} \; \begin{cases} D - B = 0 \\ D - C = 0 \end{cases}$$

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Augmented Lagrangian formula:

$$L_\delta = f(B) + g(C) + h(D) + \langle Z_1, D - B\rangle + \langle Z_2, D - C\rangle +$$

$$\frac{\delta_1}{2}\|D - B\|_F^2 + \frac{\delta_2}{2}\|D - C\|_F^2.$$

Solution to $B^{[k+1]}$

Problem:

$$B^{[k+1]} := \underset{B}{\operatorname{argmin}} \left\{ 2f(B) \; + \; \delta_1^{[k]} \left\| D^{[k]} + \frac{Z_1^{[k]}}{\delta_1^{[k]}} - B \right\|_F^2 \right\}$$

Solution:

$$B^{[k+1]} = \frac{Y + \delta_1^k D^{[k]} + Z_1^k}{1 + \delta_1^k}$$

Solution to $C^{[k+1]}$

Problem:

$$C^{[k+1]} := \underset{C}{\operatorname{argmin}} \left\{ 2g(C) \ + \ \delta_2^{[k]} \left\| D^{[k]} + \frac{Z_2^{[k]}}{\delta_2^{[k]}} - C \right\|_F^2 \right\}$$

Solution:

$$\begin{cases} S^* := \operatorname{diag}\left(\left[\, (s_1^{[k]} - \frac{\lambda_N}{\delta_2^{[k]}})_+ \ , \ldots , \ (s_p^{[k]} - \frac{\lambda_N}{\delta_2^{[k]}})_+ \, \right]^T \right) \\ C^{[k+1]} = U^{[k]} S^* {V^{[k]}}^T \end{cases}$$

Solution to $D^{[k+1]}$

Problem:

$$D^{[k+1]} := \underset{D}{\operatorname{argmin}} \left\{ 2h(D) \ + \ \delta_1^{[k]} \left\| D - K \right\|_F^2 \ + \ \delta_2^{[k]} \left\| D - L \right\|_F^2 \right\}$$

Solution:

$$D_{ij}^{[k+1]} = \operatorname{sgn}\left(Q_{ij}^{[k+1]}\right) \cdot \left(\left| Q_{ij}^{[k+1]} \right| - \frac{\lambda_L W_{ij}}{\Delta^{[k]}} \right)_+ , \quad \text{for} \quad i, j \in \{1, \ldots, p\}$$