Contents

This page intentionally left blank.

Chapter 12: Hyperbolic and Parabolic Partial Differential Equations

12.1: EXAMPLES AND CONCEPTS OF HYPERBOLIC PDE'S

In the last chapter, we discussed in some detail the heat and Laplace's equations, which are prototypes for parabolic and elliptic PDEs, respectively. We would like now to introduce some concepts and theory for the wave equation, which is the prototype for hyperbolic equations. The wave equation models many natural phenomena, including gas dynamics (in particular, acoustics), vibrating solids and electromagnetism. It was first studied in the eighteenth century to model vibrations of strings and columns of air in organ pipes. Several mathematicians contributed to these initial studies, including Taylor, Euler, and Jean D'Alembert, about whom we will say more shortly. Subsequently in the nineteenth century, the wave equation was used to model elasticity as well as sound and light waves, and in the twentieth century, it has been used in quantum mechanics and relativity and most recently in such fields as superconductivity and string theory. In general, the wave equation has a time variable t and any number of space variables JC, y, z,... and takes the form

$$u_u = c^2 \triangle u = c^2(u_{xx} + u_{yy} + \dots \qquad (1)$$

where c is a positive constant and the Laplace operator on the right is with respect to all of the space variables. Modifications of this equation have been successfully used to model numerous physical waves and wavelike phenomena. In two space variables, for example, allowing for a variable wave speed due to depth differences in an ocean, the PDE: $u_u = \nabla \cdot [H(x,y,t)\nabla u] + H_u$ has been used to model large destructive ocean waves. [1] In such an application, the function H is the depth of the ocean at space coordinates (longitude and latitude) (JC, y) and at time /. The latter term corresponds to the changes in depth due to underwater landslides. For more on this and other applications of this variable media wave equation, we mention the text [Lan-99].

[1] The symbol ∇, read as "nabla" or "del," is used to represent the gradient operator, which is the vector of all partial derivatives of a function. Thus for a function of two variables $f(x,y)$, $\nabla f = \nabla f(x,y) = (f_x(x,y), f_y(x,y))$. The large dot represents the vector dot product, so in long form: $\nabla \cdot [H(x,y,t)\nabla u] = (\partial_x, \partial_y) \cdot (Hu_x, Hu_y) = \partial_x(Hu_x) + \partial(Hu_y)$. In particular, when $H \equiv 1$ we have $\nabla \cdot [\nabla u] = \partial_x(u_x) + \partial_y(u_y) = u_{xx} + u_{yy} = \triangle u$, another way to write the Laplacian of u. Such notations are very common in the literature for partial differential equations involving several space variables.

Much of the general theory of hyperbolic PDEs is well represented by that for the **one-dimensional wave equation** $(u = u(x,t)$ depends on time / and one space variable x so we proceed now to introduce it through its historical model of a vibrating string and present some of the theory. At the end of the section we indicate some differences and similarities of higher-dimensional waves to onedimensional waves.

We consider a small segment of taut string having length As and uniform tension T that is acted on by a vertical force q, as shown in Figure 12.1.

We assume that the string is displaced only in the vertical (transverse) direction, and let $u(x,t)$ denote the y-coordinate of the string at horizontal coordinate x at the time t. If we let ρ denote the mass density (mass per unit length) of the string (assumed constant), then Newton's second law (F = ma) gives us that

$$-T sin\Theta + T sin(\Theta + \triangle\,\Theta) + q \triangle\, s\rho \triangle s u_u(x,t)$$

, where the first two terms represent the vertical component of the internal elastic forces acting on the segment of string.

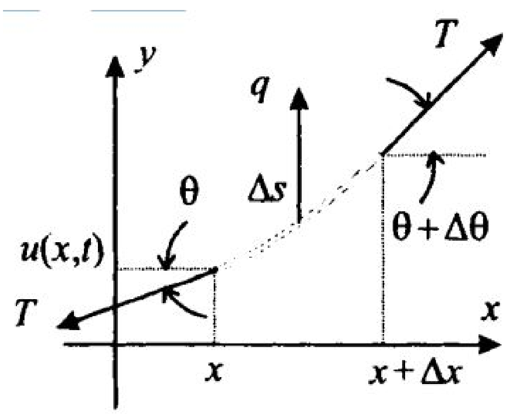

Figure 1: : A segment of a uniformly taut string having tension Tand external load g. The string is displaced vertically only, and u(x,t) is the vertical level of the string at time / and horizontal position x .

For small deflections in the string, we have $\triangle s \approx \triangle x$ and also $sin(\theta) \approx \theta \approx u_x(x,t)$ In the limit as As -» 0, this brings us to

$$T u_{xx} + q = \rho u_u, u = u()x,t \tag{2}$$

which is the **one-dimensional wave equation with external load term** q. In case q = 0, this reduces to the one-dimensional wave equation (1) with $c = (T/\rho)^{1/2}$. It turns out that this parameter c is the speed at which the wave (i.e., any solution of the equation) propagates. This will be made clear shortly. Intuitively, it makes sense that the speed of any disturbance on a string should increase along with the tension and decrease for heavier strings. For a derivation of wave equations for strings under more general hypotheses we refer to the article by S. Antman [Ant80] or Chapter 3 of the textbook by Kevorkian [Kev-00].

Figure 2: Jean Le Rond D'Alembert (1717-1783), French mathematician.

The general solution of the one-dimensional wave equation was first derived by the French mathematician Jean D'Alembert. [2] D'Alemberfs derivation is simple and elegant and the form of the solution will give many insights into qualitative aspects of wave equations. It begins by introducing the new variables:

$$\xi = x - ct, \eta = x + ct \tag{3}$$

We may now think of u as either a function of (x,t) or of (ξ, η). When we use the chain rule to translate the wave equation (1) into a PDE with respect to the new variables (ξ, η) something very nice will happen. The resulting PDE will be extremely easy to solve for the general solution. Applied using (3), the chain rule gives the following:

$$\begin{aligned} u_x &= u_\xi \xi_x + u_\eta \eta_x = u_\xi u_\eta \\ u_t &= u_\xi \xi_t + u_\eta \eta_t = -cu_\xi + cu_\eta \end{aligned} \tag{4}$$

In the same fashion, if we differentiate once again, we arrive at

$$u_{xx} = u\xi\xi + 2u_{\xi\eta} + u_{\eta\eta}, u_u = c^2(u\xi\xi - 2u_{\xi\eta} + u_{\eta\eta}) \tag{5}$$

When we substitute equations (5) into the one-dimensional wave equation (1), we obtain the following version of the wave equation in the new variables (ξ, η) :

$$u_{\xi\eta} = 0. \tag{6}$$

This PDE is very easy to solve, by "integrating" twice. Since it says that $\partial/\partial\eta(u_\xi) = 0$, we can integrate with respect to η to get $u_\xi = (F\xi)$,where

[2] Jean D'Alembert was born in Paris as an illegitimate child of a former nun while the father was out of the country. Unable to support her son, his mother left him on the steps of a church. The infant was quickly found and taken to an orphanage. He was baptized as Jean Le Rond, after the name of the church where he was found. When the infant's father returned to Paris, he arranged for Jean to be adopted by a married couple, who were friends of his. His adoptive parents brought him up well. He studied law and earned a law degree. He soon decided that mathematics was his true passion and studied it on his own. Although mostly self-taught, D'Alembert became an eminent mathematician and scholar in the same league with the likes of Euler, Laplace, and Lagrange. He made significant contributions to partial differential equations and his elegant methods, including his solution to the wave equation, very much impressed Euler. Frederick II (King of Prussia) offered D'Alembert the presidency of the prestigious Berlin Academy, a position which he declined. He was quite an eloquent and well-rounded scholar and he made significant contributions to Diderot's famous encyclopedia. Apparently, D'Alembert was prone to argumentation and his disputes with other contemporary mathematicians caused him some professional difficulties on several occasions.

$F(\xi)$ is an arbitrary function of ξ Next we integrate again, this time with respect to ξ , to conclude that

$$u(\xi,\eta) = f(\xi) + g(\eta), \tag{7}$$

where $f(\xi)$ and $g(\eta)$ are arbitrary functions of the indicated variables. (Note $f(\xi)$ is an antiderivative of $F(\xi)$) Translating back to the original variables using (3) gives us the following general solution of the wave equation:

$$u(x,t) = f(x-ct) + g(x+ct), \tag{8}$$

where f and g are arbitrary functions (with continuous second derivatives). We point out that each term in (8) represents a wave propagating along the x-axis with speed c. For example, $f(x-ct)$ is constant on lines of the form $x = ct$. As time t advances, values of x must also increase to maintain the same value of f (disturbance). Thus the first term represents a wave that propagates in the positive jc-direction with speed c (right traveling wave). Similarly, the term $g(x+ct)$ represents a left-traveling wave Both waves travel without distortion (i.e., the profile of either one of them / units of time later will be the exact same profile, but shifted to the left or right ct units along the x-axis.)

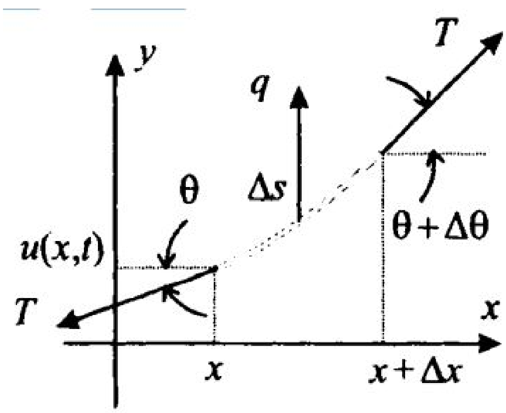

Figure 3: : A right-propagating pulse $f(x-ct)$. The general solution (8) of the onedimensional wave equation $u_{\eta} = c^2 u_{xx}$ also includes a left-propagating pulse. Both wavefronts propagate without distortion.

D'Alembert went on further with his general solution (8), formulating and solving a well-posed problem for the one-dimensional wave equation. We consider a very long string and so consider the one-dimensional wave equation on the space range $-\infty < x < \infty$ o, and the time range $0 \leqslant t < \infty$ Unlike with the heat equation, it is quite clear from (8) that merely specifying the wave profile W(JC,0) at time / = 0 is not sufficient to determine a unique solution. Indeed, the initial wave could come from a single left-moving wave, a single right-moving wave, or more generally could be made up as a superposition of two waves each moving in different directions.If we specify both the initial wave profile $u(x,0)$ and its initial velocity $u_t(x,0)$) , then this together with the wave equation will give a well-posed problem. These initial boundary conditions are often referred to as**Cauchy boundary conditions** (or **Cauchy boundary data**) Thus the **Cauchy problem for the wave equation** is summarized as follows:

$$\begin{cases} (PDE)u_{\eta} = c^2 u_{xx}, -\infty < x < \infty, 0 < t < \infty, u = u(x,t) \\ (BC's)u(x,0) = \phi(x), u_t(x,0) = v(x) - \infty < x < \infty, =\leqslant \infty \end{cases} \tag{9}$$

This highlights an important general difference between elliptic PDEs versus hyperbolic PDEs. Recall from the last chapter that for elliptic PDEs, simply specifying the value of the solution on the boundary of the domain (Dirichlet boundary conditions) resulted in a well-posed problem. For hyperbolic PDEs, more information is needed for the problem to be well posed. We now state d'Alembert's solution of this Cauchy problem:

THEOREM 12.1: (D 'Alembert 's Solution of the Cauchy Problem? Suppose that the function $\phi(x)$ has a continuous second derivative and $v(x)$ has a continuous first derivative on the whole real line. Then the Cauchy problem (9) for the onedimensional wave equation has the unique solution given by

oraz