On the number of types in sparse graphs*

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Abstract

We prove that for every class of graphs C which is nowhere dense, as defined by Nešetřil and Ossona de Mendez [25, 26], and for every first order formula $\varphi(\bar{x}, \bar{y})$, whenever one draws a graph $G \in C$ and a subset of its nodes A, the number of subsets of $A^{|\bar{y}|}$ which are of the form $\{\bar{v} \in A^{|\bar{y}|} : G \models \varphi(\bar{u}, \bar{v})\}$ for some valuation $\bar{u}$ of $\bar{x}$ in G is bounded by $O(|A|^{|\bar{x}|+\varepsilon})$, for every $\varepsilon > 0$. This provides optimal bounds on the VC-density of first-order definable set systems in nowhere dense graph classes. We also give two new proofs of upper bounds on quantities in nowhere dense classes which are relevant for their logical treatment. Firstly, we provide a new proof of the fact that nowhere dense classes are uniformly quasi-wide, implying explicit, polynomial upper bounds on the functions relating the two notions. Secondly, we give a new combinatorial proof of the result of Adler and Adler [1] stating that every nowhere dense class of graphs is stable. In contrast to the previous proofs of the above results, our proofs are completely finitistic and constructive, and yield explicit and computable upper bounds on quantities related to uniform quasi-wideness (margins) and stability (ladder indices).

Keywords Nowhere dense graphs, Stone space, first-order types, VC-density, stability, uniform quasi-wideness

1 Introduction

Nowhere dense classes of graphs were introduced by Nešetřil and Ossona de Mendez [25, 26] as a general and abstract model capturing uniform sparseness of graphs. These classes generalize many familiar classes of sparse graphs, such as planar graphs, graphs of bounded treewidth, graphs of bounded degree, and, in fact, all classes that exclude a fixed topological minor. Formally, a class C of graphs is *nowhere dense* if there is a function $t \colon \mathbb{N} \to \mathbb{N}$ such that for every $r \in \mathbb{N}$, no graph G in C contains the clique $K_{t(r)}$ on $t(r)$ vertices as *depth-r minor*, i.e., as a subgraph of a graph obtained from G by contracting mutually disjoint subgraphs of radius at most r to single vertices. The more restricted notion of *bounded expansion* requires in addition that for every fixed r, there is a constant (depending on r) upper bound on the ratio between the number of edges and the number of vertices in depth-r minors of graphs from C.

The concept of nowhere denseness turns out to be very robust, as witnessed by the fact that it admits multiple different characterizations, uncovering intricate connections to seemingly distant branches of mathematics. For instance, nowhere dense graph classes can be characterized by upper bounds on the density of bounded-depth (topological) minors [25, 26], by uniform quasi-wideness [26] (a notion introduced by Dawar [7] in the context of homomorphism preservation properties), by low tree-depth colorings [24], by generalized coloring numbers [36], by sparse neighborhood covers [15, 16], by a game called the splitter game [16], and by the model-theoretic concepts of stability and independence [1]. For a broader discussion on graph theoretic sparsity we refer to the book of Nešetřil and Ossona de Mendez [27].

The combination of combinatorial and logical methods yields a powerful toolbox for the study of nowhere dense graph classes. In particular, the result of Grohe, Kreutzer and the second author [16] exploits this combination in order to prove that a given first order sentence φ can be evaluated in time $f(\varphi) \cdot n^{1+\varepsilon}$ on n-vertex graphs from a fixed nowhere dense class of graphs C, for any fixed real $\varepsilon > 0$ and some function f. On the other hand, provided C is closed under taking subgraphs, it is known that if C is not nowhere dense, then there is no algorithm with running time of the form $f(\varphi) \cdot n^c$ for any constant c under plausible complexity assumptions [10]. In the terminology of parameterized complexity, these results show that the notion of nowhere denseness exactly characterized subgraph-closed classes where model-checking first order logic is fixed-parameter tractable, and conclude a long line of research concerning the parameterized complexity of the model checking problem for sparse graph classes (see [14] for a survey).

Summary of contribution. In this paper, we continue the study of the interplay of combinatorial and logical properties of nowhere dense graph classes, and provide new upper bounds on several quantities appearing in their logical study. Our main focus is on the notion of *VC-density* for first order formulas. This concept originates from model theory and

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aims to measure the complexity of set systems definable by first order formulas, similarly to the better-known VC-dimension. We give optimal bounds on the VC-density in nowhere dense graph classes, and in particular we show that these bounds are "as good as one could hope for".

We also provide new upper bounds on quantities related to *stability* and *uniform quasi-wideness* of nowhere dense classes. For stability, we provide explicit and computable upper bounds on the *ladder index* of any first order formula on a given nowhere dense class. For uniform quasi-wideness, we give a new, purely combinatorial proof of polynomial upper bounds on *margins*, that is, functions governing this notion. We remark that the existence of upper bounds as above is known [1, 19], but the proofs are based on nonconstructive arguments, notably the compactness theorem for first order logic. Therefore, the upper bounds are given purely existentially and are not effectively computable. Contrary to these, our proofs are entirely combinatorial and effective, yielding computable upper bounds.

We now discuss the relevant background from logic and model theory, in order to motivate and state our results.

Model theory. Our work is inspired by ideas from model theory, more specifically, from *stability theory*. The goal of stability theory is to draw certain dividing lines specifying abstract properties of logical structures which allow the development of a good structure theory. There are many such dividing lines, depending on the specifics of the desired theory. One such dividing line encloses the class of *stable structures*, another encloses the larger class of *dependent structures* (also called *NIP*). A general theme is that the existence of a manageable structure is strongly related to the non-existence of certain forbidden patterns on one hand, and on the other hand, to bounds on cardinalities of certain *type sets*. Let us illustrate this phenomenon more concretely.

For a first order formula $\varphi(\bar{x},\bar{y})$ with free variables split into $\bar{x}$ and $\bar{y}$, a *φ-ladder* of length n in a logical structure $\mathbb{A}$ is a sequence $\bar{u}_1,\ldots,\bar{u}_n,\bar{v}_1,\ldots,\bar{v}_n$ of tuples of elements of $\mathbb{A}$ such that

$$\mathbb{A} \models \varphi(\bar{u}_i,\bar{v}_j) \iff i \leqslant j \qquad \text{for all } 1 \leqslant i,j \leqslant n.$$

The least n for which there is no φ-ladder of length n is the *ladder index* of $\varphi(\bar{x},\bar{y})$ in $\mathbb{A}$ (which may depend on the split of the variables, and may be ∞ for some infinite structures $\mathbb{A}$). A class of structures C is *stable* if the ladder index of every first order formula $\varphi(\bar{x},\bar{y})$ over structures from C is bounded by a constant depending only on φ and C. This notion can be applied to a single infinite structure $\mathbb{A}$, by considering the class consisting of $\mathbb{A}$ only. We refer to the textbooks [28, 35] for an introduction to stability theory.

One of concepts studied in the early years of stability theory is a property of infinite graphs called *superflatness*, introduced by Podewski and Ziegler [29]. The definition of superflatness is the same as of nowhere denseness, but Podewski and Ziegler, instead of applying it to an infinite class of finite graphs, apply it to a single infinite graph. The main result of [29] is that every superflat graph is stable. As observed by Adler and Adler [1], this directly implies the following:

Theorem 1.1 ([1, 29])**.** *Every nowhere dense class of graphs is stable. Conversely, any stable class of finite graphs which is subgraph-closed is nowhere dense.*

Thus, the notion of nowhere denseness (or superflatness) coincides with stability if we restrict attention to subgraph-closed graph classes.

The proof of Adler and Adler does not yield effective or computable upper bound on the ladder index of a given formula for a given nowhere dense class of graphs, as it relies on the result of Podewski and Ziegler, which in turn invokes compactness for first order logic.

Cardinality bounds. One of the key insights provided by the work of Shelah is that stable classes can be characterized by admitting strong upper bounds on the cardinality of *Stone spaces*. For a first order formula $\varphi(\bar{x},\bar{y})$ with free variables partitioned into *object variables* $\bar{x}$ and *parameter variables* $\bar{y}$, a logical structure $\mathbb{A}$, and a subset of its domain B, define the set of *φ-types* with parameters from B, which are realized in $\mathbb{A}$, as follows[1]:

$$S^{\varphi}(\mathbb{A}/B) = \left\{\left\{\bar{v} \in B^{|\bar{y}|} : \mathbb{A} \models \varphi(\bar{u},\bar{v})\right\} : \bar{u} \in V(\mathbb{A})^{|\bar{x}|}\right\} \subseteq \mathcal{P}(B^{|\bar{y}|}). \tag{1}$$

Here, $V(\mathbb{A})$ denotes the domain of $\mathbb{A}$ and $\mathcal{P}(X)$ denotes the powerset of X. Putting the above definition in words, every tuple $\bar{u} \in V(\mathbb{A})^{|\bar{x}|}$ defines the set of those tuples $\bar{v} \in B^{|\bar{y}|}$ for which $\varphi(\bar{u},\bar{v})$ holds. The set $S^{\varphi}(\mathbb{A}/B)$ consists of all subsets of $B^{|\bar{y}|}$ that can be defined in this way.

Note that in principle, $S^{\varphi}(\mathbb{A}/B)$ may be equal to $\mathcal{P}(B^{|\bar{x}|})$, and therefore have very large cardinality compared to B, even for very simple formulas. The following characterization due to Shelah (cf. [34, Theorem 2.2, Chapter II]) shows that for stable classes this does not happen. A class of structures C is stable if and only if there is an infinite cardinal κ such that the following holds for all structures $\mathbb{A}$ in the elementary closure[2] of C, and all $B \subseteq V(\mathbb{A})$: *if* $|B| \leqslant \kappa$, *then* $|S^{\varphi}(\mathbb{A}/B)| \leqslant \kappa$. Therefore, Shelah's result provides an upper bound on the number of types, albeit using infinite cardinals, elementary limits, and infinite parameter sets. The cardinality bound provided by the above characterization however, does not seem to immediately translate to a result of finitary nature. As we describe below, this can be done using the notions of *VC-dimension* and *VC-density*.

[1] Here, $S^{\varphi}(\mathbb{A}/B)$ is the set of types which are *realized* in $\mathbb{A}$. In model theory, one usually works with the larger class of *complete types*. This distinction will not be relevant here.

[2] The elementary closure of C is the class of all structures $\mathbb{A}$ such that every first order sentence φ which holds in $\mathbb{A}$ also holds in some $\mathbb{B} \in C$. Equivalently, it is the class of models of the theory of C.

VC-dimension and VC-density. The notion of VC-dimension was introduced by Vapnik and Chervonenkis [6] as a measure of complexity of set systems, or equivalently of hypergraphs, and independently by Shelah [32] under the name of dependence (equivalence of the two notions was observed by Laskowski [20]).

Formally, VC-dimension is defined as follows. Let X be a set and let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a family of subsets of X. A subset $A \subseteq X$ is *shattered by* $\mathcal{F}$ if $\{A \cap F \colon F \in \mathcal{F}\} = \mathcal{P}(A)$; that is, every subset of A can be obtained as the intersection of some set from $\mathcal{F}$ with A. The *VC-dimension*, of $\mathcal{F}$ is the maximum size of a subset $A \subseteq X$ that is shattered by $\mathcal{F}$.

For a given structure $\mathbb{A}$, parameter set $B \subseteq V(\mathbb{A})$, and formula $\varphi(\bar{x}, \bar{y})$, we may consider the family $S^\varphi(\mathbb{A}/B)$ of subsets of $B^{|\bar{y}|}$ defined using equation (1). The *VC-dimension* of $\varphi(\bar{x}, \bar{y})$ on $\mathbb{A}$ is the VC-dimension of the family $S^\varphi(\mathbb{A}/V(\mathbb{A}))$. In other words, the VC-dimension of $\varphi(\bar{x}, \bar{y})$ on $\mathbb{A}$ is the largest cardinality of a finite set I for which there exist families of tuples $(\bar{u}_i)_{i \in I}$ and $(\bar{v}_J)_{J \subseteq I}$ of elements of $\mathbb{A}$ such that

$$\mathbb{A} \models \varphi(\bar{u}_i, \bar{v}_J) \iff i \in J \qquad \text{for all } i \in I \text{ and } J \subseteq I.$$

A formula $\varphi(\bar{x}, \bar{y})$ is *dependent* on a class of structures $\mathcal{C}$ if there is a bound $d \in \mathbb{N}$ such that the VC-dimension of $\varphi(\bar{x}, \bar{y})$ on $\mathbb{A}$ is at most d for all $\mathbb{A} \in \mathcal{C}$. It is immediate from the definitions that if a formula $\varphi(\bar{x}, \bar{y})$ is stable over $\mathcal{C}$, then it is also dependent on $\mathcal{C}$ (the bound being the ladder index). A class of structures $\mathcal{C}$ is *dependent* if every formula $\varphi(\bar{x}, \bar{y})$ is dependent over $\mathcal{C}$. In particular, every stable class is dependent, and hence, by Theorem 1.1, every nowhere dense class of graphs is dependent.

One of the main properties of VC-dimension is that it implies polynomial upper bounds on the number of different "traces" that a set system can have on a given parameter set. This is made precise by the well-known Sauer-Shelah Lemma, stated as follows.

Theorem 1.2 (Sauer-Shelah Lemma, [6, 31, 33])**.** *For any family $\mathcal{F}$ of subsets of a set X, if the VC-dimension of $\mathcal{F}$ is d, then for every finite $A \subseteq X$,*

$$|\{A \cap F \ : \ F \in \mathcal{F}\}| \leqslant c \cdot |A|^d,$$

where c is a universal constant.

In particular, this implies that in a dependent class of structures $\mathcal{C}$, for every formula $\varphi(\bar{x}, \bar{y})$ there exists some constant $d \in \mathbb{N}$ such that

$$|S^\varphi(\mathbb{A}/B)| \leqslant c \cdot |B|^d, \tag{2}$$

for all $\mathbb{A} \in \mathcal{C}$ and finite $B \subseteq V(\mathbb{A})$. Unlike Shelah's characterization theorem of stable classes, this result is of finitary nature: it provides quantitative upper bounds on the number of different definable subsets of a given finite parameter set. Together with Theorem 1.1, this implies that for every nowhere dense class of graphs and every first order formula $\varphi(\bar{x}, \bar{y})$, there exists a constant $d \in \mathbb{N}$ such that (2) holds.

However, the VC-dimension d may be enormous and it highly depends on $\mathcal{C}$ and the formula $\varphi(\bar{x}, \bar{y})$. This suggests investigating quantitative bounds of the form (2) for exponents smaller than the VC-dimension d, as it is conceivable that the combination of bounding VC-dimension and applying Sauer-Shelah Lemma yields a suboptimal upper bound. Our main goal is to decrease this exponent drastically in the setting of nowhere dense graph classes.

The above discussion motivates the notion of *VC-density*, a notion closely related to VC-dimension. The *VC-density* (also called the *VC-exponent*) of a set system $\mathcal{F}$ on an infinite set X is the infimum of all reals $\alpha > 0$ such that $|\{A \cap F \ : \ F \in \mathcal{F}\}| \in \mathcal{O}(|A|^\alpha)$, for all finite $A \subseteq X$. Similarly, the VC-density of a formula $\varphi(\bar{x}, \bar{y})$ over a class of structures $\mathcal{C}$ is the infimum of all reals $\alpha > 0$ such that $|S^\varphi(\mathbb{A}/B)| \in \mathcal{O}(|B|^\alpha)$, for all $\mathbb{A} \in \mathcal{C}$ and all finite $B \subseteq V(\mathbb{A})$. The Sauer-Shelah Lemma implies that the VC-density (of a set system, or of a formula over a class of structures) is bounded from above by the VC-dimension. However, in many cases, the VC-density may be much smaller than the VC-dimension. Furthermore, it is the VC-density, rather than VC-dimension, that is actually relevant in combinatorial and algorithmic applications [5, 22, 23]. We refer to [2] for an overview of applications of VC-dimension and VC-density in model theory and to the surveys [12, 22] on uses of VC-density in combinatorics.

The main result. Our main result, Theorem 1.3 stated below, improves the bound (2) for classes of sparse graphs by providing essentially the optimum exponent.

Theorem 1.3. *Let $\mathcal{C}$ be a nowhere dense class of graphs and let $\varphi(\bar{x}, \bar{y})$ be a first order formula with free variables partitioned into object variables $\bar{x}$ and parameter variables $\bar{y}$. Let $\ell = |\bar{x}|$. Then for every $\varepsilon > 0$ there exists a constant c such that for every $G \in \mathcal{C}$ and every nonempty $A \subseteq V(G)$, we have $|S^\varphi(G/A)| \leqslant c \cdot |A|^{\ell+\varepsilon}$.*

In particular, Theorem 1.3 implies that the VC-density of any fixed formula $\varphi(\bar{x}, \bar{y})$ over any nowhere dense class of graphs is $|\bar{x}|$, the number of object variables in φ.

To see that the bounds provided by Theorem 1.3 cannot be improved, consider a formula $\varphi(\bar{x}, y)$ (i.e. with one parameter variable) expressing that y is equal to one of the entries of $\bar{x}$. Then for each graph G and parameter set A, $S^\varphi(G/A)$ consists of all subsets of A of size at most $|\bar{x}|$, whose number is $\Theta(|A|^{|\bar{x}|})$. Note that this lower bound applies to any infinite class of graphs, even edgeless ones.

We moreover show that, as long as we consider only subgraph-closed graph classes, the result of Theorem 1.3 also cannot be improved in terms of generality. The following result is an easy corollary of known characterizations of obstructions to being nowhere dense.

Theorem 1.4. *Let $\mathcal{C}$ be a class of graphs which is closed under taking subgraphs. If $\mathcal{C}$ is not nowhere dense, then there is a* *formula $\varphi(x, y)$ such that for every $n \in \mathbb{N}$ there are $G \in \mathcal{C}$ and $A \subseteq V(G)$ with $|A| = n$ and $|S^{\varphi}(G/A)| = 2^{|A|}$.*

We prove in the appended full version of the paper that a similar characterization theorem can be proved for bounded expansion classes of graphs. More precisely, if $\mathcal{C}$ has bounded expansion, then there exists a constant c such that for every $G \in \mathcal{C}$ and every nonempty $A \subseteq V(G)$, we have $|S^{\varphi}(G/A)| \leqslant c \cdot |A|^{\ell}$. Conversely, if $\mathcal{C}$ has unbounded expansion, then there is a formula $\varphi(x, y)$ such that for every $c \in \mathbb{R}$ there exist $G \in \mathcal{C}$ and a nonempty $A \subseteq V(G)$ with $|S^{\varphi}(G/A)| > c|A|$.

Neighborhood complexity. To illustrate Theorem 1.3, consider the case when G is a graph and $\varphi(x, y)$ is the formula with two variables x and y expressing that the distance between x and y is at most r, for some fixed integer r. In this case, $S^{\varphi}(G/A)$ is the family consisting of all intersections $U \cap A$, for U ranging over all balls of radius r in G, and $|S^{\varphi}(G/A)|$ is called the *r-neighborhood complexity* of A. The concept of r-neighborhood complexity in sparse graph classes has already been studied before. In particular, it was proved by Reidl et al. [30] that in any graph class of bounded expansion, the r-neighborhood complexity of any set of vertices A is $\mathcal{O}(|A|)$. Recently, Eickmeyer et al. [11] generalized this result to an upper bound of $\mathcal{O}(|A|^{1+\varepsilon})$ in any nowhere dense class of graphs. Note that these results are special cases of Theorem 1.3. The study of r-neighborhood complexity on classes of bounded expansion and nowhere dense classes was motivated by algorithmic questions from the field of parameterized complexity. More precisely, the usage of this notion was crucial for the development of a linear kernel for the r-Dominating Set problem on any class of bounded expansion [9], and of an almost linear kernel for this problem on any nowhere dense class [11]. We will use the results of [9, 11, 30] on r-neighborhood complexity in sparse graphs in our proof of Theorem 1.3.

Uniform quasi-wideness. One of the main tools used in our proof is the notion of *uniform quasi-wideness*, introduced by Dawar [7] in the context of homomorphism preservation theorems. Formally, a class of graphs $\mathcal{C}$ is *uniformly quasi-wide* if for each integer $r \in \mathbb{N}$ there is a function $N \colon \mathbb{N} \to \mathbb{N}$ and a constant $s \in \mathbb{N}$ such that for every $m \in \mathbb{N}$, graph $G \in \mathcal{C}$, and vertex subset $A \subseteq V(G)$ of size $|A| \geqslant N(m)$, there is a set $S \subseteq V(G)$ of size $|S| \leqslant s$ and a set $B \subseteq A \setminus S$ of size $|B| \geqslant m$ which is r-independent in $G - S$. Recall that a set $B \subseteq V(G)$ is *r-independent* in G if all distinct $u, v \in B$ are at distance larger than r in G.

Nešetřil and Ossona de Mendez proved that the notions of uniform quasi-wideness and nowhere denseness coincide for classes of finite graphs [25]. The proof of Nešetřil and Ossona de Mendez goes back to a construction of Kreidler and Seese [18] (see also Atserias et al. [3]), and uses iterated Ramsey arguments. Hence the original bounds on the function N_r are non-elementary. Recently, Kreutzer, Rabinovich and the second author proved that for each radius r, we may always choose the function N_r to be a polynomial [19]. However, the exact dependence of the degree of the polynomial on r and on the class $\mathcal{C}$ itself was not specified in [19], as the existence of a polynomial bound is derived from non-constructive arguments used by Adler and Adler in [1] when showing that every nowhere dense class of graphs is stable. We remark that polynomial bounds for uniform quasi-wideness are essential for some of its applications: the fact that N_r can be chosen to be polynomial was crucially used by Eickmeyer et al. [11] both to establish an almost linear upper bound on the r-neighborhood complexity in nowhere dense classes, and to develop an almost linear kernel for the r-Dominating Set problem. We use this in our proof of Theorem 1.3 as well.

In our quest for constructive arguments, we give a new construction giving polynomial bounds for uniform quasi-wideness. The new proof is considerably simpler than that of [19] and gives explicit and computable bounds on the degree of the polynomial. More precisely, we prove the following theorem; here, the notation $\mathcal{O}_{r,t}(\cdot)$ hides computable factors depending on r and t.

Theorem 1.5. *For all $r, t \in \mathbb{N}$ there is a polynomial N with $N(m) = \mathcal{O}_{r,t}(m^{(4t+1)^{2rt}})$, such that the following holds. Let G be a graph such that $K_t \not\preccurlyeq_{\lfloor 9r/2 \rfloor} G$, and let $A \subseteq V(G)$ be a vertex subset of size at least $N(m)$, for a given m. Then there exists a set $S \subseteq V(G)$ of size $|S| < t$ and a set $B \subseteq A \setminus S$ of size $|B| \geqslant m$ which is r-independent in $G - S$. Moreover, given G and A, such sets S and B can be computed in time $\mathcal{O}_{r,t}(|A| \cdot |E(G)|)$.*

We remark that even though the techniques employed to prove Theorem 1.5 are inspired by methods from stability theory, at the end we conduct an elementary graph theoretic reasoning. In particular, as asserted in the statement, the proof be turned into an efficient algorithm.

We also prove a result extending Theorem 1.5 to the case where $A \subseteq V(G)^d$ is a set of *tuples* of vertices, of any fixed length d. This result is essentially an adaptation of an analogous result due to Podewski and Ziegler [29] in the infinite case, but appears to be new in the context of finite structures. This more general result turns out to be necessary in the proof of Theorem 1.3.

Local separation. A simple, albeit important notion which permeates our proofs is a graph theoretic concept of *local separation*. Let G be a graph, $S \subseteq V(G)$ a set of vertices, and let $r \in \mathbb{N}$ be a number. We say that two sets of vertices A and B are *r-separated* by S (in G) if every path from a vertex in A to a vertex in B of length at most r contains a vertex from S. Observe that taking $r = \infty$ in r-separation yields the familiar notion of a separation in graph theory. From the perspective of stability, separation (for $r = \infty$) characterizes *forking independence* in superflat graphs [17]. Therefore, r-separation can be thought of as a local analogue of forking independence, for nowhere dense graph classes.

A key lemma concerning r-separation (cf. Theorem 4.3) states that if A and B are r-separated by a set S of size s in G, then for any fixed formula $\varphi(\bar{x},\bar{y})$ of quantifier rank $O(\log r)$, the set $\{\{\bar{v} \in B^{|\bar{y}|} : G \models \varphi(\bar{u},\bar{v})\} : \bar{u} \in A^{|\bar{x}|}\}$ has cardinality bounded by a constant depending on s and φ only (and not on G, A, and B). This elementary result combines Gaifman's locality of first order logic (cf. [13]) and a Feferman-Vaught compositionality argument. This, in combination with the polynomial bounds for uniform quasi-wideness (Theorem 1.5, and its extension to tuples Theorem 3.6), as well as the previous results on neighborhood complexity [9, 11], are the main ingredients of our main result, Theorem 1.3.

A duality theorem. As an example application of our main result, Theorem 1.3, we state the following result.

Theorem 1.6. *Fix a nowhere dense class of graphs $\mathcal{C}$ and a formula $\varphi(x,y)$ with two free variables x,y. Then there is a function $f\colon \mathbb{N} \to \mathbb{N}$ with the following property. Let $G \in \mathcal{C}$ be a graph and let $\mathcal{G}$ be a family of subsets of $V(G)$ consisting of sets of the form $\{v \in V(G) \ :\ \varphi(u,v)\}$, where u is some vertex of $V(G)$. Then $\tau(\mathcal{G}) \leqslant f(\nu(\mathcal{G}))$.*

Above, $\tau(\mathcal{G})$ denotes the *transversality* of $\mathcal{G}$, i.e., the least number of elements of a set X which intersects every set in $\mathcal{G}$, and $\nu(\mathcal{G})$ denotes the *packing number* of $\mathcal{G}$, i.e., the largest number of pairwise-disjoint subsets of $\mathcal{G}$. Theorem 1.6 is an immediate consequence of the bound given by Theorem 1.3 and a result of Matoušek [23].

We remark that a similar, but incomparable result is proved by Bousquet and Thomassé [4]. In their result, the assumption on $\mathcal{C}$ is weaker, since they just require that it has *bounded distance VC-dimension*, but the assumption on $\mathcal{G}$ is stronger, as it is required to be the set of all balls of a fixed radius.

Stability. Finally, we observe that we can apply our tools to give a constructive proof of the result of Adler and Adler [1] that every nowhere dense class is stable, which yields computable upper bounds on ladder indices. More precisely, we translate the approach of Podewski and Ziegler [29] to the finite and replace the key non-constructive application of compactness with a combinatorial argument based on Gaifman's locality, in the flavor served by our observations on r-separation (Theorem 4.3). The following theorem summarizes our result.

Theorem 1.7. *There are computable functions $f\colon \mathbb{N}^3 \to \mathbb{N}$ and $g\colon \mathbb{N} \to \mathbb{N}$ with the following property. Suppose $\varphi(\bar{x},\bar{y})$ is a formula of quantifier rank at most q and with d free variables. Suppose further that G is a graph excluding K_t as a depth-$g(q)$ minor. Then the ladder index of $\varphi(\bar{x},\bar{y})$ in G is at most $f(q,d,t)$.*

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2 Preliminaries

We recall some basic notions from graph theory.

All graphs in this paper are finite, undirected and simple, that is, they do not have loops or parallel edges. Our notation is standard, we refer to [8] for more background on graph theory. We write $V(G)$ for the vertex set of a graph G and $E(G)$ for its edge set. The *distance* between vertices u and v in G, denoted $\mathrm{dist}_G(u,v)$, is the length of a shortest path between u and v in G. If there is no path between u and v in G, we put $\mathrm{dist}_G(u,v) = \infty$. The *(open) neighborhood* of a vertex u, denoted $N(u)$, is the set of neighbors of u, excluding u itself. For a non-negative integer r, by $N_r[u]$ we denote the *(closed) r-neighborhood* of u which comprises vertices at distance at most r from u; note that u is always contained in its closed r-neighborhood. The *radius* of a connected graph G is the least integer r such that there is some vertex v of G with $N_r[v] = V(G)$.

A *minor model* of a graph H in G is a family $(I_u)_{u\in V(H)}$ of pairwise vertex-disjoint connected subgraphs of G, called *branch sets*, such that whenever uv is an edge in H, there are $u' \in V(I_u)$ and $v' \in V(I_v)$ for which $u'v'$ is an edge in G. The graph H is a *depth-r minor* of G, denoted $H \preccurlyeq_r G$, if there is a minor model $(I_u)_{u\in V(H)}$ of H in G such that each I_u has radius at most r.

A class $\mathcal{C}$ of graphs is *nowhere dense* if there is a function $t\colon \mathbb{N} \to \mathbb{N}$ such that for all $r \in \mathbb{N}$ it holds that $K_{t(r)} \not\preccurlyeq_r G$ for all $G \in \mathcal{C}$, where $K_{t(r)}$ denotes the clique on $t(r)$ vertices.

A set $B \subseteq V(G)$ is called *r-independent* in a graph G if $\mathrm{dist}_G(u,v) > r$ for all distinct $u,v \in B$. A class $\mathcal{C}$ of graphs is *uniformly quasi-wide* if for every $r \in \mathbb{N}$ there is a number $s \in \mathbb{N}$ and a function $N\colon \mathbb{N} \to \mathbb{N}$ such that for every $m \in \mathbb{N}$, graph $G \in \mathcal{C}$, and vertex subset $A \subseteq V(G)$ of size $|A| \geqslant N(m)$, there is a set $S \subseteq V(G)$ of size $|S| \leqslant s$ and a set $B \subseteq A - S$ of size $|B| \geqslant m$ which is r-independent in $G - S$. Recall that Nešetřil and Ossona de Mendez proved [25] that nowhere dense graph classes are exactly the same as uniformly quasi-wide classes.

As mentioned in the introduction, Kreutzer, Rabinovich and the second author proved that for each radius r, we may always choose the function N to be a polynomial [19]. However, the proof of Kreutzer et al. [19] relies on non-constructive arguments and does not yield explicit bounds on the degree of N. In the next section, we discuss a further strengthening of this result, by providing explicit, computable bounds on N and s.

3 Bounds for uniform quasi-wideness

In this section we prove Theorem 1.5, which strengthens the result of Kreutzer et al. [19] by providing an explicit polynomial N and bound s. We remark that the result of Kreutzer et al. is sufficient to prove our main result, Theorem 1.3, but is required in our proof of Theorem 1.7, which is the effective variant of the result of Adler and Adler, Theorem 1.7. Our proof follows the same lines as the original proof of Nešetřil and Ossona de Mendez [25], with the difference that in the key technical lemma (Lemma 3.5 of [25]) we improve the bounds significantly by replacing a Ramsey argument with a refined combinatorial analysis (Theorem 3.1 below). The new argument essentially originates in the concept of *branching index* from stability theory. Due to space constraints we present only the proof of the key lemma and refer to the appended full version for a complete proof of the main theorem.

Lemma 3.1. *Let $\ell, m, t \in \mathbb{N}$ and assume $\ell \geqslant t^8$. If G is a graph and A is a 1-independent set in G with at least $(m+\ell)^{2t}$ elements, then at least one of the following conditions hold:*

- *$K_t \preccurlyeq_4 G$,*
- *A contains a 2-independent set of size m,*
- *some vertex v of G has at least $\ell^{1/4}$ neighbors in A.*

Moreover, if $K_t \not\preccurlyeq_4 G$, the structures described in the other two cases (a 2-independent set of size m, or a vertex v as above) can be computed in time $O_t(|A| \cdot |E(G)|)$.

We remark that a statement similar to that of Theorem 3.1 can be obtained by employing Ramsey's theorem, as has been done in [25]. This, however, does not give a bound which is polynomial in $m + \ell$, and thus cannot be used to prove Theorem 1.5. The remainder of this section is devoted to the proof of Theorem 3.1. We will need one more technical lemma.

Lemma 3.2 (★). *Let G be a graph such that $K_t \not\preccurlyeq_4 G$ for some $t \in \mathbb{N}$ and let $A \subseteq V(G)$ with $|A| \geqslant t^8$. Assume furthermore that every pair of elements of A has a common neighbor in $V(G) \setminus A$. Then there exists a vertex v in $V(G) \setminus A$ which has at least $|A|^{1/4}$ neighbors in A.*

We proceed with the proof of Theorem 3.1. The idea is to arrange the elements of A in a binary tree and prove that provided A is large, this tree contains a long path. From this path, we will extract the set B. In stability theory, similar trees are called *type trees* and they are used to extract long indiscernible sequences, see e.g. [21].

We will work with a two-symbol alphabet $\{D, S\}$, for *daughter* and *son*. We identify words in $\{D, S\}^*$ with *nodes* of the infinite rooted binary tree. The *depth* of a node w is the length of w. For $w \in \{D, S\}^*$, the nodes wD and wS are called, respectively, the *daughter* and the *son* of w, and w is the *parent* of both wS and wD. A node w' is a *descendant* of a node w if w' is a prefix of w (possibly $w' = w$). We consider finite, labeled, rooted, binary trees, which are called simply trees below, and are defined as follows. For a set of labels U, a (U-labeled) *tree* is a partial function $\tau \colon \{D, S\}^* \to U$ whose domain is a finite set of nodes, called the *nodes of* τ, which is closed under taking parents. If v is a node of τ, then $\tau(v)$ is called its *label*.

Let G be a graph, $A \subseteq V(G)$ be a 1-independent set in G, and $\bar{a}$ be any enumeration of A, that is, a sequence of length $|A|$ in which every element of A appears exactly once. We define a binary tree τ which is labeled by vertices of G. The tree is defined by processing all elements of $\bar{a}$ sequentially. We start with τ being the tree with empty domain, and for each element a of the sequence $\bar{a}$, processed in the order given by $\bar{a}$, execute the following procedure which results in adding a node with label a to τ.

When processing the vertex a, do the following. Start with w being the empty word. While w is a node of τ, repeat the following step: if the distance from a to $\tau(w)$ in the graph G is at most 2, replace w by its son, otherwise, replace w by its daughter. Once w is not a node of τ, extend τ by setting $\tau(w) = a$. In this way, we have processed the element a, and now proceed to the next element of $\bar{a}$, until all elements are processed. This ends the construction of τ. Thus, τ is a tree labeled with vertices of A, and every vertex of A appears exactly once in τ.

Define the *depth* of τ as the maximal depth of a node of τ. For a word w, an *alternation* in w is any position α, $1 \leqslant \alpha \leqslant |w|$, such that $w_\alpha \neq w_{\alpha-1}$; here, w_α denotes the αth symbol of w, and w_0 is assumed to be D. The *alternation rank* of the tree τ is the maximum of the number of alternations in w, over all nodes w of τ.

Lemma 3.3 (★). *Let $h, t \geqslant 2$. If τ has alternation rank at most $2t-1$ and depth at most $h-1$, then τ has fewer than h^{2t} nodes.*

Lemma 3.4. *Suppose that $K_t \not\preccurlyeq_2 G$. Then τ has alternation rank at most $2t-1$.*

Proof. Let w be a node of τ with at least $2k$ alternations, for some $k \in \mathbb{N}$. Suppose $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k$ be the first $2k$ alternations of w. By the assumption that $w_0 = D$ we have that w contains symbol S at all positions α_i for $i = 1, \ldots, k$, and symbol D at all positions β_i for $i = 1, \ldots, k$. For each $i \in \{1, \ldots, k\}$, define $a_i \in V(G)$ to be the label in τ of the prefix of w of length $\alpha_i - 1$, and similarly define $b_i \in V(G)$ to be the label in τ of the prefix of w of length $\beta_i - 1$. It follows that for each $i \in \{1, \ldots, k\}$, the following assertions hold: the nodes in τ with labels $b_i, a_{i+1}, b_{i+1}, \ldots, a_k, b_k$ are descendants of the son of the node with label a_i, and the nodes with labels $a_{i+1}, b_{i+1}, \ldots, a_k, b_k$ are descendants of the daughter of the node with label b_i.

Claim 1. For every pair a_i, b_j with $1 \leqslant i \leqslant j \leqslant k$, there is a vertex $z_{ij} \notin A$ which is a common neighbor of a_i and b_j, and is not a neighbor of any b_s with $s \neq j$.

Proof. Note that since $i \leqslant j$, the node with label b_j is a descendant of the son of the node with label a_i, hence we have $\mathrm{dist}_G(a_i, b_j) \leqslant 2$ by the construction of τ. However, we also have $\mathrm{dist}_G(a_i, b_j) > 1$ since A is 1-independent. Therefore $\mathrm{dist}_G(a_i, b_j) = 2$, so there is a vertex z_{ij} which is a common neighbor of a_i and b_j. Suppose that z_{ij} was a neighbor of b_s, for some $s \neq j$. This would imply that $\mathrm{dist}_G(b_j, b_s) \leqslant 2$, which is impossible, because the nodes with labels b_s and b_j in τ are such that one is a descendant of the daughter of the other, implying that $\mathrm{dist}_G(b_s, b_j) > 2$. ⌟

Note that whenever $i \leqslant j$ and $i' \leqslant j'$ are such that $j \neq j'$, the vertices z_{ij} and $z_{i'j'}$ are different, because z_{ij} is adjacent to b_j but not to $b_{j'}$, and the converse holds for $z_{i'j'}$. However, it may happen that $z_{ij} = z_{i'j}$ even if $i \neq i'$. This will not affect our further reasoning.

For each $j \in \{1, \ldots, k\}$, let B_j be the subgraph of G induced by the set $\{a_j, b_j\} \cup \{z_{ij} \colon 1 \leqslant i \leqslant j\}$. Observe that B_j is connected and has radius at most 2, with b_j being the central vertex. By Theorem 3.4 and the discussion from the previous paragraph, the graphs B_j for $j \in \{1, \ldots, k\}$ are pairwise disjoint. Moreover, for all $1 \leqslant i \leqslant j \leqslant k$, there is an edge between B_i and B_j, namely, the edge between $z_{ij} \in B_j$ and $a_i \in B_i$. Hence, the graphs B_j, for $j \in \{1, \ldots, k\}$, define a depth-2 minor model of K_k in G. Since $K_t \not\preccurlyeq_2 G$, this implies that $k < t$, proving Theorem 3.4. □

We continue with the proof of Theorem 3.1. Fix integers $\ell \geqslant t^8$ and m, and define $h = m + \ell$. Let A be a 1-independent set in G of size at least h^{2t}.

Suppose that the first case of Theorem 3.1 does not hold. In particular $K_t \not\preccurlyeq_2 G$, so by Theorem 3.4, τ has alternation rank at most $2t - 1$. From Theorem 3.3 we conclude that τ has depth at least h. As $h = m + \ell$, it follows that either τ has a node w which contains at least m letters D, or τ has a node w which contains at least ℓ letters S.

Consider the first case, i.e., there is a node w of τ which contains at least m letters D, and let X be the set of all vertices $\tau(u)$ such that uD is a prefix of w. Then, by construction, X is a 2-independent set in G of size at least m, so the second case of the lemma holds.

Finally, consider the second case, i.e., there is a node w in τ which contains at least ℓ letters S. Let Y be the set of all vertices $\tau(u)$ such that uS is a prefix of w. Then, by construction, $Y \subseteq A$ is a set of at least ℓ vertices which are mutually at distance exactly 2 in G. Since $K_t \not\preccurlyeq_4 G$ and $\ell \geqslant t^8$, by Theorem 3.2 we infer that there is a vertex $v \in G$ with at least $\ell^{1/4}$ neighbors in Y. This finishes the proof of the existential part of Theorem 3.1. We defer the proof of the algorithmic statement of the lemma to the appendix.

3.1 Uniform quasi-widness for tuples

We now formulate and prove an extension of Theorem 1.5 which applies to sets of tuples of vertices, rather than sets of vertices. This more general result will be used later on in the paper. The result and its proof are essentially adaptations to the finite of their infinite analogues introduced by Podewski and Ziegler (cf. [29], Corollary 3), modulo the numerical bounds.

We generalize the notion of independence to sets of tuples of vertices. Fix a graph G and a number $r \in \mathbb{N}$, and let $S \subseteq V(G)$ be a subset of vertices of G. We say that vertices u and v are *r-separated* by S in G if every path of length at most r connecting u and v in G passes through a vertex of S. We extend this notion to tuples: two tuples $\bar{u}, \bar{v}$ of vertices of G are *r-separated* by S every vertex appearing in $\bar{u}$ is r-separated by S from every vertex appearing in $\bar{v}$. Finally, if $A \subseteq V(G)^d$ is a set of d-tuples of vertices, for some $d \in \mathbb{N}$, then we say that A is *mutually r-separated* by S in G if any two distinct $\bar{u}, \bar{v} \in A$ are r-separated by S in G.

With these definitions set, we may introduce the notion of uniform quasi-wideness for tuples.

Definition 3.5. Fix a class C and numbers $r, d \in \mathbb{N}$. For a function $N \colon \mathbb{N} \to \mathbb{N}$ and number $s \in \mathbb{N}$, we say that C satisfies property $\mathrm{UQW}_r^d(N, s)$ if the following condition holds:

> *for every $m \in \mathbb{N}$ and every subset $A \subseteq V(G)^d$ with $|A| \geqslant N(m)$, there is a set $S \subseteq V(G)$ with $|S| \leqslant s$ and a subset $B \subseteq A$ with $|B| \geqslant m$ which is mutually r-separated by S in G.*

We say that C satisfies property UQW_r^d if C satisfies property $\mathrm{UQW}_r^d(N, s)$ for some $N \colon \mathbb{N} \to \mathbb{N}$ and $s \in \mathbb{N}$. If moreover one can take N to be a polynomial, then we say that C satisfies property PUQW_r^d.

When $d = 1$, we omit it from the superscripts. Note that there is a slight discrepancy in the definition of uniform quasi-wideness and the property of satisfying UQW_r, for all $r \in \mathbb{N}$. This is due to the fact that in the original definition, the set B must be disjoint from S, whereas in the property UQW_r, some vertices of S may belong to B. This distinction is inessential when it comes to dimension 1, since $|S| \leqslant s_r$ for some constant s_r, so passing from one definition to the other requires modifying the function N_r by an additive constant s_r. In particular, a class of graphs C is uniformly quasi-wide if and only if it satisfies UQW_r, for all $r \in \mathbb{N}$. However, generalizing to tuples of dimension d requires the use of the definition above, where the tuples in B are allowed to contain vertices which occur in S.

The following result provides a generalization of Theorem 1.5 to higher dimensions.

Theorem 3.6 (⋆). *If C is a nowhere dense class of graphs, then for all $r, d \in \mathbb{N}$, the class C satisfies PUQW_r^d. More precisely, for any class of graphs C and numbers $r, t \in \mathbb{N}$, if $K_t \not\preccurlyeq_{18r} G$ for all $G \in C$, then for all $d \in \mathbb{N}$ the class C satisfies $\mathrm{UQW}_r^d(N_r^d, s_r^d,)$ for some number $s_r^d \in \mathbb{N}$ and polynomial $N_r^d \colon \mathbb{N} \to \mathbb{N}$ that can be computed given r, t, and d.*

4 Bounds on the number of types

In this section we prove Theorem 1.3 and Theorem 1.4. Let us quickly recall the required notions from logic.

4.1 Logical notions

Formulas. All formulas in this paper are first order formulas on graph, i.e., they are built using variables (denoted x, y, z, etc.), atomic predicates $x = y$ or $E(x, y)$, where the latter denotes the existence of an edge between two nodes, quantifiers $\forall x$, $\exists x$, and boolean connectives $\vee, \wedge, \neg$. Let $\varphi(\bar{x})$ be a formula with free variables $\bar{x}$. (Formally, the free variables form a set. To ease notation, we identify this set with a tuple by fixing any its enumeration.) If $\bar{w} \in V^{|\bar{x}|}$ is a tuple of vertices of some graph $G = (V, E)$ (treated as a valuation of the free variables $\bar{x}$), then we write $G, \bar{w} \models \varphi(\bar{x})$ to denote that the valuation $\bar{w}$ satisfies the formula φ in the graph G.

We will consider also *colored graphs*, where we have a fixed set of colors Λ and every vertex is assigned a subset of colors from Λ. If $C \in \Lambda$ is a color then the atomic formula $C(x)$ holds in a vertex x if and only if x has color C.

Finally, we will consider *formulas with parameters* from a set A, which is a subset of vertices of some graph. Formally, such formula with parameters is a pair consisting of a (standard) formula $\varphi(\bar{x}, \bar{y})$ with a partitioning of its free variables into $\bar{x}$ and $\bar{y}$, and a valuation $\bar{v} \in A^{|\bar{y}|}$ of the free variables $\bar{y}$ in A. We denote the resulting formula with parameters by $\varphi(\bar{x}, \bar{v})$, and say that its free variables are $\bar{x}$. For a valuation $\bar{u} \in A^{|\bar{x}|}$, we write $G, \bar{u} \models \varphi(\bar{x}, \bar{v})$ iff $G, \bar{u}\bar{v} \models \varphi(\bar{x}, \bar{y})$. Here and later on, we write $\bar{u}\bar{v}$ for the concatenation of tuples $\bar{u}$ and $\bar{v}$.

Types. Fix a formula $\varphi(\bar{x}, \bar{y})$ together with a distinguished partitioning of its free variables into *object variables* $\bar{x}$ and *parameter variables* $\bar{y}$. Let $G = (V, E)$ be a graph, and let $A \subseteq V$. If $\bar{u} \in V^{|\bar{y}|}$ is a tuple of nodes of length $|\bar{y}|$, then the *φ-type of $\bar{u}$ over A*, denoted $\mathrm{tp}^{\varphi}_{G}(\bar{u}/A)$, is the set of all formulas $\varphi(\bar{x}, \bar{v})$, with parameters $\bar{v} \in A^{|\bar{y}|}$ replacing the parameter variables $\bar{z}$, such that $G, \bar{u} \models \varphi(\bar{x}, \bar{v})$.

For a fixed formula $\varphi(\bar{y}, \bar{z})$, graph $G = (V, E)$ and sets $A, W \subseteq V$, define $S^{\varphi}(W/A)$ as the set of all φ-types of tuples from W over A in G; that is,

$$S^{\varphi}(W/A) = \left\{ \mathrm{tp}^{\varphi}_{G}(\bar{u}/A) \; : \; \bar{u} \in W^{|\bar{y}|} \right\}.$$

Although not visible in the notation, the set $S^{\varphi}(W/A)$ depends on the chosen partitioning $\bar{x}, \bar{y}$ of the free variables of φ. In case $W = V(G)$ we write $S^{\varphi}_{d}(G/A)$ instead of $S^{\varphi}_{d}(W/A)$. Note that this definitions differs syntactically from the one given in section 1, as here $S^{\varphi}(G/A)$ consists of φ-types, and not of subsets of tuples. However, it is easy to see that there is a one-to-one correspondence between the two notions. The following lemma is immediate.

Lemma 4.1. *Let G be a graph and let $A \subseteq B \subseteq V(G)$. Then for each formula $\varphi(\bar{x}, \bar{y})$, it holds that $|S^{\varphi}(G/A)| \leqslant |S^{\varphi}(G/B)|$.*

4.2 Locality

We will use the following intuitive notion of functional determination. Suppose X, A, B are sets and we have two functions: $f : X \to A$ and $g : X \to B$. We say that $f(x)$ *determines* $g(x)$ for $x \in X$ if for every pair of elements $x, x' \in X$ the following implication holds: $f(x) = f(x')$ implies $g(x) = g(x')$. Equivalently, there is a function $h : A \to B$ such that $g = h \circ f$.

Recall that if A, B, S are subsets of vertices of a graph G and $r \in \mathbb{N}$, then A and B are r-separated by S in G if every path from A to B of length at most r contains a vertex from S.

The following lemma is the main result of this subsection.

Lemma 4.2 ($\star$). *For any given numbers q and d one can compute numbers p and r with the following properties. Let $G = (V, E)$ be a fixed graph and let $A, B, S \subseteq V$ be fixed subsets of its vertices such that A and B are r-separated by S in G. Then, for tuples $\bar{u} \in A^{d}$, the type $\mathrm{tp}^{q}(\bar{u}/B)$ is determined by the type $\mathrm{tp}^{p}(\bar{u}/S)$.*

Corollary 4.3 ($\star$). *For every formula $\varphi(\bar{x}, \bar{y})$ and number $s \in \mathbb{N}$ there exist numbers $T, r \in \mathbb{N}$, where r is computable from φ and T is computable from φ and s, such that the following holds. For every graph G and vertex subsets $A, B, S \subseteq V(G)$ where S has at most s vertices and r-separates A from B, we have $|S^{\varphi}(A/B)| \leqslant T$.*

4.3 Bounds on the number of types

We now come to the proof of Theorem 1.3. In the proof, we will will first enlarge the set A to a set B, called an *r-closure of A* (where r is chosen depending on φ), such that the connections of elements from $V(G) - B$ toward B are well controlled. This approach was first used in Drange et al. [9] in the context of classes of bounded expansion, and then for nowhere dense classes in Eickmeyer et al. [11]. We start by recalling these notions.

Let G be a graph and let $B \subseteq V(G)$ be a subset of vertices. For vertices $v \in B$ and $u \in V(G)$, a path P leading from u to v is called *B-avoiding* if all its vertices apart from v do not belong to B. Note that if $u \in B$, then there is only one B-avoiding path leading from u, namely the one-vertex path where $u = v$. For a positive integer r and $u \in V(G)$, the *r-projection* of u on B, denoted $M^{G}_{r}(u, B)$, is the set of all vertices $v \in B$ such that there is a B-avoiding path of length at most r leading from u to v. Note that for $u \in B$, we have $M^{G}_{r}(u, B) = \{b\}$. Equivalently, $M^{G}_{r}(u, B)$ is the unique inclusion-minimal subset of B which r-separates u from B. We will use the following result from [11].

Lemma 4.4 ([11]). *Let $\mathcal{C}$ be a nowhere dense class. Then for every $r \in \mathbb{N}$ and $\delta > 0$ there is a constant $c \in \mathbb{N}$ such that for every $G \in \mathcal{C}$ and $A \subseteq V(G)$ there exists a set B, called an* r-closure *of A, with the following properties:*

1. *$A \subseteq B \subseteq V(G)$;*
2. *$|B| \leqslant c \cdot |A|^{1+\delta}$; and*
3. *$|M^{G}_{r}(u, B)| \leqslant c \cdot |A|^{\delta}$ for each $u \in V(G)$.*

Moreover, for every set $X \subseteq V(G)$, it holds that

1. $\left| \left\{ M^{G}_{r}(u, X) \; : \; u \in V(G) \right\} \right| \leqslant c \cdot |X|^{1+\delta}$.

We note that in [9, 11] projections on B are defined only for vertices outside of B. However, adding singleton projections for vertices of B to the definition only adds $|B|$ possible projections of size 1 each, so this does not influence the validity of the above results.

We proceed with the proof of Theorem 1.3. Let us fix: a nowhere dense class of graphs $\mathcal{C}$, a graph $G \in \mathcal{C}$, a vertex subset $A \subseteq V(G)$, a real $\varepsilon > 0$, and a first order formula $\varphi(\bar{x}, \bar{y})$, where $\bar{x}$ is the distinguished ℓ-tuple of object variables. Our goal is to show that $|S^{\varphi}(G/A)| = O(|A|^{\ell+\varepsilon})$.

In the sequel, d denotes a positive integer depending on $\mathcal{C}, \ell, \varphi$ only (and not on G, A and ε), and will be specified later. We may choose positive reals δ, ε_1 such that $(\ell + \varepsilon_1)(1+\delta) \leqslant \ell + \varepsilon$ and $\varepsilon_1 > \delta(d+\ell) > \delta\ell$, for instance as follows: $\varepsilon_1 = \varepsilon/2$ and $\delta = \frac{\varepsilon}{4d+4\ell}$. The constants hidden in the $O(\cdot)$ notation below depend on $\varepsilon, \delta, \varepsilon_1, \mathcal{C}, \ell$ and φ, but not on G and A. By *tuples* below we refer to tuples of length ℓ.

Let q be the quantifier rank of φ and let p, r be the numbers obtained by applying Theorem 4.2 to q and ℓ. Let B be an r-closure of A, given by Theorem 4.4. By Theorem 4.4, the total number of distinct r-projections onto B is at most $O(|B|^{1+\delta})$, and each of these projections has size $O(|B|^{\delta})$. The first step is to reduce the statement to the following claim.

Claim 1. If X is a set of tuples with pairwise different φ-types over B, then $|X| = O(|B|^{\ell+\varepsilon_1})$.

Claim 1 implies that $|S^{\varphi}(G/B)| = O(|B|^{\ell+\varepsilon_1})$, which is bounded by $O(|A|^{(\ell+\varepsilon_1)(1+\delta)})$ since $|B| = O(|A|^{1+\delta})$.

As $(\ell + \varepsilon_1)(1 + \delta) \leqslant \ell + \varepsilon$, this shows that $|S^{\varphi}(G/B)| = O(|A|^{\ell+\varepsilon})$. Then Theorem 4.1 implies that also $|S^{\varphi}(G/A)| = O(|A|^{\ell+\varepsilon})$, and we are done. Therefore, it remains to prove Claim 1.

For a tuple $\bar{w} = w_1 \ldots w_\ell \in V(G)^\ell$, define its *projection* to be the set $C_1 \cup \ldots \cup C_\ell \subseteq B$ where $C_i = M_r^G(w_i, B)$. Note that there are at most $O(|B|^{\ell(1+\delta)})$ different projections of tuples in total, and each projection has size $O(|B|^{\delta})$. To prove Claim 1, we consider the special case when all the tuples have the same projection, say $C \subseteq B$, and obtain a stronger conclusion, for $\varepsilon_2 := \varepsilon_1 - \delta\ell > 0$.

Claim 2. If Y is a set of tuples with pairwise different φ-types over B, and each $u \in V$ has the same projection $C \subseteq B$, then $|Y| = O(|B|^{\varepsilon_2})$.

Since there are at most $O(|B|^{\ell(1+\delta)})$ different projections in total and $\ell(1 + \delta) + \varepsilon_2 = \ell + \varepsilon_1$, Claim 1 can be proved by summing the bound given by Claim 2 through all different projections C. It therefore remains to prove Claim 2.

We apply Theorem 3.6 to the set of ℓ-tuples Y, for m being the largest integer such that $|Y| \geqslant N_{2r}^{\ell}(m)$. As a conclusion, we obtain a set $Z \subseteq Y$ of m tuples that is mutually $2r$-separated by S in G, for some set of vertices $S \subseteq V(G)$ of size $s := s_{2r}^{\ell}$. Let d be the degree of the polynomial $N_{2r}^{\ell}(\cdot)$ obtained from Theorem 3.6. Note that $s = O(1)$ and $|Y| = O(m^d)$.

Claim 3. It holds that $|Z| = O(|C|)$.

We first show how Claim 3 implies Claim 2. Since $m = |Z| = O(|C|)$, and $|C| = O(|B|^{\delta})$, it follows that $|Y| = O(m^d) = O(|B|^{d\delta})$. As $\delta(d+\ell) > \varepsilon_1$, this implies that $d\delta < \varepsilon_2$, yielding Claim 2. We now prove Claim 3.

Let $Z_0 \subseteq Z$ be the set of those tuples in Z which are r-separated by S from B in G, and let $Z_1 = Z - Z_0$ be the remaining tuples. Since tuples from Z_0 have pairwise different φ-types over B, and each of them is r-separated by S from B in G, by Theorem 4.3 we infer that $|Z_0| = O(1)$. On the other hand, by the definition of Z_1, with each tuple $\bar{u} \in Z_1$ we may associate a vertex $b(\bar{u}) \in C$ which is not r-separated from $\bar{u}$ by S in G. Since the set U is mutually $2r$-separated by S in G, it follows that for any two different tuples $\bar{u}, \bar{v} \in Z_1$ we have $b(\bar{u}) \neq b(\bar{v})$. Hence $b(\cdot)$ is an injection from Z_1 to C, which proves that $|Z_1| \leqslant |C|$. To conclude, we have $|Z| = |Z_0| + |Z_1| = O(1) + O(|C|) = O(|C|)$. This finishes the proof of Claim 3 and ends the proof of Theorem 1.3.

5 Bounds for stability

Adler and Adler [1], proved that every nowhere dense class of graphs is stable. In this section, we prove its effective variant, Theorem 1.7.

Proof of Theorem 1.7. Fix a formula $\varphi(\bar{x}, \bar{y})$ of quantifier rank q and a partitioning of its free variables into $\bar{y}$ and $\bar{z}$. Let $d = |\bar{x}| + |\bar{y}|$ be the total number of free variables of φ. Let $r \in \mathbb{N}$ be the number given by Theorem 4.3, which depends on φ only. Let $\mathcal{C}$ be the class of all graphs such that $K_t \not\preccurlyeq_{18r} G$. Then, by Theorem 3.6, $\mathcal{C}$ satisfies $\mathrm{UQW}_r^d(N_r^d, s_r^d)$, for some polynomial $N_r^d \colon \mathbb{N} \to \mathbb{N}$ and number $s = s_r^d \in \mathbb{N}$ computable from d, t, r. Let T be the number given by Theorem 4.3 for φ and s. Finally, let $\ell = N_r^d(2T + 1)$. We show that every φ-ladder in a graph $G \in \mathcal{C}$ has length smaller than ℓ.

For the sake of contradiction, assume that there is a graph $G \in \mathcal{C}$ and tuples $\bar{u}_1, \ldots, \bar{u}_\ell \in V(G)^{|\bar{x}|}$ and $\bar{v}_1, \ldots, \bar{v}_\ell \in V(G)^{|\bar{y}|}$ which form a φ-ladder in G, i.e., $\varphi(\bar{u}_i, \bar{v}_j)$ holds in G if and only if $i \leqslant j$. Let $A = \{\bar{u}_i\bar{v}_i \ : \ i = 1, \ldots, \ell\} \subseteq V(G)^d$. Note that $|A| = \ell \geqslant N_r^d(2T + 1)$, since tuples $\bar{u}_i$ have to be pairwise different.

Applying property $\mathrm{UQW}_r^d(N_r^d, s_r^d)$ to the set A, radius r, and target size $m = 2T + 1$ yields a set $S \subseteq V(G)$ with $|S| \leqslant s$ and a set $B \subseteq A$ with $|B| \geqslant 2T + 1$ of tuples which are mutually r-separated by S in G. Let $J \subseteq \{1, \ldots, \ell\}$ be the set of indices corresponding to B, i.e., $J = \{j \colon \bar{u}_j\bar{v}_j \in B\}$.

Since $|J| = 2T + 1$, we may partition J into $J_1 \uplus J_2$ with $|J_1| = T + 1$ so that the following condition holds: for each $i, k \in J_1$ satisfying $i < k$, there exists $j \in J_2$ with $i < j < k$. Indeed, it suffices to order the indices of J and put every second index to J_1, and every other to J_2. Let X be the set of vertices appearing in the tuples $\bar{u}_i$ with $i \in J_1$, and let Y be the set of vertices appearing in the tuples $\bar{v}_j$ with $j \in J_2$. Since the tuples of B are mutually r-separated by S in G, it follows that X and Y are r-separated by S. As $|J_1| = T + 1$, by Theorem 4.3 we infer that there are distinct indices $i, k \in J_1$, say $i < k$, such that $\mathrm{tp}^{\varphi}(\bar{u}_i/Y) = \mathrm{tp}^{\varphi}(\bar{u}_k/Y)$. This implies that for each $j \in J_2$, we have $G, \bar{u}_i, \bar{v}_j \models \varphi(\bar{x}, \bar{y})$ if and only if $G, \bar{u}_k, \bar{v}_j \models \varphi(\bar{x}, \bar{y})$. However, there is an index $j \in J_2$ such that $i < j < k$, and for this index we should have $G, \bar{u}_i, \bar{v}_j \models \varphi(\bar{x}, \bar{y})$ and $G, \bar{u}_k, \bar{v}_j \not\models \varphi(\bar{x}, \bar{y})$ by the definition of a ladder. This contradiction concludes the proof. □

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