Derivation of momentum conservation law

based on the lecture of prof. Andrzej Styczek

Wojciech Sadowski

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1 Basics

What is a fluid? We will consider a fluid as a continuous substance. That is, the fluid fills fully available space. Therefore, *we are neglecting the discrete nature of matter.* We can do that, since the amount of fluid molecules in small volume of matter is gargantuan (remember the Avogadro number $N_a \approx 6.023 \times 10^{26}\,\mathrm{kmol}^{-1}$).

It is painfully difficult to describe the motion of each and every molecule constituting the fluid. Apart from the obvious problem of integrating all of the trajectories we also encounter a problem of determining an accurate initial conditions for every molecule. In such systems, a small error in determining these starting parameters will unfortunately lead to considerable inaccuracies further down the line.

Hence, we neglect the discrete nature of matter. The medium, i.e. fluid, consists of continuous set of ideal particles[1]. So the movement of the medium is *de facto* the movement of the points, and such description can be used for solids (a continuous medium retaining some shape or natural configuration), fluids and "things" in between. Fluids do not have this natural, reference configuration: the relative positions of the points can be arbitrary and changing. Sometimes we might assume that volume of fluid has to be conserved (or that the density is constant) but that is, of coarse, a simplification. A useful and accurate one in situations of non-varying thermodynamic state of the medium.

If this "reference shape" does not exist, than the deformation (or strain) of the fluid can be completely arbitrary. *Water spilled on the floor will deform greatly, covering the area and seeping into the cracks, potentially making you call the handyman to fix the broken flooring.* So what creates the internal forces inside the fluid? One obvious answer is the change of volume, but what else?

We will consider fluids in which the forces depends on the rate of deformation. These forces clearly will "want" to stop, or counteract, the deformation occurring in the fluid. As we have already discussed, the deformation itself does not seem to be important.

However, it is important to indicate, that this is only a simplified model, not existing in nature. In the same way as each fluid is really compressible, this

[1] A point that might have mass or charge or some other property. From Wikipedia: "an idealization of particles [...]. Its defining feature is that it lacks spatial extension; being dimensionless, it does not take up space."

assumption is a useful lie. For the most general fluid we could write

$$f(t, \boldsymbol{\tau}, \dot{\boldsymbol{\tau}}, \boldsymbol{D}, \dot{\boldsymbol{D}}, \dots) = 0,$$

and call it *constitutive equation*, i.e. a relation defining our material. It describes connection between the time t, stresses $\boldsymbol{\tau}$, deformation $\boldsymbol{D}$, the rates of those things (indicated by the same letters with a dot accent) and potentially other parameters.

As you can see, it can be bad. The parameters in the above relations could depend on each other or they could depend on each other non-locally (i.e. point a depends on the state in point b). Or even worse, they might be influenced by the history of the those parameters. Fortunately, for most of classical use cases we can confine ourselves to much simpler relations.

We will say:

(i) only $\boldsymbol{\tau}$ and $\dot{\boldsymbol{D}}$ are important;

(ii) the relation between them is defined in the same place and time;

(iii) we should be able two solve for $\boldsymbol{\tau}$, that is write $\boldsymbol{\tau} = g(\dot{\boldsymbol{D}})$;

(iv) the relationship should not have any preferential direction, that is, for example, the fluid should not behave differently along the x or y axis (in practice, this will require that the $\boldsymbol{\tau}$ and $\dot{\boldsymbol{D}}$ are *symmetric*).

What are stresses? We know what a *force* is. Intuitively, when thinking about fluids or materials, the stress is a force acting on a surface. We also know that we can add forces, so we can safely assume that the sum of forces acting on parts of the surface have to add up to the force acting on the whole surface. Let's define force differential $\mathrm{d}\mathbf{F}$ acting on a small part of the surface $\mathrm{d}S$, in a following way

$$\mathrm{d}\mathbf{F} = \mathbf{f}\,\mathrm{d}S\,,$$

so that $\mathbf{f}$ becomes a force per unit surface, so surface force density.

Let us consider a Cartesian coordinate system (O, x_1, x_2, x_3) and a small tetrahedron with walls $\mathrm{d}S_1$, $\mathrm{d}S_2$, $\mathrm{d}S_3$ and closed by the wall $\mathrm{d}S$ (see Figure 1). There can be infinite amount of such tetrahedrons, however the numbered walls $\mathrm{d}S_i$ are always laying on the same planes, perpendicular to the axis of the system. On the other hand, wall $\mathrm{d}S$ changes its position along with the plane its laying on and the surface area.

If this volume is in equilibrium, than the forces acting on the walls, $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$ and $\mathbf{f}$ respectively, add up to $\mathbf{0}$

$$\mathbf{f}_1\,\mathrm{d}S_1 + \mathbf{f}_2\,\mathrm{d}S_2 + \mathbf{f}_3\,\mathrm{d}S_3 + \mathbf{f}\,\mathrm{d}S = \mathbf{f}_i\,\mathrm{d}S_i + \mathbf{f}\,\mathrm{d}S = \mathbf{0}. \tag{1}$$

Let us divide the equation by $\mathrm{d}S$

$$\mathbf{f}_i \frac{\mathrm{d}S_i}{\mathrm{d}S} + \mathbf{f} = \mathbf{0}.$$

We do that because we notice that the fraction $\mathrm{d}S\,/\,\mathrm{d}S_i$ is equal to the cosine of the angle between the normal of $\mathrm{d}S$ and the i-th axis of the coordinate system (believe me, or prove it if you don't). We can also say that $\cos(\boldsymbol{n}, x_i) = n_i$ and we will use that later.

In Equation 1 we didn't assume anything about the directions of the forces, however, if the tetrahedron is in equilibrium, than the sum of $\mathbf{f}_i$ has to be opposite to $\mathbf{f}$. Hence, we can flip it noting that if we change direction, the sign changes as well, i.e. $\mathbf{f}(\mathbf{n}) = -\mathbf{f}(-\mathbf{n})$. After substituting our "revolutionary" findings about the $\mathrm{d}S\,/\,\mathrm{d}S_i$ ratio equal to the cosine as well, we are left with the following relation

$$\mathbf{f} = \mathbf{f}_i \frac{\mathrm{d}S_i}{\mathrm{d}S} = \mathbf{f}_i \cos(\boldsymbol{n}, x_i) = \mathbf{f}_i n_i$$

Now, considering that each force has three components, which for example for $\mathbf{f}_1$ we will denote as f_{11}, f_{12} and f_{13}, we can also write the equation for k-th component of $\mathbf{f}$

$$f_k = n_1 f_{1k} + n_2 f_{2k} + n_3 f_{3k} = n_\alpha f_{\alpha k}.$$

So, we actually have arrived at something! We have *nine* numbers giving together the components of $f_{\alpha k}$, which describe the forces acting on the surface closing our small tetrahedron. Furthermore, if you shrink the tetrahedron to a point (we already used differentials of surfaces), what we wrote above still holds! This means that, the force density acting on a surface in the fluid depends on those nine numbers and the orientation of the surface.

We will call those numbers stresses. Make a though experiment: imagine a force acting in a chosen direction, and a surface perpendicular to this force. This is aligned with the classical definition of pressure, right? Now, rotate this surface, so that the force vector lies on the surface. Force acting along the surface can be understood as shear. That's how scissors work! Now, rotate the surface keeping the normal as an axis. The force is now "shearing" the surface in a different direction, so for one force vector we can have one normal (perpendicular to the surface) stress and we observed two shear stresses. With that (hopefully) intuitive explanation we can say that among the nine numbers, we will have *three* normal stress components and *six* components related to shear stress.

The only problem is the fact that our notation is cumbersome; we have two f symbols. Let's set the magic nine numbers (they will get a name soon, I

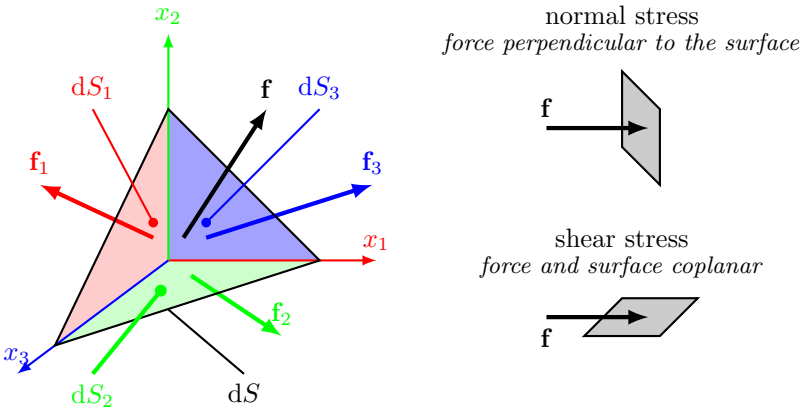

Figure 1: Elementary tetrahedron

promise!) as τ_{ij}, giving us:

$$f_i = n_\alpha \tau_{\alpha i}$$

Stress tensor Okay, so as I said, the nine-number-thingy needs a name. You saw the paragraph title, you might already now what it is. That said, this is written in style of a lecture, so please make a dramatic pause in your brain before reading the end of the next sentence. We will call τ_{ij} the *stress tensor*.

But what is a tensor, you might ask? My colleague said at some point[2] that a tensor is just a "glorified vector". This is not really true, but it is a useful starting point for a discussion. In fact, some vectors *are* tensors. Generally, a tensor is a "geometrical thing", that is, it describes some effect independent on the coordinate system. Velocity vector is a tensor or order 1, as it describes that an object moves in a specific direction with a specific velocity[3]. If we change the basis (coordinate system), the representation, that is the components of the vector will change, but the vector itself still points in the same direction. *It is a geometric concept.* Similarly to our velocity example, stress tensor also describe a distribution of forces inside the material. In nature there are no principal, favoured directions, so the object describing it should be a tensor.

So, there is some geometric transformation that tells us how to change the components of a tensors, and tensors are objects that obey this transformation. This paraphrase of a mathematical mathematical definition is frankly quite unhelpful[4]. So we will cheat a little and describe a tensor with as much verbosity as is necessary for our use case.

Let's define some operations first and show them on the example of the stress tensor, which we will represent here as

$$\boldsymbol{\tau} = \tau_{ij} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j.$$

We used the versors $\hat{\mathbf{e}}_i$ defining our chosen coordinate system. They are multiplied in a weird way, its neither scalar nor vector product. This operation is also sometimes written with the $\otimes$ symbol ($\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$) and is called tensor or outer product.

The scalar product $\mathbf{n} \cdot \boldsymbol{\tau}$ can be written easily with index notation. We know that $\mathbf{n} = n_k \hat{\mathbf{e}}_k$, so building on that we can write

$$\mathbf{n} \cdot \boldsymbol{\tau} = n_k \hat{\mathbf{e}}_k \cdot (\tau_{ij} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j) = n_k \tau_{ij} \hat{\mathbf{e}}_k \cdot \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j = n_k \tau_{ij} (\hat{\mathbf{e}}_k \cdot \hat{\mathbf{e}}_i) \hat{\mathbf{e}}_j.$$

[2] In Dresden, on the 1st of June, during breakfast, around 9:30 AM.

[3] Not every vector is a tensor, e.g. the position vector is always dependant on the chosen coordinate system.

[4] For interested here is slightly paraphrased version of a definition, again stolen from Wikipedia: A tensor of type (p, q) is an assignment of a multidimensional array

$$T^{i_1 \dots i_p}_{j_1 \dots j_q}[\mathbf{f}]$$

to each basis $\mathbf{f} = (\mathbf{e}_1, \dots, \mathbf{e}_n)$ of an n-dimensional vector space such that, if we apply the change of basis

$$\mathbf{f} \mapsto \mathbf{f} \cdot R = \left(\mathbf{e}_i R^i_1, \dots, \mathbf{e}_i R^i_n\right)$$

then the multidimensional array obeys the transformation law

$$T^{i'_1 \dots i'_p}_{j'_1 \dots j'_q}[\mathbf{f} \cdot R] = \left(R^{-1}\right)^{i'_1}_{i_1} \cdots \left(R^{-1}\right)^{i'_p}_{i_p} T^{i_1, \dots, i_p}_{j_1, \dots, j_q}[\mathbf{f}] R^{j_1}_{j'_1} \cdots R^{j_q}_{j'_q}.$$

We can think a little about the term $\hat{\mathbf{e}}_k \cdot \hat{\mathbf{e}}_i$. It is a dot product of two versors. It can be either equal to 1 if they are the same ($k = i$) or 0 if they are different. There is a term for that, called *Kronecker delta*, defined as

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

So we can write that our product is equal to $n_k \tau_{ij} \delta_{ki} \hat{\mathbf{e}}_j$. It does not help us yet, but if we said that k has to be equal to i, otherwise the terms in the sum will be 0, we might as well write i instead of k[5], i.e.

$$\mathbf{n} \cdot \boldsymbol{\tau} = n_i \tau_{ij} \hat{\mathbf{e}}_j = f_j \hat{\mathbf{e}}_j.$$

In the above, we eliminated δ_{ii} as it is always equal to 1. In fact, δ_{ij} is sometimes called an "index renamer" as it renames index j to i or *vice versa*.

Okay, so what have we learned? The dot product of a tensor with a vector gives us a vector. Will multiplication from the left, which we performed above, be different from the one from the right, that is $\boldsymbol{\tau} \cdot \mathbf{n}$? Yes[6]. That said, it would be the same if $\tau_{ij} = \tau_{ji}$.

So a lot of hints have been dropped. The tensor behaves a little bit like a matrix. In fact the matrix is often used to *represent* a tensor in a chosen coordinate system, but the tensor itself, like a velocity vector, is independent from the chosen coordinate system[7]. This matrix can be multiplied with a vector to give another vector. If the matrix is symmetric, product from both sides will return vectors with the same components. Kronecker delta is also a tensor itself, represented by a unit matrix $\boldsymbol{I}$.

Strain-rate (deformation-rate) tensor It was mentioned, that we are interested in a tensor defining a rate of deformation, or what is usually called *strain-rate tensor* in literature. We should derive it. Let's start with a series expansion of velocity field around coordinate $\mathbf{r}$:

$$v_i(t, \mathbf{r} + \mathrm{d}\mathbf{x}) = v_i(t, \mathbf{r}) + \frac{\partial v_i}{\partial x_k} \, \mathrm{d}x_k + \dots$$

If $\mathrm{d}x \to \mathbf{0}$, we disregard the higher order terms and rewrite the above as

$$v_i(t, \mathbf{r} + \mathrm{d}\mathbf{x}) = v_i(t, \mathbf{r}) + \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) \mathrm{d}x_k + \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} - \frac{\partial v_k}{\partial x_i}\right) \mathrm{d}x_k \, .$$

The first term on the right describes simple translation of the fluid parcel. The third (similar to the expression for the curl of velocity field) defines rotation. Both of these can be applied to the rigid, unchanging in shape, fluid parcel. We are left with the second one, describing the change of velocity due to "non-rigid" strain or deformation of the fluid volume. We can write this change of velocity mathematically as

$$\mathrm{d}v_k^{\text{strain}} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) \mathrm{d}x_k = \dot{D}_{ik} \, \mathrm{d}x_k \tag{2}$$

The tensor $\dot{\mathbf{D}}$ is the strain-rate tensor and very importantly it is *symmetric*!

[5] If you are confused try it out on paper, write every term, eliminate the ones equal to 0 and collapse the term to the indexed expression.

[6] I am to lazy to write it down, but it might be a good training for you.

[7] Stress tensor has rank 2; we used two versors, so called diad, to define it.

2 Road towards momentum equation

Force acting on a surface Armed with the stress tensor we can now compute the force acting on any closed surface:

$$\mathbf{F} = \oint_S \mathbf{f}\,\mathrm{d}S = \oint_S \mathbf{n}\boldsymbol{\tau} = \hat{\mathbf{e}}_\mathbf{i} \oint_S n_j \tau_{ji}\,\mathrm{d}S$$

Now we can use the Greens theorem[8] to convert the surface integral to a volume one. Or simply, when mathematicians aren't looking, it is a theorem about changing S for Ω and n_j for $\partial/\partial x_j$. Naturally, S is the surface of Ω. Let's do the deed

$$\mathbf{F} = \hat{\mathbf{e}}_\mathbf{i} \oint_S n_j \tau_{ji}\,\mathrm{d}S = \hat{\mathbf{e}}_\mathbf{i} \int_\Omega \frac{\partial \tau_{ji}}{\partial x_j}\,\mathrm{d}\Omega = \int_\Omega \boldsymbol{\nabla}\cdot\boldsymbol{\tau}\,\mathrm{d}\Omega$$

Body force We have written an expression for surface forces. What about the forces acting from the *inside*, like gravity? Here we can say

$$\mathbf{F} = \int_\Omega \mathbf{f}\,\mathrm{d}m$$

Pardon my messy notation, but now $\mathbf{f}$ is the *volume* density of force. We also wrote $\mathrm{d}m$ which is the mass differential. It is simply given as $\rho\,\mathrm{d}\Omega$, where the first in the product is the density of the material and the second is the volume differential. So

$$\mathbf{F} = \int_\Omega \rho\mathbf{f}\,\mathrm{d}\Omega$$

Derivation starts We finally start deriving the momentum equation of a continous medium. We start from basic principles

$$\frac{\mathrm{d}}{\mathrm{d}t}\text{momentum} = \text{force}$$

We first need to handle the momentum part. We divide the fluid into small, moving parts. All the parts give the sum of the whole momentum, so we write it as an integral

$$\text{momentum} = \int_\Omega \mathrm{d}(\text{momentum})\,.$$

We will handle the forces in a second. The differential of linear momentum can be simply expressed as $\mathbf{v}\,\mathrm{d}m = \rho\mathbf{v}\,\mathrm{d}\Omega$, where ρ is the density. We already wrote the expression for the forces so if we assemble everything we get this

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho\mathbf{v}\,\mathrm{d}\Omega = \int_\Omega \boldsymbol{\nabla}\cdot\boldsymbol{\tau} + \rho\mathbf{f}\,\mathrm{d}\Omega \tag{3}$$

This is a simple equation, but it does not give us much insight. On the other hand it describes the movement of *everything*. One important thing to note is that the volume Ω is completely arbitrary. *We can chose it as we see fit.* We will use this property later to obtain differential versions of the equations.

[8] In polish it is called Green-Gauss-Ostrogracki (GGO) theorem.

Reynolds transport theorem We are now standing in front of a difficult problem. We have to differentiate an integral, which usually is not a sophisticated issue. However, in our case, the integration domain Ω is not a rigid body. Instead, it deforms and changes shape as the time progresses and the fluid flows, i.e. $\Omega = \Omega(t)$. In short it depends on *the variable with respect to witch we differentiate.* Fortunately, we have a tool for that: Reynolds Transport Theorem.

Theorem 1 (Reynolds transport theorem, RTT)**.** *Given:*

(i) a material volume Ω, whose shape and position are varying with time, i.e. $\Omega = \Omega(t)$;

(ii) an extensive variable $\Phi = \Phi(t)$ with its intensive counterpart $\phi = \phi(t, \mathbf{x})$ defined inside the material volume.

The total derivative of Φ with respect to time can be formulated as:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega(t)} \phi \,\mathrm{d}\Omega = \int\limits_{\Omega(t)} \frac{D\phi}{Dt} + \phi \boldsymbol{\nabla} \cdot \mathbf{v} \,\mathrm{d}\Omega \tag{4}$$

As can be seen above, the theorem connects the time derivative of an integral similar to ours with the integral of derivatives. The expression $D\phi/Dt$ is a substantial derivative, defined as

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + v_i \frac{\partial \phi}{\partial x_i}$$

To showcase how it works, we will use RTT to derive the continuity equation.

Detour for conservation of mass The mas is conserved. This can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega} \mathrm{d}m = \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega} \rho \,\mathrm{d}\Omega = 0$$

We see the time derivative of an integral. The alarm bells have sounded. The war horns are sounding the music of battle. The ghost of all scientists pushing the boundaries of fluid dynamics, since human first managed to produced moonshine by heating up a piece of copper pipe, are eagerly awaiting our next move. We were preparing for this moment for the last 10 minutes while reading last paragraph about RTT. *We what to do.*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega} \rho \,\mathrm{d}\Omega = \int\limits_{\Omega} \frac{D\rho}{Dt} + \rho \boldsymbol{\nabla} \cdot \mathbf{v} \,\mathrm{d}\Omega = 0$$

The expression under the integral can be further expanded to

$$\int\limits_{\Omega} \frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\rho + \rho \boldsymbol{\nabla} \cdot \mathbf{v} \,\mathrm{d}\Omega$$

and if we note that

$$(\mathbf{v} \cdot \boldsymbol{\nabla})\rho + \rho \boldsymbol{\nabla} \cdot \mathbf{v} = v_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial v_i}{\partial x_i} = \frac{\partial \rho v_i}{\partial x_i} = \boldsymbol{\nabla} \cdot (\rho \mathbf{v})$$

we arrive at (almost) final result

$$\int_{\Omega} \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) \, \mathrm{d}\Omega = 0$$

Now, we will use an observation about the integration region from the previous paragraphs, that is, it can be chosen arbitrarily. We haven't made any assumptions about it, so intuitively it shouldn't matter.

Let's think about for a second. For the sake of shortening the notation lets denote $\partial\rho/\partial t + \boldsymbol{\nabla} \cdot (\rho \mathbf{v})$ as X. For a completely arbitrary region Ω, the integral $\int_\Omega X \, \mathrm{d}\Omega$ is always equal to zero. This we know from the derived mass conservation equation. Of coarse, it will be true when $X = 0$ everywhere. Still, can it be the case that that $X \neq 0$ but the integral is equal to 0?

For that to be true, the region Ω should be composed of Ω^+, a part in which $X > 0$, and Ω^- where $X < 0$. Together, positive contribution of Ω^+ and negative from Ω^- give null value of the integral.

But wait! We said, that the Ω can be chosen arbitrarily. So what if we chose $\Omega = \Omega^+$? We know that the integral has to be zero, but that is impossible! We established that in this region X is strictly positive and, therefore, the integral has to be positive as well. We have reach a result confiction with our assumptions: *this option is impossible.* We can safely say, that the only way for the integral to be null for each region, is for the integrated expression to equal to 0 in each point in the region. So we can simply write

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0$$

which competes the derivation of mass conservation law.

Back to momentum We use RTT agian, this time on Equation 3. We get:

$$\int_{\Omega} \frac{\partial \rho \mathbf{v}}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}\mathbf{v}) \, \mathrm{d}\Omega = \int_{\Omega} \boldsymbol{\nabla} \cdot \boldsymbol{\tau} + \rho \mathbf{f} \, \mathrm{d}\Omega$$

The product of velocity $\mathbf{v}\mathbf{v}$ is a tensor. We can also write it as $v_i v_j$. We note that we can actually accumulate the terms into one integral equal to zero. We saw something like that before, right? We *again* know what to do! After dropping the integration we are left with:

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}\mathbf{v}) = \boldsymbol{\nabla} \cdot \boldsymbol{\tau} + \rho \mathbf{f} \tag{5}$$

which is a general, differential form of the momentum conservation law. However, in order to move forward we have to actually handle the issue mentioned in the very beginning: defining a description for the stress tensor of our fluid, or the *constitutive relation.*

3 Fluid constitutive relations

We should now try to derive the formulas for $\boldsymbol{\tau}$ so we can plug them into the momentum equation, be done with this derivation and go home. Let's start

with a simple model of an *inviscid* fluid. If the fluid has no viscosity, than pressure should be the only thing contributing to the stress tensor. *For the sake of later discussion, please note that this is the same situation as in the non-moving viscous fluid.* We remember Pascal's law

> "Any two points at the same elevation in a continuous mass of the same static fluid will be at the same pressure"[9]

here stated assuming hydrostatic distribution of pressure. If we don't have any favoured direction it can be rephrased as

> "The pressure applied to any part of the enclosed liquid will be transmitted equally in all directions through the liquid"[10]

We can than represent the force density acting at $\mathrm{d}S$ as

$$\mathbf{f} = -\mathbf{n}p,$$

where p represents pressure. We assume that the normal vector $\mathbf{n}$ is directed outside of a fluid parcel that we are considering. The minus in front of the expression, is there due to the fact that the surface exerts the pressure inwards. If we note that $\mathbf{f} = \mathbf{n} \cdot \boldsymbol{\tau}$ we can now say

$$\mathbf{f} = \mathbf{n} \cdot \boldsymbol{\tau} = -\mathbf{n}p = \mathbf{n} \cdot (-p\mathbf{I}) \rightarrow \boldsymbol{\tau} = -p\mathbf{I}$$

or in the index notation $\tau_{ij} = -p\delta_{ij}$.

Including strain-rate Let's generalise. As we mentioned before, we are looking for relation of the form $\boldsymbol{\tau} = g(\dot{\mathbf{D}})$. We would like to do the power expansion of g[11]. We will have a constant part (independent of the function argument), linear, quadratic and so on. In our case, the constant part can be conveniently expressed as $g(\dot{\mathbf{D}})_{\dot{\mathbf{D}}=\mathbf{0}}$. This situation is exactly like the one before, we have no deformation, so the fluid is probably not moving. So we can write:

$$\boldsymbol{\tau} = -p\mathbf{I} + \text{linear term w.r.t. } \dot{\mathbf{D}} + \text{quadratic term w.r.t. } \dot{\mathbf{D}} + \dots$$

We have the constant term and now we need to determine the other ones.

Eigenvalue problem for a tensor For every tensor, it can "act" on a vector to produce another vector, as in $M_{\alpha\beta}a_\beta = b_\alpha$ or $\mathbf{Ma} = \mathbf{b}$. If we have selected a special vector $\mathbf{a}$, the resultant vector $\mathbf{b}$ might be collinear with $\mathbf{a}$. This property is a geometric fact, independent of the chosen coordinate system. So if we set this special vector as $\mathbf{q}$ and following is true

$$\mathbf{Mq} = \lambda\mathbf{q} \quad \text{or} \quad (\mathbf{M} - \lambda\mathbf{I})\mathbf{q} = \mathbf{0},$$

than both λ, called an *eigenvalue*, and $\mathbf{q}$, the *eigenvector* do not depend on the coordinate system. For $\mathbf{q}$ not to be a trivial $\mathbf{0}$, following determinant has to vanish

$$\det(\mathbf{M} - \lambda\mathbf{I}) = 0.$$

[9] White, F.M. (2016) Fluid mechanics. Eighth edition. New York, NY: McGraw-Hill Education.

[10] Wikipedia (Accessed 01.06.2023), Pascal's law

[11] Power series around 0 looks like that: $k(x) = k(0) + ax + bx^2 + \dots$

If we actually spell the determinant out we will get the third order equation for λ:

$$-\lambda^3 + I_1(\mathbf{M})\lambda^2 - I_2(\mathbf{M})\lambda + I_3(\mathbf{M}) = 0. \tag{6}$$

It is called *characteristic equation* of a tensor (or a matrix). The scalars I_1, I_2 and I_3 depend on $\mathbf{M}$. We established that eigenvectors do not depend on the coordinate system, they "zero" Equation 6, so the coefficients of the equation, that is I_1, I_2 and I_3, also have to be independent from the chosen system. They are called *invariants* and are given by the following relations[12]:

$$\begin{aligned}
I_1 &= \operatorname{tr}(\mathbf{M}) = M_{11} + M_{22} + M_{33} = \lambda_1 + \lambda_2 + \lambda_3 \\
I_2 &= \frac{1}{2}\left((\operatorname{tr}(\mathbf{M}))^2 - \operatorname{tr}\left(\mathbf{M}^2\right)\right) \\
&= M_{11}M_{22} + M_{22}M_{33} + M_{11}M_{33} - M_{12}M_{21} - M_{23}M_{32} - M_{13}M_{31} \\
&= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \\
I_3 &= \det(\mathbf{M}) = -M_{13}M_{22}M_{31} + M_{12}M_{23}M_{31} + M_{13}M_{21}M_{32} \\
&\quad - M_{11}M_{23}M_{32} - M_{12}M_{21}M_{33} + M_{11}M_{22}M_{33} = \lambda_1\lambda_2\lambda_3
\end{aligned}$$

Naturally, remembering those formulas is not useful, but it is good to remember that they exist. We will now introduce a theorem that will do a lot of work for us.

Theorem 2 (Cayley–Hamilton theorem)**.** *Every square matrix over a commutative ring (such as the real or complex numbers or the integers) satisfies its own characteristic equation.*

In practice, this means that the following holds

$$-\mathbf{M}^3 + I_1(\mathbf{M})\mathbf{M}^2 - I_2(\mathbf{M})\mathbf{M} + I_3(\mathbf{M})\mathbf{I} = 0.$$

We essentially substituted $\mathbf{M}$ into Equation 6. What can now rearrange and obtain closed expression for $\mathbf{M}^3$

$$\mathbf{M}^3 = -I_1(\mathbf{M})\mathbf{M}^2 + I_2(\mathbf{M})\mathbf{M} - I_3(\mathbf{M})\mathbf{I} \tag{7}$$

and $\mathbf{M}^4$ and so on.

Constitutive relation: coefficient of the power series So from what we have said before, we can say that all powers of the series describing $\boldsymbol{\tau} = g(\dot{\mathbf{D}})$ above the second one are not linearly independent from each other. So we can safely limit the expansion to first two and another constant term. W can write

$$\boldsymbol{\tau} = -p\mathbf{I} + \gamma_1\dot{\mathbf{D}} + \gamma_2\mathbf{I} + \gamma_3\dot{\mathbf{D}}\dot{\mathbf{D}}$$

Couple of things happened here. First we substituted Equation 7 into g instead of the expansion. We modified the coefficients, as they have to account for each power of the series, but they will still depend on invariants (due to coordinate system independence) and other parameters not connected to the strain-rate tensor. Additionally, we assumed that the eigenvectors of $\boldsymbol{\tau}$ and $\dot{\mathbf{D}}$ are collinear[13], so γ coefficients have to be scalars.

[12] Shamelessly stolen from Wikipedia (accessed 01.06.2023), Invariants of tensors

[13] I have actually not found a good book describing why this is the case... If you are curious and found a good physical resining behind it, please let me know.

Next, we will assume that the fluid is a *Newtonian*, that is linear fluid fluid. We just say that $\gamma_3 \equiv 0$ and "yeet" the quadratic part out. This is naturally not always the case. The next element of the puzzle is γ_2. It came from Equation 7, from the term with I_1 invariant. Since we have a linear fluid, and the relation should be coordinate-independent, γ_3 can only depend linearly on I_1. Similarity, γ_1 can't depend on $\dot{\mathbf{D}}$, or the fluid won't be linear... We are left with a following expression

$$\boldsymbol{\tau} = -p\mathbf{I} + 2\mu\dot{\mathbf{D}} + \xi I_1(\dot{\mathbf{D}})\mathbf{I} \tag{8}$$

The coefficients μ and ξ[14] are called *dynamic viscosity coefficient* and *second viscosity coefficient* respectively. More about those later. The model that we have wrote is a good description for gases and "normal" fluids. For polymers, resins, colloids, etc. we would have to use nonlinear model. But usually the second power is skipped anyway, and the non-linearity is hidden in the μ or ξ. For example, human blood, often simulated to design blood pumps or study flow in arteries is a non-Newtonian shear-thinning fluid: the viscosity decreases with increased shear-rate.

Last thing to determine, is what is I_1? We remember that it was given as

$$I_1(\dot{\mathbf{D}}) = \dot{D}_{11} + \dot{D}_{22} + \dot{D}_{33} = \dot{D}_{ii}$$

. The strain-rate tensor is given as Equation 2, so plugging that into the above we get

$$I_1(\dot{\mathbf{D}}) = \frac{\partial v_i}{\partial x_i}$$

or simply $\boldsymbol{\nabla} \cdot \mathbf{v}$. We now have almost complete formula for the constitutive relation

$$\boldsymbol{\tau} = -p\mathbf{I} + 2\mu\dot{\mathbf{D}} + \xi(\boldsymbol{\nabla} \cdot \mathbf{v})\mathbf{I}$$

Stokes' hypothesis We can introduce a coefficient called *bulk viscosity coefficient* given as

$$\zeta = \xi + \frac{2}{3}\mu.$$

It is customary[15] to set the bulk viscosity to 0. This is called *Stokes' hypothesis*. Let's think a little what this coefficient represents.

> "[...] the isotropic part of the complete stress tensor contains a viscous term $[\zeta(\boldsymbol{\nabla} \cdot \mathbf{v})\mathbf{I}]$ that is additive to the pressure term $[-p\mathbf{I}]$. We may then interpret $\zeta(\boldsymbol{\nabla} \cdot \mathbf{v})\mathbf{I}$ as the difference between the *thermodynamic pressure* and the opposite of the average of the normal stresses acting on any three orthogonal planes passing through a point in the fluid, which is usually referred to as the *mechanical pressure*. This difference is generally considered to be due to the time lag with which the thermodynamic equilibrium condition is reached in a motion that implies an isotropic dilatation of a fluid element."[15]

[14] Often written with a symbol λ but we already have the eigenvalues...

[15] Buresti, G. A note on Stokes' hypothesis. Acta Mech 226, 3555–3559 (2015). `https://doi.org/10.1007/s00707-015-1380-9`

So the Stokes' hypothesis, essentially states that the mechanical pressure is equivalent to thermodynamic pressure. This assumption, and following simplification of equations is useful for compressible fluids (for incompressible it does not matter), but is it accurate? Well, the scientific community does not seem to have a totally clear answer to that.

Usually it is assumed that, that for monoatomic gases, $\zeta = 0$. This is derived on the theoretical basis, but it can also be showed that for physically realistic fluids it can't be equal to zero (references of the paper[15]). Generally, experimental investigations have said that the bulk viscosity is not negligible, and often of the same magnitude as the dynamic viscosity. On the other hand, we have an ample database of results from the simulations of compressible flows, that assumed $\zeta = 0$ and provided good and useful results.

> "A reasonable explanation of this circumstance may be obtained from a deeper analysis of the contribution of the stress tensor term [...]. The effect of $\zeta(\boldsymbol{\nabla} \cdot \mathbf{v})\mathbf{I}$ is thus perfectly additive to that of the thermodynamic pressure; in other words, this term is associated with the same deformation–isotropic dilatation of a fluid element– that is connected with the thermodynamic pressure, which, however, is generally larger than $\zeta(\boldsymbol{\nabla} \cdot \mathbf{v})\mathbf{I}$ (in absolute value) by several orders of magnitude.[...] Indeed, rather than putting $\zeta = 0$, we may simply assume that the absolute value of $\zeta(\boldsymbol{\nabla} \cdot \mathbf{v})\mathbf{I}$ is negligible compared to the thermodynamic pressure, i.e., that the following relation holds:
>
> $$|\zeta(\boldsymbol{\nabla} \cdot \mathbf{v})\mathbf{I}| \ll p$$
>
> If this different point of view is adopted, there are indeed good reasons for the modified formulation [...] of Stokes' hypothesis to be a largely acceptable approximation. As a matter of fact, only in very particular conditions will the term $\zeta(\boldsymbol{\nabla} \cdot \mathbf{v})\mathbf{I}$ be comparable to the thermodynamic pressure. This may happen, for instance, when the fluid is characterized by large values of ζ (e.g., CO_2), and the motion is such that extremely large values of $\boldsymbol{\nabla} \cdot \mathbf{v}$ occur, as happens in hypersonic flows or in flows through shock waves [...].
>
> There is at least another situation in which, although the above conditions do not occur, neglecting the term $\zeta(\boldsymbol{\nabla} \cdot \mathbf{v})\mathbf{I}$ leads to results that are in contradiction with experimental evidence. In fact, as may be derived from the internal energy balance, the damping of acoustic waves at very high frequencies cannot be justified without reintroducing the viscous stresses connected with an isotropic dilatation. Indeed, an acoustic wave is associated with an oscillatory isotropic change in volume between opposite values. [...] Indeed, measuring the attenuation of high-frequency waves (ultrasounds) is one of the few methods to determine experimentally the bulk viscosity. "[15]

As stated above, the justification of Stokes' hypothesis is complex and this simplification need not always apply.

4 Momentum equation

Inviscid fluid After establishing the formula for $\boldsymbol{\tau}$ we can finally plug it in and reach the "complete" form of momentum equation. It will be easier if we write Equation 5 again:

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}\mathbf{v}) = \boldsymbol{\nabla} \cdot \boldsymbol{\tau} + \rho \mathbf{f}.$$

For now, we assume that the fluid is inviscid. As we have mentioned, this means that the only quantity contributing to the stresses is the pressure p, resulting in $\tau_{ij} = -p\delta_{ij}$. We need to compute the divergence of that:

$$\boldsymbol{\nabla} \cdot \boldsymbol{\tau} = \hat{\mathbf{e}}_i \frac{\partial \tau_{ij}}{\partial x_j} = -\hat{\mathbf{e}}_i \frac{\partial p \delta_{ij}}{\partial x_j} = -\hat{\mathbf{e}}_i \frac{\partial p}{\partial x_i} = -\boldsymbol{\nabla} p.$$

We again used the fact that δ_{ij} is the "ranamer" of indices. We plug that into the general momentum equation and we reach the conservation law from the Euler system of equations:

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}\mathbf{v}) = -\boldsymbol{\nabla} p + \rho \mathbf{f}. \tag{9}$$

Viscous fluid We