Formulas for finite differences

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1 Basics

Taylor series

$$u(x+h) = u(x) + \frac{h}{1!}\frac{\partial u}{\partial x} + \frac{h}{2!}\frac{\partial^2 u}{\partial x^2} + \frac{h}{3!}\frac{\partial^3 u}{\partial x^3} + \dots$$

2 One dimension

Forward difference

$$u(x+h) = u(x) + \frac{h}{1!}\frac{\partial u}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2} + + \frac{h^3}{3!}\frac{\partial^3 u}{\partial x^3} + \mathcal{O}\big(h^4\big)$$

We divide by h:

$$\frac{u(x+h)}{h} = \frac{u(x)}{h} + \frac{\partial u}{\partial x} + \frac{h}{2!}\frac{\partial^2 u}{\partial x^2} + \frac{h^2}{3!}\frac{\partial^3 u}{\partial x^3} + \mathcal{O}\big(h^3\big)$$

Rearange:

$$\frac{\partial u}{\partial x} = \frac{u(x+h) - u(x)}{h} - \frac{h}{2!}\frac{\partial^2 u}{\partial x^2} - \frac{h^2}{3!}\frac{\partial^3 u}{\partial x^3} + \mathcal{O}\big(h^3\big) \tag{1}$$

Finally:

$$\frac{\partial u}{\partial x} = \frac{u(x+h) - u(x)}{h} + \mathcal{O}(h)$$

Backward difference

$$u(x-h) = u(x) - \frac{h}{1!}\frac{\partial u}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{3!}\frac{\partial^3 u}{\partial x^3} + \mathcal{O}\big(h^4\big)$$

We divide by h:

$$\frac{u(x-h)}{h} = \frac{u(x)}{h} - \frac{\partial u}{\partial x} + \frac{h}{2!}\frac{\partial^2 u}{\partial x^2} + - \frac{h^2}{3!}\frac{\partial^3 u}{\partial x^3} + \mathcal{O}\big(h^3\big)$$

Rearange:

$$\frac{\partial u}{\partial x} = \frac{u(x) - u(x-h)}{h} + \frac{h}{2!}\frac{\partial^2 u}{\partial x^2} - \frac{h^2}{3!}\frac{\partial^3 u}{\partial x^3} + \mathcal{O}\big(h^3\big) \tag{2}$$

Finally:

$$\frac{\partial u}{\partial x} = \frac{u(x) - u(x-h)}{h} + \mathcal{O}(h)$$

Central difference Take a sum of forward and backward, Equation 1 and Equation 5:

$$2\frac{\partial u}{\partial x} = \frac{u(x+h) - u(x-h)}{h} - \frac{2h^2}{3!}\frac{\partial^3 u}{\partial x^3} + \mathcal{O}\big(h^3\big) \tag{3}$$

Divide by two:

$$\frac{\partial u}{\partial x} = \frac{u(x+h) - u(x-h)}{2h} - \frac{1h^2}{3!}\frac{\partial^3 u}{\partial x^3} + \mathcal{O}\big(h^3\big) \tag{4}$$

Finally:

$$\frac{\partial u}{\partial x} = \frac{u(x+h) - u(x-h)}{2h} + \mathcal{O}\big(h^2\big) \tag{5}$$

Central difference of second derivative It is not exactly central difference of a central difference. That would have a wide stencil. Bad. Instead try writing at halfs.

$$u''(x) = \frac{u'(x+h/2) - u'(x-h/2)}{h}$$

$$u'(x \pm h) = \frac{u(x \pm h/2 + h/2) - u(x \pm h/2 - h/2)}{h}$$

Pluging in...

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

3 Solving systems of equations

Gauss seidel formula From Wikipedia. We have

$$A = \underbrace{\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{L_*} + \underbrace{\begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{U}.$$

We do:

$$A\mathbf{x} = \mathbf{b} \tag{6}$$
$$(L_* + U)\mathbf{x} = \mathbf{b} \tag{7}$$
$$L_*\mathbf{x} + U\mathbf{x} = \mathbf{b} \tag{8}$$
$$L_*\mathbf{x} = \mathbf{b} - U\mathbf{x} \tag{9}$$

The Gauss–Seidel method now solves the left hand side of this expression for $\mathbf{x}$, using previous value for $\mathbf{x}$ on the right hand side. Analytically, this may be written as:

$$\mathbf{x}^{(k+1)} = L_*^{-1}\left(\mathbf{b} - U\mathbf{x}^{(k)}\right).$$

However, by taking advantage of the triangular form of L_*, the elements of $\mathbf{x}^{(k+1)}$ can be computed sequentially for each row i using forward substitution:

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right), \quad i = 1, 2, \ldots, n$$