Miscellaneous

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Contents

1 Strain

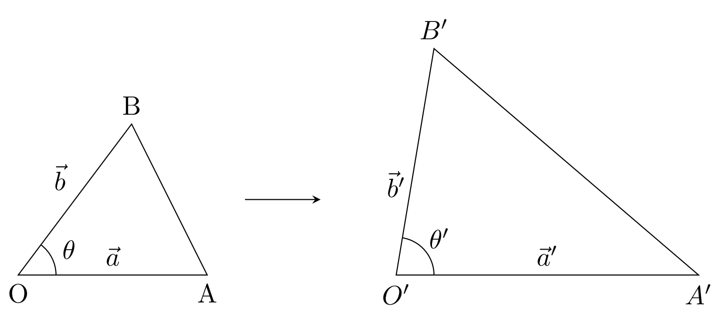
Definition 1.1. Let $\vec{u}$ be a desplacement field. The elements of **strain tensor** is defined as

$$\varepsilon_{ij} \equiv \frac{1}{2}\left(\frac{\mathrm{d}u_i}{\mathrm{d}x_j} - \frac{\mathrm{d}u_j}{\mathrm{d}x_i}\right) \tag{1}$$

Definition 1.2. The **principal strains** are defined as eigenvalues of strain tensor.

1.1 Triangle mesh (2D strain)

Let the infinitesimal triangular surface OAB be deformed into an infinitesimal triangular surface $O'A'B'$.



When considering strain, the rigid body transformation part can be ignored, so it can be assumed that O and O' are the same point, and O, A and A'

are colinear, and OAB and $OA'B'$ are in the same plane. Then we define the x_0-axis as the OA direction and the x_1-axis so that the x_0x_1 plane contains the triangles OAB and $OA'B'$. In this case, the derivatives of the deformation vector are derived from the linear equations

$$\vec{a}' = \mathbb{U}\vec{a}, \qquad \vec{b}' = \mathbb{U}\vec{b} \tag{2}$$

where $\mathbb{U}_{ij} = \partial u_i / \partial x_j$, and become

$$\begin{aligned} \frac{\partial u_0}{\partial x_0} = \frac{|\vec{a}'| - |\vec{a}|}{|\vec{a}|}, \qquad & \frac{\partial u_0}{\partial x_1} = \frac{b_0' - (1 + \partial u_0/\partial x_0) b_0}{b_1} \\ \frac{\partial u_1}{\partial x_0} = 0, \qquad & \frac{\partial u_1}{\partial x_1} = \frac{b_1' - b_1}{b_1} \end{aligned} \tag{3}$$

The normal strains and shear strains are described as follows:

$$\begin{aligned} \varepsilon_{00} &= \frac{|\vec{a}'| - |\vec{a}|}{|\vec{a}|} \\ \varepsilon_{11} &= \frac{b_1' - b_1}{b_1} \\ \varepsilon_{10} = \varepsilon_{01} &= \frac{b_0' - (1 + \varepsilon_{00}) b_0}{2b_1} \end{aligned} \tag{4}$$

The principal strains are defined as the eigenvalues of the strain tensor, so the principal strains are as follows:

$$\varepsilon_{\max}, \varepsilon_{\min} = \frac{1}{2}\left(\varepsilon_{00} + \varepsilon_{11} \pm \sqrt{(\varepsilon_x + \varepsilon_y)^2 - 4(\varepsilon_{00}\varepsilon_{11} - \varepsilon_{01}\varepsilon_{10})}\right) \tag{5}$$

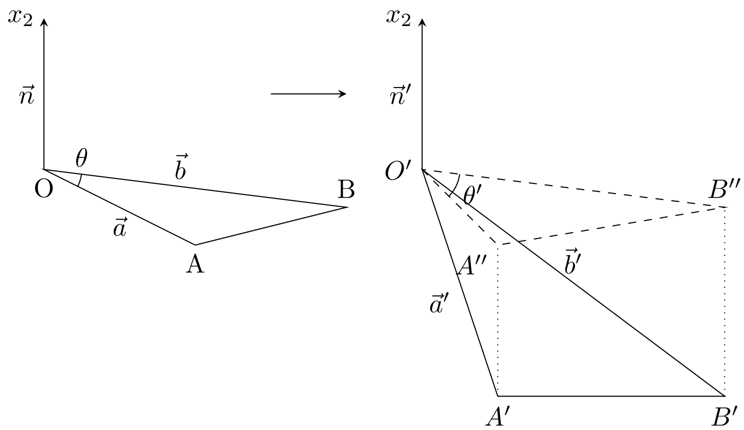
1.1.1 Shader program

Refer to **SampleCode/PrincipalStrainShader** for a sample code of shaders that calculates the 2D principal strains from the original and current vertex positions, and color meshes based on their principal strains.

1.2 Triangle mesh (3D strain)

Mathematically, a 2D object generally has a 2x2 strain tensor. However, Although 3D model meshes are treated as 2D planes, the objects they represent originally have a thickness in the third dimension. Therefore, considering the strain in the vertical direction of the surface by assuming that the mesh has a small thickness may allow for a more realistic calculation of strain.

Considering a infinitesimal triangle surface OAB which the vertex normal of the vertex O is parallel to the normal of this surface. Let this surface OAB be deformed into an infinitesimal triangular surface $O'A'B'$. Here, the vertical strain is assumed to be obtained from the displacement of each edge in the direction normal of the vertex normal of O'. Since the rigid body transformation part can be ignored, we can assume that the projection of the edge $O'A'$ onto a x_0x_1-plane is parallel to the edge OA (let this direction be x_0-direction).



In this case, we can perform the same discussion as in the 2D case regarding the derivatives of the displacement vector in the x_0y_0 plane by corresponding the projections $O'A'', O'B''$ of the edges OA', OB' onto the x_0x_1-plane to $\vec{a}', \vec{b}'$:

$$\begin{aligned} \frac{\partial u_0}{\partial x_0} &= \frac{a_0' - a_0}{a_0}, \qquad & \frac{\partial u_0}{\partial x_1} &= \frac{b_0' - (1 + \partial u_0/\partial x_0)b_0}{b_1} \\ \frac{\partial u_1}{\partial x_0} &= 0, \qquad & \frac{\partial u_1}{\partial x_1} &= \frac{b_1' - b_1}{b_1} \end{aligned} \tag{6}$$

and other coordinates are derived as

$$\begin{aligned} \frac{\partial u_2}{\partial u_0} &= \frac{a_2'}{a_0}, \qquad & \frac{\partial u_2}{\partial u_1} &= \frac{b_2' - \partial u_2/\partial x_0 b_0}{b_1} \\ \frac{\partial u_i}{\partial x_2} &= 0 \end{aligned} \tag{7}$$

The normal strains and shear strains are described as follows:

$$\varepsilon = \begin{pmatrix} \frac{a_0' - a_0}{a_0} & \frac{b_0' - (1 + \partial u_0/\partial x_0)b_0}{2b_0} & \frac{a_2'}{2a_0} \\ \frac{b_0' - (1 + \partial u_0/\partial x_0)b_0}{2b_0} & \frac{b_1'}{b_1} & \frac{b_2' - \partial u_2/\partial x_0 b_0}{2b_1} \\ \frac{a_2'}{2a_0} & \frac{b_2' - \partial u_2/\partial x_0 b_0}{2b_1} & 0 \end{pmatrix} \tag{8}$$