$$\gdef\dif{\mathrm{d}}$$ $$\gdef\pardif#1#2{\frac{\partial #1}{\partial #2}}$$

Mathematics for physics

Differentiation and partial differentiation

Lagrange multiplier

1. If there is a condition $$ g(\bm{x})=c\mathrm{\quad where\quad}c\mathrm{\ \ is\ constant} $$ on a variable $\bm{x} = (x_1,\ldots, x_n)$, find the extrema of $f(\bm{x})$. Here, assume that both functions f, g belong to C^1.

   ▶ Answer

   Under the condition $g(\bm{x})$, a infinitely small variations in variables $\dif \bm{x}$ satisfies equation: $$ \nabla g(\bm{x})\cdot\dif\bm{x} = \sum_{i=1}^n\pardif{g}{x_i}\dif x_i = 0 $$ so the change in f with respect to these variations in variables can be describe as follows: $$ \begin{aligned} \dif f &= \nabla f\cdot\dif\bm{x}\ &=\sum_{i=1}^{n}\pardif{f}{x_i}\dif x_i\ &=\sum_{i=1}^{n-1}\pardif{f}{x_i}\dif x_i + \pardif{f}{x_n}\left(\pardif{g}{x_n}\right)^{-1}\sum_{i=1}^{n-1}\pardif{g}{x_i}\dif x_i\ &=\sum_{i=1}^{n-1}\left(\pardif{f}{x_i} + \lambda\pardif{g}{x_i}\right)\dif x_i \end{aligned} $$ where $\lambda = \pardif{f}{x_n}\left(\pardif{g}{x_n}\right)^{-1}$. Since $\dif f=0$ on the constrained extrema of f for the arbitrary infinitesimal variations $\dif x_1,\ldots ,\dif x_{n-1}$, the coefficients of $\dif x_i (i=1,\ldots, n-1)$ must be zero: $$ \pardif{f}{x_i} + \lambda\pardif{g}{x_i} = 0\mathrm{\quad for\quad}x=1,\ldots,n-1 $$ From the definition of λ, this equation holds for all i between 1 and n inclusive.

   Therefore, the extrema of a function $f(\bm{x})$ subject to constraints $g(\bm{x}) = c$ can be obtained by solving the extrema problem for $\tilde{f} = f - \lambda g$.