Mathematics for physics

@t-34400

June 16, 2023

1 Differentiation and partial differentiation

1.1 Lagrange multiplier

Problems 1.1. If there is a condition

$$g(\boldsymbol{x}) = c \quad \text{where} \quad c \text{ is constant} \tag{1}$$

on a variable $\boldsymbol{x} = (x_1, \ldots, x_n)$, find the extrema of $f(\boldsymbol{x})$. Here, assume that both functions f, g belong to C^1.

Solutions 1.1. Under the condition $g(\boldsymbol{x})$, a infinitely small variations in variables $\mathrm{d}\boldsymbol{x}$ satisfies equation:

$$\nabla g(\boldsymbol{x}) \cdot \mathrm{d}\boldsymbol{x} = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} \mathrm{d}x_i = 0 \tag{2}$$

so the change in f with respect to these variations in variables can be describe as follows:

$$\begin{aligned}
\mathrm{d}f &= \nabla f \cdot \mathrm{d}\boldsymbol{x} && (3)\\
&= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{d}x_i && (4)\\
&= \sum_{i=1}^{n-1} \frac{\partial f}{\partial x_i} \mathrm{d}x_i + \frac{\partial f}{\partial x_n} \left(\frac{\partial g}{\partial x_n} \right)^{-1} \sum_{i=1}^{n-1} \frac{\partial g}{\partial x_i} \mathrm{d}x_i && (5)\\
&= \sum_{i=1}^{n-1} \left(\frac{\partial f}{\partial x_i} + \lambda \frac{\partial g}{\partial x_i} \right) \mathrm{d}x_i && (6)
\end{aligned}$$

where $\lambda = \frac{\partial f}{\partial x_n} \left(\frac{\partial g}{\partial x_n} \right)^{-1}$. Since $\mathrm{d}f = 0$ on the constrained extrema of f for the arbitrary infinitesimal variations $\mathrm{d}x_1, \ldots, \mathrm{d}x_{n-1}$, the coefficients of $\mathrm{d}x_i (i = 1, \ldots, n-1)$ must be zero:

$$\frac{\partial f}{\partial x_i} + \lambda \frac{\partial g}{\partial x_i} = 0 \quad \text{for} \quad x = 1, \ldots, n-1 \tag{7}$$

From the definition of λ, this equation holds for all i between 1 and n inclusive.

Therefore, the extrema of a function $f(\boldsymbol{x})$ subject to constraints $g(\boldsymbol{x}) = c$ can be obtained by solving the extrema problem for $\tilde{f} = f - \lambda g$.

Problems 1.2. If there are two conditions:

$$\sum_{i=1}^{N} p_i = 1 \tag{8}$$

$$\sum_{i=1}^{N} p_i \epsilon_i = E \text{ (constant)} \tag{9}$$

on the variables $p_i\{i = 1, \ldots, N\}$, find the stationary points of the following value:

$$S = -\sum_{i=1}^{N} p_i \log p_i \tag{10}$$

Solutions 1.2. Let $\tilde{f}$ be

$$\tilde{f} = -\sum_{i=1}^{N} p_i \log p_i - \alpha \sum_{i=1}^{N} p_i - \beta \sum_{i=1}^{N} p_i \epsilon_i \tag{11}$$

where α, β are the Lagrange multipliers. At the stationary points,

$$\frac{\partial \tilde{f}}{\partial p_i} = 0 \tag{12}$$

and then

$$-\log p_i - 1 - \alpha - \beta \epsilon_i = 0 \tag{13}$$

threfore

$$p_i = e^{-\alpha - 1} e^{-\beta \epsilon_i} \tag{14}$$

By determining α based on the first condition, we obtain

$$p_i = \frac{e^{-\beta \epsilon_i}}{\sum_{i=1}^{N} e^{-\beta \epsilon_i}} \tag{15}$$

β can be determined based on the second condition

$$\frac{\sum_{i=1}^{N} \epsilon_i e^{-\beta \epsilon}}{\sum_{i=1}^{N} e^{-\beta \epsilon}} = E \tag{16}$$

1.1.1 Jacobian

Problems 1.3. The Jacobian matrix is defined as:

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \equiv \left| \frac{\partial \boldsymbol{f}}{\partial x_1}, \ldots, \frac{\partial \boldsymbol{f}}{\partial x_n} \right| = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_1} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_n} \end{vmatrix} \tag{17}$$

where $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function such that each of its first-order partial derivatives exist on $\mathbb{R}^n$.

Show the equation:

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{f}} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \tag{18}$$

where $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^l$ are functions such that each of their first-order partial derivatives exist on $\mathbb{R}^n$ or $\mathbb{R}^m$.

Solutions 1.3.

$$\begin{aligned} \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{f}} &= \left| \frac{\partial \boldsymbol{f}}{\partial x_1}, \ldots, \frac{\partial \boldsymbol{f}}{\partial x_n} \right| \left| \frac{\partial \boldsymbol{g}}{\partial f_1}, \ldots, \frac{\partial \boldsymbol{g}}{\partial f_m} \right| && (19) \\ &= \left| \sum_i^m \frac{\partial f_i}{\partial x_1} \frac{\partial \boldsymbol{g}}{\partial f_1}, \ldots, \sum_i^m \frac{\partial f_i}{\partial x_n} \frac{\partial \boldsymbol{g}}{\partial f_1} \right| && (20) \\ &= \left| \frac{\partial \boldsymbol{g}}{\partial x_1}, \ldots, \frac{\partial \boldsymbol{g}}{\partial x_m} \right| && (21) \\ & && (22) \end{aligned}$$

Problems 1.4. Prove the following equation:

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial y}{\partial z}\right)_x = \left(\frac{\partial u}{\partial x}\right)_z \left(\frac{\partial y}{\partial z}\right)_u \tag{23}$$

Solutions 1.4.

$$\text{(LHS)} = \frac{\partial(u,y)}{\partial(x,y)} \frac{\partial(y,x)}{\partial(z,x)} = -\frac{\partial(u,y)}{\partial(z,x)} \tag{24}$$

$$\text{(RHS)} = \frac{\partial(y,z)}{\partial(x,z)} \frac{\partial(y,u)}{\partial(z,u)} = -\frac{\partial(y,u)}{\partial(x,z)} \tag{25}$$