

Mathematics for physics

@t-34400

May 14, 2023

1 Differentiation and partial differentiation

1.1 Lagrange multiplier

Problems 1.1. If there is a condition

$$g(\mathbf{x}) = c \quad \text{where } c \text{ is constant} \quad (1)$$

on a variable $\mathbf{x} = (x_1, \dots, x_n)$, find the extrema of $f(\mathbf{x})$. Here, assume that both functions f, g belong to C^1 .

Solutions 1.1. Under the condition $g(\mathbf{x})$, a infinitely small variations in variables $d\mathbf{x}$ satisfies equation:

$$\nabla g(\mathbf{x}) \cdot d\mathbf{x} = \sum_{i=1}^n \frac{\partial g}{\partial x_i} dx_i = 0 \quad (2)$$

so the change in f with respect to these variations in variables can be describe as follows:

$$df = \nabla f \cdot d\mathbf{x} \quad (3)$$

$$= \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \quad (4)$$

$$= \sum_{i=1}^{n-1} \frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial x_n} \left(\frac{\partial g}{\partial x_n} \right)^{-1} \sum_{i=1}^{n-1} \frac{\partial g}{\partial x_i} dx_i \quad (5)$$

$$= \sum_{i=1}^{n-1} \left(\frac{\partial f}{\partial x_i} + \lambda \frac{\partial g}{\partial x_i} \right) dx_i \quad (6)$$

where $\lambda = \frac{\partial f}{\partial x_n} \left(\frac{\partial g}{\partial x_n} \right)^{-1}$. Since $df = 0$ on the constrained extrema of f for the arbitrary infinitesimal variations dx_1, \dots, dx_{n-1} , the coefficients of dx_i ($i = 1, \dots, n-1$) must be zero:

$$\frac{\partial f}{\partial x_i} + \lambda \frac{\partial g}{\partial x_i} = 0 \quad \text{for } i = 1, \dots, n-1 \quad (7)$$

From the definition of λ , this equation holds for all i between 1 and n inclusive.

Therefore, the extrema of a function $f(\mathbf{x})$ subject to constraints $g(\mathbf{x}) = c$ can be obtained by solving the extrema problem for $\tilde{f} = f - \lambda g$.

Problems 1.2. If there are two conditions:

$$\sum_{i=1}^N p_i = 1 \quad (8)$$

$$\sum_{i=1}^N p_i \epsilon_i = E \text{ (constant)} \quad (9)$$

on the variables $p_i \{i = 1, \dots, N\}$, find the stationary points of the following value:

$$S = - \sum_{i=1}^N p_i \log p_i \quad (10)$$

Solutions 1.2. Let \tilde{f} be

$$\tilde{f} = - \sum_{i=1}^N p_i \log p_i - \alpha \sum_{i=1}^N p_i - \beta \sum_{i=1}^N p_i \epsilon_i \quad (11)$$

where α, β are the Lagrange multipliers. At the stationary points,

$$\frac{\partial \tilde{f}}{\partial p_i} = 0 \quad (12)$$

and then

$$-\log p_i - 1 - \alpha - \beta \epsilon_i = 0 \quad (13)$$

therefore

$$p_i = e^{-\alpha-1} e^{-\beta \epsilon_i} \quad (14)$$

By determining α based on the first condition, we obtain

$$p_i = \frac{e^{-\beta \epsilon_i}}{\sum_{i=1}^N e^{-\beta \epsilon_i}} \quad (15)$$

β can be determined based on the second condition

$$\frac{\sum_{i=1}^N \epsilon_i e^{-\beta \epsilon_i}}{\sum_{i=1}^N e^{-\beta \epsilon_i}} = E \quad (16)$$