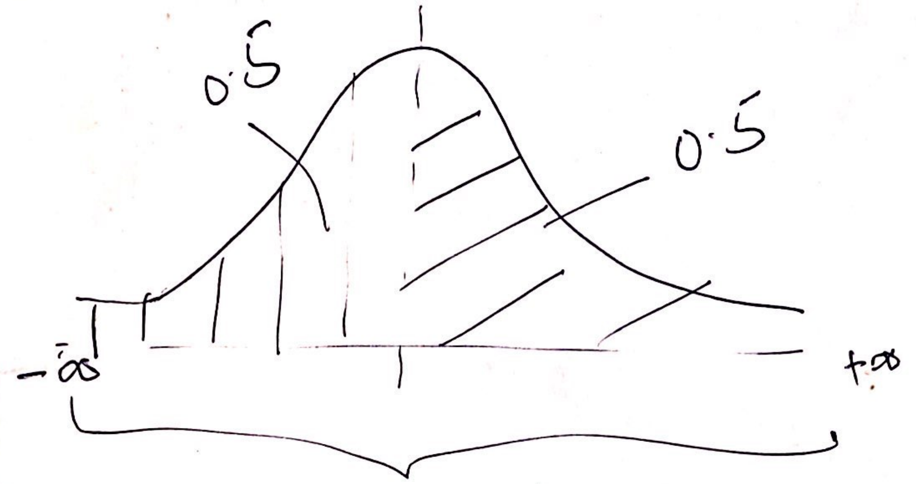
Normal Distribution



$A = 1$;

Probability Function,

$$f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < +\infty$$

Mean = Expectation = μ; $\qquad N(\mu, \sigma^2)$;

σ^2 = Variance;

Ex 4-10: Assume the current measurements in a strip of wire follow a normal distribution with a mean of 10 milliamperes and a variance of 4 milliamperes. What is the probability that a measurement exceeds 13 milliamperes?

$\mu = 10$, $\sigma^2 = 4$, $\sigma = 2$;

$P(Z$

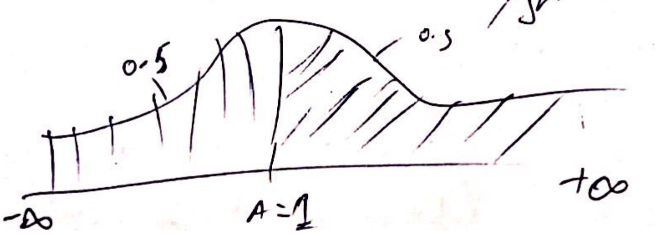
$P(X > x) = 1 - P(X \le x)$

P.d.f.

$f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

→ Probability function for a normal distribution



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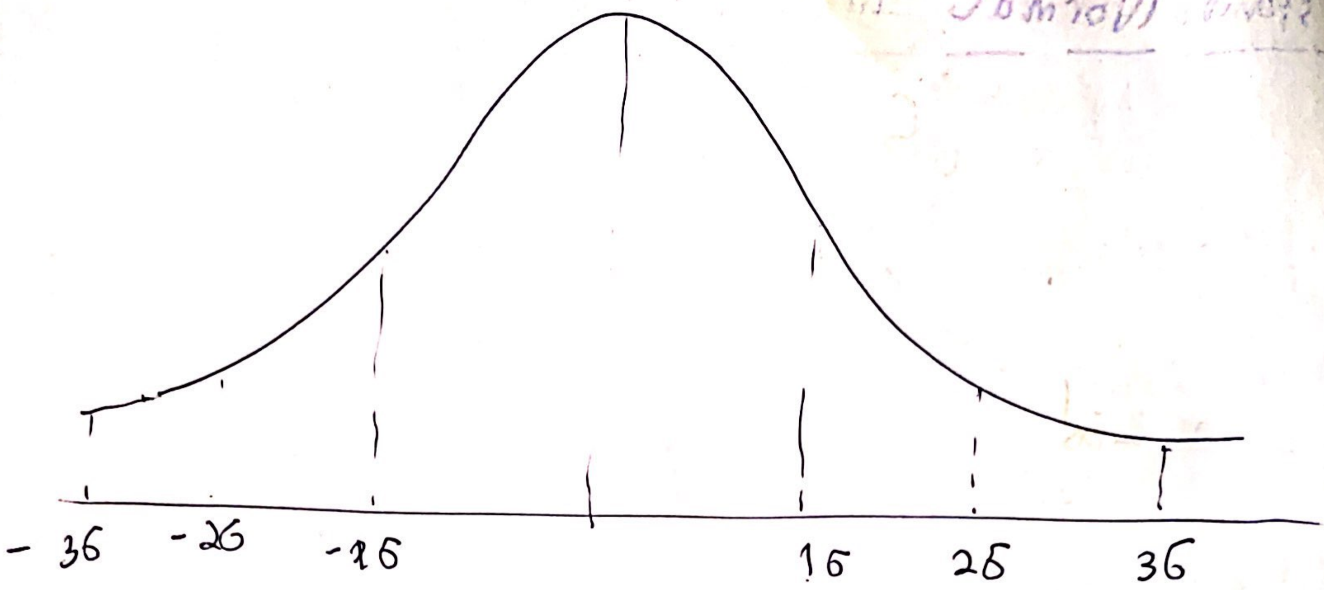


$(x-\mu)$

$$P(a < x < b) = P(a \le x < b)$$

$$= P(a < x \le b) = P(a \le x \le b);$$

$$P(a < x < b)$$

$\mu - \sigma$ $\boxed{\mu \pm \sigma}$ $\mu \pm 2\sigma$ $\mu \pm 3\sigma$

$$P((\mu-\sigma) < x < (\mu+\sigma)) = 68.27\% = 0.6827$$

$$P((\mu-2\sigma) < X < (\mu+2\sigma)) = 95.45\% = 0.9545$$

$$P((\mu-3\sigma) < x < (\mu+3\sigma)) = 99.73\% = 0.9973$$

standardized ! — Cummulative of Normal;

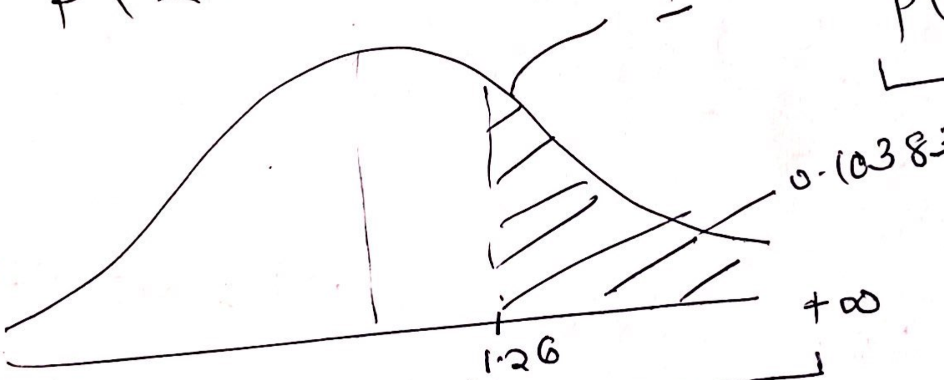
$\mu = 0, \quad \sigma^2 = 1.$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} \qquad Z = \frac{x-\mu}{\sigma}.$$

$\phi(Z)$

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}z^2};$$

① $P(Z > 1.26)$ $\qquad$ $\underbrace{P(X \leq x)}$.



$\underbrace{\hspace{3cm}}_{P(Z \leq 1.26)} \quad \underbrace{\hspace{3cm}}_{P(Z > 1.26)}$

$P(Z \leq 1.26) + P(Z > 1.26) = 1$

$P(Z > 1.26) = 1 - P(Z \leq 1.26)$

$P(1.26)$ —— $(1.26 < x < -\infty) = P(x \leq 1.26)$

$= 0.89617$

$P(Z > 1.26) = 1 - 0.89617$

$= 0.10383$

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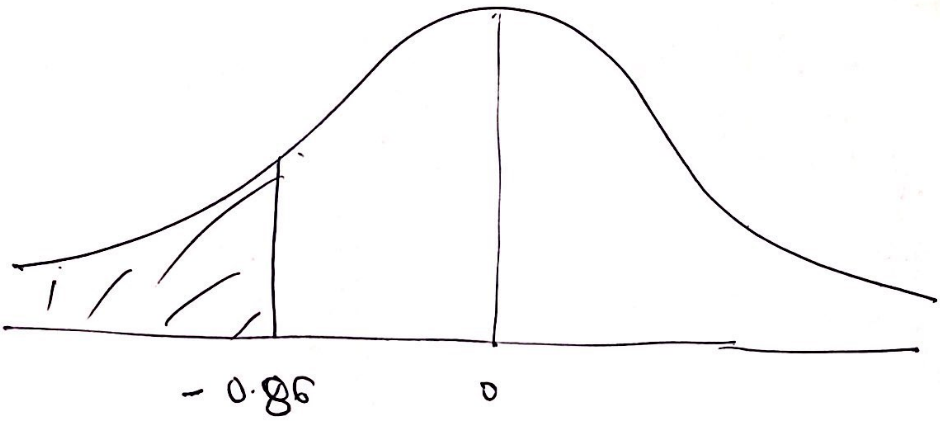
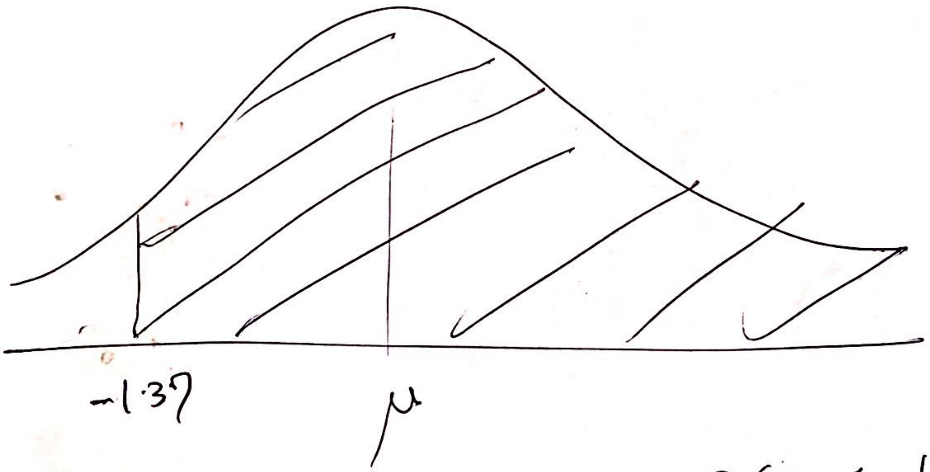
2) $P(Z<-0.86) = 0.19489$



3) $P(Z>-1.37)$

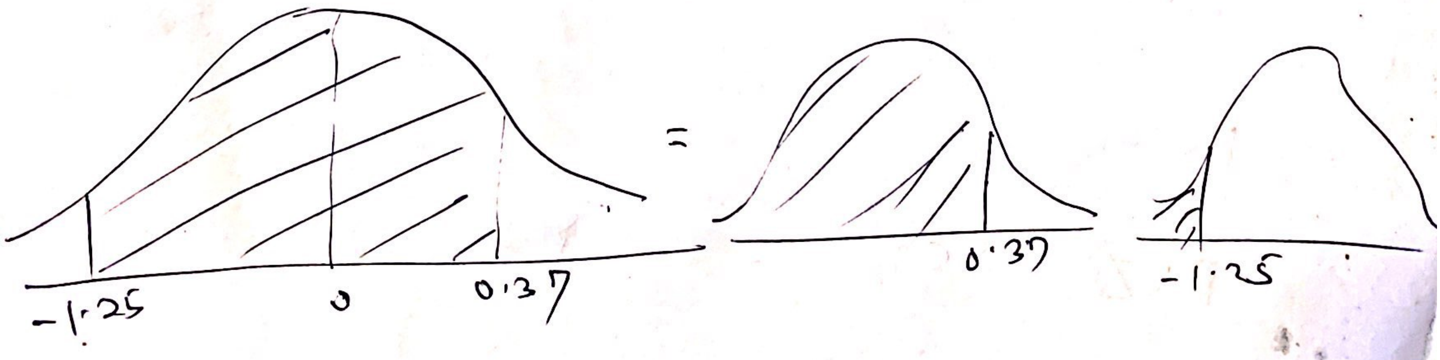


$P(Z>-1.37) = 1-P(Z<-1.37) = 0.9147$

4) $P(-1.25<Z<0.37)$

$= P(Z\le 0.37) - P(Z\le -1.25)$

$= 0.53866$



$\mu = 10$, $\sigma^2 = 4$; $\sigma = 2$; $z = \frac{x - \mu}{\sigma}$

$p(\cancel{X} > 13)$

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad z = \frac{x-\mu}{\sigma}}$$

$$Z = \frac{X - \mu}{\sigma}$$

$$\boxed{Z = \frac{x - \mu}{\sigma}}$$

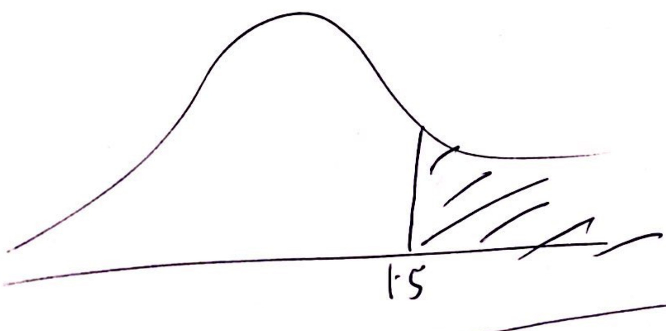
$$Z = \frac{X - 10}{2}$$

$$p(X > 13) = p(Z > z)$$

$$z = \frac{13 - 10}{2} = 3/2 = 1.5$$

$$p(Z > 1.5) = 1 - p(Z \leq 1.5)$$

$$= 0.06681 = 6.681\%$$



Ex: Find the moment generating function of a RV X that is binomially distributed.

Binomial distribution

I $f(x)$

$$M(t) = E(e^{tx}) = \sum_{X} e^{tx} f(x) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$f(x) = \binom{n}{x} p^x q^{n-x}.$$

$$= \sum_{x=0}^{n} e^{tx} \cdot \binom{n}{x} p^x q^{n-x}.$$

$$= \sum_{x=0}^{n} \binom{n}{x} (p \cdot e^t)^x q^{n-x} = (q+pe^t)^n$$

$$(q+pe^t)^n = \binom{n}{0} q^n (pe^t)^0 + \binom{n}{1} q^{n-1} (pe^t)^1 + \binom{n}{2} q^{n-2} (pe^t)^2 + \dots + \binom{n}{x} q^{n-x} (pe^t)^x$$

$$= \sum_{x=0}^{n} \binom{n}{x} q^{n-x} (pe^t)^x$$

II Binomial distribution

2 outcomes (Failure or Success)

$$\boxed{\binom{n}{r} x^r y^{n-r}}$$

$(y+x)^n$

$$X \begin{cases} 0 & \text{— Failure} \\ 1 & \text{— Success} \end{cases}$$

$$M(t) = \cancel{\sum} E(e^{tx}) = \sum e^{tx} \cdot f(x)$$

x	0	1
$f(x)$	q	p

$$M(t) = \cancel{e^{t(0)}} \cdot q + e^{t(1)} \cdot p$$

$$= q + pe^t$$

Prove that the mean and variance of a binomially distributed RV are, respectively, $\mu = np$ and $\sigma^2 = npq$.

For a binomially distributed RV

X	0	1
$f(x)$	q	p

$$\mu = \text{mean} = E(X) = \sum x \cdot f(x)$$

$$= \cancel{0 \cdot q} + 1 \cdot p = p$$

for n-trials

$$\mu = np;$$

$$\sigma^2 = E(X-\mu)^2 = E(X^2) - (E(X))^2$$

$$\sum (x-\mu)^2 f(x).$$

$$= (0-p)^2 \cdot q + (1-p)^2 \cdot p$$

$$= p^2 q + q^2 p$$

$$= pq(p+q)$$

$$= pq$$

$p+q = 1$

$1-p = q$

for n-trials $= npq$

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$$\sigma^2 = E(X^2) - (E(X))^2$$

$$E(X^2) = \sum x^2 f(x)$$

$$= 0^2 \cdot q + 1^2 \cdot p = p.$$

$$\sigma^2 = E(X^2) - (E(X))^2$$

$$= p - (p^2)$$

$$= p - p^2$$

$$= p(1-p)$$

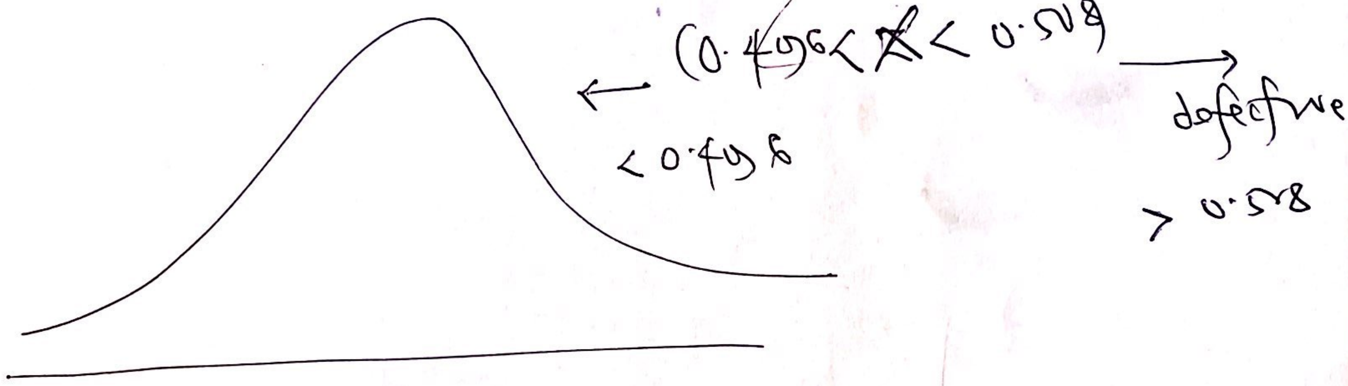
$$= pq$$

for n-trials,

$$\sigma^2 = npq,$$

The mean inside diameter of a sample of 200 washers produced by a machine is 0.502 inches and the standard deviation is 0.005 inches. The purpose for which these washers are intended allows a maximum tolerance in the diameter of 0.496 to 0.508 inches; otherwise the washers are considered defective. Determine the % of defective washers produced by the machine, assuming the diameters are normally distributed.

$n = 200, \quad \mu = 0.502; \quad \sigma = 0.005;$



$\underbrace{X < 0.496}_{\text{defective}} \quad \underbrace{0.496 < X < 0.508}_{\text{Non-defective}} \quad \underbrace{X > 0.508}_{\text{defective}}$

standardized normal variable, Z

$$Z = \frac{X - \mu}{\sigma}$$

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$X \sim 0.496$ ——— 0.508

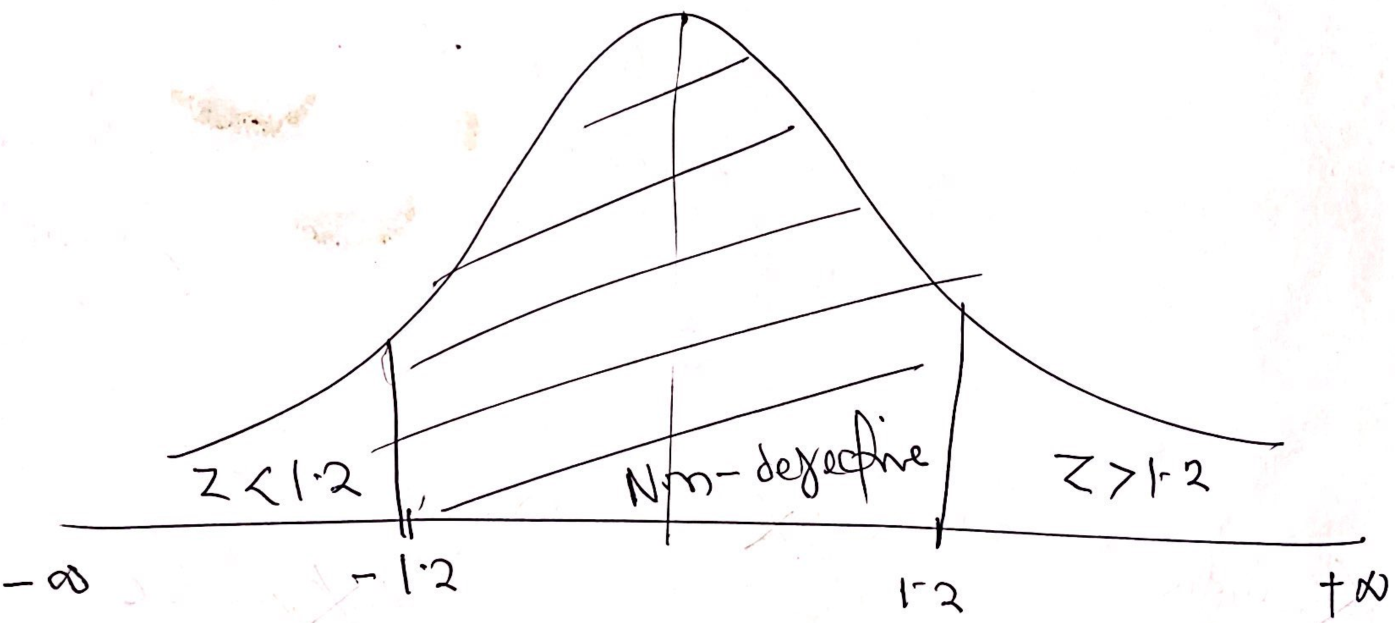
$Z \sim$

$Z = \frac{X-\mu}{\sigma}$

$Z = \frac{0.496-0.502}{0.005}$ $\qquad \frac{0.508-0.502}{0.05}$

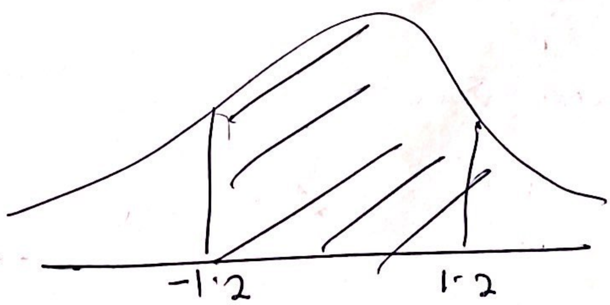
1.2

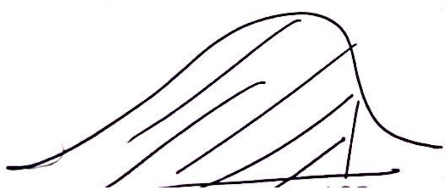
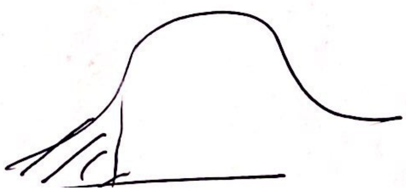
$Z = -1.2$



$P(\text{defective}) = 1 - P(\text{non-defective})$.

$P(\text{non-defective})$



$=$  $-$ 

$P(\text{non-defective}) = P(1.2) - P(-1.2)$

$-\infty < Z < 1.2$ $\qquad$ $-\infty < Z < (-1.2)$

$= 0.88493 - 0.11507$

$= 0.76986$

$P(\text{defective}) = 1 - 0.76986$

$= \cancel{0.2341}\ 0.23014$

$\% = 0.23014 \times 100\%$

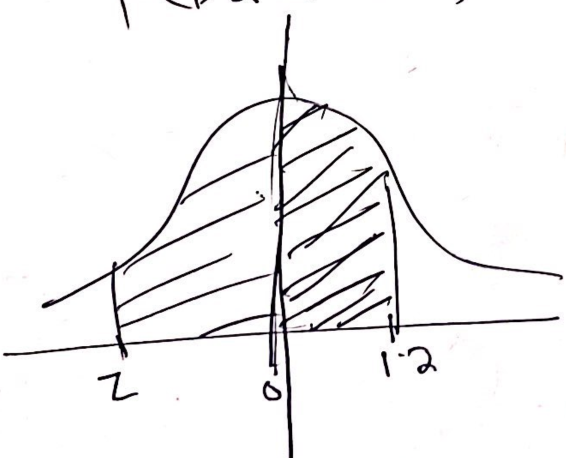
$= \underline{23.014\%}$

From table

$P(\cancel{1.2}\ Z \le 1.2) = 0.3849$

$P(Z < a) = (0 < Z < a)$

$P(-1.2) = P(1.2)$



$P(-1.2 < Z < 1.2) = 2(0.3849)$

$= 0.7698$

Find the moment generating function for the general normal distribution.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Solutn:

$$M(t) = E(e^{tx}) = \sum e^{tx} \cdot f(x) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$M(t) = \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \cdot e^{tx}\, dx \qquad \frac{x-\mu}{\sigma} = v \qquad e^{\frac{-v^2}{2}}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx - \frac{(x-\mu)^2}{2\sigma^2}} \cdot dx$$

$$\cancel{= \frac{1}{\sigma\sqrt{2\pi}} \cdot \left. \frac{e^{tx - \frac{(x-\mu)^2}{2\sigma^2}}}{t - \frac{1}{2\sigma^2} \cdot 2(x-\mu) \cdot 1} \right|_{-\infty}^{\infty}}$$

$$\cancel{= \frac{1}{\sigma\sqrt{2\pi}} \left[\left(\frac{e^{t(\infty) - \frac{(\infty-\mu)^2}{2\sigma^2}}}{t - (\infty-\mu)} \right) - \left(\frac{e^{t(-\infty) - \frac{(-\infty-\mu)^2}{2\sigma^2}}}{t - (-\infty-\mu)} \right) \right]}$$

$$\frac{x-\mu}{\sigma} = v$$

$$\cancel{\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{v^2}{2}}} \cdot e^{tx}$$

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot e^{tx} \, dx$$

$$\frac{x-\mu}{\sigma} = v \quad ; \quad x = \cancel{\frac{v+\sigma}{\mu}} \; v\sigma+\mu$$

$\sigma, \mu = \text{constant}$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\cdot v^2} e^{(v\sigma+\mu)t} \cdot \sigma \, dv$$

$x = v\sigma + \mu$

$dx = \sigma \, dv$

$$\frac{\cancel{\sigma}}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{v^2}{2}} e^{(v\sigma+\mu)t} \, dv$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma v+\mu)-\frac{v^2}{2})} \, dv$$

$$\cancel{= \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{\sigma v+\mu-\frac{v^2}{2}}}{\sigma - \frac{2v}{2}}} \Bigg|_{-\infty}^{\infty} \cancel{= \frac{1}{2\pi}\left(\frac{e^{\sigma v+\mu-\frac{v^2}{2}}}{\sigma - v}\right|_{-\infty}^{\infty}\Bigg)}$$

4

$$= \frac{1}{\sqrt{2\pi}} \int e^{\frac{-v^2}{2} + \sigma t v + \mu t} \, dv = \frac{1}{\sqrt{2\pi}} \int e^{(v-\sigma t)^2 - (2\mu t + \sigma^2 t^2)} \, dv$$

$$= \frac{1}{\sqrt{2\pi}} \int e^{(v-\sigma t)^2} \cdot \underbrace{e^{-(2\mu t + \sigma^2 t^2)}}_{\text{constant}} \, dv$$

$$= \frac{e^{-(2\mu t + \sigma^2 t^2)}}{\sqrt{2\pi}} \int e^{(v-\sigma t)^2} \, dv$$

$$v - \sigma t = w$$
$$dv = dw$$

$$= \frac{e^{-(2\mu t + \sigma^2 t^2)}}{\sqrt{2\pi}} \int e^{w^2} \, dw$$

$$\left(\quad \right) \cdot \left. \frac{e^{w^2}}{2w} \right|_{-\infty}^{\infty}$$

$$e^{-\frac{w^2}{2}} \, dw$$

$$\boxed{w = \frac{x-\mu}{\sigma}}$$

$$w = v - \sigma t$$
$$v = \frac{x-\mu}{\sigma}$$

$$w = \frac{x-\mu}{\sigma} - \sigma t$$

$$\frac{e^{\infty^2}}{2(\infty)} - \frac{e^{-(\infty)^2}}{2(-\infty)}$$

$$\frac{e^{\infty^2}}{2(\infty)}$$

$$\frac{\infty}{\infty} = \frac{0}{\infty}$$

$$\int_{-\infty}^{\infty} f(x) \, dx = 1;$$

$$\underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx = 1;}$$

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