28/01/2020

ENGINEERING STATISTICS

Statistics is concerned with abstracting data, classifying it and then comparing it with data obtained from similar sources so that plans and control mechanisms can be implemented.

Tasks of a statistician

1. Measure accurately
2. Arrange problems in quantitative terms.
3. Prepare the ground for logical inference.

Example:

1 2 1 3 3 4 3 3 5 2 4 4 5
1 3 3 4 3 2 2 4

x	tally	frequency	Cummulative
1	III	3	3
2	IIII	4	7
3	~~IIII~~ I	6	13
4	~~IIII~~	5	18
5	II	2	20
		20	

$$\text{Mean} = \frac{\sum fx}{\sum f} = \frac{(1)(3)+(2)(4)+(3)(6)+(4)(5)+(5)(2)}{20}$$

$$= \frac{3 + 8 + 18 + 20 + 10}{20} = \frac{59}{20}$$

Mean $\bar{x} = 2.95$

Mode = 3

$$\text{Median} = \frac{\Sigma f + 1}{2}^{th} = \frac{20 + 1}{2} = \frac{21}{2}$$

$$= 10.5$$

Bearing Design

No of hrs of failure

A	B
120	110
150	160
210	350
260	410
270	430
510	500
870	520
980	530
1140	570
Ave. Life = 476 hrs	Ave. life = 398 hrs

PROBABILITY.

$$\text{Prob} = \frac{\text{No of Successful Occurrence}}{\text{Total no of trials}}$$

PROPERTIES OF PROBABILITY.

① $P(x) \not< 0$

i.e $0 \leq P(x) \leq 1$

② For a certain event i;

$P(x) = 1$

③ For mutually exclusive events.

A, B

$P(A+B) = P(A) + P(B)$

④ For events A and B to be independent

$P(A \text{ and } B) = P(A) \times P(B)$.

Practice question.

0, 1, 2, ... 30

0, 1,

$\Sigma fx / \Sigma f$

Mean $\bar{x} = 0+1+2+3+4+5+6+7+8+9+10+11+12+13+14$
$+15+16+17+18+19+20+21+22+23+24+25$
$+26+27+28+29+30$

$$= \frac{464}{31} = 14.97$$

Mode = 0

$$\text{Median} = \frac{\sum f + 1}{2} \text{th} = \frac{31+1}{2} = 16$$

X	Tally	frequency	cf
0		1	1
1		1	2
2		1	3
3		1	4
4		1	5
5		1	6
6			7
7			8
8			9
9			10
10			11
11			12
12			13
13			14
14			
15			
16			
17			
1'		1	31
20			
30			

CUMMULATIVE DISTRIBUTION FUNCTION (CDF)

F(x) can be defined as

$$F(x \leq a) = \sum_{i=-\infty}^{a} p(x_i)$$

where $x_i \leq a$.

for Continuous System,

$$F(x \leq a) = F(a) = \int_{\infty}^{a} p(x)\, dx$$

$$= \int_{-\infty}^{a} f(x)\, dx = P(x \leq a)$$

eg $P(x \leq 4) = P_1 + P_2 + P_3 + P_4$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 2/3$$

04/02/2020

PROBABILITY DISTRIBUTION

Distribution - means of arrangement

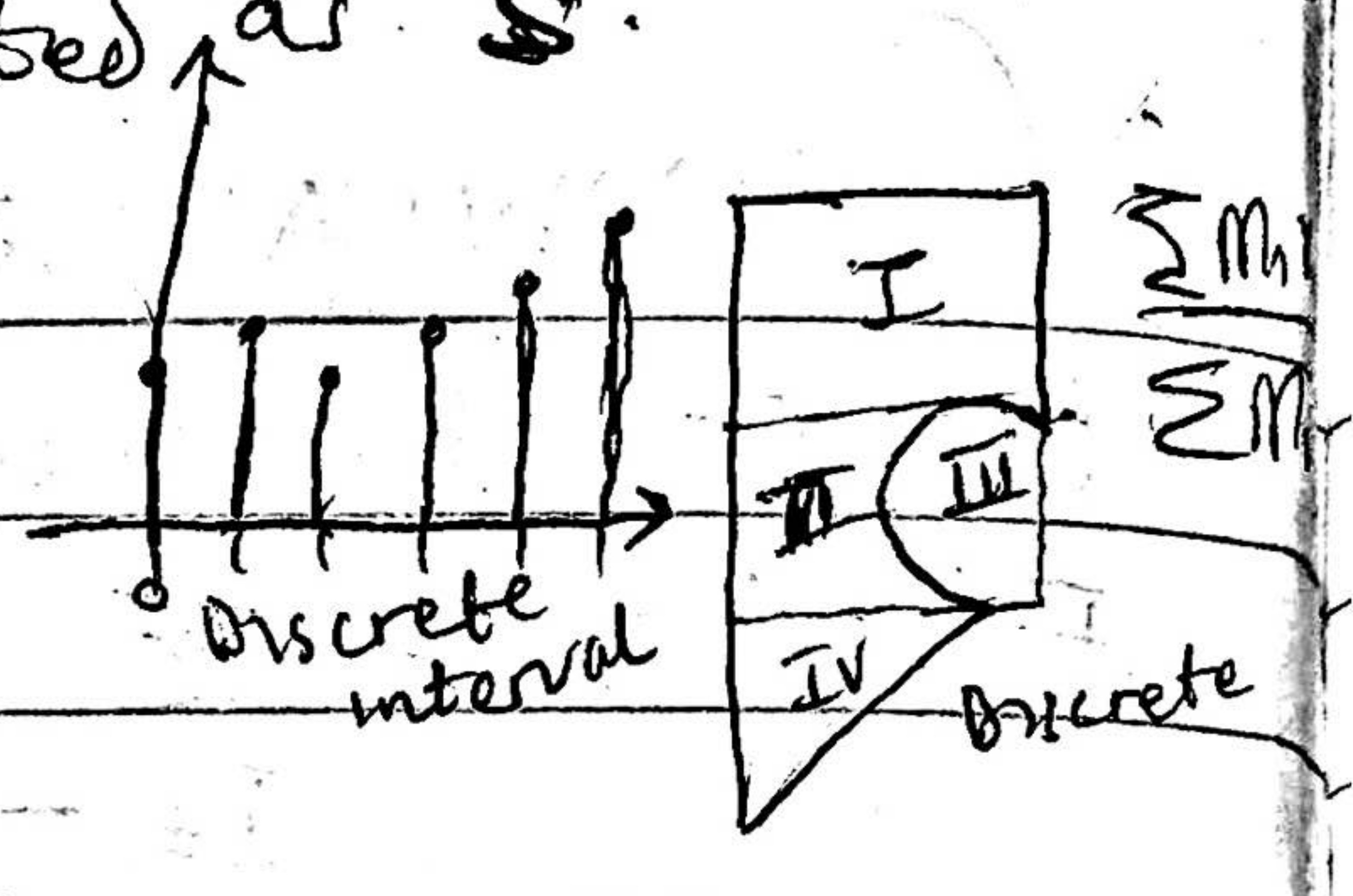
Probability ← uncertainty

Terms like ; - Events

- Outcome
- Random variable
- Trial

1) — Sample space : Set of all the possible outcomes of a random experiment. Denoted as S.

RANDOM VARIABLE :



Random Variable

Discrete RV ($\sum_{i=1}^{N}$)
(i) Expectation = Average mean (Σ)
ii) Moment ~~of~~ skewness / Kurtosis (Σ)
iii) Distribution of RV

Binomial, Poisson, Bernoulli

Continuous RV ($\int_a^b$)
(i) Expectation
(ii) Moment
(iii) Distributions

Normal, Standard Normal, Uniform, Hyper geometric

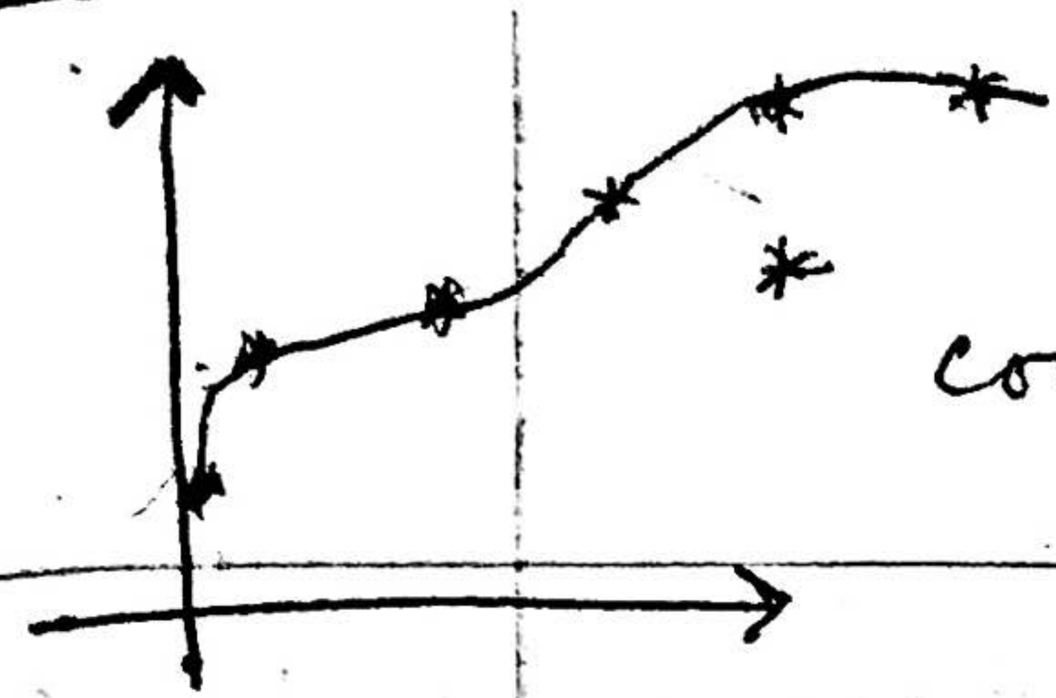
AXIOMS OF PROBABILITY.

(i) $0 \leq P_r(x) \leq 1$

$$P_r(X=4) = \frac{\text{No of A}}{\text{Total}}$$

(ii) $A_1 \quad A_2 \quad A_3$

$P_r(A_1 \cup A_2 \cup A_3 \cup \ldots)$
$= P_r(S) = 1$.



* Download Probability distribution table.

18/02/2020

DISCRETE PROBABILITY DISTRIBUTION

Die {1, 2, 3, 4, 5, 6}.

x	1	2	3	4	5	6
$f(x)$	1/6	1/6	1/6	1/6	1/6	1/6

Probability mass/density function.

$P(X = 1) = \frac{1}{6}$

$$\mu = \sum \frac{fx}{N} = \sum x \cdot (f/N)$$

$P_r = \frac{\text{No. of occurence}}{\text{Total number of outcomes}}$ ~~over~~

DISCRETE PROBABILITY DISTRIBUTION

Let x be a discrete random variable and suppose that the possible values that we can take are;

$x_1, x_2, x_3, x_4, \ldots, x_i$

Given that the probability values are assumed to be uncertain, then we can defined pdf

$P(X = x_i) = f(x_i)$

The distributi

Where $f(x_k)$ $K = 1, 2, \ldots$

The distribution of $f(x_i)$ will therefore have the following properties;

① $f(x) \geq 0$

② $\sum_i f(x) = 1$

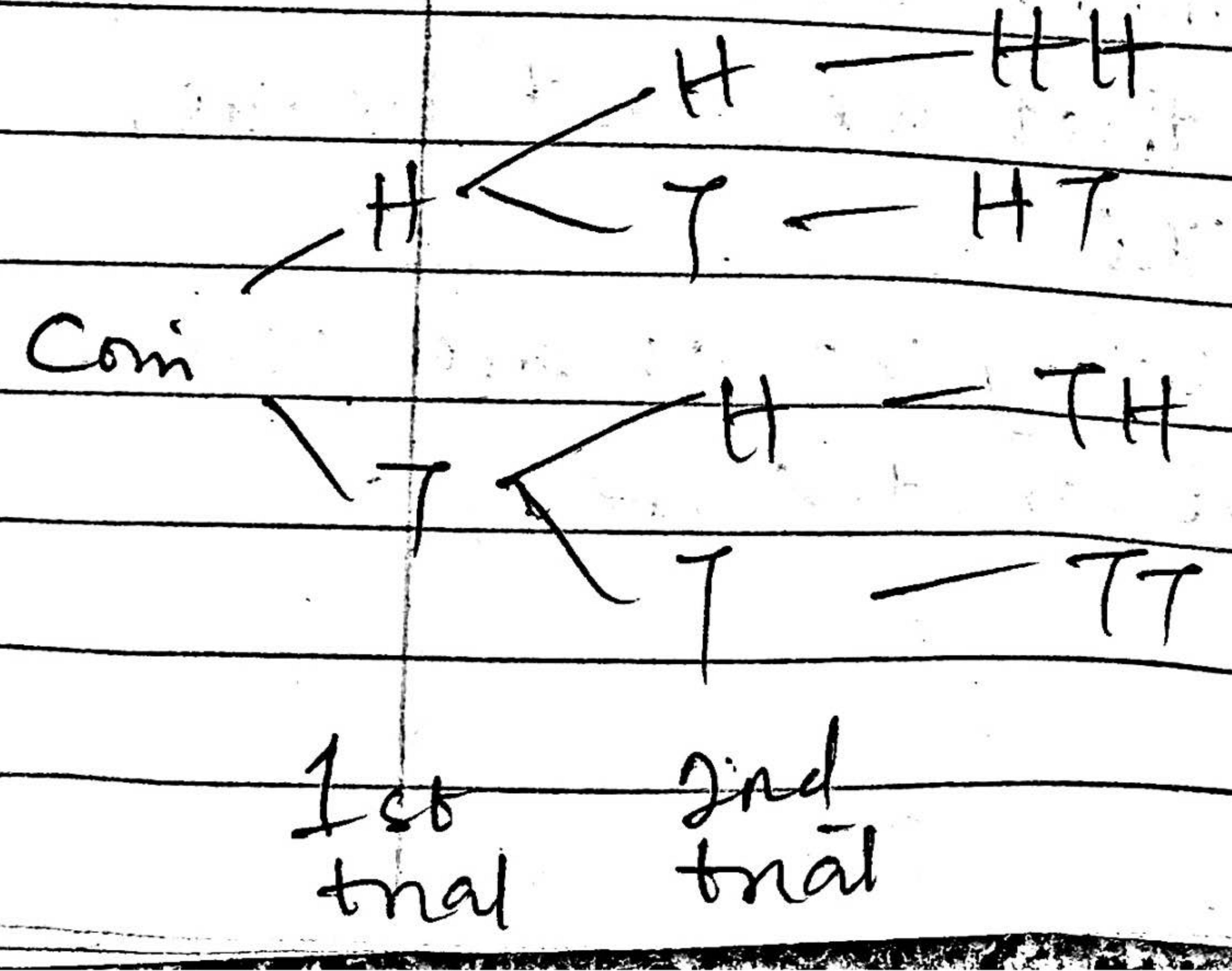
Example 1: Find the probability distribution corresponding to the random experiment described below;

Suppose that a coin is tossed twice and S represents the number of heads that shows

① Develop a sample space in this experiment.

② Develop a probability distribution for the experiment.

Soln.



① $S = \{HH, HT, TH, TT\}$

X – No of Heads

X – $\{0, 1, 2\}$.

X	x_1 0	x_2 1	x_3 2
$f(x)$	$1/4$	$1/2$	$1/4$

$f(x_i) = P(X = x_i)$

$f(x_1) = P(X = x_1) = P(X = 0)$.

$= \frac{1}{4}$

$f(x_2) = P(X = x_2) = P(X = 1)$.

$= P(TH \text{ or } HT)$

$= P(TH) \overset{+}{\vee} Pr(HT)$

$= 1/4 + 1/4 = 2(1/4) = 1/2$

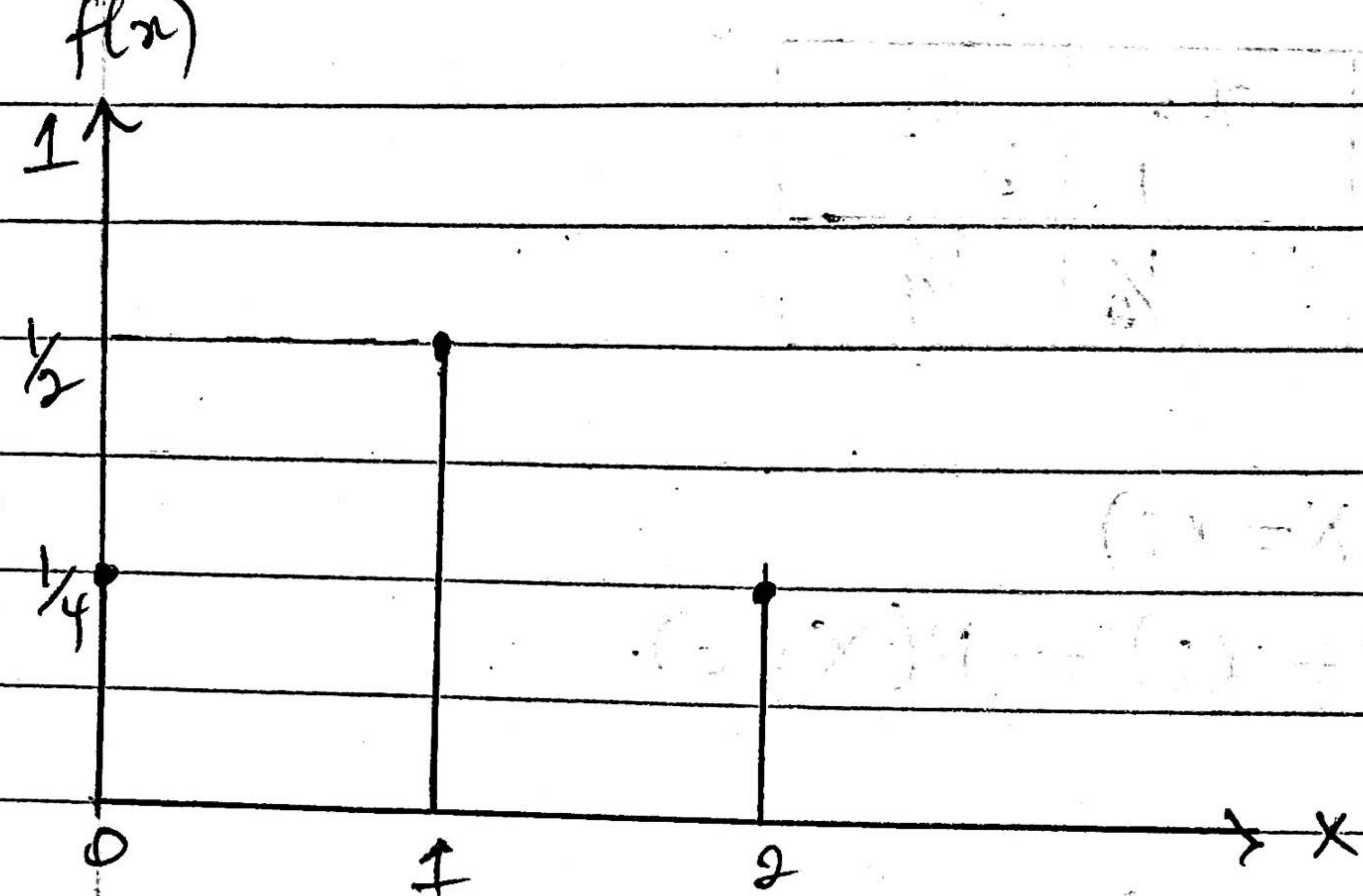
$f(x_3) = P(X = x_3) = P(X = 2)$

$= P(HH) = 1/4$

② For a die;

$S = \{1, 2, 3, 4, 5, 6\}$.

X	1	2	3	4	5	6
$f(x)$	1/6	1/6	1/6	1/6	1/6	1/6



PROBABILITY DISTRIBUTION OF A CUMULATIVE FREQUENCY CURVE:

	f	F	
$-\infty$			
x_1 —	1	1	——
x_2 —	2	$1+2=3$	$-\infty \leq x \leq x_2$
	4	$1+2+4=7$	$-\infty \leq x \leq x_3$
	6		
$+\infty$ —	—	— — —	— — — $-\infty \leq x \leq +\infty$

X	$f(x)$	$F(x)$
1	$1/6$	$1/6$
2	$1/6$	$1/6 + 1/6 = 2(1/6) = 1/3$
3	$1/6$	$2(1/6) + 1/6 = 3/6 = 1/2$
4	$1/6$	$2/6 + 1/6 = 4/6 = 2/3$
5	$1/6$	$4/6 + 1/6 = 5/6$
6	$1/6$	$5/6 + 1/6 = 6/6 = 1$

$$\left\{ \sum_{x_i} F(x) = 1 \right\}$$ Cummulative discrete probability distribution.

X	$f(x)$	$F(x)$
0	$1/4$	$1/4$
1	$2(1/4) = 1/2$	
2		

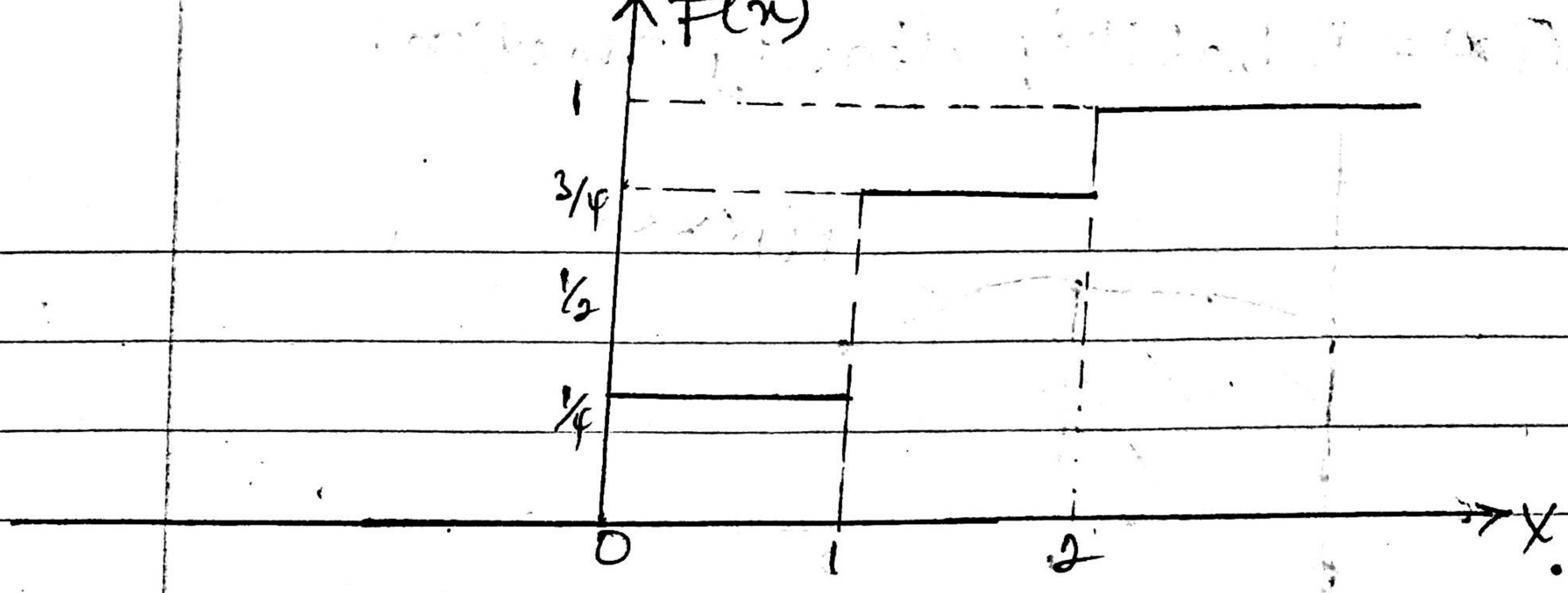
The Cummulative distribution function for a discrete random variable X can be defined as

$F(x) = P(X \leq x); \quad -\infty \leq x \leq \infty$

Where x is any real number btw $-\infty$ and $+\infty$

The distribution function $f(x)$ has the following properties;

(1) $F(x)$ is non-decreasing i.e



~~CONTINUOUS~~ CONTINUOUS RANDOM VARIABLE

A non-discrete random variable X is said to continuous if the distribution function can be represented as;

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(v)\,dv.$$

$(-\infty < x < \infty)$

where;

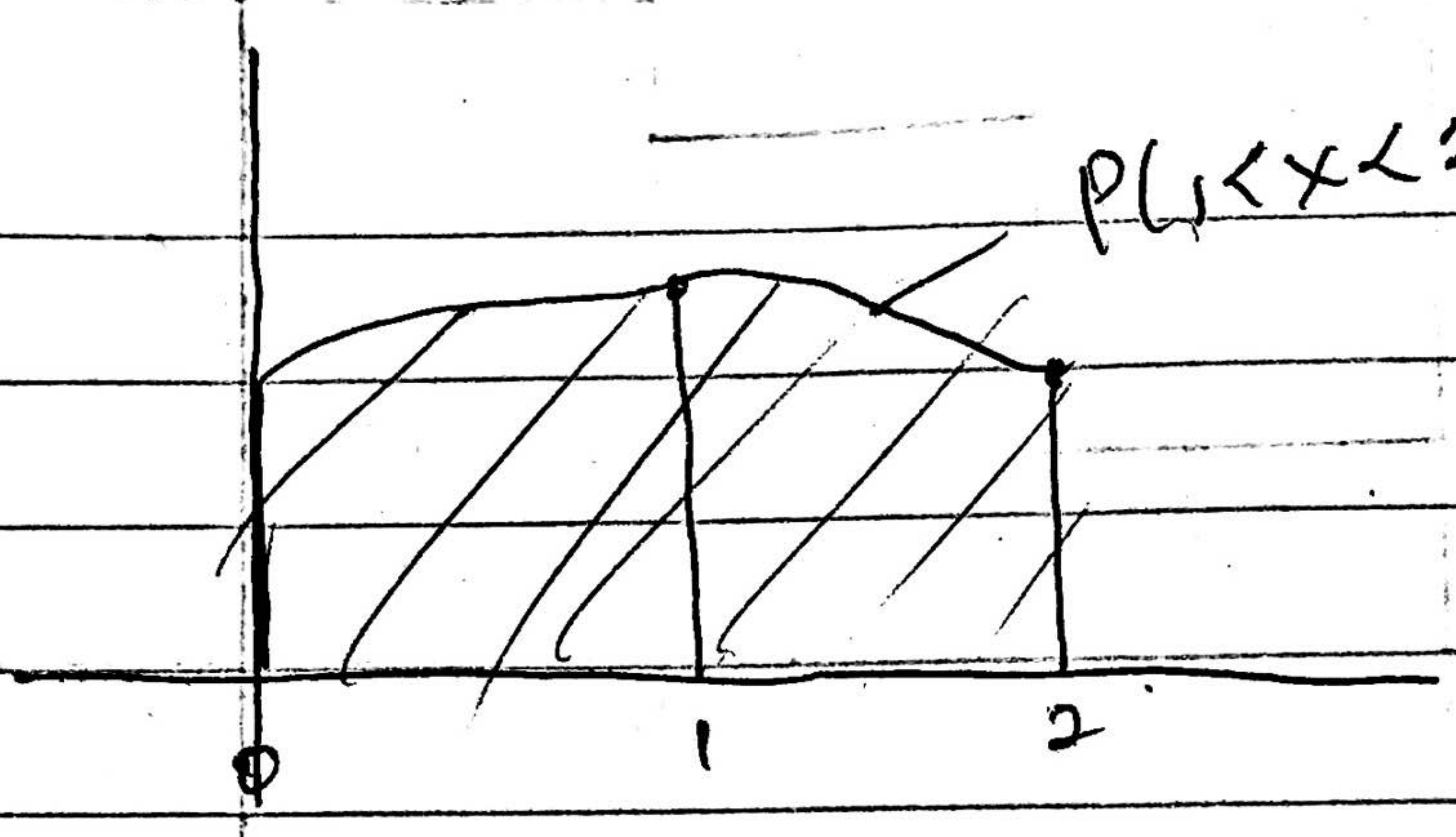
1. $f(v) \geqslant 0$
2. $\int_a^b f(v)\,dv = 1$, $I\,[a, b]$

→ Probability; $P(a < X < b)$.

Given the distribution below that;

$$f(x) = \begin{cases} cx^2, & 0 < x < 3 \\ 0, & \text{otherwise} \cdot \begin{Bmatrix} -\infty < x < 0 \\ 3 < x < \infty \end{Bmatrix} \end{cases}$$

$f(x)$ = Probability density function.



Find C ?

Soln

$$\int_{-\infty}^{\infty} f(v)\,dv = 1$$

$$\int_{-\infty}^{0} f(v)\,dv + \int_{0}^{3} f(v)\,dv + \int_{3}^{\infty} f(v)\,dv = 1$$

$$\cancel{\int_{-\infty}^{0} 0\,dv}^{\,0} + \int_{0}^{3} c x^2\,dx + \cancel{\int_{3}^{\infty} 0\,dv}^{\,0} = 1$$

$$c \cdot \frac{x^3}{3}\Big|_0^3 = 1$$

$$\frac{c}{3}\left(3^3 - 0^3\right) = 1$$

$$9c = 1$$

$$\therefore c = \frac{1}{9}$$

$$f(x) = \begin{cases} \frac{1}{9}x^2, & 0<x<3 \\ 0, & \text{otherwise} \end{cases} = \int_4^6 f(x)\,dx = \int 0\,dx \quad \text{——} \quad (1)$$

$P(1 < x < 0)$

$P(4 < x < 6)$

$$P(a < x < b) = \int_a^b f(x)\,dx\,;$$

(ii) $P(1 < x < 2) = \int_1^2 f(v)\,dv$

$$= \int_1^2 \tfrac{1}{9} x^2\,dx = \frac{x^3}{27}\Big|_1^2 = \frac{1}{27}(2^3 - 1^3)$$

$$= \frac{1}{27}(8-1) = \tfrac{7}{27}\,;$$

Find the cummulative distribution for the example above.

$P(0 \leq$

$$F(x) = \begin{cases} 0 & , x < 0 \\ x^3/27 & , 0 \leq x < 3 \\ 1 , & , x \leq 3 \end{cases}$$

$\boxed{0 \leq x \leq 3} \neq [-\infty \leq x \leq +\infty]$

$$P(0 \leq x < 3) = \int_0^{x\text{ - less than 3}} \frac{x^2}{9}\,dx = \frac{x^3}{27} \left[\text{i.e. } \frac{1}{9} \cdot \frac{x^3}{3}\Big|_0^x = \frac{x^3}{27}\right]$$

Example: $P(1 < X \leq 2)$.

$= P(1 < X < 2) + P(X = 2)$.

$P(1 < X \leq 2) = F(2) - F(1)$

$F(x) = P(X \leq x)$

$F(2) = F(x|_{x=2}) = 2^3/27 ; = \frac{8}{27}$

$F(1) = f(x|_{x=1}) = \frac{1^3}{27} = 1/27$

$\therefore P(1 < x < 2) = P(1 \leq X \leq 2) = \frac{8}{27} - \frac{1}{27} = 7/27$

$F(x)$ — CRV $\qquad f(x)$ — DRV

$F(x) = P(a < X < b) = \int_a^b f(x)\,dx$

$F(x) = \int_a^b f(x)\,dx$

By differentiating both sides;

$\frac{d}{dx} F(x) = \frac{d}{dx}\int_a^b f(x)\,dx$

$= \int_a^b \cancel{\partial}/\cancel{\partial x}\, f(x)\,\cancel{dx}$

EXPECTATION OF A RANDOM VARIABLE

Expectation of A discrete random variable

$\mu = \sum \frac{f \cdot x}{N} = \sum \left(f/_N\right) x = \sum P_x x$

x	1	2	3
f	2	6	4

$N = \sum f = 12$

$p = \frac{2}{12} : \frac{6}{12} : 4/12$

Variance, σ =

For a coin; $S\{H, T\}$

X — No of Head.

x	0	1
f		

For a die; $\{1, 2, 3, 4, 5, 6\}$

x	1	2	3	4	5	6
f	1	1	1	1	1	1
$P(x) = f/N$	1/6	1/6	1/6	1/6	1/6	1/6

$\sum f = 6$

$\sum fx = 21$

$$\bar{x} = \frac{\sum fx}{\sum f = N} = \frac{1}{N}\sum fx = \sum (f/N)\, x$$

$\mu = 21/6$

$$\mu = E(X) = \sum_{i=1}^{6} P(x) \cdot x_i$$

$$= p(x_1) \cdot x_1 + p(x_2)\, x_2 + p(x_3) \cdot x_3 + \ldots + p(x_6) \cdot x_6$$

$$= \left(\frac{1}{6}\right) \cdot 1 + \left(\frac{1}{6}\right) \cdot 2 + \left(\frac{1}{6}\right) \cdot 3 + \left(\frac{1}{6}\right) \cdot 4 + \left(\frac{1}{6}\right) \cdot 5 + \left(\frac{1}{6}\right) \cdot 6 = \frac{1+2+3+4+5+6}{6}$$

$$= \frac{21}{6}$$

$$\boxed{\begin{array}{l} \text{Average mean} = \text{Expectation} \\ E(X) = \sum_{i=0}^{N} p(x_i) \cdot x_i \end{array}}$$

|02|2026

EXPECTATION OF CONTINUOUS RV.

Average / Mean $\mu = \sum x \frac{F}{N} = \sum x \cdot P(x)$

where $p(x) = \frac{f}{N}$

The expectation of a discrete RV

$E(x) = \sum x_i \, p(x_i)$.

where $p(x_i)$ is the probability mass/density function;

where as;

$$\sigma^2 = \frac{\sum f(x) - \frac{\sum f(x^2)}{N}}{N}$$

$$\sigma^2 = \sum \left(\frac{f}{N} \cdot x\right)^2 - \frac{\sum f(x^2)}{N^2}$$

Hence, the variance is defined as;

$$\sigma^2 = \sum \frac{f(x-\mu)^2}{N}$$

$$\sigma^2 = \sum \frac{F}{N} (x-\mu)^2$$

$$\sigma^2 = \sum (x-\mu)^2 \, P(x)$$

$$\sigma^2 = E(x-\mu)^2 = \sum (x-\mu)^2 \, p(x)$$

Properties of Expectation

① $E(c) = c$

expectation of any constant is that constant.

② $E(1) = 1$ ③ $E(cX) = c\,E(X)$

④ $E(cX + uY)$
$= E(cX) + E(uY)$
$= cE(X) + uE(Y)$.

EXPECTATION OF CONTINUOUS RV & DISCRETE R

Average ; mean $\mu = \sum x \frac{f}{N} = \sum x . p(x)$

where $p(x) = f/N$.

The expectation of a discrete RV.

$$E(X) = \sum x_i\, p(x_i)$$

where $p(x_i)$ is the probability mass/density function where as

$$\sigma^2 = \sum f(x)^2 - \frac{\sum f(x^2)}{N^2}$$

Hence, the variance is defined as;

$$\sigma^2 = \sum \frac{f(x-\mu)^2}{N}$$

$$\sigma^2 = \sum f/N\,(x-\mu)^2$$

$\sigma^2 = \sum (x-\mu)^2 P(x)$

$\sigma^2 = E(x-\mu)^2 = \sum (x-\mu)^2 P(x)$.

Properties of Expectation :

① $E(c) = c$

expectation of any constant is that constant.

② $E(1) = 1$

③ $E(cX) = c\,E(X)$

④ $E(cX + uY)$

$= E(cX) + E(uY)$

$= cE(X) + uE(Y)$

From, $\sigma^2 = E(x-\mu)^2 = \sum (x-\mu)^2 \cdot P(x)$.

$(x^2 - 2\mu x + \mu^2)$

$= \sum x_i\, P(x) = 2\mu x_i\, p(x_i) + \mu^2 p(x_i)$.

$= \sum x_i^2 p(x_i) - \sum 2\mu x_i P(x_i) + \sum \mu^2 P(x_i)$

$= \sum x_i^2 p(x_i) - 2\mu \sum x_i P(x_i) + \mu^2 \sum p(x_i)$

$\mu = E(X)$

$E(X^2) = \sum x_i^2 P(x_i)$

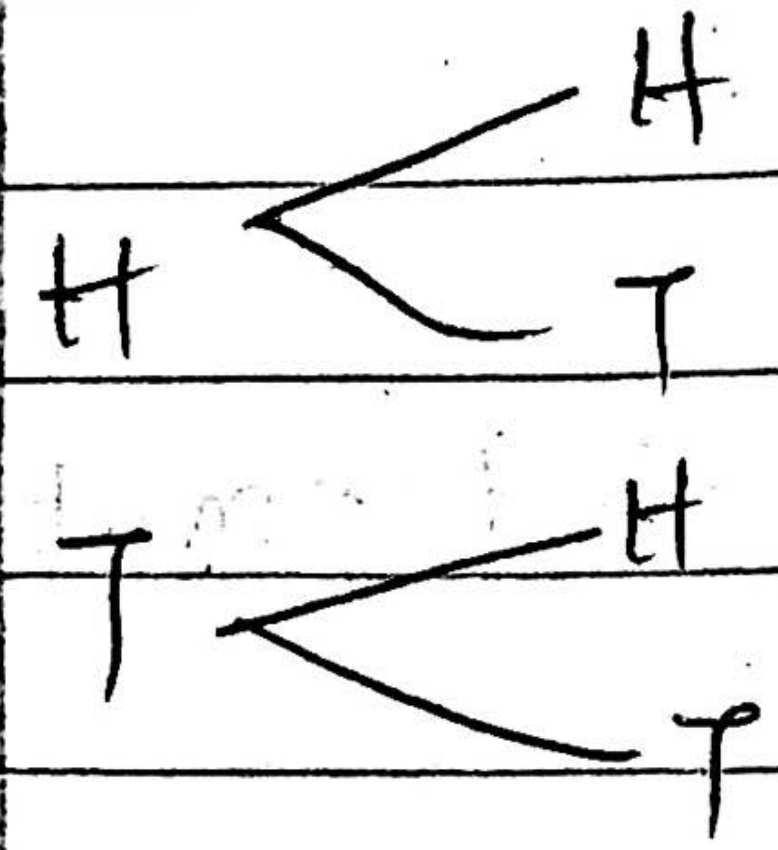
$= E(X^2) - 2\mu \cdot E(X) + \mu^2 E(1)$

$= E(X^2) - 2E(X) \cdot E(X) + (E(X))^2 \cdot 1$

$= E(x^2) - 2(E(x))^2 + (E(x))^2$

$\sigma^2 = E(x-\mu)^2 = E(x^2) - (E(x))^2$

Example:



$\mu = E(x) = \sum x\, p(x)$

$= x_1 p(x_1) + x_2 p(x_2) + x_3 p(x_3)$

$= 0(1/4) + 1(1/2) + 2(1/4)$

$= \frac{1}{2} + \frac{1}{2} = 1.$

$S = \{HH, HT, TH, TT\}.$ $E(x) = \mu = 1$

X = random variable = No of heads

$\sigma^2 = \sum (x-\mu)^2 p(x) =$

x	0	1	2
$P(x)$	1/4	1/2	1/4

$P(X=x) = P(X=0) = 1/4$

$P(X=1) = Pr(HT \text{ or } TH) = Pr(HT) \vee Pr(TH)$

$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$

$P(X=2) = 1/4$

25/02/2020

JOINT PROBABILITY DISTRIBUTION

If x and y are two ~~variable~~ discrete random variables, we ~~can~~ therefore define the joint probability distribution of the following as;

$$P(X=x, Y=y) = f(x,y)$$

where $f(x,y$ is the joint probability mass/density function.

The joint probability density function will therefore have the following properties;

① $f(x,y) \geq 0$ $\qquad [f(x) \geq 0]$

② $\sum_{x_i=0} \sum_{y_j=0}^{m} f(x_i, y_j) = 1$ $\qquad [\sum f(x) = 1]$

Suppose ~~that~~ x can assume any one of m discrete values (i.e $x_1, x_2, x_3, \ldots x_m$)

and y can assume any one of n discrete values (i.e $y_1, y_2, y_3, \ldots y_n$).

The Probability ~~that~~ of the event that $X = x_j$, $Y = y_k$ is given as;

$$P(X=x_j, Y=y_k) = f(x_j, y_k)$$

Hence, the joint probability function for X & Y ~~can~~ are be represented by ~~the~~ joint probability table that would be shown below;

The probability that $X = x_j$ is obtained by adding all entries to the rows corresponding to x_i and that is given by;

$$P(X = x_j) = f_1(x_j) = \sum_{x=1}^{t} f(x_j, y_k)$$

$X \backslash Y$	y_i	y_2	~~y_3~~ y_3	y_n	Totals
x_1	$f(x_1, y_1)$	$f(x_1, y_2)$	$f(x_1, y_3)$	$f(x_1, y_n)$	$f_1(x_1) = \sum_{x=1} f(x_1, y_j)$
x_2	$f(x_2, y_1)$	$f(x_2, y_2)$	$f(x_2, y_3)$ ...	$f(x_2, y_n)$	$f_2(x_2) = \sum_{x=2} f_1(x_2, y_j)$
⋮					⋮
x_m	$f(x_m, y_1)$	$f(x_m, y_2)$	$f(x_m, y_3)$ ...	$f(x_m, y_n)$	$f_m(x_n) = \sum_{x=m} f(x_m, y_j)$
Totals →	$f_2(y_1)$	$f_2(y_2)$	$f_2(y_3)$ ..	$f_2(y_n)$	1 ← Grand total

where $f_1(x_1) = \sum_{x=1} f(x_i, y_i)$ $\qquad f_2(y_1) = \sum_{y=1} f(x_i\, y_i)$

$f_1(x_2) = \sum_{x=2} f(x_2, y_j)$ $\qquad f_2(y_2) = \sum_{y=2} f(x_i, y_2)$

$f_1(x_m) = \sum_{x=m} f(x_m, y_j)$ $\qquad f_2(y_n) = \sum_{y=n} f(x_i, y_n)$

The totality of the table gives:

$$\sum_{x} \sum f(x, y) = 1$$

Because the probabilities of $P(X=x_i)$ and $P(Y=y_j)$ are obtained from the margins of the table, they are therefore called the marginal probability function of X and Y respectively i.e $P(X=x_i)$ and $P(Y=y_i)$.

Note that

$$\sum_{j=1}^{m} f_1(x_j) = f_1(x_1) + f_1(x_2) + f_1(x_3) + \dots$$

$$= 1.$$

This is the summation over the first rows.

$$\sum_{K=1}^{n} f_2(y_K) = f_2(y_1) + f_2(y_2) + f_2(y_3) + \dots$$

$$= 1.$$

This is the summation of

$$\sum_{x=1}^{m} \sum_{y=1}^{n} f(x,y) = 1$$

E.g

The joint probability distribution of two random variables X and Y is given by $f(x,y) = c(2x+y)$ where x and y can assume all integers such as ; $0 \leq x \leq 2$, $0 \leq y < 3$ and $f(x,y) = 0$ otherwise;

$$f(x,y) \begin{cases} c(2x+y) & 0 \leq x \leq 2 \\ & 0 \leq y \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

① Find the value of constant C.
② $P(X=2, Y=1$
③ $P(X \geq 1, Y \leq 2)$
④ What is the marginal probability.

X \ Y	0	1	2	3	
	0	c	2c	3c	$f_1(x_1) = 6c$
	2c	3c	4c	5c	$f_1(x_2) = 14c$
	4c	5c	6c	7c	$f_1(x_3) = 22c$

$f_2(y_1) = 6c$ $f_2(y_2) = 9c$ $f_2(y_3) = 12c$ $f_2(y_4) = 15c$

$f(x_1, y_1)$

$P(X=0, Y=0) = f(0,0) = C(2.0+0) = 0$

$P(X=0, Y=1) = f(0,1) = c(2.0+1) = c$

$P(X=0, Y=2) = f(0,2) = c(2.0+2) = 2c$

$P(X=0, Y=3) = f(x_1, y_4) = f(0,3) = C(2\times 0+3) = 3c$

$$\sum_x \sum_y f(x,y) = \sum_{j=1}^{m} f_1(x_j) = \sum_{k=1}^{n} f_2(y_k) \ldots$$
$$= 1$$

$$\sum_{x=0}^{2} \sum_{y=0}^{3} = f(0,0) + f(0,1) + f(0,2) + (0.3) + f(1,0) + f(1,1) + f(1,2) + f(1,3) + f(2,0) + f(2,1) + f(2,2) + f(2,2) \ldots$$
$$= 0 + c + 2c + 3c + 2c + 3c + 4c + 5c + 4c + 5c + 6c + 7c = 1$$

$42c = 1$

$c = 1/42$ // OR

a₁) $\sum_{j=0}^{2} f_1(x_j) = f_1(x_1) + f_1(x_2) + f_1(x_3)$

$= 6c + 14c + 22c = 1$

$42c = 1$

$c = \frac{1}{42}$

$\sum_{k=0}^{3} f_2(y_k) = f_2(y_1) + f_2(y_2) + f_2(y_3) + f_1(y_4)$

$= 6c + 9c + 12c + 15c = 1$

$47c = 1$

$c = 1/42$ //

Hence, we can rewrite the table replacing the value of $c = 1/42$

$x \backslash y$	0	1	2	3	
0	0	$c = 1/42$	$2c = 1/21$	$1/14$	$f_1(x_1) = 1/7$
1	$1/21$	$1/14$	$2/21$	$5/42$	$f_1(x_2) = 14/42$
2	$2/21$	$5/42$	$1/7$	$1/6$	$f_1(x_3) = 22/42$
	$1/7$	$9/42$	$\frac{12}{42}$	$15/42$	

② $P(X=2, Y=1) = f(x_3, y_2)$

$= f(x_3=2, y_2=1)$

where $f(x,y) = c(2x+y) = \frac{2x+y}{42}$

$f(x,y) = c(2x+y) = \frac{2x+y}{42}$

$f(x_3=2, y_2=1) = \frac{2(2)+1}{42} = 5/42$

which can easily be picked from the table (Row 3, Column 2).

③ $P(X \geq 1, Y \leq 2)$

i.e $\sum_{x=0}^{2} \sum_{y=0}^{2} = f(1,0) + f(1,1) + f(1,2) + f(2,0)$

$+ f(2,1) + f(2,2)$

We can pick it from the table.

$= 1/21 + 1/14 + 2/21 + 2/21 + 5/42$

$= 18/42$.

The marginal probability of x and y are;

$P(X=x) = f_1(x_j)$ → Marginal probability distribution for x.

$$= \begin{cases} 6c & ; x=0 \\ 14c & x=1 \\ 22c & x=2 \end{cases}$$

$P(Y=y) = f_2(y_k)$ — Marginal probability distribution for Y.

$$\begin{cases} 6c & ; y=0 \\ 9c & ; y=1 \\ 12c & ; y=2 \\ 13c & ; y=3 \end{cases}$$

Continuous Random Variable.

The joint probability distribution of a continuous random variable can be derived by analogy with the discrete random variable, such that the following properties would hold.

① $f(x,y) \geq 0$ $\quad [f(x) \geq 0]$

② $\displaystyle\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} f(x,y)\,dx\,dy = 1$ $\quad [\sum_x f(x) = 1]$

$\left[DRV \rightarrow \sum_{x=1}^{m}\sum_{y=1}^{n} f(x,y) = 1 \right.$

Integrate the internal integral first].

$P(a<X<b, c<Y<d)$

$$= \int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dx\,dy$$

So; the joint distribution function of a continuous random variable,

$$P(X \leq x, Y \leq y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dx\,dy$$

$\downarrow$ $\qquad$ $\downarrow$

$1[-\infty, x]$ $\quad$ $1[-\infty, y]$

Hence, the Marginal joint distribution function can be expressed as marginal joint distribution ~~for~~ fn for x.

$$P(X \le x) = F_1(x)$$
$$= \int_{-\infty}^{x} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy.$$

Marginal joint dist fn for y.

$$P(Y \le y) = F_2(y).$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{y} f(x, y)\, dx\, dy.$$

Example:

The joint density function for two continuous RV ~~x~~ X and Y is given as.

$$f(x,y) = \begin{cases} cxy & 0 < x < 4,\ 1 < y < 5 \\ 0 & \text{otherwise} \end{cases}$$

solution

$$\begin{cases} -\infty < x < 0 & 4 < x < +\infty \\ -\infty < y < 1, & 5 < y < +\infty \end{cases}$$

① Find the value of constant c

② Find the probability of $1 < x < 2,\ 2 < y < 3)$

ie $P(1 < x < 2,\ 2 < Y < 3)$

ⓒ $P(X \geq 3,\ Y \leq 2)$

~~Marginal dif~~ ~~Marginal diff~~

Soln

① Recall for discrete $= \sum_{x} \sum_{y} f(x,y) = \sum_{x} f_1(x_j) = \sum_{y} f_2(y_n) = 1$

Marginal diff. ↑ Marginal diff ↑

for continues:

$$\int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} f(x,y)\,dx\,dy = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_1(x_j)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{y} f_2(y_i)\,dx\,dy = 1$$

$$\int_{x=0}^{4} \int_{y=1}^{5} c\,xy\,dy\,dx = 1 \longrightarrow \int$$

Handling the linear Integral, we have:

$$\int_{y=1}^{5} Cxy\,dy = Cx \left.\frac{y^2}{2}\right|_{y=1}^{5} = \frac{Cx}{2}(5^2 - 1^2) = \frac{Cx}{2} \cdot 24$$

$$= 12Cx.$$

Integral

$$\int_{x=0}^{4} 12Cx\,dx = 12C \left.\frac{x^2}{2}\right|_{0}^{4} = 6C(4)^2 = 96C$$

$$\therefore \int_{x=1}^{4} \int_{y=1}^{5} f(x,y)\,dy\,dx = 1$$

i.e $96C = 1$, $C = 1/96$

Hence; $f(x,y) = \begin{cases} xy/96 & 0<x<4, \ 1<y<5 \\ 0 & \text{otherwise.} \end{cases}$

ⓐ $P(1<X<2, \ 2<Y<3)$

$$= \int_{x=1}^{2} \int_{y=2}^{3} xy/96 \, dy \, dx$$

$$= \int_{x=1}^{2} 5x/192 \, dx = \frac{5}{192}\left[x^2/2\right]_1^2$$

$$= 5/192\left[4/2 - 1/2\right] = 15/192\times2$$

ⓒ $P(X \geq 3, \ Y \leq 3)$

$$= \int_{x=3}^{4} \int_{y=1}^{2} xy/96 \, dy \, dx.$$

For the marginal joint distribution.

$$P(X \leq x) = \int_{x=-\infty}^{x} \int_{-\infty}^{\infty} f(x,y) \, dy \, dx.$$

First replace with a dummy variable $x=u$, $y=v$

$$= \int_{u=0}^{x} \int_{v=1}^{5} f(u,v) \, dv \, du.$$

$$f(u,v) = f(x,y)\Big|_{\substack{x=u \\ y=v}} = uv/96$$

$$\int_{v=1}^{5} \frac{uv}{96}\,dv = \frac{u}{96}\frac{v^2}{2}\Big|_{v=1}^{5} = \frac{u}{192}\left[5^2-1^2\right] = \frac{24u}{192} = \frac{u}{8}$$

Outer Integral

$$\int_{u=0}^{x} u/8\,du = \frac{u^2}{16}\Big|_{u=0}^{x} = \frac{x^2}{16} - \frac{0^2}{16} = \frac{x^2}{16}$$

$$P(X \leq x) = F_1(x) = \frac{x^2}{16}$$

$$f_1(x) = \begin{cases} 0 \; ; & x < 0 \\ x^2/16 & 0 < x \leq 4 \\ 0 \; ; & x \geq 4 \end{cases}$$

For marginal joint distribution of Y

$$P = (Y \leq y) = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{y} f(x,y)\,dy\,dx = \int_{u=-\infty}^{\infty=4}\int_{v=\infty}^{y} f(u,v)\,dv\,du$$

$$= \int_{u=0}^{4}\int_{v=1}^{y} uv/96\,dv\,du = \frac{y^2-1}{24}$$

From Last Class

Expectation = average mean $\qquad P(x)$

$E(x) = \sum x \cdot p(x) \qquad = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$

$\mu = \sum x \sum f/N = \sum x \cdot \sum f/N = \sum x \cdot p(x)$

hence, $P(x) = \sum f/N$.

$P(x)$ is ~~as~~ discrete function as $f(x)$ is to continuous function.

$\sigma^2 = E(x-\mu)^2 = E(x^2) - [E(x)]^2$

$E(x^2) = \sum x^2 p(x) \longrightarrow \int_{-\infty}^{\infty} x^2 P(x)\, dx$

<u>Moments In terms of expectation:</u>

Moments $\begin{cases} r\text{th} \longrightarrow \text{expectation} \\ \text{central} \end{cases}$

The ~~rate~~ rth moment of a RV x about the mean is also Known as the rth central moment and is defined as;

$$\mu_r = E(x-\mu)^r \text{ where } r = 0, 1, 2, \dots, n$$

for $r = 0$, the zeroth Central moments ~~is~~ μ_0 is

$\mu_0 = E(x-\mu)^0 = E(1) = 1$

for $r = 1$, first central moment μ_1

$\mu_1 = E(x-\mu)^1$ $\quad [E(x+Y] = E(x) + E(Y)]$

Linearity

$\mu_1 = E(x) - E(\mu)$

$= E(x) - \mu E(1) = \mu - \mu \cdot 1 = 0$

$\mu_1 = 0$

for $r = 2$ second central moment μ_2 is

$\mu_2 = E(x-\mu)^2 = $ Variance

$= E[x^2] - [E(x)]^2$

$= E(x^2) - \mu^2$ $\quad [E(x^2) = \mu_2 + \mu^2]$

For $r = 3$

$\mu_3 = E(x-\mu)^3$

$= E(x^3 - 3x^2\mu + 3x\mu^2 - \mu^3)$

$= E(x^3) - 3\mu E(x^2) + 3\mu^2 E(x) - \mu^3 E(1)$

$\mu_3 = E(x^3) - 3\mu(\mu_2 + \mu^2) + 3\mu^2 \cdot \mu - \mu^3$

gotten from $r=2$ central moment whose $\mu_2 = E(x^2) - \mu^2$

$\mu_1 = E(x^3) - 3\mu\mu_2 - 3\mu^3 + 3\mu^3 - \mu^3$

$\mu_3 = E(x^3) - 3\mu\mu_2 - \mu^3$

26/02/2020

Moments of ~~Generating Functions~~ In terms of Expectation cont'd

$\mu_r = E[(x-\mu)^r]$ — rth central moment

Central = Average = μ.

$r=0, \quad \mu_0 = 1$

$r=1 \quad \mu_1 = 0;$

$r=2, \quad \mu_2 = \sigma^2 = E(x^2) - \mu^2$

discrete RV

$\mu_r = E([X-\mu]^r) = \sum_x (x-\mu)^r f(x)$

Continuous RV

$\mu_r = E((X-\mu)^r) = \int_{-\infty}^{\infty} (x-\mu)^r f(x);$

rth moment about origin

(standardized Normal distribution).

$\mu = 0$

$\mu_r' = E(X^r)$ {the rth central moment abo the origin).

The rth central moment in discrete sense is equivalent to

$\rightarrow \mu_r = \sum (x-\mu)^r f(x)$

$(x-\mu)^r$ ——— expansion

$(x-\mu)^r = (x+(-\mu))^r \qquad (x+y)^n$

We can expand this by using our pascal triangle or binomial expansion;

$${}^rC_0 x^r(-\mu)^0 + {}^rC_1 x^{r-1}(-\mu)^1 + {}^rC_2 x^{r-2}(-\mu)^2 + {}^rC_3 x^{r-3}(-\mu)^3 + \dots + {}^rC_r x^{r-r}(-\mu)^r$$

$$= x^r - \binom{r}{1} x^{r-1}\mu + \binom{r}{2} x^{r-2}\mu^2 - \binom{r}{3} x^{r-3}\mu^3 + \dots + (-1)^r \mu^r$$

$$\therefore \mu_r = \underbrace{\sum x^r f(x)}_{E(X^r) = \mu_r'} - \binom{r}{1}\mu \sum x^{r-1} f(x) + \binom{r}{2}\mu^2 \sum x^{r-2} f(x) + \dots + (-1)^r \mu^r \sum 1 \cdot f(x).$$

When $r=0$, $\mu_0 = \sum x^0 f(x) = \sum 1 \cdot f(x) = E(1) = 1$

$r=1$; $\mu_1 = \sum x f(x) - \binom{1}{1}\mu \sum x^0 f(x)$

$$= \sum x f(x) - \mu \sum 1 f(x).$$

$\mu_1 = E(X) - \mu \cdot 1 = E(x) - \mu = 1$

$r=2$; $\mu_2 = E(X^2) - \mu^2$; $\mu_r' = E(X^r)$;

$E(X^2) = \mu_2'$;

$$\boxed{\mu_2 = \mu_2' - \mu^2}$$

$r=3$, $\mu_3 = \sum x^3 f(x) - \binom{3}{1}\mu \sum x^2 f(x) + \binom{3}{2}\mu^2$
$\sum x f(x) - \binom{3}{3}\mu^3 \sum f(x)$.

$= E(x^3) - 3\mu E(x^2) + 3\mu^2 E(x) - \mu^3$

$\therefore \mu_3 = \mu_3' - 3\mu \cdot \mu_2' + 3\mu^2 \cdot \mu - \mu^3$

$\mu^3 = \mu_3 - 3\mu \mu_2' + 2\mu^3$.

$r=4$, $\mu_4 = \sum x^4 f(x) - \binom{4}{1}\mu \sum x^3 f(x)$
$+ \binom{4}{2}\mu^2 \sum x^2 f(x) - \binom{4}{3}\mu^3 \sum x f(x) + \binom{4}{4}$
$\mu^4 \sum 1 f(x)$.

$\mu_4 = E(x^4) - 4\mu E(x^3) + 6\mu^2 E(x)^2 - 4\mu^3 E(x) + \mu^4$
$\mu_4 = \mu_4' - 4\mu \mu_3' + 6\mu^2 \mu_2' - 4\mu^3 \cdot \mu + \mu^4$
$\therefore \mu_4 = \mu_4' - 4\mu \mu_3' + 6\mu^2 \mu_2' - 3\mu^4$,

Moment Generating Functions.

The Moment generating function of a random variable x is defined as ;

$$M_x(t) = E(e^{tx})$$

We can therefore define this moment in terms of discrete & continuous RV.

In terms of discrete RV

$$M_x(t) = E(e^{tx}) = \sum_x e^{tx} f(x)$$

In terms of continuous RV :

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx.$$

Using Taylor's series expansion ;

$$f(x + \cancel{x}) = f(a) + \cancel{x} f'(a) + \cancel{x}^2 \frac{f''(a)}{2!}$$

$$f(x) = e^{tx}, \; f'(x) = te^{tx}$$

$$f'' = t^2 e^{tx}, \quad f''' = t^3 e^{tx}$$

$$e^{tx} = 1 + tx + \frac{t^2 x^2}{2!} + \frac{t^3 x^3}{3!} + \ldots ④$$

$$\therefore M_x(t) = \sum \left(1 + tx + \frac{t^2 x^2}{2!} + \frac{t^3 x^3}{3!} + \dots + \frac{t^n x^n}{n!}\right) f(x)$$

$$\cancel{E(x+X)} = \sum 1 f(x) + t \sum x f(x) + \frac{t^2}{2!} \sum x^2 f(x) + \frac{t^3}{3!} \sum x^3 f(x)$$

$$M_x(t) = 1 + \mu t + \frac{t^2}{2!} \mu_2' + \frac{t^3}{3!} \mu_3'$$

$$M_x(t) = 1 + \mu t + \frac{\mu_2' t^2}{2!} + \frac{\mu_3' t^3}{3!} + \dots + \mu_r' t^r / r!$$

Example: Find the first four moments about
(a) Origin (i.e first moment about origin)
(b) Mean

for a continuous random variable X with a density function of

$$f(x) = \begin{cases} \frac{4x(9-x^2)}{81}, & 0 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

Soln

$$\mu_r = E(x-\mu)^r, \quad \mu_r' = E(x^r).$$

(a) $\mu_1' = E(X^1) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{x=0}^{3} x \cdot \frac{4x(9-x^2)}{81}\,dx$

$= 8/5$

$\mu_2' = E(X^2) = \int_0^3 x^2 f(x)\,dx = \int_0^3 x^2 \cdot \frac{4x(9-x^2)}{81}\,dx$

$= 3$

$\mu_3' = E(X^3) = \int_0^3 x^3 f(x)\,dx = \int_0^3 x^3 \cdot \frac{4x(9-x^2)}{81}\,dx$

$= 216/35$

$\mu_4' = E(X^4) = \int_0^3 x^4 f(x)\,dx = \int_0^3 x^4 \frac{4x(9-x^2)}{81}\,dx$

$= 27/2$

(b) $\mu_r = E(X-\mu)^r$.

$\mu_1 = E(X-\mu) = E(X) - E(\mu) = 0$

$\mu_2 = E(X^2) - \binom{2}{1}\mu \cdot E(X) + \binom{2}{2}\mu^2 E(1)$.

$= E(X^2) - 2\mu^2 + 1\mu^2$

$\mu_2 = E(X^2) - \mu^2 = \sigma^2$

$\mu_2 = \mu_2' - \mu^2 = 3 - (8/5)^2$

$$\mu_3 = E(x^3) - \binom{3}{1}\mu E(x^2) + \binom{3}{2}\mu^2 E(x) - \binom{3}{3}\mu^3 E(1)$$

$$\mu_3 = \mu_3' - 3\mu \cdot \mu_2' + 3\mu^3 - \mu^3.$$

$$\mu_3 = \mu_3 - 3\mu\mu_2' + 2\mu^3$$

$$= \frac{216}{35} - 3 \cdot \frac{8}{5} \cdot 3 + 2 \cdot \left(\frac{8}{5}\right)^3$$

03/03/2020.

SAMPLING & ESTIMATION THEORIES

Distribution.

Discrete distribution.

(i) Bernoulli

(ii) Binomial

(iii) Poisson

Continuous Distribution.

(iv) Exponential distribution.

(v) Uniform.

(vi) Normal.

CONTINUOUS DISTRIBUTION

$P_x(K) = P(X=K) = p^K(1-p)^{1-K} \quad \forall K = 0, 1$

where $0 \leq p \leq 1$ [i.e the probability of a certain event is one]

cdf $\quad F_x(x) = \begin{cases} 0 & x<0 \\ 1-p & 0 \leq x \leq 1 \\ 1 & x \geq 1 \end{cases}$

$\mu_x = E(X) = P$.

Sampling means the aspect of production that has distribution and represent the distribution properly.

Distribution function is also called cummulative distribution function.

$P(X=0) = 0.5 \quad P(X=1) = 0.2 \quad P(X=2) = 0.1$

$P(X=3) = 0.05 \quad P(X=4) = 0.02 \quad P(X=5) = 0.03$

Expected value of the above probabilities:

$E(X) = (0.02) \times 70 + 0.03(65) + \ldots (0.5) \times (20)$

$\mu_x = E(X) = P$.

$\sigma_x^2 = P(1-P)$.

Q can be takes as $P(1-P)$.

P – number of success.

Q – number of failures.

Binomial probability

$P_x(x) = P(X=K) = \binom{n}{p}^{pK} (1-P)^{n-K}$.

~~$K = 0, 1, \ldots$~~

Binomial Probability Distribution.

$$P_X(X) = P(X = K) = \binom{n}{K} p^K (1-p)^{n-K}$$

$$K = 0, 1, \ldots, n$$

where $\binom{n}{K} = \frac{n!}{(n-K)!\,K!}$

$$F_X(x) = \sum_{K=0}^{n} \binom{n}{K} p^K (1-p)^{n-K} \quad n \leq x \leq n+1$$
(cdf)

Poisson Distribution.

$P_X(K) = P(X = K)$

$X = f(x) = 2x + 1$

$x = 1 \quad X = 3$

Recall the relationship

$$f(x) = \frac{dF_X(x)}{dx}$$

$f(x) = \sqrt{x}$

$x = 4 \quad X = \pm 2$

$$F_X(x) = e^{-\lambda} \sum_{K=0}^{n} \frac{\lambda^K}{K!}$$

Uniform Distribution.

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x \le a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x \geq b \end{cases}$$

$$\mu_x = E(X) = \frac{a+b}{2}$$

$$\sigma_x^2 = Var(X) = \frac{(b-a)^2}{12}$$

Exponential Distribution.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x < 0 \end{cases}$$

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\mu_X = E(X) = 1/\lambda$$

$$\sigma_X^2 = Var(X) = 1/\lambda^2$$

Normal Distribution:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\sigma(x-\mu)^2/2\sigma^2}.$$

$$F_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-(z-\mu)^2/2\sigma^2} \, dz$$

$$F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\mu}{\sigma}} e^{-\zeta^2/2} \, d$$

$$\mu_X = E(X) = \mu$$
$$\sigma_X^2 = Var(X) = \sigma^2$$

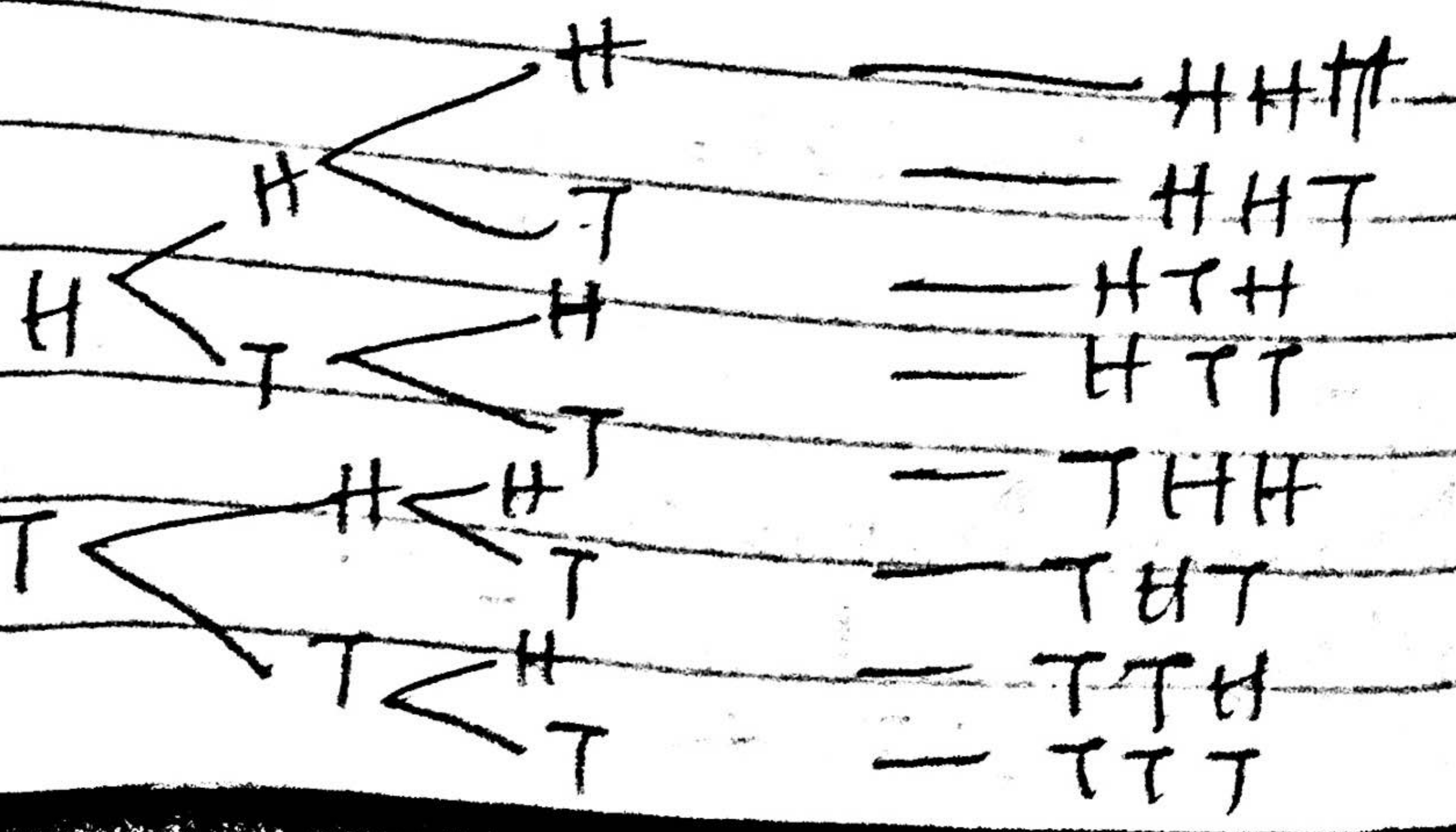
Example;

Consider the experiment of tossing a coins three times. Let X be the random variable given the numbers of heads obtained. We assume that the tosses is three times ___ and independent and the probability of a ___ is p.

1. What is the range of X
2. Find the probabilities that $P(X=0)$, $P(X=1)$, $P(X=2)$, $P(X=3)$

Soln.

Sample space;



① $R_X = \{0, 1, 2, 3\}$.

② $P(H) = p$, $P(T) = 1-p$.

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

X = number of heads.

(i) $P(X=0) = \frac{1}{8}$ $P(\{TTT\}) = (1-p)^3$.

(ii) $P(X=1) = 3/8$ $P(HTT) + P(THT) + P(TTH)$
$= 3(1-p)^2 \cdot p$

(iii) $P(X=2) = 3/8$ $P(HHT) + P(HTH) + P(TTH)$
$= 3(1-p) \cdot p^2$

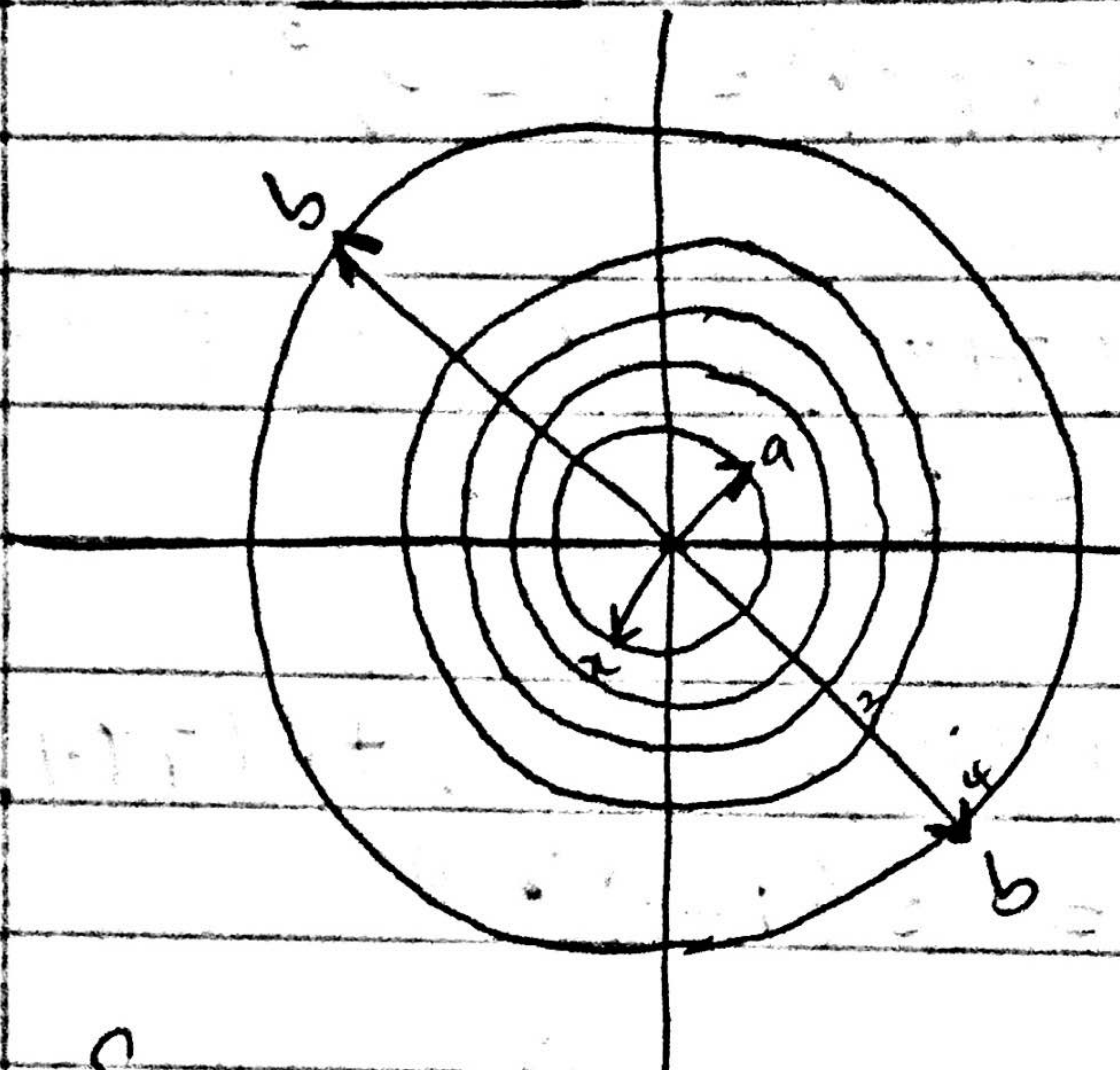
(iv) $P(X=3) = 1/8 = P(HHH) = p^3$

The above distribution is Known as Binomial distribution.

Example: Consider the experiment of throwing a dart onto a circular plate with unit radius let x be the random variable representing the distance of the point where the dart land's on the origin of the plate. Assume that always land on the plate and that the dart is equally likely to land anywhere on the plate ① What is the range of X

(ii) Find
(a) probability of x is less than a $P(x<a)$
(b) $P(a<x<b)$
(c) Sketch $F_x(x)$ ——→ Assignment.

Soln



(i) $R_x = \{x : 0 \leq x \leq 1\}$.

(ii) $P(x<a) = \dfrac{\cancel{\pi} a^2}{\cancel{\pi}} = a^2$.

(iii) $P(a<x<b) = \dfrac{\pi(b^2-a^2)}{\pi} = b^2 - a^2$.

$F(x) \leq F(y)$ iff $x \leq y$.

(2) $\lim_{x \to -\infty} F(x) = 0$; $\lim_{x \to \infty} F(x) = 1$

(3) $F(x)$ is continuous

E.g

If x takes ~~any~~ a finite discrete number of values such that $x = 1, 2, 3, 4, 5, 6, \ldots$

then, the cumulative distribution function is:

$$F(x) = \begin{cases} 0, & -\infty < x < x_1 \\ f(x_1), & -\infty < x < x_2 \\ f(x_1) + f(x_2), & -\infty < x < x_3 = \begin{cases} -\infty < x < x_2 + \\ x_2 < x < x_3 \end{cases} \\ f(x_1) + f(x_2) + f(x_3) + \ldots & -\infty < x < x_2 + \ldots \\ \vdots & \\ f(x_1) + f(x_2) + f(x_3) + \ldots f(x_n) & \end{cases}$$

$F(x) = P(X \leq x)$

x	0^{x_1}	1^{x_2}	2^{x_3}
$F(x)$	$1/4$	$1/2$	$1/4$

$$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ 0 + 1/4 = 1/4 & -\infty < x < x_1 = 1 \\ 1/4 + 2(1/4) = 3/4 & -\infty < x < x_2 = 2 \\ 3/4 + 1/4 = 4/4 = 1 & -\infty < x < x_\infty = +\infty \end{cases}$$