MOMENT OF GENERATING FUNCTIONS

Expectation

$E(X) = \mu$ — moment of generating function

$\mu_r = \left[E(x-\mu)^r\right]$ — rth central moment

$r = 0, 1, 2, 3, 4, \ldots$

$r = 0, \quad \mu_0 = E(x-\mu)^0 = E(1) = 1$, — zeroth central moment;

$r = 1, \quad \mu_1 = E(x-\mu)^1 = E(x-\mu);$

$= E(x) - E(\mu)$

$= E(x) - \mu E(1)$

$= \mu - \mu$

$\mu_1 = 0$ — First central moment

$r = 2,$

$$\mu_2 = E(x-\mu)^2 = E(x^2 - 2x\mu + \mu^2) = \sigma^2$$

Second central moment.

$$= E(x^2) - 2\mu E(x) + \mu^2$$

$$= E(x^2) - \mu^2$$

$r = 3$

$$\mu_3 = E(x-\mu)^3 = E\left(x^3 - 3\mu^2 \boxed{E(X)} - 3\mu \boxed{E(X^2)} + \mu^3\right)$$

$$\mu_3 = \cancel{E(x^3 - 3\mu^2 x - 3\mu x^2 + \mu^3)}$$

$$= E(x^3 - 3\mu x^2 + 3\mu^2 x - \mu^3)$$

$$\mu_3 = E(x^3) - 3\mu E(x^2) + 3\mu^2 E(x) - \mu^3 (E(1))$$

$$= E(x^3) - 3\mu \cdot E(x^2) + 3\mu^2 \cdot \mu - \mu^3$$

$$\boxed{\mu_3 = E(x^3) - 3\mu E(x^2) + 2\mu^3}$$

Third central moment;

Q: <u>DRV</u>

$$\mu_r = E(x-\mu)^r = \sum_{x=0}^{\infty} (x-\mu)^r f(x)$$

<u>CRV</u>

$$\mu_r = \int_{-\infty}^{\infty} (x-\mu)^r f(x)\, dx .$$

Rth moment about the origin (Raw moment)

$$\mu'_r = E(X)^r \qquad \mu = 0$$

$$\mu_r = E(x-\mu)^r \, ;$$

$$\mu'_r = \mu_r \Big|_{\mu = 0} = E(x-0)^r = E(X)^r .$$

$$\mu_3 = E(X^3) - 3\mu E(X^2) + 2\mu^3$$

$$\mu_4 = E(x-\mu)^4 = E(x^4 + 4(x^3)(-\mu)^1 + 6x^2(-\mu)^2 + 4x(-\mu)^3 + 1(-\mu)^4)$$

$$= E(x^4 + 4\mu x^3 + 6\mu^2 x^2 - 4\mu^3 x + \mu^4)$$

$$= E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 4\mu^3 \underbrace{E(X)}_{\mu} + \mu^4$$

$$\mu_4 = E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) \cancel{4\mu^3} - 3\mu^4 ;$$

$$\mu'_2 = E(x^2) , \quad \mu'_3 = E(x^3)$$

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$$\mu_2 = \underbrace{E(X^2)}_{\mu_2'} - \mu^2 ;$$

$$\mu_3 = \underbrace{E(X^3)}_{\mu_3'} - 3\mu \underbrace{E(X^2)}_{\mu_2'} + 2\mu^3 ;$$

$$\mu_4 = \underbrace{E(X^4)}_{\mu_4'} - 4\mu \underbrace{E(X^3)}_{\mu_3'} + 6\mu^2 \underbrace{E(X^2)}_{\mu_2'} - 3\mu^4 ;$$

$$\mu_5 = E(X^5) - 5\mu E(X^4) + 10\mu^2 E(X^3) - 10\mu^3 E(X^2) + 5\mu^5 ;$$

$$\mu_0 = 1 ;$$

$$\mu_1 = 0 ;$$

$$\mu_2 = \mu_2' - \mu^2 ;$$

$$\mu_3 = \mu_3' - 3\mu\mu_2' + 2\mu^3 ;$$

$$\mu_4 = \mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4 ;$$

$$\mu_5 = \mu_5' - 5\mu\mu_4' + 10\mu^2\mu_3' - 10\mu^3\mu_2' + 4\mu^5$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$$\mu_r = \mu_r' - \binom{r}{1}\mu\mu_{r-1}' + \binom{r}{2}\mu^2\mu_{r-2}' + \cdots$$
$$\cdots + (-1)^n\binom{r}{n}\mu^n\mu_{r-n}'.$$

1 Moment of generating function,

$$M_x(t) = E(e^{tx});$$

DRV

$$M_x(t) = E(e^{tx}) = \sum e^{tx} f(x)$$

CRV

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$\longrightarrow$

$f(x)\ e^{tx}$

$f(0) = e^0 = 1$

$f'(0) = te^{tx} = te^0 = t;$

$f''(0) = t^2 e^{tx} = t^2 e^0 = t^2$

e^{tx}

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0).$$

$$e^{tx} = 1 + x \cdot t + \frac{x^2}{2!} \cdot t^2 + \frac{x^3}{3!} \cdot t^3 + \cdots$$

$$M_x(t) = E(e^{tx})$$

$$= E\left(1 + tx + \frac{t^2 x^2}{2!} + \frac{t^3 x^3}{3!} + \cdots\right)$$

$$= E(1) + E(tx) + E\left(\frac{t^2 x^2}{2!}\right) + E\left(\frac{t^3 x^3}{3!}\right)$$

$$= 1 + tE(x) + \frac{t^2}{2!} E(x^2) + \frac{t^3}{3!} E(x^3)$$

$$M_x(t) = 1 + \mu t + \frac{\mu_2' t^2}{2!} + \frac{\mu_3' t^3}{3!} + \dots$$

MOMENT OF GENERATING FUNCTION

Rth central moment (mean — deviation from the mean)

$$\mu_r = E(X-\mu)^r \qquad X-\mu$$

rth moment about the origin

$$\mu'_r = E(X)^r$$

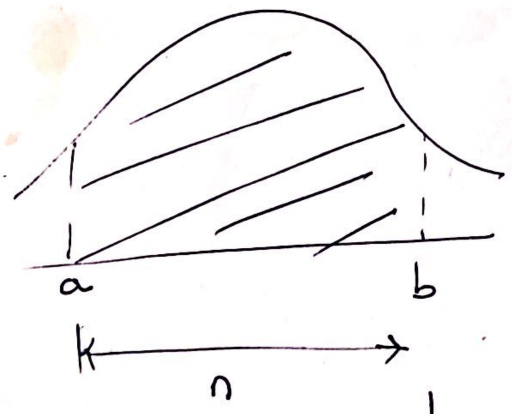
Special Continuous RV

$f(x)$ — probability function

$$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx \, ;$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx \, ; \qquad \frac{1}{n}$$

I. Continuous Uniform distribution



$f(x) = \frac{1}{n} =$ — Range.

$f(x) = \frac{1}{\;}$, $a \le x \le b$

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$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx \qquad f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

$$= \cancel{\int_{-\infty}^{a} x \cdot 0 \cdot dx} + \int_{a}^{b} x \cdot \frac{1}{b-a}\, dx + \cancel{\int_{b}^{+\infty} x \cdot 0\, dx} \qquad \begin{array}{l} -\infty < x < a \\ b < x < \infty \end{array}$$

$$= \int_{a}^{b} x \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \int_{a}^{b} x\, dx$$

$$= \frac{1}{b-a} \cdot \frac{x^2}{2} \Big|_{a}^{b}$$

$$= \frac{1}{2(b-a)} \cdot b^2 - a^2$$

$$= \frac{1}{2\cancel{(b-a)}} \cdot \cancel{(b-a)}(b+a)$$

$$\mu = E(X) = \frac{a+b}{2}$$

Variance,

$$\sigma^2 = E(X-\mu)^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

$$= \int_a^b \left(x - \left[\frac{a+b}{2}\right]\right)^2 \cdot \frac{1}{b-a}\, dx$$

$$\frac{1}{b-a}\int_a^b \left(\left(\frac{x}{2}\right)^2 - x(a+b) - \left(\frac{a+b}{2}\right)^2 \right) dx$$

$$\sigma^2 = E(x-\mu)^2 = E(x^2) - \left(E(x)\right)^2$$

$$E(x^2) = \int_a^b x^2 \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a}\int_a^b x^2\, dx$$

$$= \frac{1}{b-a} \cdot \frac{x^3}{3}\Big|_a^b = \frac{1}{3(b-a)} \cdot b^3 - a^3$$

$$E(x^2) = \frac{b^3 - a^3}{3(b-a)}$$

$$\sigma^2 = \frac{b^3 - a^3}{3(b-a)} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{b^3 - a^3}{3(b-a)} - \frac{(a+b)^2}{4}$$

$\cancel{(b+a)}$

$\cancel{b)}$ $$\frac{4(b^3-a^3) - 3(b-a)(b+a)^2}{12(b-a)}$$

$$= \frac{4b^3 - 4a^3 - 3(b-a)(a^2+2ab+b^2)}{12(b-a)}$$

$$= \frac{\cancel{4b^3} - \cancel{4a^3} - \cancel{3ba^2} - 3\cancel{b^2}a - \cancel{3b^3} + \cancel{3a^3} + \cancel{3ba^2} + 3ab^2}{12(b-a)}$$

$$= \frac{b^3-a^3}{12(b-a)} = \frac{\cancel{(b-a)}(b+a)^2}{12\cancel{(b-a)}}$$

$$(b-a)(b^2+ab+a^2) = b^3 + \cancel{ab^2} + \cancel{a^2b} - \cancel{ab^2} + \cancel{a^2b} - a^3$$

$$6^2 =$$

$$= \frac{\cancel{4b^3} - \cancel{4a^3} - 3a^2b - 6ab^2 - \cancel{3b^3} + \cancel{3a^3} + 6a^2b + 3ab^2}{12(b-a)}$$

$$= \frac{b^3 - a^3 + 3a^2b - 3ab^2}{12(b-a)}$$

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$$= \frac{b^3 - 3b^2a + 3ba^2 - a^3}{12(b-a)}$$

1
1 1
1 2 1
1 3 3 1 — $(\quad)^3$

$(b-a)^3$

$$(b-a)^3 = 1b^3 + 3b^2(-a)^1 + 3b(-a)^2 + 1\cdot b^0(-a)^3$$

$$(b+(-a))^3 = b^3 - 3b^2a + 3ba^2 - a^3$$

$$\boxed{\sigma^2 = \frac{(b-a)^3}{12(b-a)} = \frac{(b-a)^2}{12}}$$

Schaums: A RV X has a density function given by:

$$f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0 & x < 0 \end{cases}$$

find: (a) moment generating function

(b) the first four moments about the origin.

$n=1$, $x=2$, $n=3$,

Soln:

(a) $m(t) = E(e^{tX}) =$

$$E(X) = \int_{-\infty}^{\infty} X \cdot f(x)\, dx$$

$$E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot 2e^{-2x}\, dx$$

$$= 2\int_{0}^{\infty} e^{(t-2)x}\, dx$$

$$= 2 \cdot \frac{e^{(t-2)x}}{(t-2)} \Bigg|_{0}^{\infty}$$

$$= \frac{2}{t-2}\left(e^{(t-2)\infty} - e^{(t-2)0}\right)$$

$$= \frac{2}{t-2}\left(\cancel{e^{-\infty}}^{\,0} - \cancel{e^{0}}^{\,1}\right) = \frac{2}{t-2}(0-1)$$

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$$M(t) = \frac{2}{2-t} \; ; \qquad t < 2$$

$$2 - t \not< 0 \qquad t \not< 2$$

$$2 - t > 0$$

$$-t > -2$$

$$t < +2$$

$$\frac{2}{2-t} \quad \frac{\frac{2}{2}}{\frac{2-t}{2}} = \frac{1}{\frac{2}{2} - t/2} = \frac{1}{1-(t/2)}$$

$$\ln x \,, \; \frac{1}{1-x} = 1 + \frac{x}{1} + \frac{x^2}{4!} + \frac{x^3}{8} + \frac{x^4}{16} + x^5/32 + \cdots$$

$$\frac{1}{1-x} = \boxed{1 + x + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8}}$$

$$\frac{1}{1-x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{4} + \frac{x^4}{8} + \frac{x^5}{16} + \cdots$$

$$\frac{1}{1-(x/2)} = 1 + x/2 + \frac{x^2}{4} + \frac{x^3}{8} + \frac{x^4}{16} + x^5/32 + \cdots$$

$$M(t) = \frac{1}{1-(t/2)} = 1 + t/2 + \frac{t^2}{4} + \frac{t^3}{8} + \frac{t^4}{16} + \frac{t^5}{32} +$$

$$M(t) = 1 + \mu t + \frac{\mu_2' t^2}{2!} + \frac{\mu_3' t^3}{3!} + \frac{\mu_4' t^4}{4!} + \dots$$

$$= 1 + t/2 + t^2/4 + t^3/8 + \frac{t^4}{16}$$

$\mu = 1/2$, $\frac{\mu_2'}{2!} = \frac{1}{\cancel{4}_2}$, $\mu_2' = 1/2$;

$\frac{\mu_3'}{3!} = \frac{1}{\cancel{8}_4}$ $\quad \mu_3' = 3/4$,

$\cancel{6}$ 3

$\frac{\mu_4'}{4!} = \frac{1}{\cancel{16}_2}$ $\quad \mu_4' = 3/2$

$\cancel{4} \times 3 \times \cancel{2}$

$\therefore \mu = 1/2$, $\mu_2' = 1/2$, $\mu_3' = 3/4$, $\mu_4' = 3/2$

$\mu_r = E(X)^r$

$$\mu_1 = E(X) = \int_0^\infty x f(x)\,dx$$

$$= \int_0^\infty x \cdot 2e^{-2x}\,dx$$

$$= 2\int_0^\infty x e^{-2x}\,dx$$

$$u = x \qquad dv = e^{-2x}\,dx$$

$$\underline{dx} = dx \qquad v = \frac{e^{-2x}}{-2}$$

$$\int u\,dv = uv - \int v\,dx$$

$$= \frac{xe^{-2x}}{-2} + \int \frac{e^{-2x}}{+2}\,dx$$

$$= \left.\frac{xe^{-2x}}{-2} + \frac{e^{-2x}}{-4}\right|_0^\infty$$

$$= \left(\frac{\cancel{\infty e^{-\infty}}}{-2} - \frac{\cancel{e^{-\infty}}}{-4}\right) - \left(\frac{\cancel{0e^{-0}}}{2} - \frac{e^{-0}}{4}\right)$$

$$= -\left(0 - 1/4\right) = 1/4.$$

$$\mu_1 = 2\int_0^\infty x e^{-2x}\,dx = 2(1/4) = 1/2;$$

$$\mu_2' = E(X)^2 = \int x^2 f(x)\,dx$$

$$= \int x^2 \cdot 2e^{-2x}\,dx = 2\int x^2 \cdot e^{-2x}\,dx$$

$u = x^2$, $dv = e^{-2x}\,dx$

$du = 2x$ $\quad v = \frac{e^{-2x}}{-2}$

$$\int u\,dv = uv - \int v\,du$$

$$= \frac{x^2 \cdot e^{-2x}}{-2} - \int -\frac{e^{-2x}}{2} \cdot 2x$$

$\frac{1}{2}\underbrace{\int x e^{-2x}}_{1/4} = 1/8$

$$= \frac{x^2 e^{-2x}}{-2} + \int \frac{2x \cdot e^{-2x}}{2}$$

$u = 2x$, $dv = \frac{e^{-2x}}{2}$

$du = 2$, $v = \frac{e^{-2x}}{-4}$

$$= \frac{x^2 e^{-2x}}{-2} + \left[uv - \int v\,du\right]$$

$$= \frac{x^2 e^{-2x}}{-2} + \left[2x \cdot \frac{e^{-2x}}{-4} - \int \frac{e^{-2x}}{-\cancel{4}2} \cdot \cancel{2}\right]$$

$$= \frac{x^2 e^{-2x}}{-2} + \left[\cancel{2}x \cdot \frac{e^{-2x}}{-\cancel{4}2} - \int \frac{e^{-2x}}{-2}\right.$$

$$= \frac{x^2 e^{-2x}}{-2} + \left[\frac{x \cdot e^{-2x}}{-2} + \int \frac{e^{-2x}}{2}\right.$$

$$= \frac{x^2 e^{-2x}}{-2} + \left[\frac{x \cdot e^{-2x}}{-2} + \frac{e^{-2x}}{-4}\right]_0^\infty$$

$$x^2 e^{-2x} + x \cdot e^{-2x} + e^{-2x} \Big|^\infty$$

$$= \frac{x^2 \cdot e^{-2x}}{-2} + \frac{x \cdot e^{-2x}}{-2} - \frac{e^{-2x}}{4} \Big|_0^{\infty}$$

$$\boxed{= \frac{x^2 \cdot e^{-2x}}{-2} + \cancel{}}$$

$$= \frac{x^2 \cdot e^{-2x}}{-2} + \frac{e^{-2x}}{-2}\left[x - \frac{1}{2}\right]\Big|_0^{\infty}$$

$$= \frac{e^{-2x}}{-2}\left[x^2 + x - \frac{1}{2}\right]\Big|_0^{\infty}$$

$$= \cancel{\frac{e^{-2(\infty)}}{-2}\left[\infty^2 + \infty - \frac{1}{2}\right]} - \frac{e^{-2(0)}}{-2}\left[0^2 + 0 + \frac{1}{2}\right]$$

$$= \cancel{} = \frac{+e^{-2(0)}}{+2}\left[\cancel{0}\tfrac{1}{2}\right] = \frac{1}{2}\left[+\frac{1}{2}\right] = \frac{1}{4}$$

$=$ Recall that $2\int x^2 \cdot e^{-2x}\,dx$

Where $\int x^2 \cdot e^{-2x}\,dx = 1/4$

$$\therefore = 2\left(\frac{1}{4}\right) = \frac{1}{2}$$

$$\mu_3' = E(X)^3 = \int x^3 \, 2e^{-2x} \, dx$$

$$= 2 \int x^3 e^{-2x} \, dx$$

$$u = x^3 \qquad dv = e^{-2x} \, dx$$

$$du = 3x^2 dx \quad v = \frac{e^{-2x}}{-2}$$

$$2\left(x^3 \frac{e^{-2x}}{-2} - \int \frac{e^{-2x}}{-2} \cdot 3x^2 \, dx \right)$$

$$\left(\frac{x^3 e^{-2x}}{2} + \frac{3}{2} \int \underbrace{x^2 e^{-2x} \, dx}_{1/2} \right)$$

$$2\left(\left(\frac{-\cancel{\infty^3 e^{-\infty}}}{2} \right)^{\infty} - \left(\frac{-\cancel{0^3 e^{-0}}}{2} \right)^{0} + \frac{3}{2}\left(\frac{1}{2}\right) \right)$$

$$\cancel{2}\left(\frac{3}{2} \times \frac{1}{\cancel{2}} \right) = \frac{3}{2}$$

$$\mu_3' = \frac{3}{2};$$

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19 : Find the first four moments (a) about the origin (b) about the mean, for a RV X having density function;

$$f(x) = \begin{cases} \dfrac{4x(9-x^2)}{81} & , 0 \le x \le 3. \\ 0 & , \text{otherwise} \end{cases}$$

$$\mu'_r = E(x)^r , \qquad \mu_r = E(x-\mu)^r$$

about the origin — moment about the mean

(a) $$\mu'_1 = E(x) = \int_0^3 \frac{4x(9-x^2)}{81} \cdot x \, dx = \mu$$

$$= \frac{4}{81}\int_0^3 9x^2 - 9x^3 \, dx$$

$$= \frac{4}{81} \cdot \left(\frac{9x^3}{3} - \frac{9x^4}{4} \Big|_0^3 \right) = \frac{4}{81}\left(\cancel{\frac{9}{3}(3^3 - 0^3) - \frac{9}{4}(3^4 - 0^4)} \right)$$

$$= \cancel{\frac{4}{81}\left(\frac{9}{3}(3^3) - \frac{9}{4}(3^4) \right)} \quad \frac{4}{81}\left(\left[\frac{9}{3}(3^3) - \frac{9}{4}(3^4)\right] - \left[\frac{9}{3}(0^3) - \frac{9}{4}(0^4)\right] \right)$$

$$= \frac{4}{81}\left(\frac{9 \cdot 3^3 \cdot 4 - 9 \cdot 3^4 \cdot 3}{12} \right)$$

$$= \frac{4}{81}\left(\frac{9}{3} \cdot 3^3 - \frac{9}{4} \cdot 3^4 \right) = \frac{4}{81} \cdot 3^4 \left(\frac{1}{3} - \frac{3}{4} \right)$$

$$= 12\left(\frac{4-9}{12}\right) = 1.6 = \frac{16}{} = (8/5),$$

$$\mu_2' = E(X^2) = \int_0^3 \frac{4x(9-x^2)}{81} \cdot x^2\,dx = 3$$

$$\mu_3' = E(X^3) = \int_0^3 \frac{4x(9-x^2)}{81} \cdot x^3\,dx =$$

(b) $\mu_r = E(X-\mu)^r$

$\mu = E(X) = \mu_1' = 8/5;$

$$\mu_1 = E(X-8/5) = \int_0^3 (x-8/5)\,\frac{4x}{81}(9-x^2)\,dx = 0$$

$$\mu_2 = E(X-8/5)^2 = \int_0^3 (x-8/5)^2 \cdot \frac{4x}{81}(9-x^2)\,dx$$

$$\mu_3 = E(X-8/5)^3 = \int_0^3 (x-8/5)^3 \cdot \frac{4x}{81}(9-x^2)\,dx$$

$$\mu_4 = E(X-8/5)^4 = \int_0^3 (x-8/5)^4 \cdot \frac{4x}{81}(9-x^2)\,dx$$

$\mu_1 = 0;$

$$\mu_2 = \mu_2' - \mu^2 = 3 - (8/5)^2 =$$

$$\mu_3 = \mu_3' - 3\mu\,\mu_2' + 2\mu^3 = \frac{216}{35} - 3\left(\frac{8}{5}\right)(3) + 2\left(\frac{8}{5}\right)^3$$

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MEASURE OF OTHER CENTRAL TENDENCIES

1. MODE

2. MEDIAN

$$P(X < x) \leq 1/2 ;$$

3. MEAN DEVIATION

$$M.D(X) = E(X-\mu) ; = \sum (x-\mu)\, f(x) ;$$

4. SKEWNESS

$$\alpha_3 = \frac{E(X-\mu)^3}{\sigma^3} = \frac{\mu_3}{\sigma^3} ;$$

5. KURTOSIS

$$\alpha_4 = \frac{E(X-\mu)^4}{\sigma^4} = \frac{\mu_4}{\sigma^4} ;$$

3.17: A RV X can assume the values of 1 or -1 with probability 1/2 each. Find:

(a) the moment generating function;

(b) first 4 moments about the origin.

X	1	-1
$f(x)$	1/2	1/2

(a) $M(t) = E(e^{tX}) = \sum e^{tx} \cdot f(x)\, dx$

$= e^{t(1)} \cdot (1/2) + e^{t(-1)} \cdot (1/2)$

$M(t) = \frac{1}{2}(e^{t} + e^{-t})$

$e^{t} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \dots$

$e^{-t} = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \dots$

$e^{t} + e^{-t} = 2 + 2\left(\frac{t^2}{2!}\right) + 2\left(\frac{t^4}{4!}\right) + 2\left(\frac{t^6}{6!}\right) + \dots$

$\frac{1}{2}(e^{t} + e^{-t}) = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \dots$

$$M(t) = 1 + \mu t + \mu'_2 \frac{t^2}{2!} + \mu'_3 \frac{t^3}{3!} + \mu'_4 \frac{t^4}{4!} + \dots$$

(b)

$$\mu'_r = E(x-\mu)^r = \sum (x-\mu)^r f(x);$$

$$M(t) = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \dots$$

$$\mu = 0; \quad \mu'_2 = 1, \quad \mu'_3 = 0; \quad \mu'_4 = 1; \quad \mu'_5 = 0,$$

$$\mu'_6 = 1;$$

ALT

$$\mu'_r = E(x)^r$$

(1)

$$\mu'_1 = E(x) = \mu = \sum x f(x)$$

$$= 1\left(\tfrac{1}{2}\right) - 1\left(\tfrac{1}{2}\right) = 0$$

$$\mu'_2 = E(x^2) = \sum x^2 f(x)$$

$$= 1^2\left(\tfrac{1}{2}\right) + (-1)^2\left(\tfrac{1}{2}\right) = \tfrac{1}{2} + \tfrac{1}{2} = 1,$$

$$\mu'_3 = E(x^3) = \sum x^3 f(x)$$

$$= 1^3\left(\tfrac{1}{2}\right) + (-1)^3\left(\tfrac{1}{2}\right) = 1\left(\tfrac{1}{2}\right) - 1\left(\tfrac{1}{2}\right) = 0$$

$$\mu'_4 = \sum x^4 f(x) = 1^4(1/2) + (-1)^4(1/2) = 1(1/2) + 1(1/2)$$
$$= 1/2 + 1/2 = 1$$

$$\mu'_5 = \sum x^5 f(x) = 1^5(1/2) + (-1)^5(1/2) = 1(1/2) - 1(1/2)$$
$$= 1/2 - 1/2 = 0$$

$$\mu'_6 = \sum x^6 f(x) = 1^6(1/2) + (-1)^6(1/2) = 1(1/2) + 1(1/2)$$
$$= 1/2 + 1/2 = 1$$

(e) For the example, find the characteristic function of the RV.

The characteristic function is defined as:

$$E(e^{iwX});$$

$$E(e^{iwX}) = \sum e^{iwx} f(x),$$

$$= e^{iw(1)}(1/2) + e^{iw(-1)} \cdot 1/2$$

$$= \frac{1}{2}(e^{iw} + e^{-iw})$$

$$= \underline{\cos w}$$

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$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$
$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$$

$$\frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta.$$

$$\therefore \frac{e^{iw} + e^{-iw}}{2} = \cos w \longrightarrow$$

3.21: find the characteristic function of the RV having density function given by:

$$f(x) = \begin{cases} 1/2a, & |x| < a \quad [-a<x<a] \\ 0, & \text{otherwise,} \end{cases}$$

Soln:

xteristic function $= E(e^{iwx}) = \int_{-\infty}^{\infty} e^{iwx} f(x)\,dx$

$$= \int_{-a}^{a} e^{iwx} \cdot \frac{1}{2a}\,dx = \frac{1}{2a}\int_{-a}^{a} e^{iwx}\,dx$$

$$= \frac{1}{2a}\left(\frac{e^{iwx}}{iw}\Big|_{-a}^{a}\right) = \frac{1}{i2aw}\left(e^{iwa} - e^{-iwa}\right)$$

$$= \frac{1}{i2aw}\left(e^{iwa} - e^{-iwa}\right)$$

$$2i \sin wa = e^{iwa} - e^{-iwa}$$

$$\sin wa = \frac{e^{iwa} - e^{-iwa}}{2i}$$

$$= \frac{1}{wa}\left(\frac{e^{iwa} - e^{-iwa}}{2i}\right)$$

$$= \frac{\sin wa}{wa}$$

3.22 Find the characteristic function of the RV X having density function $f(x) = ce^{-a|x|}$, $-\infty < x < \infty$, where $a > 0$, and c is a suitable constant.

$$f(x) = \begin{cases} c e^{-a|x|}, & -\infty < x < \infty \\ \end{cases}$$

$$P(S) = 1, \quad \sum f(x) = 0; \quad \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} c e^{-a|x|} \, dx = 1.$$

$$= \int_{-\infty}^{0} c e^{-a(-x)} \, dx + \int_{0}^{\infty} c e^{-a(+x)} \, dx = 1;$$

$0 > x > \infty -$

-ve +ve

$$c\left[\int_{-\infty}^{0} e^{ax} \, dx + \int_{0}^{\infty} e^{-ax} \, dx\right] = 1$$

$$c\left[\frac{e^{ax}}{a}\Bigg|_{-\infty}^{0} + \frac{e^{-ax}}{-a}\Bigg|_{0}^{\infty}\right] = 1$$

$$c\left[\frac{1}{a}\left(e^{a(0)} - e^{a(-\infty)}\right) - \frac{1}{a}\left(e^{-a(\infty)} - e^{-a(0)}\right)\right] = 1$$

$$\frac{c}{a}\left[\left(\overset{1}{\cancel{e^{0}}} - \overset{0}{\cancel{e^{-\infty}}}\right) - \left(\overset{0}{\cancel{e^{-\infty}}} - \overset{1}{\cancel{e^{0}}}\right)\right] = 1$$

$$\frac{c}{a}\left[1 - 0 - (0 - 1)\right] = 1$$

$$\frac{c}{a}\left(1 - 0 - 0 + 1\right) = 1; \quad \frac{2c}{a} = 1;$$

$$c = a/2;$$

characteristics function,

$$E(e^{iwx}) = \int_{-\infty}^{\infty} e^{iwx} f(x)\, dx$$

$$= \cancel{\int_{-\infty}^{0} \frac{a}{2} e^{-a(-x)}} \cancel{+} \cancel{\int_{0}^{\infty}}$$

$$f(x) = \begin{cases} \frac{a}{2} e^{-a|x|} \end{cases} = \begin{cases} \frac{a}{2} e^{-a(-x)}, & -\infty < x < 0 \\ \frac{a}{2} e^{-a(+x)}, & 0 < x < \infty \end{cases}$$

$$E(e^{iwx}) = \int_{-\infty}^{\infty} e^{iwx} f(x)\, dx$$

$$= \int_{-\infty}^{0} e^{iwx} \cdot \frac{a}{2} e^{ax}\, dx + \int_{0}^{\infty} e^{iwx} \cdot \frac{a}{2} e^{-ax}\, dx$$

$$\frac{a}{2}\left[\int_{-\infty}^{0} e^{(iw+a)x}\, dx + \int_{0}^{\infty} e^{(iw-a)x}\, dx\right]$$

$$= \frac{a}{2}\left[\frac{e^{(iw+a)x}}{(iw+a)}\Bigg|_{-\infty}^{0} + \frac{e^{(iw-a)x}}{(iw-a)}\Bigg|_{0}^{\infty}\right]$$

$$= \frac{a}{2(iw+a)}\left(\cancel{e^{(iw+a)0}} - \cancel{e^{(iw+a)(-\infty)}}\right) + \frac{a}{2(iw-a)}\left(\cancel{e^{(iw-a)\infty}} - \cancel{e^{(iw+a)0}}\right)$$

3.31 :- For the RV of 3.18, ⓐ find $P(|X-\mu| > 1)$

ⓐ from 3.18,

$$f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$\mu = 1/2$

$P|X - 1/2| \cancel{<} 1$;

$$\mu = E(X) = \int x f(x) = \int_0^\infty 2xe^{-2x}\,dx$$

$$= 2xe^{-2x}\Big|_0^\infty = \int xe^{-2x}$$

$u = x \quad dv = e^{-2x}$

$du = dx \quad v = \frac{e^{-2x}}{-2}$

$$\int u\,dv = uv - \int v\,du$$

$$= \frac{xe^{-2x}}{-2} + \int \frac{e^{-2x}}{2}\,.dx$$

$$= \frac{xe^{-2x}}{-2} - \frac{e^{-2x}}{-4}\Big|^\infty$$

$$= \frac{a}{2(i\omega+a)}(1-0) + \frac{a}{2(i\omega-a)}(0-1)$$

$$= \frac{a}{2}\left(\frac{1}{i\omega+a} - \frac{1}{i\omega-a}\right) = \frac{a}{2}\left(\frac{i\omega-a-(i\omega+a)}{(i\omega+a)(i\omega-a)}\right)$$

$$\frac{a}{2}\left(\frac{\cancel{i\omega}-a-\cancel{i\omega}+a}{(i\omega)^2-a^2}\right)$$

$$= \frac{a}{2}\left(\frac{-2a}{-\omega^2-a^2}\right) = \frac{a}{\cancel{+2}}\left(\frac{+\cancel{2}a}{a^2+\omega^2}\right)$$

$$= \frac{a^2}{a^2+\omega^2}$$

$$\left(-\frac{\infty e^{-\infty}}{-2} - \frac{e^{-\infty}}{4}\right) - \left(\frac{0 e^{-0}}{-2} - \frac{e^{-0}}{4}\right) = 1/4$$

$$\mu = 2\int x e^{-2x}\,dx = 2\left(1/4\right) = 1/2$$

$$P\left|X - 1/2\right| \cancel{<} 1$$

$$+\left(x - 1/2\right) \cancel{<} 1$$

$+1/2$ $\quad -1/2$

$1 + 1/2 = 3/2$

$1 + 1/2 = 1/2$

$$\boxed{P(a < X < b)}$$

$|x|$

$$P\left|X - 1/2\right| < 1$$

$$P\left(X - 1/2\right) < \pm 1$$

$X - 1/2 < 1$ $\qquad$ $X - 1/2 < -1$

$X < 1 + 1/2 = 3/2$ $\qquad$ $X < -1 + 1/2$

$X < -1/2$

$$P\left|X-\tfrac{1}{2}\right|<\tfrac{1}{2}$$

$$= P\left(-\tfrac{1}{2}<X<\tfrac{3}{2}\right)$$

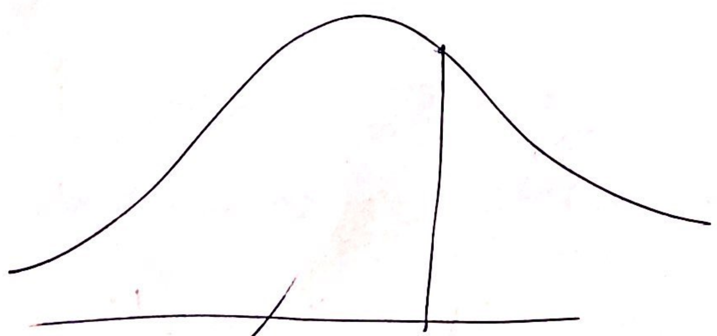
$$= \int_{-1/2}^{3/2} f(x)\,dx$$

$$= \cancel{\int_{-1/2}^{0} 0\,dx}^{\,0} + \int_{0}^{3/2} 2e^{-2x}\,dx$$

$$= \frac{\cancel{2}\,e^{-2x}}{-\cancel{2}}\Bigg|_{0}^{3/2} = -\left(e^{-\cancel{2}(\frac{3}{\cancel{2}})} - e^{-2(0)}\right)$$

$$= -\left(e^{-3}-1\right)$$

$$= 1 - e^{-3}$$



$$P\left(\left|X-\tfrac{1}{2}\right| \geq 1\right) = 1 - P\left(\left|X-\tfrac{1}{2}\right|<1\right) = 1-(1-e^{-3}) = e^{-3}$$