PS 1

DA COSTA, Thomas

1 Problem

$$\max_{\{c_t,k_{t+1},i_t\}_{t=1}^T} w(c_t) = \max_{\{c_t,k_{t+1},i_t\}_{t=1}^T} E_0 \sum_{t=0}^{T} \beta^t u(c_t) \tag{1}$$
$$\text{s.t.} \tag{2}$$
$$c_t + i_t \leq A_t k_t^\alpha \tag{3}$$
$$k_{t+1} = (1-\delta)k_t + i_t \tag{4}$$
$$k_0 \text{ given} \tag{5}$$
$$\text{for } u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma} \tag{6}$$

2 Question 1

u is continuously differentiable, strictly increasing, strictly concave, bounded, and satisfies the Inada conditions.

The feasibility constraint can be written by combining 3 and 4, giving:

$$\Rightarrow c_t = A_t k_t^\alpha + (1-\delta)k_t - k_{t+1} \tag{7}$$

The First Order Condition (the Euler equation for capital) comes from the derivative of $w(c_t)$ with respect to k_{t+1}:

$$\frac{\partial w(c_t(A_t,k_t,k_{t+1}))}{\partial k_{t+1}} = 0 \tag{8}$$
$$\Rightarrow E_0(\beta^t \frac{\partial u(c_t(A_t,k_t,k_{t+1}))}{\partial k_{t+1}} + \beta^{t+1} \frac{\partial u(c_{t+1}(A_{t+1},k_{t+1},k_{t+2}))}{\partial k_{t+1}}) = 0 \tag{9}$$
$$\Rightarrow E_0 \beta^t (\frac{\partial u(c_t)}{\partial c_t} \frac{\partial c_t}{\partial k_{t+1}} + \beta \frac{\partial u(c_{t+1})}{\partial c_{t+1}} \frac{\partial c_{t+1}}{\partial k_{t+1}}) = 0 \tag{10}$$

Based on the equation (7), we have $\frac{\partial c_t}{\partial k_{t+1}} = -1$ and $\frac{\partial c_{t+1}}{\partial k_{t+1}} = A_{t+1}\alpha k_{t+1}^{\alpha-1} + (1-\delta)$. Thus:

$$\frac{\partial u(c_t)}{\partial c_t} = \beta \frac{\partial u(c_{t+1})}{\partial c_{t+1}} [A_{t+1}\alpha k_{t+1}^{\alpha-1} + (1-\delta)] \tag{11}$$

The transversality condition is:

$$\lim_{T\to\infty} \frac{\partial w(c_t)}{\partial c_t} k_{T+1} = 0 \tag{12}$$

3 Question 2

The steady-state levels are reached at $t = t^*$ such that $c_{t^*+1} = c_{t^*} = c^*$ and $k_{t^*+1} = k_{t^*} = k^*$. If $\forall\, t, A_t = 1$, the steady-state thus impose from (4):

$$k^* = \frac{i_t}{\delta} \tag{13}$$
$$k^* = \frac{(k^*)^\alpha - c^*}{\delta} \tag{14}$$
$$\delta k^* = (k^*)^\alpha - c^* \tag{15}$$
$$\Rightarrow c^* = (k^*)^\alpha - \delta k^* \tag{16}$$

The equation (11) gives the expression of k^*, knowing that in the steady-state, $u(c_{t^*}) = u(c_{t^*+1})$:

$$1 = \beta[\alpha(k^*)^{\alpha-1} + (1-\delta)] \tag{17}$$
$$\Rightarrow k^* = (\frac{\beta^{-1} + \delta - 1}{\alpha})^{\frac{1}{\alpha-1}} \tag{18}$$

4 Question 3

An analytical solution is a policy function $c_t = g(k_t, \Theta)$ and a law of motion $k_{t+1} = h(k_t, \Theta)$, with Θ the parameters of the model. Let us choose $\sigma = 1 \to u(c_t) = \log(c_t)$ and $\delta = 1$. We can write the Euler equation as:

$$c_t^{-1} = \beta c_{t+1}^{-1} [A_{t+1}\alpha k_{t+1}^{\alpha-1}] \tag{19}$$

And the feasibility constraint as:

$$c_t = A_t k_t^\alpha - k_{t+1} \tag{20}$$

Combining (19) and (20), we get:

$$c_{t+1} = c_t(\alpha\beta)A_{t+1}k_{t+1}^{\alpha-1} \tag{21}$$

$$A_{t+1}k_{t+1}^{\alpha} - k_{t+2} = [A_t k_t^{\alpha} - k_{t+1}](\alpha\beta)A_{t+1}k_{t+1}^{\alpha-1} \tag{22}$$

If we still assume that $\forall t, A_t = 1$, we get:

$$k_{t+1}^{\alpha} - k_{t+2} = [k_t^{\alpha} - k_{t+1}](\alpha\beta)k_{t+1}^{\alpha-1} \tag{23}$$

$$k_{t+1}^{\alpha} - k_{t+2} = [k_t^{\alpha} k_{t+1}^{\alpha-1} - k_{t+1}^{\alpha}](\alpha\beta) \tag{24}$$

We set $\alpha\beta = \theta$.

$$k_{t+1}^{\alpha} - k_{t+2} = \theta k_t^{\alpha} k_{t+1}^{\alpha-1} - \theta k_{t+1}^{\alpha} \tag{25}$$

$$\Rightarrow (1+\theta)k_{t+1}^{\alpha} - k_{t+2} = \theta k_t^{\alpha} k_{t+1}^{\alpha-1} \tag{26}$$

$$\Rightarrow (1+\theta) - \frac{k_{t+2}}{k_{t+1}^{\alpha}} = \theta k_t^{\alpha} k_{t+1}^{-1} \tag{27}$$

$$\Rightarrow (1+\theta) - \frac{k_{t+2}}{k_{t+1}^{\alpha}} = \theta(\frac{k_{t+1}}{k_t^{\alpha}})^{-1} \tag{28}$$

We set $z_t = \frac{k_{t+1}}{k_t^{\alpha}}$, and we get:

$$(1+\theta) - z_{t+1} = \theta z_t^{-1} \tag{29}$$

$$\Rightarrow z_{t+1} = 1 + \theta - \theta z_t^{-1} \tag{30}$$

An unstable difference equation for z_t must remain at its rest-point, $\forall t, z_{t+1} = z_t = z^*$. It leads to:

$$z^* = 1 + \theta - \theta(z^*)^{-1} \tag{31}$$

$$\Rightarrow (z^*)^2 - (1+\theta)z^* + \theta = 0 \tag{32}$$

$$\Delta = (1+\theta)^2 - 4\theta \tag{33}$$

$$= 1 + 2\theta + \theta^2 - 4\theta \tag{34}$$

$$= 1 - 2\theta + \theta^2 \tag{35}$$

$$\Rightarrow \Delta = (1-\theta)^2 \geq 0 \tag{36}$$

The roots of the polynomial (32) thus are:

$$z^* = \frac{(1+\theta) \pm (1-\theta)}{2} \begin{cases} z_+^* = 1 \\ z_-^* = \theta \end{cases} \tag{37}$$

A strictly positive consumption is only possible if $k_t^\alpha > k_{t+1}$ (based on (20)), which is equivalent to $z_t < 1$. Thus, the only stable solution is $z_-^* = \theta = \alpha\beta$. We eventually get the law of motion for capital:

$$k_{t+1} = \alpha\beta k_t^\alpha \tag{38}$$

And the policy function for consumption (from (20)):

$$c_t = k_t^\alpha - \alpha\beta k_t^\alpha = (1-\alpha\beta)k_t^\alpha \tag{39}$$

5 Question 4

We set $\beta = 0.99$, $\alpha = 0.36$, $\delta = 0.025$. $\sigma = 1$ implies that $u(c_t) = \ln(c_t)$. The social planner programme is to maximize the utility function for $T \in \{50, 100, 200\}$.

$$\begin{aligned} \max_{\{k_t, c_t\}_{t=0}^T} \sum_{t=0}^T \beta^t \ln(c_t) \\ k_{t+1} = (1-\delta)k_t + i_t \\ k_0 = 0.9k^* \\ c_t + i_t \leq A_t k_t^\alpha \end{aligned}$$

The feasibility contraint is:

$$c_t = A_t k_t^\alpha + (1-\delta)k_t - k_{t+1} \tag{40}$$

The optimality condition in finite time is:

$$k_{T+1} = 0 \tag{41}$$

Because the problem is similar to 2, we can use the same Euler equation (11) to solve the problem.

$$c_{t+1} = c_t\beta[A_{t+1}\alpha k_{t+1}^{\alpha-1} + (1-\delta)] \tag{42}$$

With the law of motion for capital:

$$k_{t+1} = (1-\delta)k_t + A_t k_t^\alpha - c_t \tag{43}$$

With the initial condition $k_0 = 0.9k^*$, we can solve the problem by finding the right c_0 that will lead to $k_{T+1} = 0$.