Quantitative Macroeconomics

DA COSTA, Thomas
MESSAL, Amine

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Problem

$$\max_{\{c_t, k_{t+1}, i_t\}_{t=1}^{T}} w(c_t) = \max_{\{c_t, k_{t+1}, i_t\}_{t=1}^{T}} E_0 \sum_{t=0}^{T} \beta^t u(c_t) \tag{1}$$

$$\text{s.t.} \tag{2}$$

$$c_t + i_t \leq A_t k_t^{\alpha} \tag{3}$$

$$k_{t+1} = (1-\delta) k_t + i_t \tag{4}$$

$$k_0 \text{ given} \tag{5}$$

$$\text{for } u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma} \tag{6}$$

Question 1

u is continuously differentiable, strictly increasing, strictly concave, bounded, and satisfies the Inada conditions.

Feasibility constraint

The feasibility constraint can be written by combining 3 and 4, giving:

$$\Rightarrow c_t = A_t k_t^{\alpha} + (1-\delta) k_t - k_{t+1} \tag{7}$$

Euler Equation

The First Order Condition (the Euler equation for capital) comes from the derivative of $w(c_t)$ with respect to k_{t+1}, where we integrated the feasibility constraint into $w(c_t)$.

$$\frac{\partial w(c_t(A_t,k_t,k_{t+1}))}{\partial k_{t+1}} = 0 \tag{8}$$

$$\Rightarrow E_0(\beta^t \frac{\partial u(c_t(A_t,k_t,k_{t+1}))}{\partial k_{t+1}} + \beta^{t+1}\frac{\partial u(c_{t+1}(A_{t+1},k_{t+1},k_{t+2}))}{\partial k_{t+1}}) = 0 \tag{9}$$

$$\Rightarrow E_0\beta^t(\frac{\partial u(c_t)}{\partial c_t}\frac{\partial c_t}{\partial k_{t+1}} + \beta\frac{\partial u(c_{t+1})}{\partial c_{t+1}}\frac{\partial c_{t+1}}{\partial k_{t+1}}) = 0 \tag{10}$$

Based on the equation 7, we have $\frac{\partial c_t}{\partial k_{t+1}} = -1$ and $\frac{\partial c_{t+1}}{\partial k_{t+1}} = A_{t+1}\alpha k_{t+1}^{\alpha-1} + (1-\delta)$. Thus:

$$\frac{\partial u(c_t)}{\partial c_t} = \beta\frac{\partial u(c_{t+1})}{\partial c_{t+1}}[A_{t+1}\alpha k_{t+1}^{\alpha-1} + (1-\delta)] \tag{11}$$

Transversality condition

The transversality condition is:

$$\lim_{T\to\infty}\frac{\partial w(c_t)}{\partial c_t}k_{T+1} = 0 \tag{12}$$

Question 2

The steady-state levels are reached at $t = t^*$ such that $c_{t^*+1} = c_{t^*} = c^*$ and $k_{t^*+1} = k_{t^*} = k^*$. If $\forall t, A_t = 1$, the steady-state thus impose from 4:

$$k^* = \frac{i_t}{\delta} \tag{13}$$

$$k^* = \frac{(k^*)^\alpha - c^*}{\delta} \tag{14}$$

$$\delta k^* = (k^*)^\alpha - c^* \tag{15}$$

$$\Rightarrow c^* = (k^*)^\alpha - \delta k^* \tag{16}$$

The equation 11 gives the expression of k^*, knowing that in the steady-state, $u(c_{t^*}) = u(c_{t^*+1})$:

$$1 = \beta[\alpha(k^*)^{\alpha-1} + (1-\delta)] \tag{17}$$

$$\Rightarrow k^* = (\frac{\beta^{-1}+\delta-1}{\alpha})^{\frac{1}{\alpha-1}} \tag{18}$$

Question 3

An analytical solution is a policy function $c_t = g(k_t, \Theta)$ and a law of motion $k_{t+1} = h(k_t, \Theta)$, with Θ the parameters of the model. Let us choose $\sigma = 1 \rightarrow u(c_t) = \log(c_t)$ and $\delta = 1$. We can write the Euler equation as:

$$c_t^{-1} = \beta c_{t+1}^{-1}[A_{t+1}\alpha k_{t+1}^{\alpha-1}] \tag{19}$$

And the feasibility constraint as:

$$c_t = A_t k_t^{\alpha} - k_{t+1} \tag{20}$$

Combining 19 and 20, we get:

$$c_{t+1} = c_t(\alpha\beta)A_{t+1}k_{t+1}^{\alpha-1} \tag{21}$$

$$A_{t+1}k_{t+1}^{\alpha} - k_{t+2} = [A_t k_t^{\alpha} - k_{t+1}](\alpha\beta)A_{t+1}k_{t+1}^{\alpha-1} \tag{22}$$

If we still assume that $\forall t, A_t = 1$, we get:

$$k_{t+1}^{\alpha} - k_{t+2} = [k_t^{\alpha} - k_{t+1}](\alpha\beta)k_{t+1}^{\alpha-1} \tag{23}$$

$$k_{t+1}^{\alpha} - k_{t+2} = [k_t^{\alpha}k_{t+1}^{\alpha-1} - k_{t+1}^{\alpha}](\alpha\beta) \tag{24}$$

We set $\alpha\beta = \theta$.

$$k_{t+1}^{\alpha} - k_{t+2} = \theta k_t^{\alpha}k_{t+1}^{\alpha-1} - \theta k_{t+1}^{\alpha} \tag{25}$$

$$\Rightarrow (1+\theta)k_{t+1}^{\alpha} - k_{t+2} = \theta k_t^{\alpha}k_{t+1}^{\alpha-1} \tag{26}$$

$$\Rightarrow (1+\theta) - \frac{k_{t+2}}{k_{t+1}^{\alpha}} = \theta k_t^{\alpha}k_{t+1}^{-1} \tag{27}$$

$$\Rightarrow (1+\theta) - \frac{k_{t+2}}{k_{t+1}^{\alpha}} = \theta\left(\frac{k_{t+1}}{k_t^{\alpha}}\right)^{-1} \tag{28}$$

We set $z_t = \frac{k_{t+1}}{k_t^{\alpha}}$, and we get:

$$(1+\theta) - z_{t+1} = \theta z_t^{-1} \tag{29}$$

$$\Rightarrow z_{t+1} = 1 + \theta - \theta z_t^{-1} \tag{30}$$

An unstable difference equation for z_t must remain at its rest-point, $\forall t, z_{t+1} = z_t = z^*$. It leads to:

$$z^* = 1 + \theta - \theta(z^*)^{-1} \tag{31}$$

$$\Rightarrow (z^*)^2 - (1+\theta)z^* + \theta = 0 \tag{32}$$

$$\Delta = (1+\theta)^2 - 4\theta \tag{33}$$
$$= 1 + 2\theta + \theta^2 - 4\theta \tag{34}$$
$$= 1 - 2\theta + \theta^2 \tag{35}$$
$$\Rightarrow \Delta = (1-\theta)^2 \geq 0 \tag{36}$$

The roots of the polynomial 32 thus are:

$$z^* = \frac{(1+\theta) \pm (1-\theta)}{2} \begin{cases} z_+^* = 1 \\ z_-^* = \theta \end{cases} \tag{37}$$

A strictly positive consumption is only possible if $k_t^\alpha > k_{t+1}$ (based on 20), which is equivalent to $z_t < 1$. Thus, the only stable solution is $z_-^* = \theta = \alpha\beta$. We eventually get the law of motion for capital:

$$k_{t+1} = \alpha\beta k_t^\alpha \tag{38}$$

And the policy function for consumption (from 20):

$$c_t = k_t^\alpha - \alpha\beta k_t^\alpha = (1-\alpha\beta) k_t^\alpha \tag{39}$$

Question 4

We solve the finite-horizon problem. Starting at $k_0 = 0.9k^*$. We consider T = 50, 100, 200 and impose the optimality condition for $k_T + 1$.

Algorithm

We first initialize the parameters and variables of the model (see fig. 1). We then write a program that solves as a function of c_0 for the terminal capital $k_T + 1(c_0)$ (fig. 2). We add a multiple shooting algorithm using the bisection method, solving for c_0 that achieves $k_{T+1}(c_0) = k_{T+1}$ (when `question = 4`, see figure 3). We eventually plot the results for different values of T (fig 4).

Results

The results can be observed figure 5, where the trends of consumption and capital over $T+1$ periods have been plotted.

We control by pritting the values of K over time that the transversality assumption is verified, that is, $k_{T+1} = 0$. It can be observed in table 1.

```
% Define parameters

par.alpha = 0.36; % capital share
par.beta = 0.99; % discount factor
par.delta = 0.025;
T = [50, 100, 200];
par.eps = 1e-9;

% steady-state
kstar = ((par.delta - 1 + 1/par.beta) / par.alpha)^(1/(par.alpha - 1));
cstar = kstar^par.alpha - par.delta * kstar;

%k_0 definition
k_0 = 0.9 * kstar;
```

Figure 1: Initialization of parameters and calibration of initial conditions.

```
function [c,k, out] = NCGM(c_0, k_0, T, par)

    % Exogenous productivity shifter (we set A = 1 for all t)
    % Warning : for endogenous productivity shifter, RGM function should be
    % adapted to add the computation of A before the one of (k,c) in the loop
    A = ones(1,T+1);

    c = zeros(1, T+1);
    k = zeros(1, T+1);

    c(1) = c_0;
    k(1) = k_0;

    out = false;

    for t = 2:T+1
        %k or c should not be negative
        k(t) = (1 - par.delta)*k(t-1) + A(t-1)*k(t-1)^par.alpha - c(t-1);
        c(t) = c(t-1) * par.beta * (A(t) * par.alpha * k(t)^(par.alpha - 1) + 1 - par.delta);
        if k(t) < -par.eps || c(t) < -par.eps % par.eps insted of 0 to avoid looping indefinitely
            out = true;
            c(t+1:end) = 0;
            k(t+1:end) = 0;
            break;
        end
    end
end
```

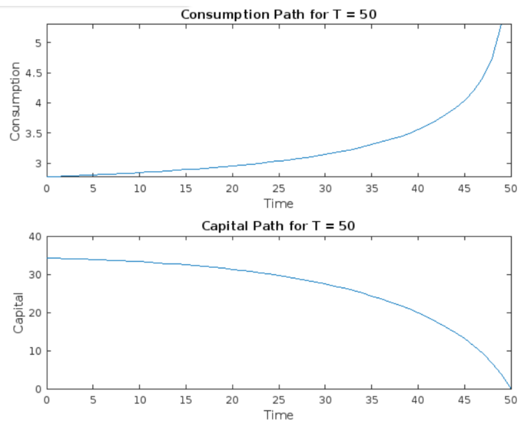
Figure 2: Algorithm providing the computation of the Ramsey Growth Model, c_0 and k_0 given.

```matlab
function [C, K] = shooting(k_0, kstar, cstar, par, T, maxiter, question)

    c_l = 0 ;
    c_u = 10 ;

    error = 1;
    i = 1;

    while abs(error) > par.eps && i <= maxiter
        i = i + 1;
        c_0 = (c_l + c_u)/2;
        [C, K, out] = NCGM(c_0, k_0, T, par);

        if question == 4
            if out == true
                error = -1;
                c_u = c_0;
            else
                error = K(T+1)/k_0;
                c_l = c_0;
            end
        elseif question == 5
            error = K(T+1) - kstar;
            if abs(error) < par.eps
                    break
            elseif error < 0
                c_u = c_0;
            else
                c_l = c_0;
            end
        else
            error("Question should either be 4 or 5");
        end


    end
    if i < maxiter
        fprintf(1, "The algorithm converged after %d iterations\n", i)
    else
        fprintf(1, "The algorithm failed to converge after %d iterations\n", maxiter)
    end
end
```

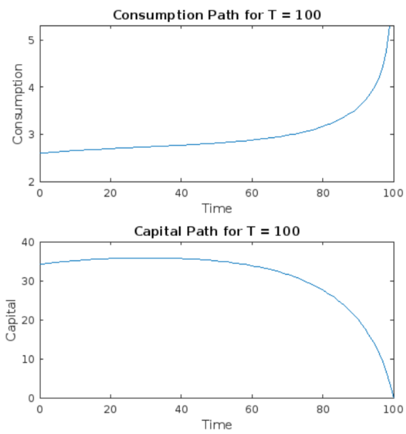
Figure 3: Algorithm using the bisection method to find c_0, given the analytical solution.

```
% Loop over all values of T
for i = 1:length(T)
    % Run the shooting algorithm
    [C, K] = shooting(k_0, kstar, cstar, par, T(i), 2000, 4);

    % Plot the sequences C(t) and K(t)
    figure;
    subplot(2, 1, 1);
    plot(0:T(i)-1, C(1:end-1));
    title(['Consumption Path for T = ', num2str(T(i))]);
    xlabel('Time');
    ylabel('Consumption');

    subplot(2, 1, 2);
    plot(0:T(i), K(1:end));
    title(['Capital Path for T = ', num2str(T(i))]);
    xlabel('Time');
    ylabel('Capital');

    % Sauvegarder la figure
    saveas(gcf, ['qu4_figure_T_', num2str(T(i)), '.png']);
end
```

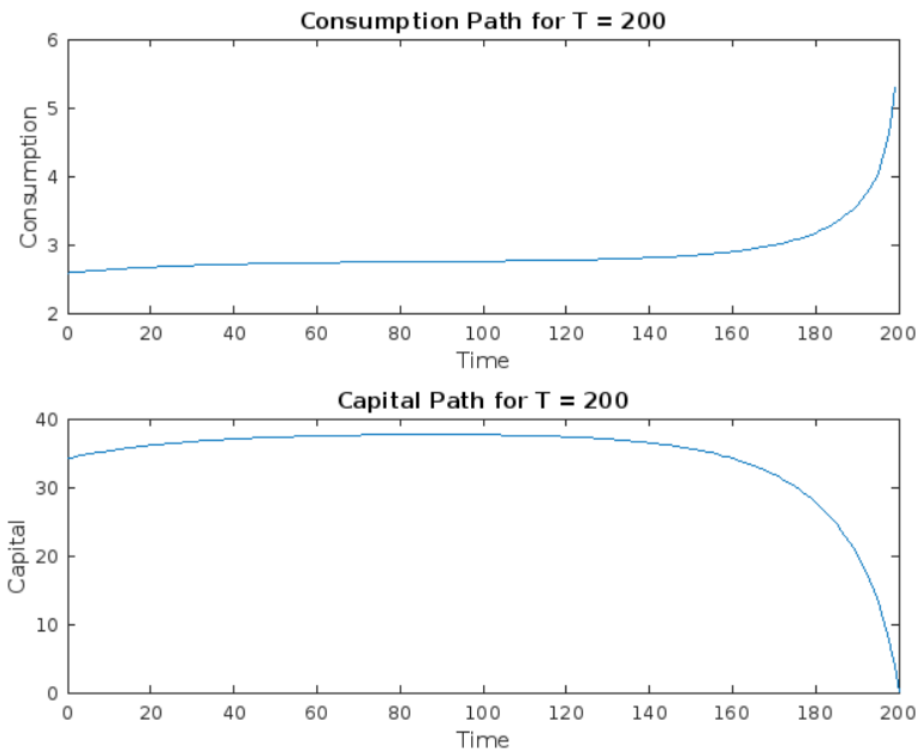
Figure 4: Algorithm for plotting the trends of consumption and capital over $T+1$ periods.



(a) $T = 50$



(b) $T = 100$



(c) $T = 200$

Figure 5: Consumption and capital over $T = 50, 100$ and 200 periods.

Table 1: Last values of K

Column 154 to 163							
35.2760	35.1461	35.0097	34.8663	34.7156	34.5571	34.3904	...
Column 164 to 173							
33.6321	33.4169	33.1903	32.9514	32.6996	32.4341	32.1541	...
Column 174 to 183							
30.8679	30.4992	30.1090	29.6956	29.2575	28.7928	28.2993	...
Column 184 to 193							
25.9887	25.3117	24.5874	23.8113	22.9778	22.0809	21.1132	...
Colonnes 194 à 201							
16.3309	14.8341	13.1721	11.3086	9.1910	6.7366	3.7955	0.0000

Question 5

Here we solve the infinite-horizon problem. Starting at $k_0 = 0.9k^*$. We consider $T = 200$ and impose that $k_{T+1} = k^*$.

Algorithm

We use the same algorithm as in figure 3 but this time setting `question = 5`. We plot our results based on the algorithm displayed figure 6.

Results

After solving for c_0 such that $k_{T+1}(c_0) = k^*$, here is the figure we obtain, ensuring that the economy converges to steady state in at most $T = 200$ periods :

```
T_5 = 200;

% Run the shooting algorithm
[C, K] = shooting(k_0, kstar, cstar, par, T_5, 2000, 5);

% Plot the sequences C(t) and K(t)
figure;
subplot(2, 1, 1);
plot(0:T_5, C);
title(['Consumption Path for T = ', num2str(T(i))]);
xlabel('Time');
ylabel('Consumption');

subplot(2, 1, 2);
plot(0:T_5, K);
title(['Capital Path for T = ', num2str(T(i))]);
xlabel('Time');
ylabel('Capital');

saveas(gcf, 'qu5_figure_T_200.png');
```

Figure 6: Algorithm for plotting the consumption and capital sequence over $T = 200$ period.

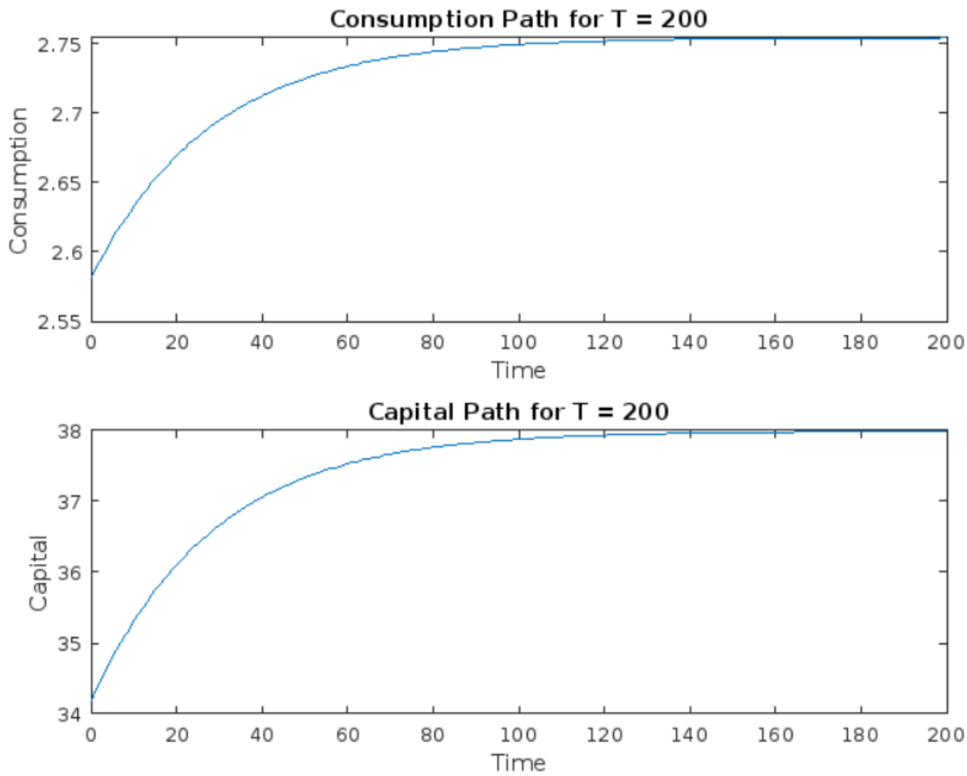


Figure 7: Consumption and capital over $T+1$ periods.