Problem Set #1

PSE Masters in Economics
Quantitative Macro, Fall 2024

Due Date: Thursday 3 October, 12 noon

Please hand in your answers, the computer programme (matlab or other) and the figures with results using file names that contain all group members' last names (e.g. SCHEIDENBERGER_SEMPE_PS_1.m).

Consider the following problem solved by a social planner for $t = 0, 1, ..., T$.

$$\max_{\{c_t, k_{t+1}, i_t\}_{t=1}^{T}} E_0 \sum_{t=0}^{T} \beta^t u(c_t)$$
$$\text{s.t.}$$
$$c_t + i_t \leq A_t k_t^{\alpha}$$
$$k_{t+1} = (1-\delta)k_t + i_t$$
$$k_0 \text{ given}$$
$$\text{for} \quad u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

where c_t is consumption, k_t the capital stock, i_t investment, A_t an exogenous productivity shifter, and $\beta, \alpha, \sigma, \delta$ parameters. Note that for $\sigma = 1$ u(c_t) $=$ $log(c_t)$.

1. Write down the first-order condition (Euler equation for capital), the feasibility constraint that links current consumption and output to capital next period (by combining the law of motion for capital and the resource constraint), and the $t = T$ conditions (transversality condition for infinite T, or condition for k_{T+1} with T finite).

2. Calculate analytically the constant "steady-state" levels when $A_t = 1 \ \forall t$ of capital $\bar{k}$ and consumption $\bar{c}$ such that $x_s = \bar{x}, = t...\infty, \ x = c, k$ satisfies feasibility and optimality.

3. Consider $\delta = 1$ and $\sigma = 1$. Solve analytically for the policy functions $c_t(k_t)$ and law of motion $k_{t+1}(k_t)$ (see the slides to get you started, noting that $z_t = \frac{k_{t+1}}{k_t^{\alpha}}$ is not predetermined but needs to remain between 0 and 1. An unstable difference equation for z_t must therefore remain at its rest-point $z_t = z_{t+1} = \bar{z}$ for all t. Choose the rest point that is compatible with positive consumption.).

Consider the following parameter values, roughly corresponding to a quarterly calibration:

β	α	σ	δ
0.99	0.36	1	0.025

Exercise 4 and 5 are intuitive, numerical ways of finding the optimum in the finite horizon problem, and the saddle-path in a phase-diagram with infinite horizon, based on finding a scalar unknown (c_0). Exercise 6 and 7 are more complicated, based on finding an unknown vector ($\{r_t\}$), but prepare you for the more general sequence-space methods in later parts of the course. Do EITHER exercise 4 and 5, OR exercise 6 and 7.

4. **Finite T** Consider a deterministic path of the finite-horizon problem, $\{k_t, c_t\}_{t=0}^{T}$ with short-hand $\{k_t, c_t\}$, starting at $k_0 = 0.9\bar{k}$. Consider $T = 50, 100, 200$ and impose the optimality condition for k_{T+1}. Write a program that solves as a function of c_0 for the terminal capital $\hat{k}_{T+1}(c_0)$ using the Euler equation for capital investment, the feasibility constraint, and the definition of the marginal return of capital. Use a multiple shooting algorithm (you can use for example the bisection method) to solve for c_0 that achieves $\hat{k}_{T+1}(c_0) = k_{T+1}$. Plot the sequences c$_t, k_t$.

   Algorithm (Pseudo code)

   (a) Choose a convergence measure, e.g. $|\theta(c_0)| = \left|\frac{\hat{k}_{t+1}(c_0) - k_{T+1}}{k_0}\right|$.

   (b) Clear workspace

   (c) Declare parameter values, choose a convergence criterion ϵ that works for your convergence measure.

   (d) Choose an initial guess for c_0 (or for bisection two values c_0^u, c_0^l, s.t. $\theta(c_0^u) < 0, \theta(c_0^l) > 0$).

   (e) Loop over T periods starting from c_0, k_0:
   $y_t = ...$; $k_{t+1} = y_t - c_t + (1-\delta)k_t$; $c_{t+1} = c_t * [xxxx]$ from the Euler Equation (fill in for [xxxx]), $t = 0, 1, 2, ...$

   (f) If $|\theta(c_0)| < \epsilon$ stop. O/w update c_0 and go back to 3.

5. **A numerical "Phase diagram"**
   Now consider a deterministic path of the infinite-horizon problem starting at $k_0 = 0.9\bar{k}$. Consider T large enough, and assume the economy converges to steady state in at most T periods. Use the same algorithm as in **3.** to solve for c_0 such that $k_{T+1} = \bar{k}$. Plot the sequences $\{c_t, k_t\}$.

6. **Towards competitive equlibrium**
Set $c_T = \bar{c}, k_0 = \bar{k}$.
a. Write a function that takes a path of the net marginal product of capital $\{r_t\}_1^{T+1}, r_t = \alpha k_{D,t}^{\alpha-1} - \delta$ as input (where $k_{D,t}$ is firms' demand for capital) and outputs a $T \times 1$ vector of excess demand for capital $\{dk\} = \{k_S\} - \{k_D\}$. For this, use the Euler equation to solve for the consumption path $\{c_t\}_0^{T-1}$, then use the resource constraint to calculate household capital supply $\{k_S\}$.

Pseudo code

(a) Declare a function that takes as inputs $\{r_t\}$, parameters $\alpha, \beta, \delta, \sigma, T$, $\bar{k}$ and k_0, and whose output is a vector $\{dk\} = \{k_{S,t} - k_t(r_t)\}$.

(b) For $t = T, T-1, ..., 1$ calculate $k_{D,t}(r_t)$ from the definition of the marginal product.

(c) For $t = T-1, T-2, ..., 0$, calculate $c_t = [xxxx]c_{t+1}$ from the Euler Equation (fill in for [xxxx]).

(d) For $t = T-1, T-1, ..., 0$, calculate $k_{S,t}$ from the budget constraint and $\{c_t\}$, given $c_T = \bar{c}$, $k_{S,T+1} = \bar{k}$.

(e) Calculate $\{dk\}$.

b. Write a program that uses Broyden's method to solve for the paths $\{r_t, k_t, c_t\}$ starting at k_0 that are consistent with $\{dk\} = 0$. Plot the sequences $\{c_t, k_t\}$. Compare them to those from the analytical solution for the case $\delta = 1$.

Algorithm (Pseudo code)

(a) Choose a convergence measure for the guess (e.g. $\theta(\{dr\}) = \frac{max(abs(\{r_t^{new} - r_t^{old}\}))}{\bar{r}}$) or the error (e.g. $\theta(\{dk\}) = \frac{max(abs(\{dk\}))}{\bar{k}}$).

(b) Clear workspace, declare parameters, choose a convergence criterion ϵ.

(c) Guess a path of interest rates $\{r_t\}$, $t = 1, ..., T$. Set $i = 1$.

(d) Calculate $\{dk\}$ using the program above.

(e) If $i = 1$, and $\theta(\{.\}) > \epsilon$, numerically calculate the N^2 elements of the Jacobian $\frac{\partial dk_t}{\partial r_{s+1}}$ for $t, s = 1, ..., T$. Use Newton's formula to update the guess $\{r_t\}$ and set $i = i + 1$.

(f) If $i > 1$, and $\theta(\{.\}) > \epsilon$, update the guess using Broyden's formula and set $i = i + 1$. Use the formula for the recursive updating of the Jacobian from the lecture slides.

(g) If $\theta(\{.\}) < \epsilon$, you have found a solution.

7. **A TFP shock**
Repeat the previous excercise starting from $k_0 = \bar{k}$ but set $\{A_t\}$ to $1 + 0.01, 1 + 0.01 \times 0.97^1, 1 + 0.01 \times 0.97^2, ...$