SOC542 Statistical Methods in Sociology II

Multiple Regression

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February 14, 2022

Plan

- Recap
- Multiple regression: An overview
- Lab: Bayesian regression in R

Recap

What we have learned so far

- Principles of frequentist inference
- Simple linear regression
- Probability and Bayesian inference

Multiple regression

OLS assumptions review

- x and y are independently and identically distributed (IID).
  - The sample x must contain some variability. Specifically, $var(x) > 0$.
- The conditional distribution of u given x has a mean of zero.
  - Errors are independent $E[u_i|x_i] = E[u_i] = 0$.
  - Errors have constant variance $var(u_i) = \sigma^2$.
  - Errors are uncorrelated.
- If these assumptions are met, then OLS is **BLUE**
  - The **Best Linear conditionally Unbiased Estimator**

Multiple regression

Simple linear regression

- Let's say we estimate a simple linear regression of the form:

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x + \hat{u}$$

- In this case, we assume that the outcome y is a linear function of a single predictor x.
- But what if we think have reason to believe that y is also a function of other predictors?

Multiple regression

Omitted variable bias

- Omitted variable bias occurs when we leave out (or *omit*) a predictor that should be in our model.
- Omitted variable bias exists when
  - x is correlated with the omitted variable z.
  - The omitted variable is a predictor of the dependent variable y.

Multiple regression

Consequences omitted variable bias

- The assumption that $E(u_i|x_i) = 0$ is violated.
  - If z is correlated with x but not included, then u captures the unmeasured effect of z *and* thus u is correlated with x.
- The slope coefficient β_1 will be *biased*.
  - The mean of the sampling distribution of the OLS estimator may not equal the true effect of x.
  - $\hat{\beta}_1 = \beta_1 + bias$
- The OLS estimator is *inconsistent* as $\hat{\beta}_1$ does not converge in probability to β_1.
  - Bias remains even with large samples.
- The greater the correlation between x and u, the greater the bias.

Multiple regression

Example

- Let's say we have a model of income as a function of age:

$$y_i = \beta_0 + \beta_1 Age + u$$

- We estimate the model and see a strong, positive relationship between age and income (i.e. $\hat{\beta}_1$ is positive)

Multiple regression

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- What other factors might be correlated with age *and* predict income?

Multiple regression

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- We estimate the model and see a strong, positive relationship between age and income (i.e. $\hat{\beta}_1$ is positive)
- What other factors might be correlated with age *and* predict income?

$$y_i = \beta_0 + \beta_1 Age + \beta_2 Education + u$$

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Example

- Let's say we have a model of income as a function of age:

$$y_i = \beta_0 + \beta_1 Age + u$$

- We estimate the model and see a strong, positive relationship between age and income (i.e. $\hat{\beta}_1$ is positive)
- What other factors might be correlated with age *and* predict income?

$$y_i = \beta_0 + \beta_1 Age + \beta_2 Education + u$$

- Education is correlated with age and predicts income. Our estimate of the effect of age is *biased* without taking education into account.

Simulating omitted variable bias

```
N <- 100
x <- rnorm(N,2,1)
z <- 0.1*x + rnorm(N,5,2)
y <- 0.8*x + -2*z + rnorm(N, 0, 1)
```

Simulating omitted variable bias

```r
m.omit <- lm(y ~ x)
m.both <- lm(y ~ x + z)
print(m.omit$coefficients)
```

```
## (Intercept)           x
##  -9.8047561   0.7414095
```

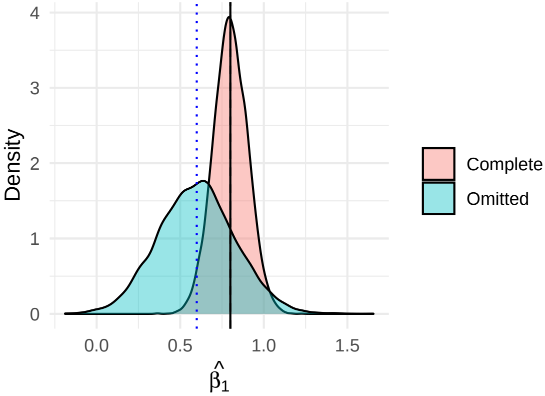
```r
print(m.both$coefficients)
```

```
## (Intercept)           x           z
##   0.1864063   0.8167683  -2.0324375
```

Simulating omitted variable bias

```r
coefs.omitted <- c()
coefs.complete <- c()
sims <- 1E4
for (i in 1:1E4) {
    x <- rnorm(N,2,1)
    z <- 0.1*x + rnorm(N,5,1)
    y <- 0.8*x + -2*z + rnorm(N, 0, 1)
    m.omit <- lm(y ~ x)
    m.both <- lm(y ~ x + z)
    coefs.omitted[i] <- m.omit$coefficients[2]
    coefs.complete[i] <- m.both$coefficients[2]
}
```

Simulating omitted variable bias



Distribution of $\hat{\beta}_1$ over 10000 simulations.

Multiple regression

The multiple regression model

- In a multiple regression model, we specify a linear relationship between an outcome and a set of k predictors.

$$E[y|x_1, x_2, ..., x_k] = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + ... + \beta_k x_k + u$$

Multiple regression

Independent variables and controls

- The predictors added to the model can be considered as additional **independent variables** or as **controls**.
- In general, we use the former term when we have a theoretical reason to be interested in a variable and the latter when we expect it to matter but are not interested in analyzing the relationship directly.
- We typically add control variables to hold constant other factors that could produce omitted variable bias if unaccounted for.

Multiple regression

Interpreting coefficients

- Consider the following population model:

$$y_i = \beta_0 + \beta_1 x + \beta_2 z + u$$

- β_1 is the effect of a unit change in x *when z is held constant*.

$$\beta_1 = \frac{\Delta y}{\Delta x}, \textit{holding z constant}$$

Multiple regression

Interpreting the intercept

- Consider the same model:

$$y_i = \beta_0 + \beta_1 x + \beta_2 z + u$$

- β_0 is the expected value of y_i when $x = 0$ and $z = 0$.

Multipe regression

The OLS estimator

- The OLS estimator minimizes the sum of the squared residuals.
- Over n observations and k predictors, we minimize the following quantity:

$$\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_{1i} - \beta_1 x_{2i} - ... - \beta_k x_{ki})^2$$

- Thus, the predicted values are

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_{1i} + \hat{\beta}_2 x_{2i} + ... + \hat{\beta}_k x_{ki}$$

- And the residuals are defined as $\hat{u}_i = y_i - \hat{y}_i$ for $i = 1, ..., n$.

Multiple regression

Model fit and the Standard Error of the Regression

- The **Standard Error of the Regression (SER)** is an estimate of the standard deviation of the error term u_i. It captures the spread of y around the regression line.
- For a single regressor,

$$SER = \sigma_{\hat{u}} = \frac{1}{n-2}\sum_{i=1}^{n} \hat{u}_i^{\,2} = \frac{SSR}{n-2}$$

- We subtract 2 because both the slope and the intercept are estimated from the data.
- In general, a smaller value indicates that the model fits the data better.

Model fit and the Standard Error of the Regression

- If we have multiple predictors we need to include a degrees of freedom adjustment:

$$SER = \sigma_{\hat{u}} = \frac{1}{n-k-1}\sum_{i=1}^{n} \hat{u}_i{}^2 = \frac{SSR}{n-k-1}$$

- The adjustment has a small effect when n is large.

Multiple regression

Model fit and R^2

- We define R^2 in the sample way as a simple regression:

$$R^2 = \frac{\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2} = \frac{ESS}{TSS}$$

$$R^2 = 1 - \frac{SSR}{TSS}$$

Multiple regression

Adjusted R^2

- R^2 increases as we add predictors because the SSR declines as long as $\hat{\beta}_k \neq 0$. It therefore provides an inflated measure of model fit.
- We can adjust for the fact that new variables are expected to increase model fit by adding a degrees of freedom correction, where k is the number of predictors:

$$Adjusted\ R^2 = 1 - \frac{n-1}{n-k-1}\frac{SSR}{TSS}$$

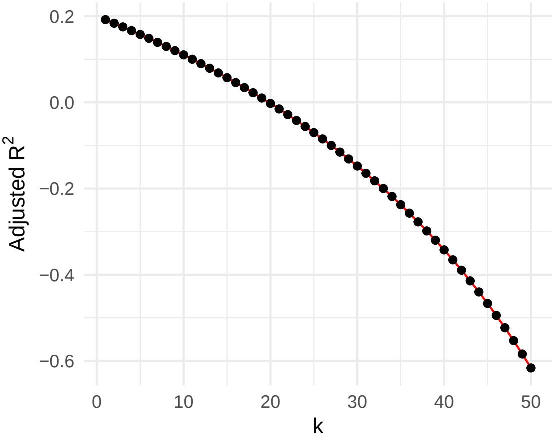
Multiple regression

Properties of adjusted R^2

- Adjusted R^2 is *always less than* R^2.
- Adding a predictor can increase Adjusted R^2, but it can decline if the change to the SSR is weaker than the offset $n - 1/n - k - 1$.
- Adjusted R^2 can be negative if the reduction in SSR does not offset $n - 1/n - k - 1$.

Multiple regression

The penalty $\frac{n-1}{n-k-1}$ increases as we add predictors



This example shows the effect of the degree of freedom adjustment, assuming $\beta_k = 0$ for all $k > 1$.

Multiple regression

Bayesian R^2

- There is no direct analogue for R^2 in Bayesian statistics
  - Recall that frequentist models assume *fixed* parameters, whereas Bayesian parameters have *distributions*.
- If we treat the Bayesian estimates as fixed, for example by taking the median of the posterior distribution $\hat{\beta}_k$, we could calculate something using the formula above, but it would not account for the *uncertainty* contained in the posterior distribution.

Multiple regression

Bayesian R^2

- Instead, we use posterior simulations to repeat the calculation across all samples from the posterior.
- Bayesian R^2 therefore has a posterior distribution.[1] We can summarize this into a single metric using the same approach as the regression coefficients, e.g. using the median of the posterior distribution.[2]

[1] "Everything that depends upon parameters has a posterior distribution" - McElreath 98.

[2] See GHV 170-171

Multiple regression

Significance tests: t-tests

- Like simple linear regression, we typically interpret the statistical significance of regression coefficients using the t-statistics.
- Typically, we are interested in testing the null hypothesis that $\beta_k = 0$. We get the t-statistic by dividing a coefficient by its standard error:

$$t = \frac{\hat{\beta}_k - 0}{SE(\hat{\beta}_k)} = \frac{\hat{\beta}_k}{SE(\hat{\beta}_k)}$$

- We can the use the t-statistic to look up the relevant *p-value*.

Multiple regression

Confidence interval

- Most regression software provides a 95% confidence interval around each estimate. For β_k this would take the following form:

$$[\hat{\beta}_j - 1.96SE(\hat{\beta}_j), \hat{\beta}_j + 1.96SE(\hat{\beta}_j)]$$

- Recall that only 5% of the probability density of a t-distribution is greater than $|1.96|$.

Multiple regression

Joint tests

- The F-statistic is used to test a **joint hypothesis**.
- If we consider a two variable example, we might test the following *null hypothesis*:

$$H_N : \beta_1 = 0, \beta_2 = 0$$

- A joint test has q restrictions. In this case, $q = 2$.
- The *alternative hypothesis* H_A is that one or more of the q restrictions does not hold.

Multiple regression

Joint tests and the F-statistic

- Since we expect the predictors to have a *joint sampling distribution*, we cannot conduct a joint test by summarizing a series of paired tests (e.g. a t-test for every predictor) because the t-statistics are not independent.
- Instead, we must calculate the F-statistic. Where $q = 2$ it is defined as:

$$F = \frac{1}{2}\left(\frac{t_1^2 + t_2^2 - 2\hat{\rho}_{t_1,t_2} t_1 t_2}{1 - \hat{\rho}_{t_1,t_2}^2}\right)$$

Multiple regression

Joint tests and the F-statistic

- If $\hat{\rho}_{t_1,t_2} = 0$ the equation simplifies to the average of the squared t-statistics:

$$F = \frac{1}{2}(t_1^2 + t_2^2)$$

- The p-value can then be derived from the relevant *F-distribution*, where $F \sim F_{q,\infty}$
- Typically, we use an F-test to test the restriction that $\beta_1 = 0, \beta_2 = 0, ..., \beta_k = 0$.

Multiple regression

Joint tests and the F-statistic

- If we assume the residuals are *homoskedastic*, we can test the restriction $\beta_1 = 0, \beta_2 = 0, ..., \beta_k = 0$ using the following formula:

$$F_0 = \frac{(SSR_r - SSR_u)/q}{SSR_u/(n - k + 1)}$$

- The SSR_r is obtained from *restricted* model where we calculate the SSR assuming the null hypothesis is true. The SSR from the fitted model, SSR_u, is known as the *unrestricted* SSR$>$.
- The test statistic is assessed using an *F-distribution* with q degrees of freedom and $n - k + 1$ observations.
- In most cases the homoskedasticity assumption is likely violated, so we use the more complicated formula from the previous slide, known as the *heteroskedasticity robust* F-statistic.

Multiple regression

Interpreting regression output

```
Call:
lm(formula = y ~ x + z)

Residuals:
    Min      1Q  Median      3Q     Max
-2.5202 -0.4995  0.1230  0.6302  1.9269

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.74462    0.48670   1.530    0.129
x            0.60325    0.08670   6.958 4.08e-10 ***
z           -2.06166    0.08673 -23.772  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.8768 on 97 degrees of freedom
Multiple R-squared:  0.8598,	Adjusted R-squared:  0.8569
F-statistic: 297.4 on 2 and 97 DF,  p-value: < 2.2e-16
```

Multiple regression

Bayesian approaches

- "We have essentially no interest in using hypothesis tests for regression because we almost never encounter problems where it would make sense to think of the coefficients as being exactly zero" - GHV 147
- Bayesian regression is assessed by analyzing the posterior distribution of parameters to understand uncertainty.
- Nonetheless, Bayesian equivalents to t-tests and F-tests can be used if desired.[3]

[3] See Kruschke and Liddell 2018.

Multiple regression

Multicollinearity

- **Multicollinearity** occurs when a predictor x is highly correlated one or more other predictors z.
  - **Perfect multicollinearity** arises when $cor(x, z) = 1$ or $cor(x, z) = -1$.
    - Usually due to some type of misspecification. e.g. accidentally including the same variable twice.
  - **Imperfect multicollinearity** means that two or more regressors are highly correlated.

Multiple regression

Multicollinearity and its implications

- Assume the following model and that x and z are highly correlated:

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i + \hat{\beta}_2 z_i + u_i$$

- The variance of $\hat{\beta}_1$[4] is inversely proportional to $1 - \rho^2_{x,z}$, where $\rho_{x,z}$ is the correlation between x and z.
    - If $\rho_{x,z}$ is large, then this term is small and thus the variance is large.
- Multicollinearity *increases variance* and *reduces precision*, potentially making β_1 **non-identifiable**.

[4] The same issue also applies to $\hat{\beta}_2$

Simulating multicollinearity

```r
N <- 100
x <- rnorm(N,2,1)
x2 <- rnorm(N,0,1)
z <- 0.7*x + rnorm(N,0,1)
y <- 0.5*x + -0.5*x2 + 0.5*z + rnorm(N, 1, 1)
```

Simulating multicollinearity

```r
m1 <- summary(lm(y ~ x + x2))
m2 <- summary(lm(y ~ x + x2 + z))
round(m1$coefficients,2) # omitted variable bias
```

```
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)     0.72       0.26    2.80     0.01
## x               0.95       0.11    8.35     0.00
## x2             -0.57       0.11   -4.99     0.00
```

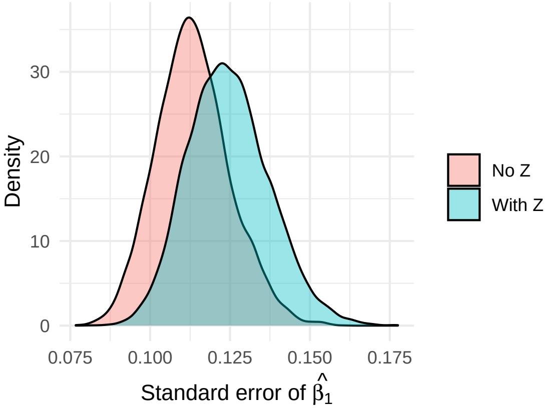
```r
round(m2$coefficients,2) # multicolinearity
```

```
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)     0.73       0.25    2.97        0
## x               0.67       0.14    4.79        0
## x2             -0.59       0.11   -5.40        0
## z               0.42       0.13    3.21        0
```

Simulating multicollinearity

```
se.omitted <- c()
se.complete <- c()
sims <- 1E4
for (i in 1:1E4) {
    x <- rnorm(N,2,1)
    x2 <- rnorm(N,0,1)
    z <- 0.7*x + rnorm(N,0,1)
    y <- 0.5*x + -0.5*x2 + 0.5*z + rnorm(N, 1, 1)
    m.omit <- summary(lm(y ~ x + x2))
    m.complete <- summary(lm(y ~ x + x2 + z))
    se.omitted[i] <- m.omit$coefficients[2,2]
    se.complete[i] <- m.complete$coefficients[2,2]
}
```

Simulating multicollinearity



Distribution of standard error over 10000 simulations.

Multiple regression

Fixing multicollinearity

- In general, multicollinearity is less severe than omitted variable bias.
  - The inflated variance will lead to more Type II errors than Type I errors.
  - Omitted variable bias can produce Type I errors, sign errors, and magnitude errors.

Fixing multicollinearity

- *Solution 1*: Use more data. If we have a larger sample then we might be able to learn from additional variation in x and z.
- *Solution 2*: If we are only concerned about x then we could exclude z. But this risks omitted variable bias if z is also a predictor of y.
- *Solution 3*: Transform or combine predictors (e.g. factor analysis).

Multiple regression

Revisiting our assumptions

- $E(u_i|x_{1i}, x_{2i}, ..., x_{ki}) = 0$
- All $y_i, x_{1i}, x_{2i}, .., x_{ki}$ are IID.
- Large outliers are unlikely.
- No perfect multicollinearity.

Multiple regression

Causality and variable selection

- While the topics of omitted variable bias and multicollinearity are commonly discussed in the context of multiple regression, we generally pay less attention to the causal relations between predictors that can impact our inferences.

Multiple regression

Spurious relationships and confounding

- Sometimes we observe **spurious** relationships in regression models where a correlation between two variables exists, despite the absence of any causal relationship.
  - e.g. Finding that hurricanes with female names *caused* more deaths than male named hurricanes.
- Sometimes this occur purely due to chance but it can be due to **confounding**: a confounding variable z influences both x and y.
- Adding more predictors can often help to reduce the risk of spurious associations.
  - If we control for the confounder z, the spurious relationship between y and x disappears.

Multiple regression

Masked relationships

- Assume a true relationship between y and x.
- We estimate a model $y = \beta_0 + \beta_1 x$.
- The results do not show evidence of an association (i.e. $\hat{\beta}_1 \approx 0$).
- We estimate a second model including a new predictor z.
- Controlling for z allows us to observe a relationship between y and x.

Multiple regression

Colliders

- A more complicate issue occurs when z is a common cause of y and x.
- In this case, controlling for z can introduce bias.
- Collider bias is rarely considered in sociological research but has recently been receiving more attention from methodologists [5].

[5] See McElreath Chapter 6; Cunningham Chapter 3; Morgan, Stephen L., and Christopher Winship. 2014. *Counterfactuals and Causal Inference*. 2nd ed. Cambridge University Press.

Multiple regression

OLS in matrix form

- Ordinary least squares estimates can be computed directly using matrix multiplication, where X is a matrix of predictors and the first column is a vector of 1s and y is the outcome.

$$\hat{\beta} = X^T X^{-1} X^T y$$

Multiple regression

OLS in matrix form

```
Intercept <- rep(1,N)
X <- cbind(Intercept,x,x2,z)

Betas <- solve(t(X) %*% X) %*% (t(X)  %*% y)
print(t(Betas))
```

```
##      Intercept         x         x2         z
## [1,]  1.001835 0.5775868 -0.6165326 0.3963238
```

```
m <- lm(y ~ x + x2 + z)
print(m$coefficients)
```

```
## (Intercept)           x          x2           z
##   1.0018350   0.5775868  -0.6165326   0.3963238
```

`t()` is the transpose operation, `solve()` finds the inverse of a matrix, and `%*%` is the matrix multiplication operator.

Multiple regression

OLS in matrix form

- This demonstrates how OLS estimates are derived directly from algebraic manipulation of the data.
- OLS is a special case. Other approaches we will encounter in a few weeks require more complicated *maximum likelihood estimation*.
- Bayesian regression with uniform priors will converge to the least squares solution, despite a radically different estimation procedure.

Next week

Non-linear predictors

- Dummy variables
- Categorical variables
- Non-linear transformations