SOC542 Statistical Methods in Sociology II

Introduction to Bayesian statistics

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Plan

- Probability review
- Bayes' theorem and its applications
- Comparing Bayesian and Frequentist approaches
- Bayesian estimation
- Lab: Bayesian regression in R

Probability review

Simple probability

- $P(A)$ refers to the probability of an event A
  - e.g. $P(A) = 0.5$ when referring to the probability of receiving a heads on a fair coin toss.
  - e.g. $P(B) = \frac{1}{6}$ is the probability of rolling six with a fair die.
- In each case, we have a *random process* with a set of possible outcomes (e.g. heads or tails) referred to as the *sample space*.

Probability review

Simple probability

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- What is the probability of a sequence of N heads?
  - $P(A)^N$
- In this case, $P(A)$ becomes vanishingly small as $n \rightarrow \infty$
  - $0.5^{10} = 0.00098 = \frac{1}{1024}$

Probability review

Simple probability

- We can easily use simulations to verify our calculation. In this case, I use the `rbinom` function to simulate 1024 sequences of 10 tosses of a fair coin.

```
sims <- rbinom(1024, 10, 0.5)
print(length(sims[sims >= 10]))
```

```
## [1] 0
```

Probability review

Independence

- Assume we roll a single die and flip a single coin. What is the probability of rolling a six and getting a tails?

Probability review

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$$P(A, B) = P(A)P(B) = \frac{1}{2} * \frac{1}{6} = \frac{1}{12}$$

Probability review

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- The two events are independent of one another, so the *joint probability* is simply the product of the probabilities of the two events.

Probability review

Conditional probability and independence

- $P(A)$ and $P(B)$ are independent *if and only if* $P(A|B) = P(A)$.
  - e.g. The number we rolled on the die has no effect on the outcome of the coin toss.

Probability review

Conditional probability and independence

- Consider a deck of 52 standard playing cards. What is the probability of randomly drawing an Ace?[1]

[1] Example from Cunningham 2021, p. 17.

Probability review

Conditional probability and independence

- Consider a deck of 52 standard playing cards. What is the probability of randomly drawing an Ace?

$$P(Ace) = 4/52 = 1/13$$

- Let's assume we pick an Ace and put it to the side. What's the probability we get another Ace?

Probability review

Conditional probability and independence

- Consider a deck of 52 standard playing cards. What is the probability of randomly drawing an Ace?

$$P(Ace) = \frac{4}{52} = \frac{1}{13}$$

- Let's assume we pick an Ace and put it to the side. What's the probability we get another Ace?
- Wrong answer: $P(Ace_2) = \frac{4}{52} = \frac{1}{13}$.

Probability review

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- Let's assume we pick an Ace and put it to the side. What's the probability we get another Ace?
- Wrong answer: $P(Ace_2) = \frac{4}{52} = \frac{1}{13}$.
- Correct answer: $P(Ace_2) = P(Ace_2|Ace_1) = 3/51 = 0.059$.
- This is an example of *conditional probability* since $P(Ace_2|Ace_1) \neq P(Ace_1)$.

Probability review

Conditional probability and independence

- We can express a conditional probability as:

$$P(A|B) = \frac{P(B, A)}{P(B)}$$

- The probability of A conditional on B is the **joint probability** of A and B, divided by the **marginal probability** of B.
- The denominator is the sum of over possible joint probabilities of B and A, $\sum_{A^*} P(B, A^*)$.
  - The $*$ denotes that A^* may take multiple values.

Probability review

Conditional probability and independence

- If two events are independent, then $P(A|B) = P(A)$.
- To reject independence, we need to show that $P(A, B) \neq P(A)P(B)$

Probability review

Bayes' theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Probability review

Bayes' theorem

- What's the probability it is going to rain given that we can see clouds?

$$P(Rain|Cloud) = \frac{P(Cloud|Rain)P(Rain)}{P(Cloud)}$$

Probability review

Bayes' theorem

- Let's say we live in England...
  - $P(Cloud) = 0.7$
  - $P(Rain) = 0.3$
  - $P(Cloud|Rain) = 1$

$$P(Rain|Cloud) = \frac{P(Cloud|Rain)P(Rain)}{P(Cloud)} = \frac{1 * 0.3}{0.7} = \frac{0.3}{0.7} \approx 0.429$$

Probability review

Deriving Bayes' theorem

- Start with the definition of conditional probability:

$$P(A|B) = \frac{P(B,A)}{P(B)}$$

- Multiply each side by $P(B)$:

$$P(A|B)P(B) = P(B,A)$$

- Analogously, if we start with $P(B|A)$ we can get:

$$P(B|A)P(A) = P(B,A)$$

Probability review

Deriving Bayes' theorem

- The previous example shows that the following quantities are equal:

$$P(A|B)P(B) = P(B|A)P(A)$$

- Divide both sides by $P(B)$ to get Bayes' theorem:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Bayes' theorem

COVID-19 tests

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$P(C19|+) = \frac{P(+|C19)P(C19)}{P(+)}$$

Bayes' theorem

COVID-19 tests

$$P(C19|+) = \frac{P(+|C19)P(C19)}{P(+)}$$

- $P(C19|+)$: Probability you have COVID-19 given that you test positive.
- $P(+|C19)$: Probability you test positive given that you have COVID-19.
- $P(C19)$: Probability you have COVID-19 given population infection rates.
- $P(+)$: Probability a test returns a positive result.

Bayes' theorem

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Bayes' theorem

COVID-19 tests

- Assume there is a 1% chance you have COVID-19.
- Assume a test has a false negative rate of 2%.
    - 98% of the time it correctly diagnoses COVID-19, 2% of the time it fails to detect it.
- Assume the same test has a false positive rate of 5%
    - 95% of the time it correctly rejects COVID-19 when a person is negative, 5% of the time it falsely diagnoses COVID-19.
- What is the probability you really have COVID-19 following a positive test?

Bayes' theorem

COVID-19 tests: P(+|C19)

$$P(C19|+) = \frac{P(+|C19)P(C19)}{P(+)}$$

- If we assume a false negative rate of 2%. Then the probability of a positive test given COVID-19 is $P(+|C19) = 1 - 0.02 = 0.98$.

Bayes' theorem

COVID-19 tests: P(C19)

$$P(C19|+) = \frac{P(+|C19)P(C19)}{P(+)}$$

- Assume 1% of the population has COVID-19, then $P(C19) = 0.01$.

Bayes' theorem

COVID-19 tests: P(+)

- To calculate the proportion of positive tests we need to count all the positive texts, irrespective of whether someone is positive.
- To obtain this, we can reformulate Bayes rule as

$$\frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A*)P(A*)}$$

$$\frac{P(+|C19)P(C19)}{P(+|C19)P(C19) + P(+|C19-)P(C19-)}$$

Bayes' theorem

COVID-19 tests: P(+)

$$P(C19|+) = \frac{P(+|C19)P(C19)}{P(+)}$$

- We already know the first part of the denominator, $P(+|C19)P(C19) = 0.98 * 0.01$.
- If the test has a false positive rate of 5%, $P(+|C19-) = 0.05 * (1 - 0.01)$
- Thus, we take the sum of these probabilities to get the marginal probability of a positive test: $P(+) = (0.98 * 0.01) + (0.05 * (1 - 0.01))$

Bayes' theorem

COVID-19 tests: Calculating P(C19|+)

- If we plug the numbers into Bayes' theorem we get

$$P(C19|+) = \frac{0.98 * 0.01}{0.98 * 0.01 + 0.05 * 0.99}$$

- We can use R to do the calculation for us

```
(0.98*0.01) / ((0.98*0.01) + (0.05*(1-0.01)))
```

```
## [1] 0.1652614
```

Bayes' theorem

Terminology

Posterior $\propto$ **Likelihood** x **Prior**

- In the previous example,
  - $P(C19|+)$ is the **posterior**.
  - $P(+|C19)$ is the **likelihood of the data**.
  - $P(C19)$ is the **prior**.
- The denominator $P(+)$ is ensures the result is a probability. It is sometimes referred to as the **marginal likelihood** or **normalizing constant**.

Bayes' theorem

COVID-19 tests: Tabular explanation

- The four cells in the middle of the table represent the *joint probabilities* of two events.
- The row and column totals represent the *marginal probabilities* of each event.
- θ is used to denote the parameters we are estimating.

Test result	$\theta = C19+$	$\theta = C19-$	Marginal (Test)
$+$	P($+$|C19)P(C19)	P($+$|C19-)P(C19-)	$\sum_\theta P(+\|\theta)P(\theta)$
-	P(-|C19)P(C19)	P(-|C19-)P(C19-)	$\sum_\theta P(-\|\theta)P(\theta)$
Marginal C19	P(C19$+$)	P(C19-)	1.0

Bayes' theorem

COVID-19 tests: Tabular explanation

- To calculate $P(C19|+)$ we can take the *joint probability* of C19 and a positive test and divide it by the *marginal probability* of a positive test.
- We can get the relevant values directly from the table: $0.98 * 0.01/0.06$.

Test result	$\theta = C19+$	$\theta = C19-$	Marginal (Test)
$+$	0.98*0.01	0.05*(1-0.01)	0.06
-	(1-0.98)*0.01	(1-0.05)*(1-0.01)	0.94
Marginal C19	0.01	(1-0.01)	1.0

Bayes' theorem

Changing our priors

- If there is a new surge we could update our prior and recalculate
- Let's change our prior to assume 10% COVID-19 prevalence in the population

```
(0.98*0.1) / ( (0.98*0.1) + (0.05*0.9) )
```

```
## [1] 0.6853147
```

- Now we get a much higher posterior probability.
- We could also extend this analysis by incorporating other prior information, e.g. symptoms, exposure

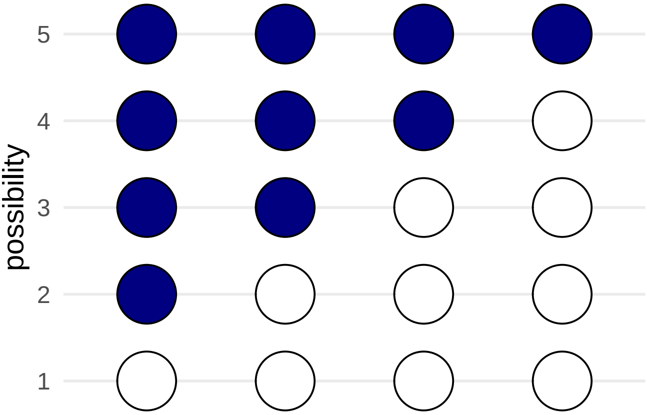
Bayesian inference as counting

McElreath's marble counting example

- Consider a bag containing four marbles
- The marbles can be white or blue
- We draw a sample of marbles from the bag (with replacement)

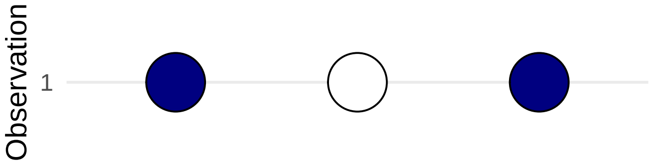
Bayesian inference as counting

Conjecture: Five possibilities



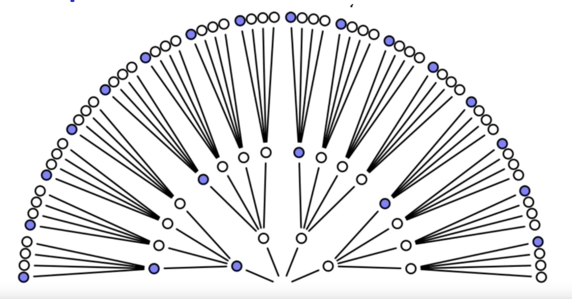
Bayesian inference as counting

A sample from the bag produces



Bayesian inference as counting

Sampling and possibilities



McElreath 2020, Fig. 2.2 (p. 22)

Bayesian inference as counting

Counting the possibilities

Conjecture	Ways to produce [B,W,B]
[W,W,W,W]	0 x 4 x 0 = 0
[B,W,W,W]	1 x 3 x 1 = 3
[B,B,W,W]	2 x 2 x 2 = 8
[B,B,B,W]	3 x 1 x 3 = 9
[B,B,B,B]	4 x 0 x 4 = 0

Bayesian inference as counting

From counts to probability

Conjecture	Propoportion B	Ways [B,W,B]	Plausibility
[W,W,W,W]	0.00	0	0.00
[B,W,W,W]	0.25	3	0.15
[B,B,W,W]	0.50	8	0.40
[B,B,B,W]	0.75	9	0.45
[B,B,B,B]	1.00	0	0.00

Bayesian inference as counting

Summary

- We enumerated the set of plausible data generating processes p
- We counted the ways we could produce the data given each value of p. This is known as the *likelihood*.
- We normalized these counts to get *posterior* probabilities, which indicate the relative plausibility of each option p.
- The most plausible value is the one that has the most ways of generating the data.

Bayesian inference as counting

Incorporating prior information

- Now let's say we pick another marble and it's blue. We can use the prior information to update our counts.

Conjecture	Ways to produce [B]	Prior counts	New counts
[W,W,W,W]	0	0	0 x 0 = 0
[B,W,W,W]	1	3	3 x 1 = 3
[B,B,W,W]	2	8	8 x 2 = 16
[B,B,B,W]	3	9	9 x 3 = 27
[B,B,B,B]	4	0	0 x 4 = 0

Bayesian inference as counting

Bayes' theorem and data analysis

- In a general sense, we can think about Bayesian inference as calculating the posterior distribution in the following way:

$$Posterior = \frac{Probability\ of\ the\ data\ * Prior}{Average\ probability\ of\ the\ data}$$

Bayesian inference

"Bayesian inference is reallocation of credibility across possibilities" - John Kruscke[2]

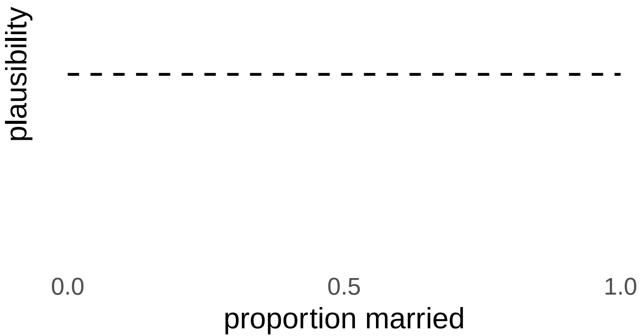
[2] Chapter 2 of Kruschke's 2015 book *Doing Bayesian Data Analysis* provides an outline of his argument and is available online.

Bayesian inference for a continuous parameter

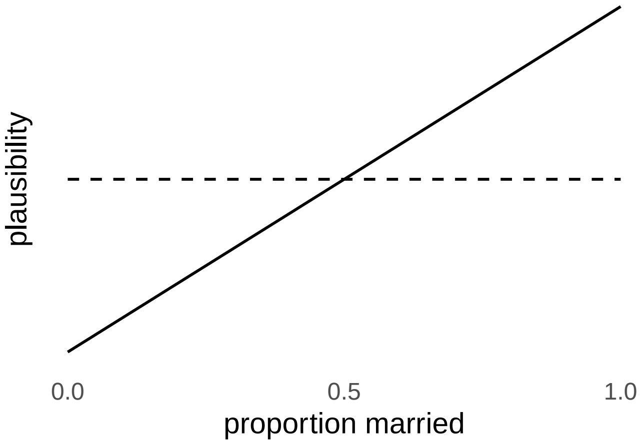
Estimating the marriage rate

- Assume a demographer is interested in estimating the marriage rate in the population.
- The demographer starts out with a "flat" prior
  - The marriage rate could be anywhere from 0 (nobody is married) to 1 (everybody is married).
- The demographer samples people at random and asks them their marital status.

Assume zero knowledge with a flat (uniform) prior

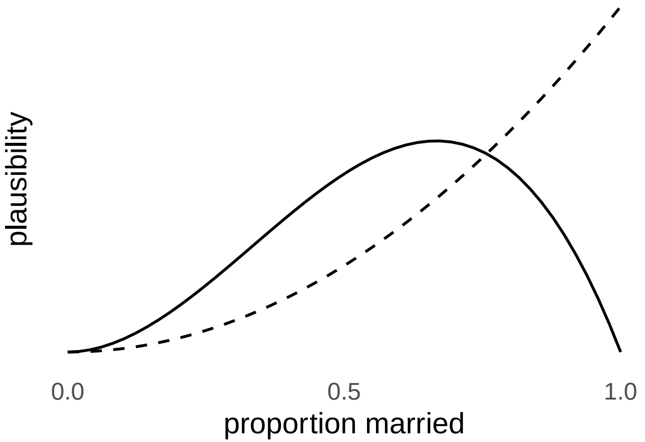


First observation: Married

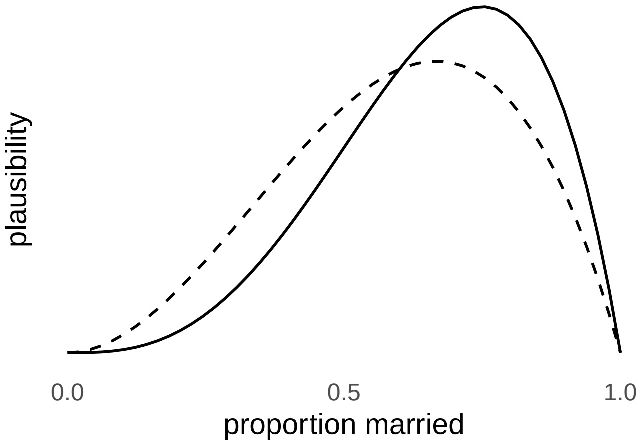


Second observation: Married

Third observation: Single



Fourth observation: Married

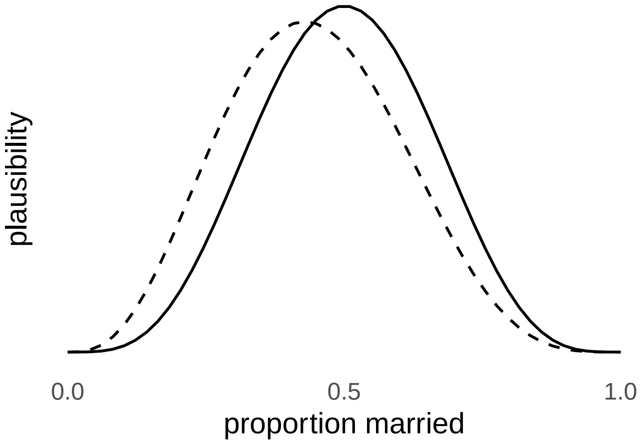


Fifth observation: Single

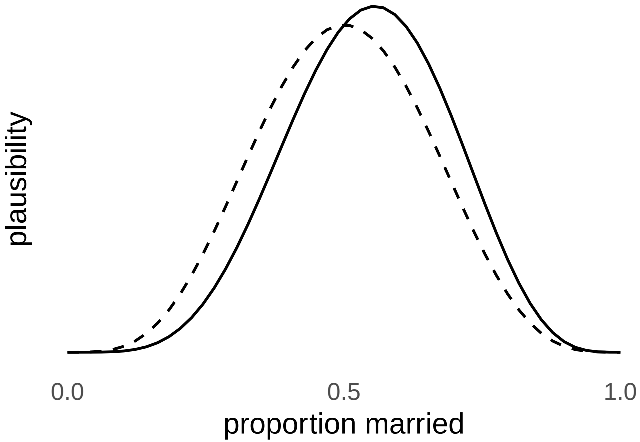
Sixth observation: Single

Seventh observation: Single

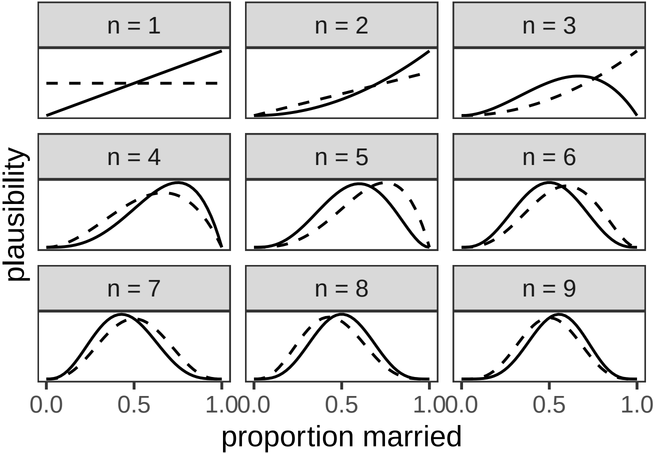
Eighth observation: Married



Nineth observation: Single



Priors and Posteriors



Bayesian Updating

- This example demonstrates the concept of **Bayesian updating**
  - We use new information to update our beliefs
- Each time we update we use the previous **posterior** as the new **prior**!

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Bayesian Updating

- This example demonstrates the concept of **Bayesian updating**
  - We use new information to update our beliefs
- Each time we update we use the previous **posterior** as the new **prior**!
- Most of the time we use all our data at once to get the final posterior rather than iteratively updating.
- Bayesian updating is order invariant: we will get the same result regardless of the way observations are ordered.

Formalizing the marriage model

Writing down a model

- This problem can be represented using the Beta-Binomial model.[3]

$$Marriage \sim Binomial(N, p)$$
$$p \sim Beta(a, b)$$

[3] Chapter 3 of *Bayes Rules!* for an extended discussion.

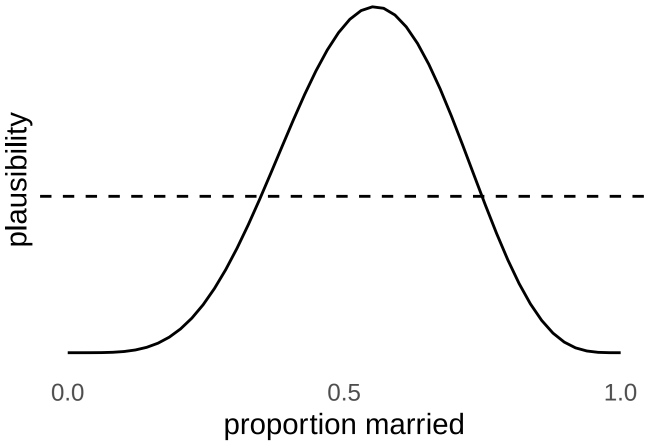
Formalizing the marriage model

Writing down a model

- The goal of this analysis is to produce an estimate of p, the probability of marriage.
- Our prior for p is represented by $Beta(a, b)$
  - The Beta distribution is bounded to [0,1]
  - It is equivalent to a *Uniform* distribution when $a = b = 1$, representing complete uncertainty

Formalizing the marriage model

Prior and posterior distributions



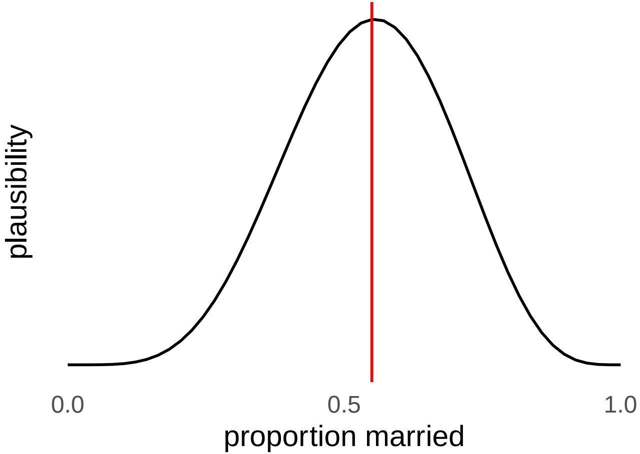
Formalizing the marriage model

Understanding the posterior distribution

- Unlike previous discrete examples, our estimate of p is represented by the entire posterior distribution
- In this case, the posterior distribution is simple to calculate
  - $Beta(1,1) \rightarrow Beta(1 + married, 1 + N) \rightarrow Beta(6, 10)$
  - This is due to **conjugacy**, as the prior and posterior belong to the same family of distributions
- As our models get more complex, we need to use more sophisticated approaches to estimate posterior distributions

Formalizing the marriage model

Summarizing the posterior distribution



Bayesian Regression

- We can extend this approach to linear regression, where outcomes are modeled using the Normal distribution.

$$y_i \sim Normal(\mu_i, \sigma)$$
$$\mu_i = \beta_0 + \beta_1 x_i$$

$$\beta_0 \sim Normal(0, 1)$$
$$\beta_1 \sim Normal(0, 1)$$

$$\sigma \sim Uniform(0, 1)$$

Bayesian Regression

- In this case, we make the *assumption* that y_i is normally distributed and that we can express its mean in terms of x (recall that $E[y|x] = \beta_0 + \beta_1 x_i$)
- After estimating a model using the data we get the *posterior* distribution for each parameter
  - We can then make statistical inferences regarding $\hat{\beta}_1$

Comparing Bayesian and Frequentist approaches

Thomas Bayes (1701-1761)

Source: Wikipedia.

Comparing Bayesian and Frequentist approaches

Pierre-Simon Laplace (1749-1827)

Source: Wikipedia.

Comparing Bayesian and Frequentist approaches

Ronald Fisher (1890-1962)

Source: Wikipedia.

Comparing Bayesian and Frequentist approaches

Historical developments

- Frequentist (or "Fisherian") statistics dominated for most of the 20th century.
- Bayesian inference critiqued as too subjective and difficult to implement
- Reversal over the past couple of decades as critiques of Bayesianism debunked, cheap compute power makes it tractable, and key tenets of Frequentist statistics are questioned (e.g. controversy over p-hacking).
- The Bayesian approach is now mainstream in statistics and much of the natural sciences, but the social sciences have been slower to adopt.[4]

[4] See Scott and Bartlett 2019.

Comparing Bayesian and Frequentist approaches

Theoretical foundations

- Frequentist
  - Long-run probabilities
  - Sampling distributions
- Bayesian
  - Probability theory

Comparing Bayesian and Frequentist approaches

Sample size

- Frequentist
  - Properties of estimators depend on minimal sample size
- Bayesian
  - No minimum sample size
  - But larger samples improve precision of estimates

Comparing Bayesian and Frequentist approaches

Point estimates

- Frequentist
  - Models produce point estimates
- Bayesian
  - No singular point estimates
    - Many different summaries of the posterior distribution are possible (e.g. mean, median, mode)

Comparing Bayesian and Frequentist approaches

P-values

- Frequentist
  - p-values used to communicate statistical significance
- Bayesian
  - Critique: p-values are based on arbitrary distributional assumptions
  - Uncertainty is captured by entire posterior distribution
  - *Bayes' Factor* is a Bayesian version of a p-value[5]

[5] See Kruschke and Liddell 2018.

Comparing Bayesian and Frequentist approaches

Confidence intervals

- Frequentist
  - Confidence intervals defined using test statistics and conventions
  - Assumption that a parameter is fixed and that interval is derived from a sample
- Bayesian
  - Critique: Frequentist conventions are arbitrary
  - Assumption that a parameter has a distribution
  - *Credible intervals* or *compatibility intervals* can be used to summarize the posterior distribution

Comparing Bayesian and Frequentist approaches

Confidence intervals: Interpretation of a 95% interval

- Frequentist
  - Over many repeat samples, 95% of calculated confidence intervals would contain the true value of the parameter
- Bayesian (assume an interval over 95% of the posterior distribution)
  - There is a 95% probability that the estimated parameter lies within the defined range, given the model and the data.
  - "What the interval indicates is a range of parameter values compatible with the model and the data." McElreath, p. 54.

Computation and Bayesian Estimation

Bayesian Estimation

- Three methods for estimating the posterior distribution
  - Analytical calculations
  - Grid and quadratic approximation
  - Markov Chain Monte Carlo

Computation and Bayesian Estimation

Analytical calculations

- For simple problems we can use calculus to provide an analytical solution for the posterior distribution
- But this approach does not scale well beyond simple problems like the marriage example

Computation and Bayesian Estimation

Grid and quadratic approximation

- Grid approximation (see McElreath 2.4.3)
  - We can approximate continuous spaces by using grids
    - But scales very poorly to complex examples
- Quadratic approximation (see McElreath 2.4.4)
  - A more robust approach that involves using distributions to approximate the posterior
  - Flexible for many regression problems but also has trouble scaling

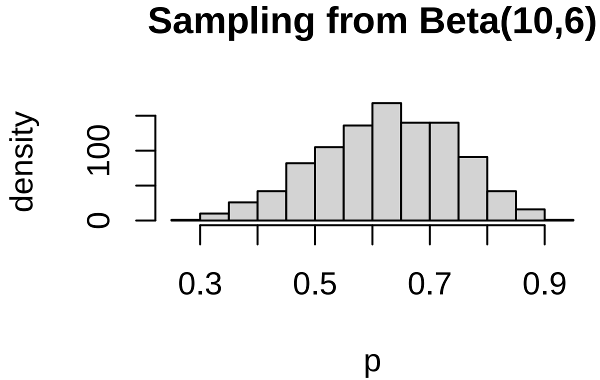
Computation and Bayesian Estimation

Markov Chain Monte Carlo (MCMC)

- Use simulation to draw samples from the posterior distribution
  - A computationally intensive approach
  - Samples provide an approximation for complex spaces
  - More efficient for complex models than quadratic approximation
- MCMC has led to major advances in Bayesian methods since the 1990s (see McElreath 2.4.5).

Computation and Bayesian Estimation

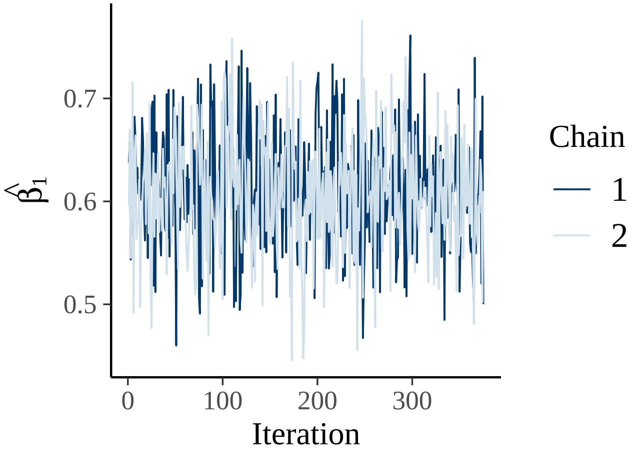
MCMC intution: Sampling from the posterior[6]



[6] Note this is a trivial case since we already know the exact posterior distribution!

Computation and Bayesian Estimation

Samples from a Markov Chain



Computation and Bayesian Estimation

`Stan` and Hamiltonian Monte Carlo

- `Stan` is a programming language developed for statistical computing
- It implements **Hamiltonian Monte Carlo (HMC)** sampling
  - A variant of MCMC methods based on Hamiltonian physics
  - Approximates the posterior by "flicking" a particle and observing its movement
- HMC is highly effective at solving complex problems[7]
  - It provides lots of useful diagnostics making it easier to debug than early MCMC approaches
  - Greater flexibility as it not require *conjugacy*

[7] See McElreath Chapter 9 and Betancourt 2018 for a more advanced conceptual overview. Link to simulation.

Bayesian Regression in R

- We will be using `stan_glm` to estimate regression models via HMC in R
- The *posterior distributions* of the parameters are analyzed to make inferences about the relationship between x and y
- We can also use the posterior to generate new data consistent with the model and to calculate new kinds of regression diagnostics

Final remarks

"All models are wrong, but some are useful" - George Box[8]

[8] This aphorism is attributed to statistician George Box. See Wikipedia for further discussion.

Next week

- Multivariate regression

Lab

- Bivariate Bayesian regression using `stan_glm`