# SOC542 Statistical Methods in Sociology II

## Count outcomes

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April 7, 2025

# Course updates

### Homework

- Homework 3 grades released
- Homework 4 released after class, due next Friday, 4/18
  - Count outcomes
  - Categorical and ordered outcomes (next week)

# Course updates

## Projects

- Preliminary results due 4/25
  - One or more tables or figures of descriptive statistics
  - One or more regression tables showing
    - Bivariate results
    - Multivariate results
  - Must include at least one figure showing estimates (e.g. coefficients, predictions, marginal effects)
  - Draft write up of methodology and results
- Presentations on 5/5

## Plan

- Count outcomes
- Poisson regression
- Overdispersion and negative-binomial regression
- Offsets
- Zero-inflated models

# Count outcomes

- Count outcomes are variables defined as *non-negative integers*.
  - Values must be 0 or greater.
  - Numbers must not contain any fractional component.

## Count outcomes

- In general, we obtain count variables by counting discrete events over space and time. Many social processes produce counts:
  - How many people currently live in a census tract?
  - How many siblings does someone currently have?
  - How many times has someone ever been arrested?
  - How many sexual partners reported in one year?

# Count outcomes

### Modeling counts using OLS

- We could treat counts like continuous variables and model them using OLS.
- Such a strategy might be appropriate if a count variable is normally distributed.
- But like the LPM, we might run into problems when making predictions.
  - Predictions not constrained to be positive or counts.

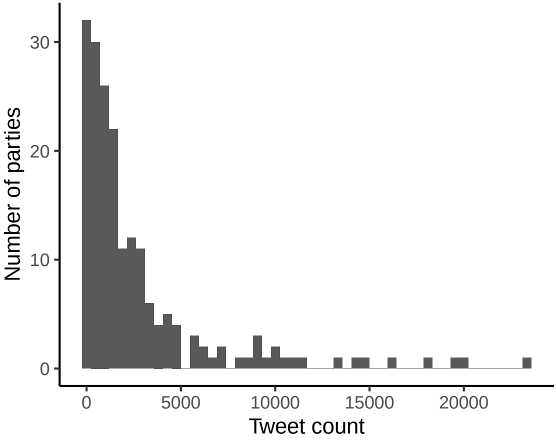
# Data

### Twitter and political parties in Europe

- Data from Twitter accounts of 190 political parties in 28 countries in Europe
- Includes cumulative number of tweets and engagements (likes, replies, retweets) from 2018
- Data on left-right ideology (0-10 scale) and % of parliamentary seats held

# Data

## Twitter and political parties in Europe



# Count outcome

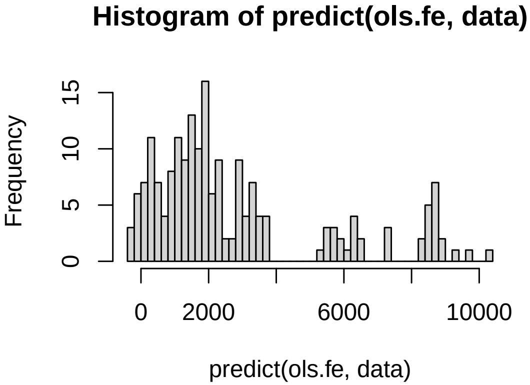
## Modeling counts using OLS

| | OLS | OLS FE | OLS FE (Log) |
|---|---|---|---|
| Seats % | 8.826 | 47.643* | 0.016 |
| | (23.091) | (21.346) | (0.009) |
| Ideology [0-10] | -79.080 | -46.490 | -0.070 |
| | (130.196) | (112.614) | (0.046) |
| Intercept | 3066.667*** | 1958.652 | 7.767*** |
| | (740.226) | (2033.170) | (0.823) |
| Num.Obs. | 190 | 190 | 190 |
| R2 | 0.002 | 0.427 | 0.428 |
| R2 Adj. | -0.008 | 0.323 | 0.324 |
| F | 0.228 | 4.112 | 4.120 |

* p $<0.05$, ** p $<0.01$, *** p $<0.001$
Country FE omitted.

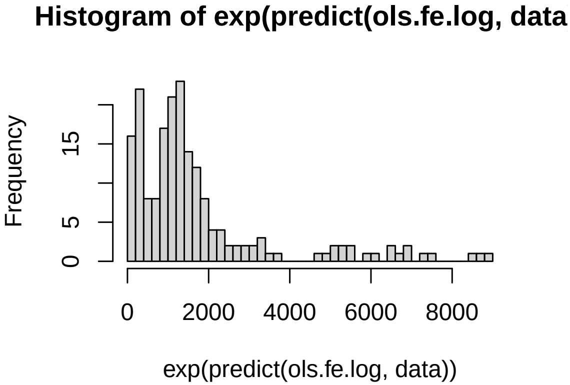
# Count outcome

## Making predictions with OLS



# Count outcome

## Making predictions with OLS



# Count outcomes

### Analyzing predictions

```r
min(predict(ols.fe, data))
```

```
## [1] -370.4861
```

```r
min(exp(predict(ols.fe.log, data)))
```

```
## [1] 26
```

# Poisson regression

## Modeling counts as Poisson processes

- The **Poisson** distribution is a discrete probability distribution that indicates the count of events in a fixed interval. These counts can be considered as rates of events per unit.[1]
- The *probability mass function* is defined by a single parameter $\lambda$, where the probability of observing $k$ events is equal to

$$P(x = k) = \frac{\lambda^k e^- \lambda}{k!}$$

- For any Poisson distributed random variable, $x$

$$E(x) = \lambda = Var(x)$$

[1] The distribution gets its name from French mathematician Siméon Denis Poisson [1781-1840].

# Poisson regression

### Modeling counts as Poisson processes

- Let's say the average number of visits to doctor each year is 1.6.
- We can model the probabilities of observing different numbers of visits given $\lambda = 1.6$:

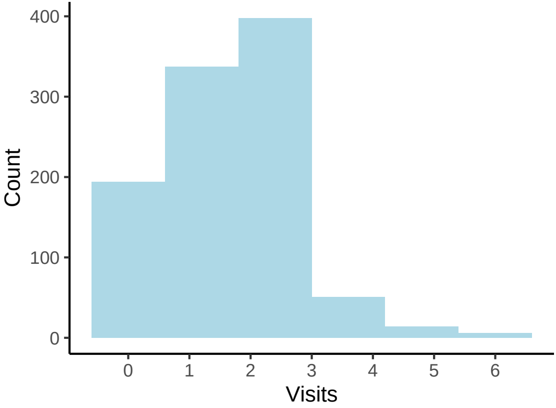
$$P(k \text{ visits a year}) = \frac{1.6^k e^{-1.6}}{k!}$$

$$P(0 \text{ visits a year}) = \frac{1.6^0 e^{-1.6}}{0!} = \frac{e^{1.6}}{1} \approx 0.2$$

$$P(1 \text{ visits a year}) = \frac{1.6^1 e^{-1.6}}{1!} = \frac{1.6e^{-1.6}}{1} \approx 0.4$$

# Poisson regression

## Poisson distributions



1000 draws from Poisson($\lambda = 1.6$)

# Poisson regression

## Poisson distributions, $E[x] = \lambda = Var(x)$

```
round(mean(x),2)
```

```
## [1] 1.59
```

```
round(var(x),2)
```

```
## [1] 1.53
```

## Poisson regression

- The Poisson regression model assumes that the outcome is Poisson distributed, conditional on the observed predictors.

$$y \sim Poisson(\lambda)$$

- To ensure that our estimates are positive, we can use a logarithmic *link function*, thus

$$y = log(\lambda) = \beta_0 + \beta_1 x_1 + \beta_2 x_1 + ... + \beta_k x_k$$

- Like logistic regression, this equation can equivalently be expressed using the *inverse* of the logarithm function:

$$\lambda = e^{\beta_0 + \beta_1 x_1 + \beta_2 x_1 + ... + \beta_k x_k}$$

# Poisson regression

## Fitting a model

```
pois <- glm(tweet_count ~ seats_per + left_right +
                  country,
         data = data, family = poisson(link = "log"))
```

# Poisson regression

| | OLS FE (Log) | Poisson |
|---|---|---|
| Seats % | 0.016 | 0.013*** |
| | (0.009) | (0.000) |
| Ideology [0-10] | -0.070 | -0.018*** |
| | (0.046) | (0.001) |
| Intercept | 7.767*** | 7.708*** |
| | (0.823) | (0.012) |
| Num.Obs. | 190 | 190 |
| R2 | 0.428 | |
| R2 Adj. | 0.324 | |
| Log.Lik. | -308.005 | -166974.776 |

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$
Country FE omitted.

# Poisson regression

### Interpretation

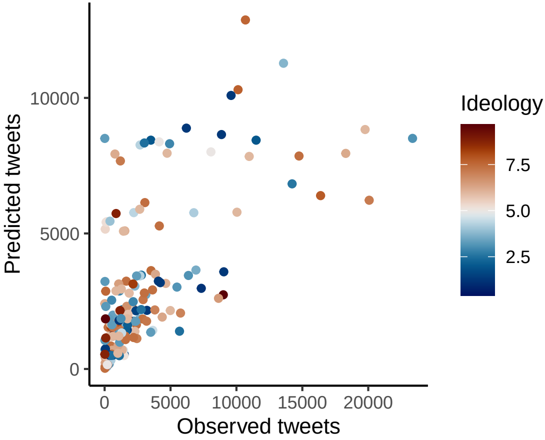
- The intercept $\beta_0$ is the *logged* average value of the outcome when all other predictors are equal to zero.
- Each coefficient $\beta_i$ indicates the effect of a unit change of $x_i$ on the *logarithm* of the outcome.
  - e.g., $\beta_{seats\%}$ implies that the expected log number of tweets increases by 0.013 in response to a 1-unit, or 1% increase in parliamentary seats held by a party.
- Coefficients can be interpreted as *multiplicative* changes after exponentiation
  - e.g., $e^{\beta_{seats\%}} = e^{0.013} \approx 1.013$. This implies that a ~1.3% increase in tweets.
  - These coefficients are sometimes referred to as **incident rate ratios (IRRs)**.

# Poisson regression

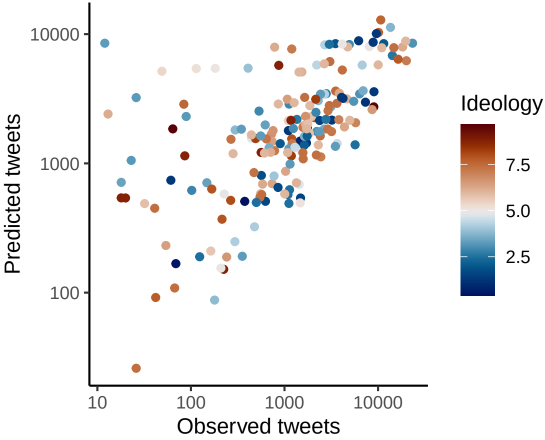
| | Poisson (Exponentiated) |
|---|---|
| Seats % | 1.014*** |
| | (0.000) |
| Ideology [0-10] | 0.983*** |
| | (0.001) |
| Intercept | 2225.822*** |
| | (25.924) |
| Num.Obs. | 190 |
| Log.Lik. | -166974.776 |
| F | 11552.594 |

* p $<$0.05, ** p $<$0.01, *** p $<$0.001
Country FE omitted.

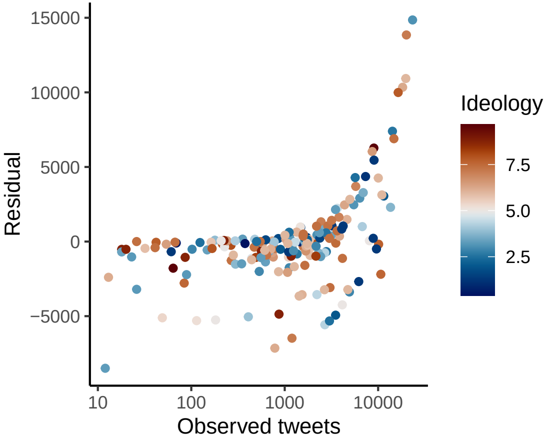
# Poisson regression



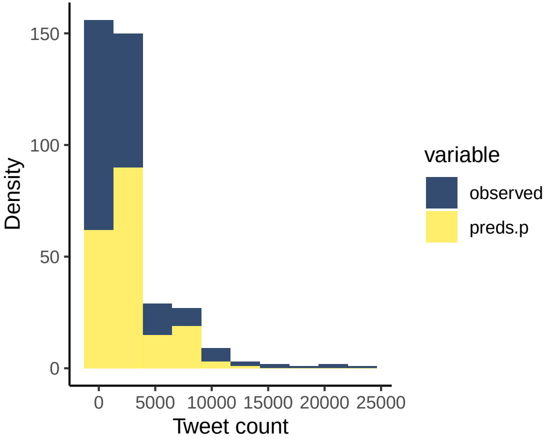
# Poisson regression



# Poisson regression



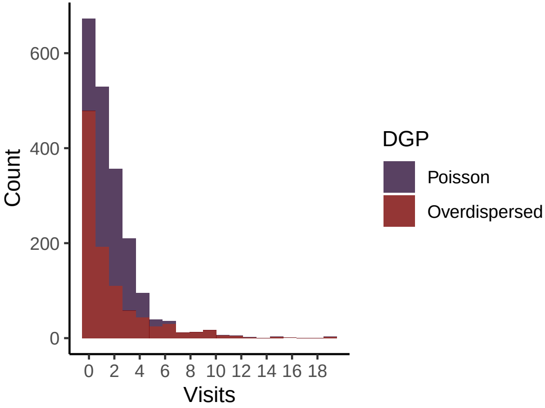
# Poisson regression



# Overdispersion

- A random variable is **overdispersed** if the observed variability is greater than the variability expected by the underlying probability model.
- **Underdispersion** occurs if the variability is lower than expected, but it is rarely an issue.

# Overdispersion




Poisson($\lambda = 1.6$) and NegBin($\mu = 1.6$, $\theta = 0.5$)

# Overdispersion

### Negative binomial distribution and regression

- The **negative binomial** distribution (aka the gamma-Poisson distribution) includes an additional parameter $\theta$ to account for dispersion, referred to as a **scale parameter**.

$$y = NegativeBinomial(\lambda, \theta)$$

- In negative binomial regression, $\theta$ is estimated from the data. The value must be positive.
  - Lower values indicate greater overdispersion.
  - Negative binomial becomes identical to Poisson as $\lim_{\theta \to \infty}$.

# Overdispersion

### Fitting a negative binomial regression

Negative binomial regression is not implemented in `glm`. Instead, we can use the `glm.nb` function from the `MASS` package.[2]

```
library(MASS)
nb <- glm.nb(tweet_count ~ seats_per + left_right + country,
         data = data)
```

[2]The "fixest" package has an implementation, "fenegbin" that is more suitable for these data as it can also cluster standard errors.

# Comparing Poisson and negative binomial regression

| | Poisson | Negative binomial |
|---|---|---|
| Seats % | 0.013*** | 0.014* |
| | (0.000) | (0.006) |
| Ideology [0-10] | -0.018*** | -0.054 |
| | (0.001) | (0.031) |
| Intercept | 7.708*** | 8.031*** |
| | (0.012) | (0.564) |
| Num.Obs. | 190 | 190 |
| AIC | 334009.6 | 3267.4 |
| BIC | 334107.0 | 3368.0 |
| Log.Lik. | -166974.776 | -1602.681 |
| F | 11552.594 | 9.208 |

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$
Country FE omitted. Exponentiated coefficients.

# Negative binomial regression

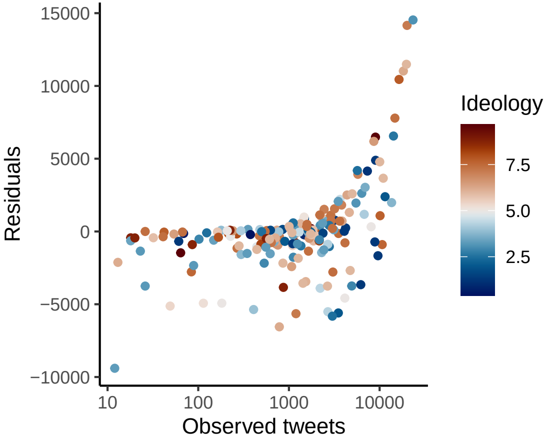
```r
nb$theta
```

```
## [1] 1.197566
```

```r
nb$SE.theta
```

```
## [1] 0.110282
```

# Negative binomial regression



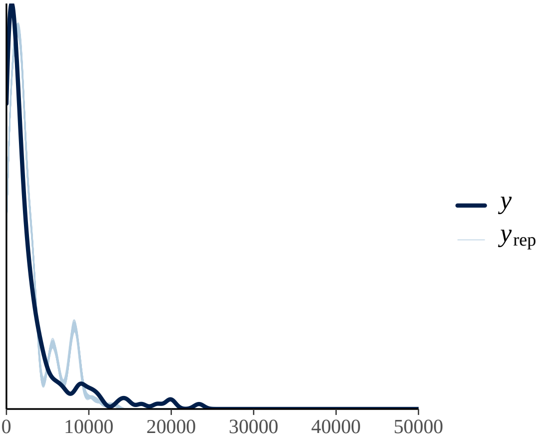
# Negative binomial regression

### Bayesian estimation

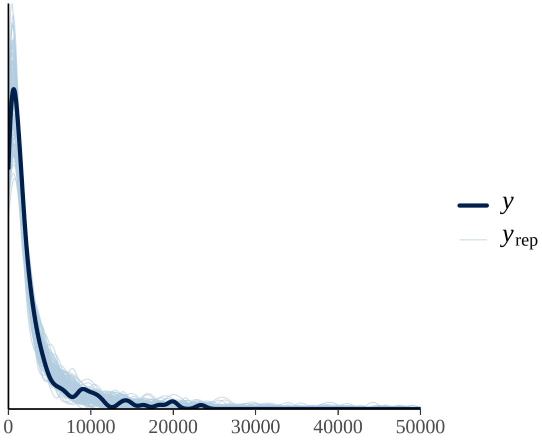
```
pois.b <- stan_glm(tweet_count ~ seats_per + left_right +
                        country,
        data = data,
        family = poisson,
       seed = 08901, chains = 1, iter = 4000, refresh = 0)

nb.b <- stan_glm(tweet_count ~ seats_per + left_right +
                      country,
        data = data,
        family = neg_binomial_2(),
       seed = 08901, chains = 1, iter = 4000, refresh = 0)
```

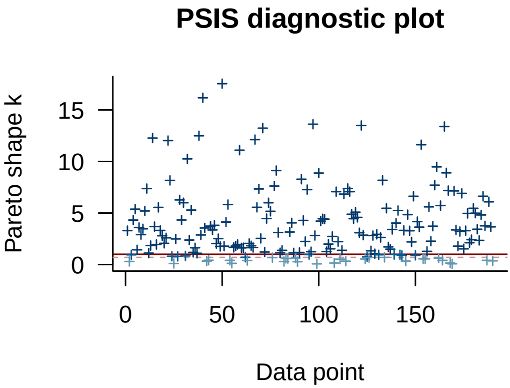
# Poisson posterior predictive check



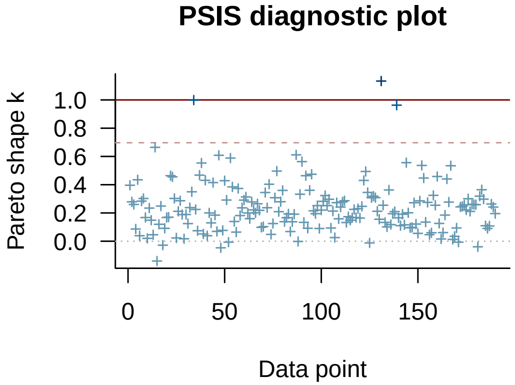
# Negative binomial posterior predictive check



# Poisson PSIS plot



# Negative binomial PSIS plot



# Negative binomial regression

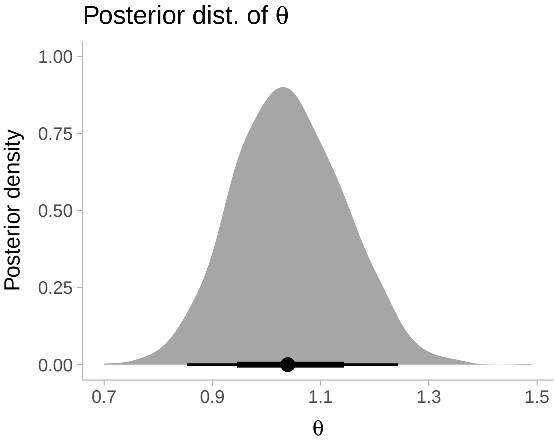
### Comparing Poisson and negative binomial models

```r
loo_compare(l.pois, l.nb)
```

```
##         elpd_diff se_diff
## nb.b          0.0     0.0
## pois.b -170636.7 22413.8
```

# Negative binomial regression

## Bayesian estimate of $\theta$



# Offsets

### Intuition

- Assume a count outcome $y$ is measured over varying time intervals $t$. The level of $y$ will vary both as a function of the underlying count process and the length of **exposure**.[3]
- We can add an **offset** to our model to account for varying exposures.
- The outcome of a model with an offset is now $\frac{y}{t}$.

[3] The same logic would apply if we measured quantities over varying spatial units, e.g. counting people in blocks versus census tracts.

# Offsets

### Explanation

- The mean of a Poisson process, $\lambda$ is implicitly $\lambda = \frac{\mu}{\tau}$, the expected number of events, $\mu$, over the duration $\tau$.
- Assume a Poisson process where $\lambda_i$ is the expected number of events for the $i^{th}$ observation. We can write the link function as

$$y = Poisson(\lambda)$$

$$log(\lambda) = log(\frac{\mu}{\tau}) = \beta_0 + \beta_1 x$$

- This can be re-written as

$$= log(\mu) - log(\tau) = \beta_0 + \beta_1 x$$

# Offsets

## Explanation

- We can think of $\tau$ as the number of **exposures** for each observation. Thus, we can write out a new model for $\mu$:

$$y \sim Poisson(\mu)$$

$$log(\mu) = log(\tau) + \beta_0 + \beta_1 x$$

# Offsets

**Example: Predicting retweet rates**

- The number of retweets depends on the number of times a party tweeted
  - No tweets, no retweets
  - More tweets, more retweets?
- Two specifications
  - No offset
  - Log(tweets) included as offset

# Offsets

### Example: Predicting retweet rates

```
nb.rt <- glm.nb(retweet_total ~
                    seats_per + left_right +  country,
                    data = data)
nb.rt.o <- glm.nb(retweet_total ~ offset(log(tweet_count)) +
                    seats_per + left_right +  country,
                    data = data)
```

# Offsets

| | NB | NB (Offset) |
|---|---|---|
| Seats % | 1.033*** | 1.033*** |
| | (0.009) | (0.007) |
| Ideology [0-10] | 0.973 | 0.977 |
| | (0.045) | (0.034) |
| Intercept | 21286.898*** | 7.528** |
| | (17921.337) | (4.739) |
| Num.Obs. | 190 | 190 |
| AIC | 4391.7 | 4243.0 |
| BIC | 4492.3 | 4343.7 |
| Log.Lik. | -2164.836 | -2090.518 |
| F | 21.379 | 13.095 |

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

# Offsets

**Using offsets**

- Offsets allow models to be interpreted as rates rather than counts.
- Always include an offset if there are differences in measurement intervals across observations.
- Offsets can also be included when intervals are constant if a rate is more informative.

# Zero-inflated models

### Intuition

- Some count outcomes have high rates of zeros. What if zeros are generated by a different process compared to non-zeros?
- **Zero-inflated models** allow us to jointly model the process determining whether counts are non-zero and the expected count for each non-zero observation.

# Zero-inflated models

### Specification

- The zero-inflated Poisson model consists of a mixture of two linear models, a logistic regression predicting the probability of a zero and a Poisson model predicting the count outcome.

$$y_i = ZIPoisson(p, \lambda)$$

$$logit(p) = \gamma_0 + \gamma_1 z$$

$$log(\lambda)\beta_0 + \beta_1 x$$

- Each model has its own set of regression parameters. These can be specified differently to model each process.

## Zero-inflated models

**Example: Books borrowed from the library**

- Are you borrowing any books from the library?
  - If so, how many?

# Zero-inflated models

## Simulating library usage (binary)

```
N <- 1000 # N

z <- rnorm(N, 0.5, 1) # Random variable determines library usage
p_lib <- 1/(1 + (exp(1)^-(z))) # Convert to probability

lib <- rep(0,N) # Generate binary library variable
for (i in 1:N) {
    lib[i] <- rbinom(1, 1, p_lib[i])
}

sum(lib)/N
```

```
## [1] 0.604
```

# Zero-inflated models

## Simulating borrowing (count)

```r
x <- rnorm(N) # Random variable for books borrowed

books <- c() # Store number of books borrowed for each student
for (i in 1:N) {
    if (lib[i] == 1) { # Borrow books if library visitor
        books[i] <- rpois(1, lambda = exp(0.5 + x[i]))
    } else {books[i] <- 0} # Otherwise zero books borrowed
}
mean(books)
```

```
## [1] 1.584
```

```r
max(books)
```

```
## [1] 52
```

# Zero-inflated models

### Two kinds of zeros

```
sum(books == 0)
```

```
## [1] 542
```

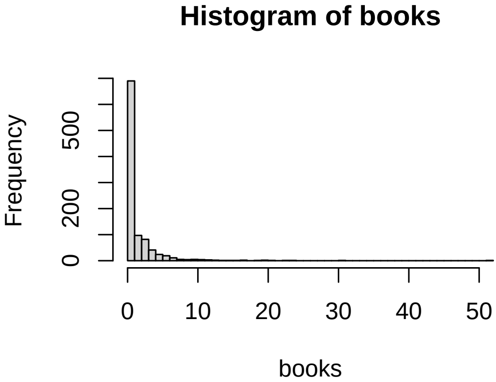
```
sum(books == 0 & lib == 1)
```

```
## [1] 146
```

```
sum(books == 0 & lib == 0)
```

```
## [1] 396
```

# Zero-inflated models



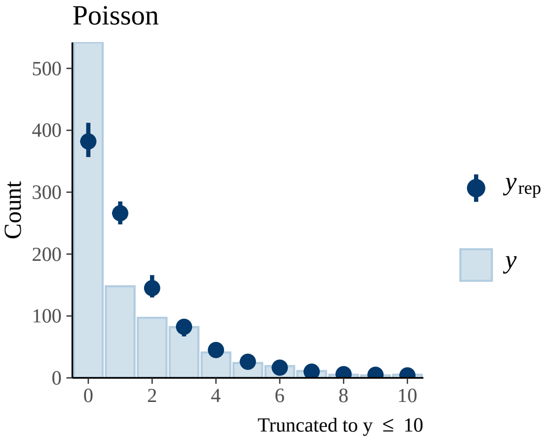
# Zero-inflated models

## Estimating a Poisson model

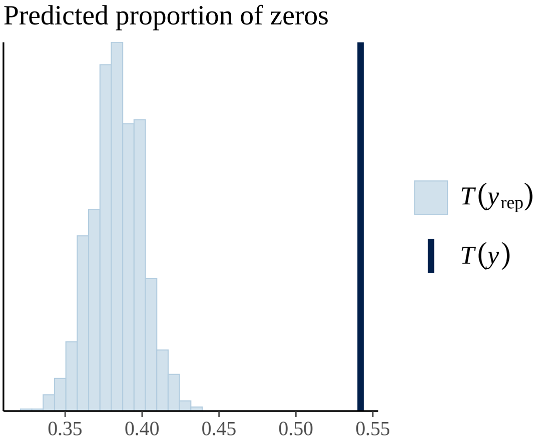
```
book.data <- as.data.frame(cbind(books, x, z))

pois.m <- stan_glm(books ~ x, data = book.data, family = poisson(),
        seed = 08901, chains = 1, refresh = 0)
```

# Poisson posterior predictive checks



# Poisson posterior predictive checks



# Zero-inflated models

### Estimating a zero-inflated Poisson model

We must use the brms library to implement Bayesian zero-inflated Poisson regression.

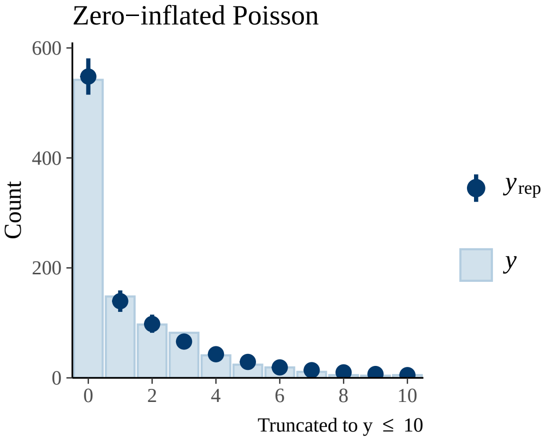
```
library(brms)
zip <- brm(bf(books ~ x,
           zi ~ z),
              data = book.data,
              family = zero_inflated_poisson(link = "log",
                                              link_zi = "logit"),
              seed = 08901, refresh = 0, chains = 1,
              backend = "cmdstanr")
```

```
## Running MCMC with 1 chain...
##
## Chain 1 finished in 6.6 seconds.
```

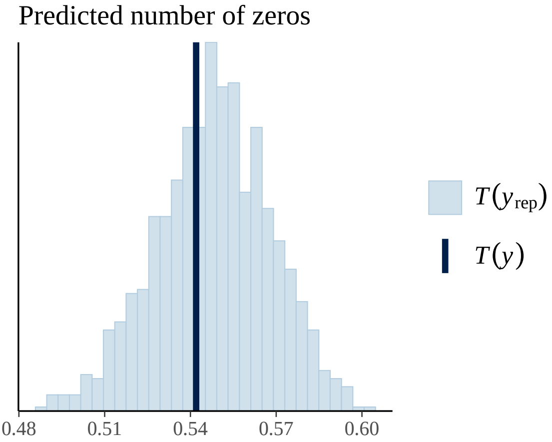
# Zero-inflated models

| | Poisson (exp.) | ZI Poisson (exp.) |
|---|---|---|
| (Intercept) | 0.991 | |
| | [0.921, 1.063] | |
| x | 2.687 | |
| | [2.557, 2.826] | |
| b_Intercept | | 1.681 |
| | | [1.561, 1.818] |
| b_x | | 2.697 |
| | | [2.559, 2.832] |
| b_zi_Intercept | | 0.906 |
| | | [0.753, 1.090] |
| b_zi_z | | 0.421 |
| | | [0.347, 0.522] |
| ELPD | -1806.5 | -1327.4 |

# Posterior predictive checks



# Posterior predictive checks



# Zero-inflated models

### Comparing standard and zero-inflated Poisson models

```
##        elpd_diff se_diff
## zip       0.0       0.0
## pois.m -479.1      45.1
```

# Summary

- Standard linear models are generally unsuitable for count data
- Poisson regression can be used for many count outcomes
- Overdispersion occurs when variation higher than expected under Poisson model
  - Negative binomial regression includes a scale parameter to model this
- Offsets transform from counts to rates and should be used when measurement intervals vary
- Zero-inflated models can decompose processes generating zeros and counts

## Next week

- Categorical outcomes
  - Multinomial and ordered logistic regression

# Lab

- Estimating and interpreting count models in R