

2 ベクトル場と微分形式

§2.1 ベクトル場の座標変換

(a) 常微分方程式の変数変換

$$\frac{dx}{dt} = V(x) \quad (2.1)$$

$$y^i = y^i(x) = y^i(x^1, \dots, x^n).$$

x^i が (2.1) を満たすとして y^i を t で微分

$$\begin{aligned} \frac{dy^i}{dt}(t) &= \sum_{j=1}^n \frac{\partial y^i}{\partial x^j}(x^1, \dots, x^n) \frac{dx^j}{dt}(t) \\ &= \sum_{j=1}^n \frac{\partial y^i}{\partial x^j}(x^1, \dots, x^n) V^j(x^1, \dots, x^n) \end{aligned} \quad (2.2)$$

(自励系)

$$\frac{dy}{dt} = \left(\sum_{j=1}^n \frac{\partial y^1}{\partial x^j} V^j, \dots, \sum_{j=1}^n \frac{\partial y^m}{\partial x^j} V^j \right) \quad (2.3)$$

例 2.1 $n=2$ $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix}$

$$r = \sqrt{x^2 + y^2} \quad \theta = \tan^{-1}(y/x).$$

$$\begin{cases} \frac{dx}{dt} = V^x(x, y) \\ \frac{dy}{dt} = V^y(x, y) \end{cases} \quad \begin{matrix} r = x^2 + y^2 \\ \frac{dr}{dt} = \frac{dx}{dt} x + \frac{dy}{dt} y \\ \frac{dr}{dt} = \frac{1}{r} \end{matrix}$$

のとき

$$\begin{aligned} \frac{dr}{dt} &= \frac{\partial r}{\partial x} V^x(r \cos \theta, r \sin \theta) + \frac{\partial r}{\partial y} V^y(r \cos \theta, r \sin \theta) \\ &= \cos \theta V^x(r \cos \theta, r \sin \theta) + \sin \theta V^y(r \cos \theta, r \sin \theta). \end{aligned}$$