$$\Phi_*\left(\frac{\partial}{\partial x^i}\right) = \sum_{j=1}^{n} \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j} = \left(\frac{\partial y^1}{\partial x^i}, \cdots, \frac{\partial y^n}{\partial x^i}\right) \tag{2.5}$$

単に $\displaystyle\frac{\partial}{\partial x^i} = \sum_{j=1}^{n} \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}$ と書く。

x と $y = \Phi(x)$ を可微分同相写像 Φ で同一視できる。

例 2.5　極座標変換 $(x, y) \to (r, \theta)$ の場合

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$$
$$= \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}$$
$$= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

逆に解いて,

$$\begin{cases} \dfrac{\partial}{\partial x} = \cos\theta\dfrac{\partial}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial}{\partial \theta}, \\ \dfrac{\partial}{\partial y} = \sin\theta\dfrac{\partial}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial}{\partial \theta} \end{cases} \quad \square$$

(d)　オイラー・ラグランジュの方程式の座標変換

$L(x^1, \cdots, x^n\ \xi^1, \cdots, \xi^n)$: $2n$変数関数

$$\frac{d}{dt}\frac{\partial L}{\partial \xi^i}(x^1, \cdots, x^n, \dot{x}^1, \cdots, \dot{x}^n) - \frac{\partial L}{\partial x^i}(x^1, \cdots, x^n, \cdots, x^n) = 0 \tag{2.6}$$

オイラー・ラグランジュ方程式

Φ : 可微分同相写像 : $y \mapsto x = \Phi(y)$