(c) オイラー・ラグランジュ方程式

$L(x_1,\cdots,x_n,y_1,\cdots,y_n)$ ：無限回微分可能

$\Omega(x_0,x_1)=\{x:[0,1]\to\mathbb{R}^n \mid x：無限回微分可能\}$

$x\in\Omega(x_0,x_1)$ に対して,

$$\mathcal{L}(x,\dot{x}) \overset{\text{def}}{=} \int_0^1 L(x_1,\cdots,x_n,\dot{x}_1,\cdots,\dot{x}_n)\,dt \qquad (1.31)$$

$$\overset{\text{def}}{=} \int_0^1 L(x,\dot{x})\,dt \qquad \square$$

・ $\mathcal{L}:\Omega(x_0,x_1)\to\mathbb{R}$ の極値について.

$\Delta x:[0,1]\to\mathbb{R}^n$, $\Delta x(0)=\Delta x(1)=0$

$x_\delta = x+\delta\Delta x$ とおく。

$$\frac{d}{d\delta}\mathcal{L}(x_\delta,\dot{x}_\delta)\Big|_{\delta=0} = \frac{d}{d\delta}\int_0^1 L(x_\delta,\dot{x}_\delta)\,dt\Big|_{\delta=0}$$

$$=\frac{d}{d\delta}\mathcal{L}(x_\delta,\dot{x}_\delta)=\frac{d}{d\delta}\int_0^1 L(x_\delta,\dot{x}_\delta)\,dt$$

$$=\int_0^1 \Bigg(\underbrace{\frac{dx_\delta}{d\delta}}_{\Delta x}\cdot\underbrace{\left(\frac{\partial L}{\partial x_{\delta 1}},\cdots,\frac{\partial L}{\partial x_{\delta n}}\right)}_{\overset{\text{def}}{=}\frac{\partial L}{\partial x_\delta}}(x_\delta,\dot{x}_\delta)+\underbrace{\frac{d\dot{x}_\delta}{d\delta}}_{\frac{d\Delta x}{dt}}\cdot\underbrace{\left(\frac{\partial L}{\partial \dot{x}_{\delta 1}},\cdots,\frac{\partial L}{\partial \dot{x}_{\delta n}}\right)}_{\overset{\text{def}}{=}\frac{\partial L}{\partial \dot{x}_\delta}}(x_\delta,\dot{x}_\delta)\Bigg)dt\Bigg|_{\delta=0} \qquad (1.32)$$

$$=\int_0^1\left(\Delta x\cdot\frac{\partial L}{\partial x}(x,\dot{x})+\frac{d\Delta x}{dt}\cdot\frac{\partial L}{\partial \dot{x}}(x,\dot{x})\right)dt$$

$$=\underbrace{\left[\Delta x\frac{\partial L}{\partial \dot{x}}\right]_0^1}_{0}-\int_0^1\left(\Delta x\cdot\frac{\partial L}{\partial \dot{x}}-\Delta x\cdot\frac{\partial L}{\partial x}\right)=\int_0^1\left(\Delta x\cdot\left(\frac{\partial L}{\partial x}-\frac{\partial L}{\partial \dot{x}}\right)\right)dt \qquad (1.33)$$