問4　$[\mathbb{V}_1 + \mathbb{V}_2, \mathbb{W}] = [\mathbb{V}_1, \mathbb{W}] + [\mathbb{V}_2, \mathbb{W}]$, $[\mathbb{V}, \mathbb{W}] = -[\mathbb{W}, \mathbb{V}]$

[証明]

$$[\mathbb{V}_1 + \mathbb{V}_2, \mathbb{W}] = \sum_{i,j} \left((V_1^i + V_2^i) \frac{\partial W^j}{\partial x^i} - W^i \frac{\partial (V_1^j + V_2^j)}{\partial x^i} \right) \frac{\partial}{\partial x^j}$$

$$= \sum_{i,j} \left(V_1^i \frac{\partial W^j}{\partial x^i} - W^i \frac{\partial V_1^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} + \sum_{i,j} \left(V_2^i \frac{\partial W^j}{\partial x^i} - W^i \frac{\partial V_2^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}$$

$$= [\mathbb{V}_1, \mathbb{W}] + [\mathbb{V}_2, \mathbb{W}] \quad \square$$

$$[\mathbb{V}, \mathbb{W}] = \sum_{i,j} \left(V^i \frac{\partial W^j}{\partial x^i} - W^i \frac{\partial V^j}{\partial x^i} \right) \frac{\partial}{\partial x^i}$$

$$= -\sum_{i,j} \left(W^i \frac{\partial V^j}{\partial x^i} - V^i \frac{\partial W^j}{\partial x^i} \right) \frac{\partial}{\partial x^i} = -[\mathbb{W}, \mathbb{V}] \quad \square$$

補題 2.55　$\mathbb{V}, \mathbb{W}$：ベクトル場

φ_t, ψ_t：$\mathbb{V}, \mathbb{W}$ に随伴する1径数変換群　のとき

次の2つは同値

(i) $[\mathbb{V}, \mathbb{W}] = 0$

(ii) $\forall p, t, s$ について　$\varphi_t(\psi_s(p)) = \psi_s(\varphi_t(p))$。

[証明] (i) ⇒ (ii)

$\mathbb{W}(t, p) = D\varphi_{-t} \mathbb{W}(\varphi_t(p))$　とする。

$$\left. \frac{d\mathbb{W}(t,p)}{dt} \right|_{t=t_0} = \lim_{\varepsilon \to 0} \frac{D\varphi_{-t_0-\varepsilon} \mathbb{W}(\varphi_{t_0+\varepsilon}(p)) - D\varphi_{-t_0} \mathbb{W}(\varphi_{t_0}(p))}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{D\varphi_{-t_0} \cdot \varphi_{-\varepsilon} \mathbb{W}(\varphi_{t_0+\varepsilon}(p)) - D\varphi_{-t_0} \mathbb{W}(\varphi_{t_0}(p))}{\varepsilon}$$

補題 [illegible] →
$$= \lim_{\varepsilon \to 0} D\varphi_{-t_0} \left(\frac{D\varphi_{-\varepsilon} \mathbb{W}(\varphi_{t_0+\varepsilon}(p)) - \mathbb{W}(\varphi_{t_0}(p))}{\varepsilon} \right)$$