$$= -\sum_{j,k,l,m,n} g^{im}\frac{\partial g_{mn}}{\partial q^k} g^{nj} g^{kl} p_l p_j - \frac{1}{2}\sum_{j,k,l} g^{ij}\frac{\partial g^{kl}}{\partial q^j} p_k p_l$$

$$= -\sum_{j,k,l,m,n} g^{im}\frac{\partial g_{mn}}{\partial q^k} g^{nj} g^{kl} p_l p_j + \frac{1}{2}\sum_{j,k,l,m,n} g^{ij} g^{km}\frac{\partial g_{mn}}{\partial q^j} g^{nl} p_k p_l$$

↲ $g^{ij}$ は対称行列

$$= -\sum_{k,m,n} g^{im}\frac{\partial g_{mn}}{\partial q^k} y^n y^k + \frac{1}{2}\sum_{j,m,n} g^{ij}\frac{\partial g_{mn}}{\partial q^j} y^m y^n$$

$$= -\sum_{j,m,n} g^{im}\frac{\partial g_{mn}}{\partial q^j} y^n y^j + \frac{1}{2}\sum_{j,m,n} g^{ij}\frac{\partial g_{mn}}{\partial q^j} y^m y^n$$ ← $k$を$j$に交換

$$= -\sum_{j,m,n} g^{ij}\frac{\partial g_{jm}}{\partial q^n} y^m y^n + \frac{1}{2}\sum_{j,m,n} g^{ij}\frac{\partial g_{mn}}{\partial q^j} y^m y^n$$ ← $j \to n \to m \to j$ を第1項で

$$= \frac{1}{2}\sum_{j,m,n} g^{ij}\left(\frac{\partial g_{mn}}{\partial q^j} - 2\frac{\partial g_{jm}}{\partial q^n}\right) y^m y^n .$$ $\because\ x^i = q^i,\quad y^i = \frac{dx^i}{dt}$

$$= \frac{1}{2}\sum_{j,m,n} g^{ij}\left(\frac{\partial g_{mn}}{\partial q^j} - 2\frac{\partial g_{jm}}{\partial q^n}\right)\frac{dq^m}{dt}\frac{dq^n}{dt} .$$

$$= \frac{1}{2}\sum_{j,m,n} g^{ij}\left(\frac{\partial g_{mn}}{\partial q^j} - \frac{\partial g_{jm}}{\partial q^n} - \frac{\partial g_{jn}}{\partial q^m}\right)\frac{dq^m}{dt}\frac{dq^n}{dt}$$ ← 2ケのうちの1ケを$m$と$n$を交換

よって

$$\frac{d^2x^i}{dt^2} + \sum_{j,m,n}\frac{1}{2}g^{ij}\left(\frac{\partial g_{jm}}{\partial q^n} + \frac{\partial g_{jn}}{\partial q^m} - \frac{\partial g_{mn}}{\partial q^j}\right)\frac{dq^m}{dt}\frac{dq^n}{dt} = 0 \qquad (3.64)$$

$$\Gamma^i_{nm} = \sum_j \frac{1}{2}g^{ij}\left(\frac{\partial g_{jm}}{\partial q^n} + \frac{\partial g_{jn}}{\partial q^m} - \frac{\partial g_{mn}}{\partial q^j}\right)\frac{dq^m}{dt}\frac{dq^n}{dt}$$

とおく。

(クリストフェルの記号)

定理 3.59　$x(t)$ が測地線

$$\Longleftrightarrow \frac{d^2x^i}{dt^2} + \sum_{m,n}\Gamma^i_{nm}\frac{dx^m}{dt}\frac{dx^n}{dt} = 0 \quad (3.66)$$

□