$$
\begin{array}{ccc}
S & & U \supset \Omega(x_0, x_1) \\
\cup & & \cup \qquad \Psi \\
\boldsymbol{l} : [0,1] \to \mathbb{R}^3 & \overset{\varphi \ 1\text{-}1}{\longleftrightarrow} & x : [0,1] \to U \\
\boldsymbol{l}(0) = A,\ \boldsymbol{l}(1) = B & \boldsymbol{l}(t) = \varphi(x(t)). & x(0) = x_0, \quad x(1) = x_1 .
\end{array}
$$

$$
\mathcal{L}(\boldsymbol{l}) = \int_0^1 \sqrt{\frac{d\boldsymbol{l}}{dt} \cdot \frac{d\boldsymbol{l}}{dt}}\, dt
$$

$$
= \int_0^1 \sqrt{\sum_{i=1}^{3} \frac{\partial \varphi}{\partial x^i} \frac{dx^i}{dt} \sum_{j=1}^{3} \frac{d\varphi}{dx^j} \frac{dx^j}{dt}}\, dt \qquad \leftarrow \text{合成関数の微分法}
$$

$$
= \int_0^1 \sqrt{\sum_{i,j} \left(\frac{dx^i}{dt} \frac{dx^j}{dt}\right) \frac{\partial \varphi}{\partial x^i} \cdot \frac{\partial \varphi}{\partial x^j}}\, dt \qquad \Delta
$$

$$
g_{ij}(x) = \frac{\partial \varphi}{\partial x^i}(x) \cdot \frac{\partial \varphi}{\partial x^j}(x) = \sum_k \frac{\partial \varphi^k}{\partial x^i}(x) \frac{\partial \varphi^k}{\partial x^j}(x) \quad : \text{第1基本形式, リーマン計量}
$$

とおくと

$$
\mathcal{L}(\boldsymbol{l}) = \int_0^1 \sqrt{\sum_{i,j} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}}\, dt \qquad (3.49)
$$

$(g_{ij})_{i=1}^{2}{}_{j=1}^{2}$ は 可逆な対称行列 $(g_{12} = g_{21})$