◦ ベクトル場の発散　$\operatorname{div} \mathbb{W} = \dfrac{\partial W^u}{\partial u} + \dfrac{\partial W^v}{\partial v}$

の座標変換について.

$\operatorname{div}(\Phi_* \mathbb{W})(\Phi(u,v))$

$$= \frac{\partial}{\partial x}\left(\frac{\partial x}{\partial u}W^u + \frac{\partial x}{\partial v}W^v\right) + \frac{\partial}{\partial y}\left(\frac{\partial y}{\partial u}W^u + \frac{\partial y}{\partial v}W^v\right)$$

$$= \frac{\partial x}{\partial u}\frac{\partial W^u}{\partial x} + \frac{\partial x}{\partial v}\frac{\partial W^v}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial W^u}{\partial y} + \frac{\partial y}{\partial v}\frac{\partial W^v}{\partial y}$$

$$+ \frac{\partial^2 x}{\partial x \partial u}W^u + \frac{\partial^2 x}{\partial x \partial v}W^v + \frac{\partial^2 y}{\partial y \partial u}W^u + \frac{\partial^2 y}{\partial y \partial v}W^v$$

$$= \frac{\partial W^u}{\partial u} + \frac{\partial W^v}{\partial v} + \left(\frac{\partial^2 x}{\partial x \partial u} + \frac{\partial^2 y}{\partial y \partial u}\right)W^u + \left(\frac{\partial^2 x}{\partial x \partial v} + \frac{\partial^2 y}{\partial y \partial v}\right)W^v$$

$$= \operatorname{div} \mathbb{W} + \left(\frac{\partial^2 x}{\partial x \partial u} + \frac{\partial^2 y}{\partial y \partial u}\right)W^u + \left(\frac{\partial^2 x}{\partial x \partial v} + \frac{\partial^2 y}{\partial y \partial v}\right)W^v \qquad (2.10)$$

極座標変換　$(x, y) = (r\cos\theta, r\sin\theta)$ の場合.　($u=r$ or $v=r$ として)

$$\frac{\partial^2 x}{\partial x \partial r} + \frac{\partial^2 y}{\partial y \partial r} = \frac{\partial}{\partial x}\cos\theta + \frac{\partial}{\partial y}\sin\theta$$

$$= \frac{\partial}{\partial x}\frac{x}{\sqrt{x^2+y^2}} + \frac{\partial}{\partial y}\frac{y}{\sqrt{x^2+y^2}} = \frac{1}{\sqrt{x^2+y^2}} \neq 0$$

$$\therefore \quad \operatorname{div} \mathbb{W}(u,v) \neq \operatorname{div}(\Phi_* \mathbb{W})(\Phi(u,v)). \quad \square$$

問2　勾配 grad の座標変換について

[解]　$y = \Phi(x)$　　$f : \underset{\mathbb{R}^n}{\overset{}{V}} \to \mathbb{R} : y \mapsto f(y)$　とすると

$V \subset \mathbb{R}^n$

「$\operatorname{grad}(f \circ \Phi(x)) = (\Phi^{-1})_*(\operatorname{grad} f)$」

$$\operatorname{grad}(f\circ\Phi) = \sum_i \frac{\partial(f\circ\Phi)}{\partial x^i}\frac{\partial}{\partial x^i} = \sum_i \sum_j \frac{\partial f}{\partial y^j}\frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial x^i}$$

$$(\Phi^{-1})_*(\operatorname{grad} f) = (\Phi^{-1})_*\left(\sum_i \frac{\partial f}{\partial y^i}\frac{\partial}{\partial y^i}\right) = \sum_i \sum_j \frac{\partial f}{\partial y^i}\frac{\partial x^j}{\partial y^i}\frac{\partial}{\partial x^j}$$