補題 2.18　任意の微分形式 u について．

$$d(du) = 0 .$$

[証明]　関数 f（微分0形式）について

$$ddf = d\left(\sum_i \frac{\partial f}{\partial x^i} dx^i\right) = \sum_i d\left(\frac{\partial f}{\partial x^i}\right) \wedge dx^i$$

$$= \sum_i \sum_j \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i$$

$$= \sum_{i<j} \left(\frac{\partial^2 f}{\partial x^j \partial x^i} - \frac{\partial^2 f}{\partial x^i \partial x^j} \right) dx^j \wedge dx^i = 0 \quad (*)$$

一般の場合は (*) と 補題 2.17 を用いて

$$dd\left(\sum_{1 \leqq i_1 < \cdots < i_k \leqq n} f_{i_1 \cdots i_k} \, dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right)$$

$$= d\left(\sum_{1 \leqq i_1 < \cdots < i_k \leqq n} df_{i_1 \cdots i_k} \wedge (dx^{i_1} \wedge \cdots \wedge dx^{i_k})\right)$$

$$= \sum_{1 \leqq i_1 < \cdots < i_k \leqq n} d\left(df_{i_1 \cdots i_k} \wedge (dx^{i_1} \wedge \cdots \wedge dx^{i_k})\right)$$

$$= \sum_{1 \leqq i_1 < \cdots < i_k \leqq n} \left(\underbrace{ddf_{i_1 \cdots i_k}}_{=0} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} - df_{i_1 \cdots i_k} \wedge d(dx^{i_1} \wedge \cdots \wedge dx^{i_k})\right)$$

$$= \sum_{1 \leqq i_1 < \cdots < i_k \leqq n} \left(- df_{i_1 \cdots i_k} \wedge \underline{d(1)} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_1}\right)$$

$$= 0$$

□

補題 2.19　$u_j = \sum_{i=1}^{n} u_{ji} \, dx^i \quad (j = 1, \cdots, n)$

：$U \subseteqq \mathbb{R}^n$ 上の微分1形式　について

$$u_1 \wedge u_2 \wedge \cdots \wedge u_n = \begin{vmatrix} u_{11} & \cdots & u_{i1} & \cdots & u_{n1} \\ \vdots & \ddots & \vdots & & \vdots \\ u_{1j} & \cdots & u_{ij} & \cdots & u_{nj} \\ \vdots & & \vdots & \ddots & \vdots \\ u_{1n} & \cdots & u_{in} & \cdots & u_{nn} \end{vmatrix} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n .$$