[証明] (ii) ⇒ (i) は (3.35) より明らか.

(i) ⇒ $$\frac{\partial}{\partial t} f(\varphi_t(p))\Big|_{t=t_0} = \lim_{\varepsilon\to 0}\frac{f(\varphi_{t_0+\varepsilon}(p)) - f(\varphi_{t_0}(p))}{\varepsilon}$$
$$= \lim_{\varepsilon\to 0}\frac{f(\varphi_\varepsilon(\varphi_{t_0}(p))) - f(\varphi_{t_0}(p))}{\varepsilon}$$
$$= \mathbb{V}(f)(\varphi_{t_0}(p)) = 0 .$$

$$\therefore\ f(\varphi_{t_0}(p)) = f(\varphi_t(p)) = f(\varphi_0(p)) = f(p)$$

$\therefore\ f\circ\varphi_t = f$. (補題 3.29 の証明と同様) ⇒ (ii)

これでこの項の～(read the rest aloud)～ 述べておこう.

補題 3.34 $$[\mathbb{V}, \mathbb{W}](f) = \mathbb{V}(\mathbb{W}(f)) - \mathbb{W}(\mathbb{V}(f)) .$$

[証明] $$\mathbb{V}(\mathbb{W}(f)) - \mathbb{W}(\mathbb{V}(f)) = \sum_{i,j} V^i\frac{\partial}{\partial x^i}\left(W^j\frac{\partial f}{\partial x^j}\right) - \sum_{i,j} W^i\frac{\partial}{\partial x^i}\left(V^j\frac{\partial f}{\partial x^j}\right)$$
$$= \sum_{i,j}\left(V^i\frac{\partial W^j}{\partial x^i}\frac{\partial f}{\partial x^j} + \underline{V^i W^j\frac{\partial^2 f}{\partial x^i\partial x^j}}\right) - \sum_{i,j}\left(W^i\frac{\partial V^j}{\partial x^i}\frac{\partial f}{\partial x^j} + \underline{W^i V^j\frac{\partial^2 f}{\partial x^i\partial x^j}}\right)$$
$$= \sum_{i,j}\left(V^i\frac{\partial W^j}{\partial x^i} - W^i\frac{\partial V^j}{\partial x^i}\right)\frac{\partial f}{\partial x^j}$$
$$= [\mathbb{V}, \mathbb{W}](f) .$$ ■

系 3.35 (ヤコビの恒等式)

$$[[\mathbb{U}, \mathbb{V}], \mathbb{W}] + [[\mathbb{V}, \mathbb{W}], \mathbb{U}] + [[\mathbb{W}, \mathbb{U}], \mathbb{V}] = 0 .$$

[証明] f: 任意の関数

$$([[\mathbb{U}, \mathbb{V}], \mathbb{W}] + [[\mathbb{V}, \mathbb{W}], \mathbb{U}] + [[\mathbb{W}, \mathbb{U}], \mathbb{V}])f$$
$$= [\mathbb{U}, \mathbb{V}](\mathbb{W}(f)) - \mathbb{W}([\mathbb{U}, \mathbb{V}](f)) + [\mathbb{V}, \mathbb{W}](\mathbb{U}(f)) - \mathbb{U}([\mathbb{V}, \mathbb{W}](f))$$
$$+ [\mathbb{W}, \mathbb{U}](\mathbb{V}(f)) - \mathbb{V}([\mathbb{W}, \mathbb{U}](f))$$