また, $|f_i|^2 = |e_i' - 0_1'|^2 = \overline{e_i'0'}^2 = \overline{e_i 0}^2 = 1$

$\therefore \quad |f_i| = 1$

よって $f_i \cdot f_j = \delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$

$\Rightarrow \quad \sum_{k=1}^{3} r_{ik} r_{jk} = \delta_{ij} \qquad (1 \leqq i, j \leqq 3)$

$\Rightarrow \quad C = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$ とおくと

$C \cdot {}^tC = I$

$\Rightarrow \quad \det(C)^2 = 1 \quad \Rightarrow \quad \det(C) = \pm 1 \quad \Delta$

合同変換は内積を変えない。すなわち,

$p, q, r \in \mathbb{R}^3$ について $f(p) = p'$ $f(q) = q'$ $f(r) = r'$

$a = q - p$, $b = r - p$, $a' = q' - p'$ $b' = r' - p'$ とおくと

$a \cdot b = a' \cdot b'$

$\because \quad \overline{qr}^2 - (\overline{pq}^2 + \overline{pr}^2) = -2a \cdot b$

$\overline{q'r'}^2 - (\overline{p'q'}^2 + \overline{p'r'}^2) = -2a' \cdot b'$

$\Rightarrow \quad a \cdot b = a' \cdot b' \quad \Delta$

$x = (x_1, x_2, x_3)$, $f(x) = x'$, $x' - 0' = \underset{(y_1, y_2, y_3)}{\underset{\shortparallel}{y}}$ とすると

$x_i = x \cdot e_i = (x - 0) \cdot (e_i - 0)$

$= (x' - 0') \cdot (e_i' - 0')$