(c) ポアソンの括弧と括弧積

$$\text{ポアソン括弧}\quad \{G, H\} \overset{\text{def}}{=} \sum_i \left(\frac{\partial G}{\partial q^i}\frac{\partial H}{\partial p^i} - \frac{\partial G}{\partial p^i}\frac{\partial H}{\partial q^i}\right) \qquad (3.36)$$

定理 3.36 (i) G が H で定まるハミルトン方程式の第1積分

$\Leftrightarrow$ (ii) $\{G, H\} = 0$

[証明]

$$\frac{dG}{dt} = \sum_i \frac{\partial G}{\partial q^i}\frac{dq^i}{dt} + \frac{\partial G}{\partial p^i}\frac{dp^i}{dt}$$

$$= \sum_i \frac{\partial G}{\partial q^i}\frac{\partial H}{\partial p^i} - \frac{\partial G}{\partial p^i}\frac{\partial H}{\partial q^i}$$

$$= \{G, H\}$$

$$= 0$$

■

補題 3.37 $\quad [\mathbb{X}_f, \mathbb{X}_g] = \mathbb{X}_{\{f, g\}}$

[証明]

$$[\mathbb{X}_f, \mathbb{X}_g] = \left[\sum_{i=1}^n \left(\frac{\partial f}{\partial q^i}\frac{\partial}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial}{\partial q^i}\right), \sum_{i=1}^n \left(\frac{\partial g}{\partial q^i}\frac{\partial}{\partial p^i} - \frac{\partial g}{\partial p^i}\frac{\partial}{\partial q^i}\right)\right]$$

$$= \sum_{i,j} \frac{\partial f}{\partial q^i}\frac{\partial^2 g}{\partial p^i \partial q^j}\frac{\partial}{\partial p^j} - \sum_{i,j} \frac{\partial f}{\partial q^i}\frac{\partial^2 g}{\partial p^i \partial p^j}\frac{\partial}{\partial q^j}$$

$$- \sum_{i,j} \frac{\partial f}{\partial p^i}\frac{\partial^2 g}{\partial q^i \partial q^j}\frac{\partial}{\partial p^j} + \sum_{i,j} \frac{\partial f}{\partial p^i}\frac{\partial^2 g}{\partial q^i \partial p^j}\frac{\partial}{\partial q^j}$$

$$- (f \Leftrightarrow g)\text{。}$$

ただし、$(f \Leftrightarrow g)$ は上の4つの総和で f と g を入れ替えたもの。 △

$$\mathbb{X}_{\{f,g\}} = \mathbb{X}_{\sum_i \left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q^i}\right)}$$

$$= \sum_{i,j} \left(\frac{\partial}{\partial q^j}\left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p^i}\right)\frac{\partial}{\partial p^j} - \frac{\partial}{\partial p^j}\left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p^i}\right)\frac{\partial}{\partial q^j}\right)$$

$$- \sum_{i,j} \left(\frac{\partial}{\partial q^j}\left(\frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q^i}\right)\frac{\partial}{\partial p^j} - \frac{\partial}{\partial p^j}\left(\frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q^i}\right)\frac{\partial}{\partial q^j}\right)$$