$$\Rightarrow \lim_{\varepsilon\to 0} \frac{\frac{\partial \varphi_{-\varepsilon}}{\partial x^i}(q) - e_i}{\varepsilon} = \frac{-\partial \mathbb{V}}{\partial x^i}(q) \qquad (2.31)$$

e_i : 第 i 成分が 1，それ以外が 0 のベクトル。

$q = p$ とし、$W^i(p)$ を掛けて i について和をとると

$$\lim_{\varepsilon\to 0} \sum_i \frac{\frac{\partial \varphi_{-\varepsilon}}{\partial x^i}(p) W^i(p) - e_i W^i(p)}{\varepsilon} = -\sum_i W^i \frac{\partial \mathbb{V}}{\partial x^i}(p)$$

$$(\text{左辺}) = \lim_{\varepsilon\to 0} \frac{D\varphi_{-\varepsilon}\mathbb{W}(p) - \mathbb{W}(p)}{\varepsilon}$$

(2.□) に代入

$$[\mathbb{V}, \mathbb{W}](p) = \sum_i \left(\frac{\partial \mathbb{W}}{\partial x^i} V^i(p) - W^i \frac{\partial \mathbb{V}}{\partial x^i}(p) \right) \qquad (2.32)$$

すなわち、

補題 2.53

$$\left[\sum_i V^i \frac{\partial}{\partial x^i}, \sum_i W^i \frac{\partial}{\partial x^i}\right] = \sum_{i,j} \left(V^i \frac{\partial W^j}{\partial x^i} - W^i \frac{\partial V^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}. \qquad \square$$

例 2.54.

$$\mathbb{V}_A(x) = \sum_i V_A^i(x) \frac{\partial}{\partial x^i} = \sum_{i,j} a_{ij} x^j \frac{\partial}{\partial x^i}$$

について。

$$[\mathbb{V}_A, \mathbb{V}_B] = \sum_{i,j,k,l} \left(a_{ik} x^k \frac{\partial (b_{jl} x^l)}{\partial x^i} - b_{ik} x^k \frac{\partial (a_{jl} x^l)}{\partial x^i} \right) \frac{\partial}{\partial x^j}$$

$$= \sum_{i,j,k} (b_{ji} a_{ik} x^k - a_{ji} b_{ik} x^k) \frac{\partial}{\partial x^j}$$

$$= \mathbb{V}_{BA-AB}.$$

$[A, B] = AB - BA$ とすると、

$$[\mathbb{V}_A, \mathbb{V}_B] = \mathbb{V}_{[B,A]} \qquad \square$$