$$= \mathbb{U}(\mathbb{V}(\mathbb{W}(f))) - \mathbb{V}(\mathbb{U}(\mathbb{W}(f))) - \mathbb{W}(\mathbb{U}(\mathbb{V}(f))) + \mathbb{W}(\mathbb{V}(\mathbb{U}(f)))$$

$$+ \mathbb{V}(\mathbb{W}(\mathbb{U}(f))) - \mathbb{W}(\mathbb{V}(\mathbb{U}(f))) - \mathbb{U}(\mathbb{V}(\mathbb{W}(f))) + \mathbb{U}(\mathbb{W}(\mathbb{V}(f)))$$

$$+ \mathbb{W}(\mathbb{U}(\mathbb{V}(f))) - \mathbb{U}(\mathbb{W}(\mathbb{V}(f))) - \mathbb{V}(\mathbb{W}(\mathbb{U}(f))) + \mathbb{V}(\mathbb{U}(\mathbb{W}(f)))$$

$$= 0.$$

$$[[\mathbb{U}, \mathbb{V}], \mathbb{W}] + [[\mathbb{V}, \mathbb{W}], \mathbb{U}] + [[\mathbb{W}, \mathbb{U}], \mathbb{V}] = \sum_i X^i \frac{\partial}{\partial x^i}$$

とおくと,

$^\exists x^i$ として,

$$X^i = [[\mathbb{U}, \mathbb{V}], \mathbb{W}] + [[\mathbb{V}, \mathbb{W}], \mathbb{U}] + [[\mathbb{W}, \mathbb{U}], \mathbb{V}]\,(x^i) = 0.$$

よって (iを1からnまでとって)

$$[[\mathbb{U}, \mathbb{V}], \mathbb{W}] + [[\mathbb{V}, \mathbb{W}], \mathbb{U}] + [[\mathbb{W}, \mathbb{U}], \mathbb{V}] = 0. \quad \blacksquare$$

問3, $f(y^1, \cdots, y^n)$: 関数, $\varphi(x^1, \cdots, x^n) = (y^1, \cdots, y^n)$: 可微分同相写像

$\mathbb{V} = \sum_i V^i \frac{\partial}{\partial x^i}$, $\varphi^* f(p) \overset{\mathrm{def}}{=} f(\varphi(p))$ のとき,

$$\mathbb{V}(\varphi^* f) = \varphi^*((\varphi_* \mathbb{V})(f))$$

[証明]
$$\varphi^*((\varphi_* \mathbb{V})(f))(x) = \varphi^*\Big(\Big(\varphi_* \sum_i V^i \frac{\partial}{\partial x^i}\Big)(f)\Big)(x)$$

$$= \Big(\sum_i \sum_j V^i \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}\Big)(f)(\varphi(x))$$

$$= \sum_i \sum_j V^i \frac{\partial y^j}{\partial x^i} \frac{\partial f}{\partial y^j}(\varphi(x))$$

$$= \sum_i V^i \frac{\partial(\varphi^* f)}{\partial x^i}(x)$$

$$= \mathbb{V}(\varphi^* f). \quad \blacksquare$$