[証明]

$$\frac{dG(q(t),p(t))}{dt} = \sum_{i=1}^{2}\left(\frac{\partial G}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial G}{\partial p_i}\frac{dp_i}{dt}\right)$$

$$= \sum_{i=1}^{2}\left(\frac{\partial G}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial G}{\partial p_i}\frac{\partial H}{\partial q_i}\right)$$

$$= \{G, H\} \qquad \Box$$

系 1.18　$G : \mathbb{R}^4 \to \mathbb{R}$: ハミルトニアンHに対するハミルトン方程式の第1積分

$$\Leftrightarrow \quad \{G, H\} = 0$$

[証明]　定義 1.12，定理 1.17　□

[定理 1.15の証明]　$r = \sqrt{q_1^2 + q_2^2}$，$V(q_1, q_2) = K(r)$ とおくと

$$\{A, H\} = \sum_{i=1}^{2}\left(\frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q_i}\right) \qquad (\text{定義})$$

$$= \sum_{i=1}^{2}\frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \sum_{i=1}^{2}\frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q_i}$$

$$= \frac{\partial A}{\partial q_1}\frac{\partial H}{\partial p_1} + \frac{\partial A}{\partial q_2}\frac{\partial H}{\partial p_2} - \frac{\partial A}{\partial p_1}\frac{\partial H}{\partial q_1} - \frac{\partial A}{\partial p_2}\frac{\partial H}{\partial q_2}$$

$$= p_2 p_1 - p_1 p_2 - q_2\frac{\partial H}{\partial q_1} + q_1\frac{\partial H}{\partial q_2}$$

$$= -q_2\frac{\partial V}{\partial q_1} - q_1\frac{\partial V}{\partial q_2}$$

$$= -q_2\frac{\partial K}{\partial r}\frac{\partial r}{\partial q_1} + q_1\frac{\partial K}{\partial r}\frac{\partial r}{\partial q_2}$$

$$= -\frac{\partial K}{\partial r}\left(q_2\frac{q_1}{r} - q_1\frac{q_2}{r}\right) = 0$$

(系 1.18)　□