ELSEVIER Journal of Pure and Applied Algebra 170 (2002) 185–242

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

Algebraic K-theory of non-linear projective spaces

Thomas Hüttemann

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany

Received 16 June 2000; received in revised form 11 May 2001
Communicated by C.A. Weibel

Abstract

In the spirit of "The Fundamental Theorem for the algebraic K-theory of spaces: I" (J. Pure Appl. Algebra 160 (2001) 21–52) we introduce a category of sheaves of topological spaces on n-dimensional projective space and present a calculation of its K-theory, a "non-linear" analogue of Quillen's isomorphism $K_i(\mathbf{P}^n_R) \cong \bigoplus_0^n K_i(R)$. © 2002 Elsevier Science B.V. All rights reserved.

MSC: 19D10; 55U35

1. Introduction

Let R denote a commutative ring. Quillen has proved [6, Section 8, Theorem 2.1] that there is an isomorphism of K-groups $K_i(\mathbf{P}^n_R) \cong \bigoplus_0^n K_i(R)$ for all $i \geqslant 0$ where $\mathbf{P}^n_R = \operatorname{Proj} R[X_0, X_1, \ldots, X_n]$ is the n-dimensional projective space over R.

This paper is concerned with an analogous result for the algebraic K-theory of spaces in the sense of Waldhausen [10]. We define a category $\mathbf{P}^n$ of "quasi-coherent sheaves" on projective n-space and a notion of twisted structure sheaves $\mathcal{O}_{\mathbf{P}^n}(j)$. If Y is a pointed space, we can form the "tensor product" $Y \wedge \mathcal{O}_{\mathbf{P}^n}(j)$.

Theorem 4.4.5. *The assignment*

$$(Y_0, Y_1, \ldots, Y_n) \mapsto \bigvee_{j=0}^{n} Y_j \wedge \mathcal{O}_{\mathbf{P}^n}(-j)$$

induces a weak homotopy equivalence $\prod_0^n A^{\mathrm{sfd}}(*) \to \Omega|h\mathcal{S}_\bullet \mathbf{P}^n|$ (*where* $A^{\mathrm{sfd}}(*)$ *is the version of Waldhausen's algebraic K-theory of spaces functor using stably finitely dominated spaces, and the target is the algebraic K-theory of non-linear projective n-space*).

PII: S0022-4049(01)00088-3

In more detail, recall that a quasi-coherent sheaf on the projective line over some ring R can be described as a diagram

$$Y_+ \xrightarrow{f_+} Y \xleftarrow{f_-} Y_-$$

where Y_+ is an $R[T]$-module, Y is an $R[T,T^{-1}]$-module, Y_- is an $R[T^{-1}]$-module, f_+ is an $R[T]$-linear map, f_- is an $R[T^{-1}]$-linear map, such that the induced diagram

$$Y_+ \otimes_{R[T]} R[T,T^{-1}] \to Y \leftarrow Y_- \otimes_{R[T^{-1}]} R[T,T^{-1}]$$

consists of isomorphisms of $R[T,T^{-1}]$-modules.

In analogy to this algebraic description Hüttemann, Klein, Vogell, Waldhausen and Williams defined a homotopy theoretic version of sheaves on the projective line. They considered diagrams $Y_+ \xrightarrow{f_+} Y \xleftarrow{f_-} Y_-$ where Y_+ is a topological space with an action of the natural numbers $\mathbf{N}$, Y is a space with an action of the integers $\mathbf{Z}$, Y_- is a space with an action of the negative integers $\mathbf{N}_-$, and f_+ (resp. f_-) is an $\mathbf{N}$-equivariant (resp. $\mathbf{N}_-$-equivariant) map such that the induced diagram

$$Y_+ \times_{\mathbf{N}} \mathbf{Z} \to Y \leftarrow Y_- \times_{\mathbf{N}_-} \mathbf{Z}$$

consists of weak homotopy equivalences. It was shown in [4] that the K-theory of the category of these "sheaves" (subject to a suitable finiteness condition) is weakly equivalent to the space $A^{\mathrm{sfd}}(*) \times A^{\mathrm{sfd}}(*)$.

Note that in algebraic geometry the description of quasi-coherent sheaves on $\mathbf{P}^1_R$ can be extended to higher dimensions: there is an equivalence of categories

$$\text{quasi-coherent sheaves on } \mathbf{P}^n_R \iff \text{certain diagrams of modules.} \qquad (*)$$

As for $n=1$ we can build analogous homotopy-theoretic gadgets on the right-hand side and prove an appropriate splitting theorem for K-theory. At present, the author is not aware of an interpretation of "non-linear sheaves" on the left-hand side of $(*)$.

1.1. Outline of the paper

In Section 2.1 we recall standard facts about equivariant spaces. Section 2.2 summarizes the construction of iterated homotopy cofibres for cubical diagrams of topological spaces; this material will be used later (Section 3.7) to define a "global sections" functor.

In Section 3.2 we discuss the monoids we need to define projective space, introduce the crucial construction of "inverting an indeterminate" (Definition 3.2.5) and prove a technical result about finiteness of equivariant spaces (Corollary 3.2.9).

The main objects under consideration are *non-linear sheaves*, introduced in Section 3.3. Such a sheaf Y is a collection of topological spaces Y^A, one for each non-empty subset $A \subseteq \{0,1,\ldots,n\}$, and a structure map $Y^A \xrightarrow{Y^b_\sigma} Y^B$ for each inclusion of sets $\sigma\colon A \subset B$, subject to the following conditions:

(1) the data given determines a commutative diagram of topological spaces;
(2) each space Y^A is equipped with an action of a certain monoid M^A (Definition 3.2.1);
(3) the structure maps are equivariant;
(4) the structure maps satisfy a certain *homotopy sheaf condition* (Definition 3.3.3).

The idea behind this definition is that if one replaces "monoid" by "monoid ring" and "equivariant space" by "module", and uses a "strict" (non-homotopic) version of the sheaf condition, the resulting category is equivalent to the category of quasi-coherent sheaves of modules on projective n-space in the sense of algebraic geometry.

The following Sections 3.4–3.8 contain the basic machinery. Section 3.4 exhibits two model structures on a category of non-linear presheaves (diagrams as above satisfying conditions (1)–(3)). These structures are auxiliary in nature and are not directly related to sheaves. Following that, we introduce the restriction of a sheaf to a lower-dimensional projective space, and twisting sheaves (Sections 3.5 and 3.6). The relevant observation is the following (Lemma 3.6.9): if X_i denotes one of the homogeneous coordinates in projective space, considered as a global section of the twisting sheaf, multiplication with X_i induces a self-map of sheaves with cokernel given by the extension by zero of a restriction of the sheaf to $\mathbf{P}^{n-1}$.

These constructions are modelled closely after the corresponding algebraic constructions. The "global sections" functor defined in Section 3.7, however, is an ad hoc definition which does not translate into the algebraic geometers' global sections functor. A similar comment applies to the notion of "spread sheaves" in Section 3.8. Both constructions are used in the proof of the splitting result; in fact, the global sections functor is used to give a homotopy equivalence $\Omega|h\mathscr{S}_\bullet\mathbf{P}^n| \to \prod_0^n A^{\mathrm{sfd}}(*)$.

Finally, Section 4 contains the K-theoretical part of this paper. Section 4.1 contains a discussion of finiteness notions for sheaves; this relies heavily on the formalism of model structures and homotopy categories. In Section 4.2 we prove that the global sections functor preserves finiteness. Sections 4.3 and 4.4 contain the main result of the paper, the splitting theorem.

Roughly speaking, the proof works as follows:

Step 1: Construct a fibration sequence (Lemma 4.4.3)

$$K(\mathbf{P}^{n,\{0\}}) \to K(\mathbf{P}^n) \xrightarrow{\Gamma} A(*)$$

where $K(\mathbf{P}^n)$ denotes the K-theory of the category of non-linear sheaves, and $K(\mathbf{P}^{n,\{0\}})$ denotes the K-theory of non-linear sheaves having contractible global sections. The map to $A(*)$ is induced by the global sections functor. This fibration sequence has a section up-to-homotopy, hence there is a weak equivalence $A(*) \times K(\mathbf{P}^{n,\{0\}}) \simeq K(\mathbf{P}^n)$.

Step 2: Construct a fibration sequence (Lemma 4.4.3)

$$K(\mathbf{P}^{n,\{0,1\}}) \to K(\mathbf{P}^{n,\{0\}}) \xrightarrow{\Gamma\circ\theta_1} A(*)$$

where $K(\mathbf{P}^{n,\{0,1\}})$ denotes the K-theory of non-linear sheaves having contractible global sections of their 0th and 1st twist. The map to $A(*)$ is induced by the global sections functor applied to the 1st twist. This fibration sequence has a section up-to-homotopy, hence there is a weak equivalence $A(*) \times K(\mathbf{P}^{n,\{0,1\}}) \simeq K(\mathbf{P}^{n,\{0\}})$.

We continue in this fashion, until we reach

Step $n+1$: Construct a fibration sequence (Lemma 4.4.3)

$$K(\mathbf{P}^{n,\{0,1,\dots,n\}}) \to K(\mathbf{P}^{n,\{0,1,\dots,n-1\}}) \xrightarrow{\Gamma\circ\theta_n} A(*)$$

where $K(\mathbf{P}^{n,\{0,1,\dots,n\}})$ denotes the K-theory of non-linear sheaves having contractible global sections of their 0th, 1st, 2nd, $\dots$, nth twist. The map to $A(*)$ is induced by the global sections functor applied to the nth twist. This fibration sequence has a section up-to-homotopy, hence there is a weak equivalence $A(*) \times K(\mathbf{P}^{n,\{0,1,\dots,n\}}) \simeq K(\mathbf{P}^{n,\{0,1,\dots,n-1\}})$. Now it turns out that the fibre of this sequence is contractible (Lemma 4.4.4). This is true since if a sheaf has contractible global sections for $n+1$ successive twists, it is very close to being the trivial sheaf (it suspends to a sheaf consisting of weakly contractible spaces). In this sense, Lemma 4.4.4 is the key to the whole splitting result. The proof is by induction on the dimension n: we restrict the sheaf to projective spaces of lower dimensions and show that all restrictions are trivial, hence the sheaf itself has to be trivial.

Assembling the weak equivalences from steps 1 to $n+1$ finally yields the desired splitting.

2. Preliminaries

This section contains a collection of various definitions and results on equivariant spaces and iterated homotopy cofibres used throughout the rest of the paper.

2.1. Equivariant spaces

To avoid some of the pathologies of set-theoretic topology we work exclusively with the model category of compactly generated spaces. The main technical result of this section is the construction of certain maps which are cofibrations of spaces but fail to be cofibrations in the equivariant sense (Lemma 2.1.3).

Let $kTop_*$ denote the category of pointed (or based) Kelley spaces (or k-spaces) in the sense of [3, Definition 2.4.21(3)]: a space Y is a Kelley space if every compactly open subset $U \subseteq Y$ is open (here U is compactly open if for all compact Hausdorff spaces K and all continuous maps $f : K \to Y$, the set $f^{-1}(U)$ is open in K). According to [3, 2.4.24] this category has a model structure where a map f is a weak equivalence if and only if it is a weak homotopy equivalence, and f is a fibration if and only if it is a Serre fibration. All k-spaces are fibrant. Cofibrations are retracts of generalized CW-inclusions [2, 8.8 and 8.9] where we have to use the pointed cells Δ^n_+. It follows that all cofibrant objects are Hausdorff. Colimits agree in $kTop_*$ and the category of pointed topological spaces. Limits can be computed by first calculating the limit in the category of topological spaces, then applying the "Kelleyfication" functor k. In particular, all smash products occurring in this paper will bear this modified product topology. The category $kTop_*$ is a proper model category; in particular, the gluing lemma holds.

A monoid is a (multiplicative) semi-group with identity element 1. A *monoid with zero* is a monoid M with a distinguished element $0 \in M$ such that $m \cdot 0 = 0 \cdot m = 0$ for all $m \in M$. A map of monoids with zero is a monoid homomorphism preserving the zero element. A *topological monoid with zero* is a monoid with zero which is also a pointed Kelley space with 0 as basepoint such that the multiplication is continuous.

We can consider S^0 as a monoid with zero, it is initial in the category of monoids with zero. If M is a topological monoid, one can add a disjoint zero element. This gives a functor $M \mapsto M_+$ left adjoint to the forgetful functor (forgetting the zero element).

Suppose M is a topological monoid with zero. A *right action* of M on a space $Y \in kTop_*$ is a continuous map $Y \wedge M \to Y$ satisfying the usual associativity and unitality condition. Similarly, we can define left actions. If M happens to be commutative every right action determines a left action and vice versa. Let M-$kTop_*$ denote the category of pointed topological spaces with a right action of M; morphisms are M-equivariant pointed continuous maps. The following proposition summarizes formal homotopical properties of equivariant spaces:

Proposition 2.1.1. (1) *The category M-$kTop_*$ has the structure of a topological model category where a map is a weak equivalence* (*resp. fibration*) *if and only if it is a weak homotopy equivalence* (*resp. fibration*) *of underlying k-spaces*, *and a cofibration if and only if it is a retract of a generalized CW-inclusion in the sense of* [2, 8.8] (*cells are of the form $\Delta^n_+ \wedge M$*). *Furthermore*, *all objects are fibrant.*

(2) *If $M = G \wedge \bar{M}$ is the smash product of a topological monoid with zero G and a discrete monoid with zero $\bar{M}$*, *the forgetful functor $(G \wedge \bar{M})$-$kTop_* \to G$-$kTop_*$ preserves cofibrations.*

(3) *If M is cofibrant as an object of $kTop_*$*, *the forgetful functor M-$kTop_* \to kTop_*$ preserves cofibrations.*

(4) *If M is cofibrant as an object of $kTop_*$*, *all cofibrant objects of M-$kTop_*$ are Hausdorff.*

From now on, "topological space", "topological monoid", etc., will always refer to k-spaces.

The category M-$kTop_*$ has a suspension functor $\Sigma Y := S^1 \wedge Y$ with M acting on Y only. Suspension preserves cofibrations, acyclic cofibrations and hence all weak equivalences between cofibrant objects. Note that this is still true if "cofibration" (and "cofibrant") refers to the underlying maps and objects in $kTop_*$.

Proposition 2.1.1 implies that an object of M-$kTop_*$ is cofibrant if and only if it is a retract of a generalized M-free pointed CW-complex. Let $\mathbf{C}(M)$ denote the full subcategory of cofibrant objects in M-$kTop_*$.

An object $Y \in M$-$kTop_*$ is called *finite* if it is obtained from a point by attaching finitely many free M-cells (in particular, Y is cofibrant). It is called *homotopy finite* if it is connected by a chain (or zigzag) of weak equivalences in M-$kTop_*$ to a finite object. By Dwyer and Spalinski [2, 5.8 and 5.11], this is equivalent to the existence of a finite object Z and a weak equivalence $Z \to Y$. If in addition Y is cofibrant, the Whitehead theorem [2, Lemma 4.24] applies: Y is homotopy finite if and only if Y is homotopy equivalent, in the strong sense, to a finite object. A space Y is called *finitely dominated* if it is a retract of a homotopy finite object. Finally, Y is said to be *stably finitely dominated* if some suspension of Y is finitely dominated. The full subcategories of $\mathbf{C}(M)$ consisting of the finite, homotopy finite, finitely dominated and stably finitely dominated objects will be denoted by $\mathbf{C}_{\mathrm{f}}(M), \mathbf{C}_{\mathrm{hf}}(M), \mathbf{C}_{\mathrm{fd}}(M)$ and $\mathbf{C}_{\mathrm{sfd}}(M)$, respectively.

Suppose Y and Z are objects of $kTop_*$ with an action of M from the right (resp. left), we can form their *tensor product* $Y \wedge_M Z \in kTop_*$ defined as the coequalizer (in $kTop_*$) of the two maps $Y \wedge M \wedge Z \to Y \wedge Z$ given by the action of M on Y and Z. If Z has an additional right $\bar{M}$-action (compatible with the left M-action), the tensor product $Y \wedge_M Z$ is an $\bar{M}$-equivariant space.

The following lemma is an exercise in general nonsense; we omit the proof.

Lemma 2.1.2. *Suppose $f : M \to \bar{M}$ is a morphism of topological monoids with zero. Then M acts on $\bar{M}$ via f from the left.*

1. *The functor $\cdot \wedge_M \bar{M} : M$-$kTop_* \to \bar{M}$-$kTop_*$ has a right adjoint $\bar{Y} \mapsto Y$ where Y is $\bar{Y}$ as a topological space, but with M acting via f.*
2. *The functor $\cdot \wedge_M \bar{M}$ preserves cofibrations and acyclic cofibrations. It maps weak equivalences between cofibrant objects to weak equivalences.*
3. *The functor $\cdot \wedge_M \bar{M}$ maps cells to cells. It restricts to a functor $\mathbf{C}_?(M) \to \mathbf{C}_?(\bar{M})$ where ? may denote any of the decorations* f, hf, fd *or* sfd.

Let $I := [0,1]_+$ denote the unit interval with a disjoint basepoint. If $Y \in M$-$kTop_*$ is cofibrant, $Y \wedge I$ is a good cylinder object for Y [2, Definition 4.2], i.e., the map $Y \vee Y \to Y \wedge I$ (inclusion of top and bottom into the cylinder) is a cofibration of M-spaces. Using this, we obtain a functorial mapping cylinder construction Z_g for maps $g \in M$-$kTop_*$. It is compatible with all finiteness notions (for cofibrant spaces) and commutes with colimits.

We will also have occasion to apply the following technical result:

Lemma 2.1.3. *Let G denote a topological monoid with zero. Suppose M is a discrete monoid with zero, and $t \in M$ is an element such that right translation by t (i.e., the map $\rho_t : M \to M$, $m \mapsto mt$) is injective. Then for each $Y \in \mathbf{C}(G \wedge M)$, the self-map $Y \xrightarrow{t} Y, y \mapsto yt$ is a cofibration in G-$kTop_*$ (though not necessarily in $(G \wedge M)$-$kTop_*$).*

Proof. Let $Q := M \setminus \rho_t(M)$ denote the subset of those elements which are not a translate of t. Assume first that Y is a finite generalized free $(G \wedge M)$-equivariant CW-complex, i.e., Y can be obtained from a point by attaching finitely many free cells. Write

$Y = \bar{Y} \cup_{\partial C} C$ where $C = \Delta^k_+ \wedge G \wedge M$ is a free cell with boundary ∂C, and consider the following diagram:

$$\begin{array}{ccccc} \Delta^k_+ \wedge G \wedge M & \longleftarrow & \partial\Delta^k_+ \wedge G \wedge M & \longrightarrow & \bar{Y} \\ {\scriptstyle t}\big\downarrow & & \big\downarrow{\scriptstyle t} & & \big\downarrow{\scriptstyle t} \\ \Delta^k_+ \wedge G \wedge M & \longleftarrow & \partial\Delta^k_+ \wedge G \wedge M & \longrightarrow & \bar{Y} \end{array}$$

We may assume by induction that the right vertical map is a cofibration. We claim that the map from the pushout of the left square into its terminal vertex is a cofibration. Indeed, the pushout and the terminal vertex differ by a one-point union of $\Delta^k_+ \wedge G$, indexed over Q, with attachment done over a one-point union of $\partial\Delta^k_+ \wedge G$, indexed over Q. (This is true since right translation by t is assumed to be injective.)

In this situation, Reedy's patching lemma [1, Lemma 3.8] asserts that the induced map from the pushout of the top row into the pushout of the bottom row is a cofibration. Hence the lemma is true for finite Y.

By transfinite induction, this proves the lemma for (not necessarily finite) generalized free CW-complexes. Finally, if Y is a retract of a generalized free CW-complex Z, the map $Y \xrightarrow{t} Y$ is a retract of the map $Z \xrightarrow{t} Z$. But the latter is a cofibration, hence so is the former. $\square$

2.2. Iterated homotopy cofibres

The iterated cofibre functor Γ (sometimes called "total cofibre functor") measures how far a cubical diagram of spaces is from being homotopy cocartesian. We will use Γ as a substitute for a global sections functor. The present section contains a description of Γ and its basic properties.

Definition 2.2.1. Assume $\mathscr{C}$ is a category. For any object $c \in \mathscr{C}$ let $\mathscr{C} \downarrow c$ denote the category of objects over c, and define $\mathscr{C} \downarrow \hat{c}$ as the full subcategory of objects over c without the identity of c. There is a functor $j : \mathscr{C} \downarrow \hat{c} \to \mathscr{C}$ defined by $(a \to c) \mapsto a$. If Y is a functor $\mathscr{C} \to G\text{-}kTop_*$ we define the *latching space of Y at c* as

$$L_c Y := \operatorname*{colim}_{\mathscr{C} \downarrow \hat{c}} Y \circ j$$

(cf. [3, 5.2.2]). Note that $L_c Y$ comes equipped with a canonical map to $Y(c)$ induced by the structure maps $Y(a) \to Y(c)$. Given an object of $\mathscr{C} \downarrow \hat{c}$, i.e., a morphism $b \to c$ in $\mathscr{C}$ different from id_c, we obtain a map $Y(b) \to L_c Y$ since $Y(b)$ appears in the diagram defining the latching space.

For a finite non-empty set N let $\langle N \rangle$ denote its power set regarded as a category with inclusions as morphisms. For all $A \subseteq N$ we identify $\langle N \rangle \downarrow \hat{A}$ with a full subcategory of $\langle N \rangle$.

Let $\mathscr{C}^{\langle N \rangle} = Func(\langle N \rangle, \mathscr{C})$ denote the category of functors $\langle N \rangle \to \mathscr{C}$. If G denotes a topological monoid with zero, we have defined $G\text{-}kTop_*^{\langle N \rangle} := Func(\langle N \rangle, G\text{-}kTop_*)$, the category of N-cubical diagrams in $G\text{-}kTop_*$. If Y is an object of $G\text{-}kTop_*^{\langle N\rangle}$ we write Y^A for $Y(A)$ and call this space the *A-component* of Y.

The latching space L_AY has a canonical map to Y^A. We say that *Y satisfies the latching space condition at A* if this map is a cofibration in $G\text{-}kTop_*$. More generally, if $f : Y \to Z$ is a map of cubes, we say that *f satisfies the latching space condition at A* if the induced map $L_AZ \cup_{L_AY} Y^A \to Z^A$ is a cofibration.

For $0 \leqslant k \leqslant \#N$ define a functor

$$e_k : G\text{-}kTop_*^{\langle N\rangle} \to G\text{-}kTop_*^{\langle N\rangle}, \quad e_k(Y)^A := \begin{cases} Cyl(L_AY \to Y^A) & \text{if } k = \#A, \\ Y^A & \text{if } k \neq \#A, \end{cases}$$

where Cyl denotes the mapping cylinder construction, and $e_k(Y)$ has the obvious structure maps: for $B = A \amalg \{i\} \subseteq N$, the map $e_k(Y)^A \to e_k(Y)^B$ is given by the structure map $Y^A \to Y^B$ of Y if neither A nor B has k elements; by the composite $Y^A \to L_BY \to Cyl(L_BY \to Y^B)$ if $\#B = k$; and by the composite $Cyl(L_AY \to Y^A) \to Y^A \to Y^B$ if $\#A = k$.

Let $\gamma : G\text{-}kTop_*^{\langle N\rangle} \to G\text{-}kTop_*$ denote the functor taking Y to the strict cofibre of the map $L_NY \to Y^N$. We define the *iterated homotopy cofibre of Y* as

$$\Gamma(Y) := \gamma \circ e_{\#N} \circ \cdots \circ e_1 \circ e_0(Y).$$

Finally, we define the Kronecker *delta cube δ_C* (for a subset $C \subseteq N$) as the functor $\delta_C : G\text{-}kTop_* \to G\text{-}kTop_*^{\langle N\rangle}$ with $\delta_C(K)^A = *$ if $A \neq C$ and $\delta_C(K)^C = K$.

Remark 2.2.2. (1) We will only be interested in $\Gamma(Y)$ if all the spaces Y^A are cofibrant in $G\text{-}kTop_*$. In this case, the following description holds: calculate the homotopy colimit C of the diagram obtained from Y by deleting the terminal vertex Y^N; there is a canonical map $C \to Y^N$, its homotopy cofibre is $\Gamma(Y)$. The functors e_j "make Y cofibrant as a cube" which guarantees that application of γ "gives the correct homotopy type". More precisely, Γ is a model for the total left derived of γ in the sense of [5] on the subcategory of objects with cofibrant components.

(2) Since both the mapping cylinder construction and formation of latching spaces are compatible with smash products, the functor Γ commutes with smash products with spaces. Explicitly, if Y is an object of $kTop_*^{\langle N\rangle}$ and $K \in G\text{-}kTop_*$, there is a canonical isomorphism of G-spaces $\Gamma(K \wedge Y) \cong K \wedge \Gamma(Y)$ where $K \wedge Y$ denotes the cubical diagram $A \mapsto K \wedge Y^A$ in $G\text{-}kTop_*$.

(3) There are natural transformations $e_k \to \mathrm{id}$ which are weak equivalences on each component. Explicitly, the A-component is given by id_{Y^A} if $\#A \neq k$, and is the projection from the mapping cylinder of $L_AY \to Y^A$ to Y^A if A has k elements. Moreover $e_0 \cong \mathrm{id}$.

(4) The functor Γ admits a recursive definition. If we write $N = M \amalg \{j\}$, we can regard an N-cube as a map "in j-direction" of two M-cubes and compute the point-wise homotopy cofibre. The iterated homotopy cofibre of the resulting M-cube is isomorphic to the iterated homotopy cofibre of the original N-cube.

The category $G\text{-}kTop_*^{\langle N\rangle}$ admits two model structures with pointwise weak equivalences. The *f-structure* is obtained from [3, 5.2.5] if $\langle N\rangle$ is considered as an inverse category equipped with degree function $d(A):=n+1-\#A$. A map of cubes is an *f-cofibration* if it is a pointwise cofibration. The *c-structure* has pointwise fibrations. The corresponding cofibrations will be denoted by *c-cofibrations*; explicitly, a map f is a c-cofibration if and only if it satisfies the latching space condition at A for all $A \subseteq N$. This follows from [3, 5.2.5] using the degree function $d(A):=\#A$ (which makes $\langle N\rangle$ into a direct category).

Corollary 2.2.3. (1) *The functor Γ maps an (acyclic) f-cofibration between f-cofibrant objects to an (acyclic) cofibration in $G\text{-}kTop_*$; in particular $\Gamma(Y)$ is cofibrant if Y is f-cofibrant.*
(2) *The functor Γ preserves weak equivalences between f-cofibrant objects.*
(3) *The functor Γ commutes with colimits.*

Proof. For (3), note that the functors e_k are compatible with colimits by construction, Moreover, γ has a right adjoint δ_N, hence commutes with colimits. To prove (1), observe that if f is an f-cofibration, the map $e_{\#N}\circ\cdots\circ e_0(f)$ is a c-cofibration. By adjointness, γ maps c-cofibrations to cofibrations. Application of Brown's lemma [2, 9.9] shows that Γ preserves all weak equivalences between f-cofibrant cubes, hence (2) holds. □

Lemma 2.2.4. *Suppose N is a finite non-empty set and Y is an f-cofibrant N-cube with trivial initial vertex. Suppose that all structure maps of Y away from the initial vertex are weak equivalences.*
(1) *For all $k\in N$, the iterated homotopy cofibre of Y is weakly equivalent to $\Sigma^n Y^{\{k\}}$ (where $n=\#N-1$).*
(2) *If Y has contractible iterated homotopy cofibre and the space $Y^{\{k\}}$ is simply connected for some $k\in N$, then the spaces Y^A are contractible for all $A\subseteq N$.*

Proof. (1) Compute homotopy cofibres in k-direction as explained in Remark 2.2.2(4). The resulting $(N\setminus\{k\})$-cube Z is weakly equivalent to $\delta_\emptyset Y^{\{k\}}$ since all structure maps starting at a non-initial vertex are weak equivalences and consequently their homotopy cofibres are weakly contractible. Next, computing in ℓ-direction (where $\ell\neq k$), we see that $\Gamma(Z)$ is weakly equivalent to the iterated homotopy cofibre of the $(N\setminus\{k,\ell\})$-cube $\delta_\emptyset(\Sigma Y^{\{k\}})$ since the homotopy cofibre of a map $K\to *$ is ΣK. Continuing in this manner, we obtain a chain of isomorphisms and weak equivalences

$$\Gamma(Y)\cong\Gamma(Z)\simeq\Gamma(\delta_\emptyset Y^{\{k\}})\simeq\Sigma^n Y^{\{k\}}.$$

(2) By part (1), the space $\Sigma^n Y^{\{k\}}$ is contractible. This implies that $Y^{\{k\}}$ is contractible since $Y^{\{k\}}$ is simply connected and cofibrant. Hence all components of Y are contractible by hypothesis on the structure maps of Y. □

3. Non-linear projective space

3.1. A short review of projective spaces

Let R be a commutative ring. Algebraic geometers define the n-dimensional projective space over R as the (projective) scheme $\mathbf{P}^n_R := \operatorname{Proj} S$ where S is the polynomial ring $S := R[X_0, X_1, \ldots, X_n]$. Note that S is a graded ring with the usual total degree of polynomials; indeterminates have degree 1.

There is another description of the scheme $\mathbf{P}^n_R$: It can be obtained by gluing certain affine schemes. Let $S^i := R[X_0, X_1, \ldots, X_n, X_i^{-1}]$ denote the ring S with X_i inverted, and let similarly S^{ij} be S with both X_i and X_j inverted. The rings S^i and S^{ij} are graded rings again (with X_i^{-1} having degree -1) and we can define R^i and R^{ij} to be their degree 0 subrings. There exist inclusion maps $R^i \to R^{ij} \leftarrow R^j$. Passage to spectra (in the sense of algebraic geometry) yields

$$\operatorname{Spec} R^i \leftarrow \operatorname{Spec} R^{ij} \to \operatorname{Spec} R^j$$

and it can be shown that the scheme obtained by gluing all the affine schemes $\operatorname{Spec} R^i$ for $i = 0, 1, \ldots, n$ along the schemes $\operatorname{Spec} R^{ij}$ is isomorphic to $\operatorname{Proj} S$. Thus $\operatorname{Spec} R^{ij}$ is the intersection of $\operatorname{Spec} R^i$ and $\operatorname{Spec} R^j$ inside $\mathbf{P}^n_R$. We call $\operatorname{Spec} R^i$ the ith canonical open set.

We can characterize the intersections of more than two of the canonical sets. Let $\langle n \rangle$ denote the set $\{0, 1, \ldots, n\}$. For $A \subseteq \langle n \rangle$ we define S^A as S with $\prod_{i \in A} X_i$ inverted (i.e., with all X_i, $i \in A$, inverted), and let R^A denote the degree 0 subring of S^A. Then we have inclusion maps $R^A \to R^B$ whenever $A \subseteq B \subseteq \langle n \rangle$. By applying the functor Spec, we obtain a collection of subschemes of $\mathbf{P}^n_R$, and for non-empty $A \subseteq \langle n \rangle$ we see that $\operatorname{Spec} R^A$ is the intersection $\bigcap_{i \in A} \operatorname{Spec} R^i$ inside $\mathbf{P}^n_R$.

The reader should note at this point that all the rings constructed from R are monoid rings of a very special kind. Let $\mathbf{N}$ denote the natural numbers including 0 considered as a monoid with respect to the sum. Then the polynomial ring $R[X]$ is, by definition, the monoid ring $R[\mathbf{N}]$ with X corresponding to $1 \in \mathbf{N}$ (the generator of $\mathbf{N}$). More generally, we have

$$R[X_0,\ X_1, \ldots, X_n] = R[\underbrace{\mathbf{N} \oplus \cdots \oplus \mathbf{N}}_{n+1 \text{ summands}}],$$

where $\oplus$ denotes the sum of abelian monoids (finite sums and products agree and are given by cartesian product of underlying sets). Inverting X_i amounts then to changing the corresponding factor $\mathbf{N}$ to $\mathbf{Z}$; thus $R[X_0, X_1, X_0^{-1}] = R[\mathbf{Z} \oplus \mathbf{N}]$ and $R[X_0, X_1, X_1^{-1}] = R[\mathbf{N} \oplus \mathbf{Z}]$.

To introduce some notation, we denote by $\tilde{M}^A_n$ the monoid described above occurring in the definition of S^A, i.e., a sum of $n+1$ copies of $\mathbf{N}$ or $\mathbf{Z}$ (a more formal definition will be given later). Then we have the short formula $S^A = R[\tilde{M}^A_n]$.

There is a similar description of the rings R^A. Let M^A_n denote the set of those elements of $\tilde{M}^A_n$ having sum zero. Then M^A_n is a submonoid of $\tilde{M}^A_n$ and $R^A = R[M^A_n]$.

We are interested in quasi-coherent sheaves on $\mathbf{P}^n_R$. Such a sheaf is determined by its sections over the canonical open sets Spec (R^i) and its behaviour on intersections of these. In more detail, suppose we have, for all non-empty $A \subseteq \langle n \rangle$, an R^A-module M^A, and for each inclusion $A \subseteq B$ an R^A-equivariant additive map $M^A \to M^B$ which becomes an isomorphism after "inverting the action of X_i for $i \in B \setminus A$". Then there is a unique quasi-coherent $\mathcal{O}_{\mathbf{P}^n_R}$-module $\mathscr{F}$ with $\Gamma(\mathscr{F}, \mathrm{Spec}(R^A)) = M^A$. The category of "diagrams" of this kind is equivalent to the category of quasi-coherent sheaves on $\mathbf{P}^n_R$.

3.2. Non-linear polynomial rings

By "forgetting the linear structure", i.e., forgetting the ring R, we pass from monoid rings to monoids (with zero) which can be thought of as "non-linear rings". We define the relevant monoids and introduce a construction to invert distinguished generators (an analogue of the algebraic process of localization). The material is applied immediately to compare different finiteness notions of equivariant spaces (Lemmas 3.2.7, 2.3.8 and Corollary 3.2.9). These results will be used later to handle finiteness conditions for sheaves.

Definition 3.2.1. Suppose N is a non-empty finite set. For $A \subseteq N$, define the monoid with zero

$$\tilde{M}^A = \tilde{M}^A_N := \{(m_i)_{i \in N} \in \mathbf{Z}^N \mid \forall i \in N \setminus A : m_i \geqslant 0\}_+.$$

The homomorphism $\deg : \tilde{M}^A \to \mathbf{Z}_+$, $(a_i)_{i \in N} \mapsto \sum_{i \in N} a_i$ is called "degree map" (here $\mathbf{Z}_+$ is the set of all integers with a disjoint basepoint considered as a monoid with zero; "multiplication" is given by the usual sum of integers).

Define subsets $\tilde{M}^A(j) = \tilde{M}^A_N(j) := \deg^{-1}(j)$ for all $j \in \mathbf{Z}$, and note that $\tilde{M}^A(0)$ is really a monoid with zero which acts on the pointed sets $\tilde{M}^A(j)$. It is convenient to introduce the notation $M^A = M^A_N := \tilde{M}^A(0)$.

There is a convenient class of finite sets, the standard sets $\langle n \rangle := \{0, 1, \ldots, n\}$ for $n \in \mathbf{Z}$, and the even-more-standard sets $[n]$ which are $\langle n \rangle$ as sets again, but equipped with the natural order. When using the standard sets, we write M^A_n for $M^n_{\langle n \rangle}$ and $M^A_{[n]}$.

We think of the above monoids as multiplicative monoids: write t_i for the element which contains a $1 \in \mathbf{Z}$ in the ith place and $0 \in \mathbf{Z}$ everywhere else, then the collection of the t_i and t_i^{-1} generates $\tilde{M}^N_N = (\mathbf{Z}^{\# N})_+$ as an abelian monoid with zero. The monoids M^N_N are generated by the compound symbols $t_i t_j^{-1}$ for $i \neq j$. The symbol t_i should be thought of as the indeterminate X_i, and M^A (resp. $\tilde{M}^A$) corresponds to the ring R^A (resp. S^A) of the previous section.

Example 3.2.2. The monoid $M^{\{0\}}_1$ is isomorphic to $\mathbf{N}_+$ (natural numbers with disjoint basepoint):

$$\mathbf{N}_+ \xrightarrow{\cong} M^{\{0\}}_1 = \{(a,b) \in \mathbf{Z} \times \mathbf{N} \mid a + b = 0\}_+, \quad t \mapsto (-t, t).$$

This corresponds, in the linear setting, to the isomorphism of rings

$$R[X] \cong R[T_0, T_0^{-1}, T_1]_0, \quad X \mapsto T_0^{-1} T_1.$$

For $n=2$, we find the following isomorphisms:

$$(\mathbf{N} \times \mathbf{N})_+ \to M_2^{\{0\}}, \quad (a,b) \mapsto (-a-b,\ a,b),$$

$$(\mathbf{Z} \times \mathbf{N})_+ \to M_2^{\{0,1\}}, \quad (a,b) \mapsto (-a-b,\ a,b),$$

$$(\mathbf{Z} \times \mathbf{Z})_+ \to M_2^{\{0,1,2\}}, \quad (a,b) \mapsto (-a-b,\ a,b).$$

Definition 3.2.3. Suppose M is a monoid with zero. A subset $I \subseteq M$ is called a (*twosided*) *ideal* of M, denoted $I \lhd M$, if $0 \in I$, and for all $(m,a) \in M \times I$ the elements $a \cdot m$ and $m \cdot a$ are contained in I.

If $I \lhd M$ we can define a new monoid with zero M/I. As a set, it is given by $(M \setminus I)_+$ (this is the quotient $M/I = M \cup_I *$ in the category of pointed sets). The monoid structure is induced by that of M.

There is an obvious map $M \to M/I$, sending $m \in M \setminus I$ to $m \in M/I$ and mapping all of $I \subseteq M$ to $* \in M/I$. It is readily verified that this map is a map of monoids with zero. Its kernel, i.e., the preimage of the zero element (not of the identity) is I. If R is a commutative ring, the module $\tilde{R}[I]$ is an ideal of the (reduced) monoid ring $\tilde{R}[M]$, and there is a canonical isomorphism $\tilde{R}[M/I] \cong \tilde{R}[M]/\tilde{R}[I]$.

Example 3.2.4. Suppose N is a finite set, $A \subset N$ a subset, and $j \in N \setminus A$. We define the ideal "generated by t_j"

$$I_{j,N}^A := \{(a_i)_{i \in N} \in M_N^A \mid a_j \neq 0\}_+$$

of M_N^A. Then there is an isomorphism $M_N^A / I_{j,N}^A \cong M_{N \setminus \{j\}}^A$, mapping $(a_i)_{i \in N}$ to $(a_i)_{i \in N \setminus \{j\}}$. In particular, M_N^A acts on $M_{N \setminus \{i\}}^A$ via the projection $M_N^A \to M_N^A / I_{j,N}^A$. We will repeatedly make use of this fact.

For the rest of this paper we will consider M^A as a topological monoid with zero having the discrete topology. Similarly, we regard $\tilde{M}^A(j)$ as an object of M^A-$kTop_*$.

From now on, let G denote a topological monoid with zero, and suppose N is a non-empty finite set with $n+1$ elements.

Definition 3.2.5. Suppose we are given a (possibly empty) subset $A \subseteq N$, an element $i \in N$, and a space $Y \in (G \wedge M^A)$-$kTop_*$. Then we can form the space

$$Y[t_i^{-1}] := Y \wedge_{M^A} M^{A \cup \{i\}} \in (G \wedge M^{A \cup \{i\}})\text{-}kTop_*$$

and call $Y[t_i^{-1}]$ obtained by inverting the action of t_i on Y. This construction is functorial in Y. More generally, given subsets A and B of N, there is a functor

$$\cdot[t_B^{-1}] = \cdot \wedge_{M^A} M^{A \cup B} : (G \wedge M^A)\text{-}kTop_* \to (G \wedge M^B)\text{-}kTop_*, \quad Y \mapsto Y[t_B^{-1}].$$

The construction of Definition 3.2.5 is the non-linear analogue of inverting indeterminates in a Laurent ring. As in the linear case, there are alternative descriptions using mapping telescopes. (For $n=1$ this yields the telescope construction of [4, 2.1].) One can check that all three constructions have the same universal property and hence are canonically isomorphic; we omit the details.

Lemma 3.2.6 (Telescope constructions). *Suppose $A \subseteq N$ is not empty.*

(1) *Write $a=\#A$. The space $Y[t_i^{-1}]=Y\wedge_{M^A} M^{A\cup\{i\}}$ is isomorphic to the colimit (in the category G-$kTop_*$) of the sequence*

$$Y \xrightarrow{t_i^a\cdot t_A^{-1}} Y \xrightarrow{t_i^a\cdot t_A^{-1}} Y \xrightarrow{t_i^a\cdot t_A^{-1}} \cdots,$$

where we have used the multi-index notation $t_A^{-1}:=\prod_{j\in A} t_j^{-1}$. In particular, the colimit admits a canonical action of $M^{A\cup\{i\}}$.

(2) *Suppose N is ordered. Write $m=\max(A)$. The space $Y[t_i^{-1}]=Y\wedge_{M^A} M^{A\cup\{i\}}$ is isomorphic to the colimit (in the category G-$kTop_*$) of the following sequence:*

$$Y \xrightarrow{t_i\cdot t_m^{-1}} Y \xrightarrow{t_i\cdot t_m^{-1}} Y \xrightarrow{t_i\cdot t_m^{-1}} \cdots .$$

In particular, the colimit admits a canonical action of $M^{A\cup\{i\}}$.

The following technical result asserts that finite equivariant spaces are sequentially small with respect to the above telescope construction.

Lemma 3.2.7 (Smallness of finite equivariant spaces). *Suppose G is cofibrant as an object of $kTop_*$. Let $A\subseteq N$, $j\in N\setminus A$ and spaces $Z\in \mathbf{C}(G\wedge M^A)$ and $Y\in \mathbf{C}_{\mathrm{f}}(G\wedge M^A)$ be given; define $a:=\#A$. Let s_0 denote the canonical $(G\wedge M^A)$-equivariant inclusion $Z\to Z[t_j^{-1}]=Z\wedge_{M^A} M^{A\amalg\{j\}}$. For any map $f: Y\to Z[t_j^{-1}]$ in $(G\wedge M^A)$-$kTop_*$ there is a $(G\wedge M^A)$-equivariant map $g: Y\to Z$ and an integer $k\geqslant 0$ such that $f=t_j^{-ak}t_A^k\circ s_0\circ g$. Moreover, k can be enlarged arbitrarily.*

Proof. We proceed by induction on the number of (equivariant) cells in Y. For $Y=*$ we can choose $k=0$ and $g=f$.

Identify $Z[t_j^{-1}]$ with the colimit of the telescope construction (Lemma 3.2.6(1)). The top row of the diagram

$$\begin{tikzcd}
Z \arrow[r, "t_j^a\cdot t_A^{-1}"] \arrow[d, "s_0"'] & Z \arrow[r, "t_j^a\cdot t_A^{-1}"] \arrow[d, "s_1"'] & \dots \arrow[r, "t_j^a\cdot t_A^{-1}"] & Z \arrow[d, "s_k"] & \dots \\
Z[t_j^{-1}] \arrow[r, equal] & Z[t_j^{-1}] \arrow[r, equal] & \dots \arrow[r, equal] & Z[t_j^{-1}] & \dots
\end{tikzcd}$$

is the telescope. The vertical arrows are the canonical maps into the colimit, where s_0 is as above. The maps in the telescope are cofibrations of G-spaces by Lemma 2.1.3. Since G is cofibrant, the telescope consists of cofibrations of Hausdorff spaces in $kTop_*$ by Lemma 2.1.1(3,4).

Let $C := \Delta^i_+ \wedge G \wedge M^A$ denote a cell with boundary ∂C, and suppose $Y = \bar{Y} \cup_{\partial C} C$. By induction, we find $\bar{k}$ and a map $\bar{g}: \bar{Y} \to Z$ with $f|_{\bar{Y}} = t_j^{-a\bar{k}} t_A^{\bar{k}} \circ s_0 \circ \bar{g} = s_{\bar{k}} \circ \bar{g}$. Let $\alpha : \Delta^i_+ \to Z[t_j^{-1}]$ denote the restriction of f to the "generating" non-equivariant cell of C. Since Z is Hausdorff (Proposition 2.1.1(4)) and Δ^i_+ is compact, this map factors through some finite stage ℓ of the telescope construction. By forcing $(G \wedge M^A)$-equivariance we obtain a map $\beta: C \to Z$ such that $f|_C = t_j^{-a\ell} t_A^{\ell} \circ s_0 \circ \beta = s_\ell \circ \beta$. Since $s_{\bar{k}+\ell}$ is injective the following diagram commutes:

$$\begin{tikzcd}
 & \bar{Y} \arrow[r, "\bar{g}"] \arrow[dr, "f|_{\bar{Y}}"'] & Z \arrow[r, "t_j^{a\ell} t_A^{-\ell}"] \arrow[d, "s_{\bar{k}}"] & Z \arrow[d, equal, "\mathrm{id}"] \\
\partial C \arrow[ur] \arrow[dr] & & Z[t_j^{-1}] & Z \arrow[l, "s_{\bar{k}+\ell}"'] \\
 & C \arrow[r, "\beta"'] \arrow[ur, "f|_C"] & Z \arrow[u, "s_\ell"'] \arrow[r, "t_j^{a\bar{k}} t_A^{-\bar{k}}"] & Z \arrow[u, equal, "\mathrm{id}"']
\end{tikzcd}$$

Let $k := \bar{k} + \ell$, and define g as the induced map from $Y = \bar{Y} \cup_{\partial C} C$ to the rightmost Z in the diagram. Then by construction $f = s_k \circ g = t_j^{-ak} t_A^k \circ s_0 \circ g$.

To enlarge k by $m \geqslant 0$, note that s_0 is M^A-equivariant, hence

$$\begin{aligned} f &= t_j^{-ak} t_A^k \circ s_0 \circ g = t_j^{-a(k+m)} t_A^{(k+m)} \circ t_j^{am} t_A^{-m} \circ s_0 \circ g \\ &= t_j^{-a(k+m)} t_A^{(k+m)} \circ s_0 \circ (t_j^{am} t_A^{-m} \circ g). \qquad \square \end{aligned}$$

Lemma 3.2.8 (Smallness of finite equivariant spaces—alternative version). *Suppose G is cofibrant as an object of kTop$_*$. Let $A \subseteq [n]$, $j \in [n] \setminus A$ and spaces $Z \in \mathbf{C}(G \wedge M^A)$ and $Y \in \mathbf{C}_{\mathrm{f}}(G \wedge M^A)$ be given; define $m := \max(A)$. Let s_0 denote the canonical $(G \wedge M^A)$-equivariant inclusion $Z \to Z[t_j^{-1}] = Z \wedge_{M^A} M^{A \amalg \{j\}}$. For any map $f: Y \to Z[t_j^{-1}]$ in $(G \wedge M^A)$-kTop$_*$ there is a $(G \wedge M^A)$-equivariant map $g: Y \to Z$ and an integer $k \geqslant 0$ such that $f = t_j^{-k} t_m^k \circ s_0 \circ g$. Moreover, k can be enlarged arbitrarily.*

Proof. This is similar to the previous lemma, except that one uses the second telescope construction (Lemma 3.2.6(2)) instead of the first. $\square$

The following corollary has been used (in the case $n = 1$) implicitly in the proof of [4, 5.2].

Corollary 3.2.9. *Suppose G is cofibrant as an object of kTop$_*$. Let $A \subseteq N$, $j \in N \setminus A$ and $Y \in \mathbf{C}_{\mathrm{f}}(G \wedge M^A_N)$ be given. If $Y[t_j^{-1}] \simeq *$, the space $\Sigma Y \vee Y$ is homotopy finite as a $(G \wedge M^A_{N \setminus \{j\}})$-space (with the restricted action); hence its retract Y is finitely dominated as a $(G \wedge M^A_{N \setminus \{j\}})$-space.*

This is non-trivial even for $N=[1]$. Consider the case $A=\{0\}$ and $j=1$. Then $M_N^A \cong \mathbf{N}_+$, and $M_{N\setminus\{j\}}^A$ is the trivial monoid. The single equivariant zero-cell $S^0 \wedge M_N^A \cong M_N^A \cong \mathbf{N}_+$ is certainly not homotopy finite as an unequivariant space (it is an infinite one point union of zero spheres).

Proof of Corollary 3.2.9. Since $Y[t_j^{-1}]$ is contractible, the inclusion $s_0: Y \to Y[t_j^{-1}]$ is null homotopic (where s_0 is the canonical inclusion as in Lemma 3.2.7). Choose a homotopy $H: Y \wedge I \to Y[t_j^{-1}]$ from s_0 to the trivial map.

The space $Y \wedge I$ is finite as a $(G \wedge M_N^A)$-space, hence we know by Lemma 3.2.7 that H factors through some finite stage of the telescope: there is a map $F: Y \wedge I \to Y$ and an integer $m \geqslant 0$ with $H = t_j^{-am} t_A^m \circ s_0 \circ F$ where $a := \# A$, and consequently $t_j^{am} t_A^{-m} \circ H = s_0 \circ F$. By choice of H, the map $s_0 \circ F$ is a homotopy from $t_j^{am} t_A^{-m} \circ s_0 = s_0 \circ t_j^{am} t_A^{-m}$ to the trivial map. But s_0 is an injective map, so F is a null homotopy of $Y \xrightarrow{t_j^{am} t_A^{-m}} Y$. So we have a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "i_0"] \arrow[d, "t_j^{am} t_A^{-m}"'] & Y \wedge I \arrow[d, "F"] & Y \arrow[l, "i_1"'] \arrow[d, "*"] \\ Y \arrow[r, equal] & Y \arrow[r, equal] & Y \end{tikzcd} \qquad (*)$$

where i_0 and i_1 denote the inclusion of Y as top and bottom into the cylinder. Application of the homotopy cofibre (mapping cone) functor to the vertical maps yields a sequence of weak equivalences in $(G \wedge M_{N\setminus\{j\}}^A)$-$kTop_*$

$$\text{hocofibre}(Y \xrightarrow{t_j^{am} t_A^{-m}} Y) \xrightarrow{\sim} \text{hocofibre}(F) \xleftarrow{\sim} \text{hocofibre}(Y \xrightarrow{*} Y) \cong \Sigma Y \vee Y.$$

On the other hand, the left vertical map in $(*)$ is a cofibration in G-$kTop_*$ by Lemma 2.1.3. Hence, the canonical map from the mapping cone into the strict cofibre $Y/t_j^{am} t_A^{-m}(Y)$ is an equivariant weak homotopy equivalence.

It remains to note that the space $Y/t_j^{am} t_A^{-m}(Y)$ is finite as a $(G \wedge M_{N\setminus\{j\}}^A)$-space. This is shown by induction on the number of cells in Y. Since formation of quotients commutes with cell attachment, it suffices to show that the cofibre of

$$M_N^A \xrightarrow{t_j^{am} t_A^{-m}} M_N^A \qquad (**)$$

is isomorphic, as an $M_{N\setminus\{j\}}^A$-space, to a finite one-point union of copies of $M_{N\setminus\{j\}}^A$; then a free $(G \wedge M_N^A)$-equivariant cell of Y gives rise to a finite one-point union of free $(G \wedge M_{N\setminus\{j\}}^A)$-equivariant cells of the quotient space. But we can partition the set M_N^A into subsets according to the value of the jth component; explicitly, we have an isomorphism

$$M_N^A \cong \bigvee_{i \geqslant 0} \tilde{M}_{N\setminus\{j\}}^A(-i) \wedge \{i\}_+$$

of $M^A_{N\setminus\{j\}}$-spaces. Each of the subsets $\tilde{M}^A_{N\setminus\{j\}}(-i)$ is (non-canonically) equivariantly isomorphic to $M^A_{N\setminus\{j\}}$, an isomorphism is given by t^i_b for any $b \in A$. Now the cofibre of the map $(**)$ is seen to be $\bigvee_{i=0}^{am-1} \tilde{M}^A_{N\setminus\{j\}}(-i)$ which is isomorphic to $\bigvee_{i=0}^{am-1} M^A_{N\setminus\{j\}}$. □

3.3. *Non-linear sheaves on projective space*

In Section 3.1 we indicated how to describe quasi-coherent sheaves by certain diagrams of modules. We want to "forget the linear structure", i.e., replace rings by monoids and modules by equivariant spaces to obtain a non-linear homotopical version of sheaves on projective space. The most important examples are the structure sheaves (consisting of the monoids M^A introduced in the previous section) and "twisted" structure sheaves (which will be introduced later). As before, we assume that G is a topological monoid with zero, and N is a non-empty finite set with $n+1$ elements.

Definition 3.3.1 (*Presheaves on projective space*). We define the category $\mathbf{pP}^N(G)$ of (*G-equivariant quasi-coherent*) *presheaves* on projective N-space to be the following subcategory of $kTop_*^{\langle N\rangle}$: objects are the functors $Y : \langle N\rangle \to kTop_*$ with $Y(\emptyset) = *$ such that for each $A \subseteq N$, the space $Y^A := Y(A)$ has a (right) $(G \wedge M^A_N)$-action, and for each morphism $\sigma : A \to B$ in $\langle N\rangle$, the associated map $Y^\flat_\sigma : Y^A \to Y^B$ is $(G \wedge M^A)$-equivariant. We will sometimes refer to the map $Y^\flat_\sigma$ as a "$\flat$-type structure map" of Y. The space Y^A is called the *A-component* of Y. A morphism $f : Y \to Z$ is a natural transformation of diagrams, consisting of $(G \wedge M^A)$-equivariant maps $f^A : Y^A \to Z^A$ called *components* of f.—If $N = [n]$ or $N = \langle n\rangle$, we write $\mathbf{pP}^n(G)$ instead of $\mathbf{pP}^N(G)$. Note that $\mathbf{pP}^0(G) = G\text{-}kTop_*$, and every choice of a bijection $N \cong \langle n\rangle$ defines an isomorphism of categories $\mathbf{pP}^N(G) \cong \mathbf{pP}^n(G)$. Using the "abstract" set N is analogous to thinking of (ordinary) projective space as a functor from abstract vector spaces to topological spaces. This abstract definition is convenient since we have to use all canonical embeddings of $\mathbf{P}^{n-1}$ into $\mathbf{P}^n$, not just the inclusion given by inclusion of the first $n-1$ coordinates. At the same time, the notation reflects functoriality in the set N (although this is not used in the present paper).

As a remark on terminology, note that a presheaf is nothing but a diagram of equivariant spaces. Its linear analogue, a diagram of modules over certain monoid rings, does not determine a presheaf in the sense of algebraic geometry. We ask the reader to apologize this abuse of language.

We will sometimes use the category $\langle N\rangle_0$ of non-empty subsets of N as indexing category (omitting the redundant one-point space corresponding to $\emptyset \subseteq N$). If $G = S^0$ (the "trivial" monoid with $0 \neq 1$), we write $\mathbf{pP}^N$ omitting G from the notation.

The category $\mathbf{pP}^N(G)$ has a zero object given by $A \mapsto *$, the constant functor with the one point space as value. We call this the zero presheaf, sometimes denoted $*$.

For any object $Y \in \mathbf{pP}^N(G)$ we define the suspension of Y, denoted $\Sigma(Y)$, as the functor $A \mapsto \Sigma(Y^A)$ with M^A acting trivially on the suspension coordinate (componentwise suspension). Similarly, we can define a mapping cylinder by applying the mapping cylinder construction componentwise.

Each of the structure maps $Y_\sigma^\flat$ has a corresponding adjoint map

$$Y_\sigma^\# : Y^A[t_B^{-1}] = Y^A \wedge_{M^A} M^B \to Y^B$$

since the functor $\cdot \wedge_{M^A} M^B : (G \wedge M^A)\text{-}kTop_* \to (G \wedge M^B)\text{-}kTop_*$ is left adjoint to the functor restricting the $(G \wedge M^B)$-action to $G \wedge M^A$ along the inclusion $M^A \to M^B$. Sometimes we will call $Y_\sigma^\#$ a "$\sharp$-type structure map" of Y. These structure maps will be used to formulate a "sheaf condition" (see below).

An object of $\mathbf{pP}^N(G)$ can be visualized as an n-simplex with a space attached to each of its faces, and maps corresponding to inclusion of faces (suppressing the redundant one-point space corresponding to $\emptyset \subseteq N$). In the case $N = \langle 1 \rangle$, a typical object Y is depicted

$$Y^{\{0\}} \to Y^{\{0,1\}} \leftarrow Y^{\{1\}}$$

(the arrows indicate $\flat$-type structure maps). For $N = \langle 2 \rangle$, we have the following picture:

$$\begin{tikzcd}
 & & Y^{\{2\}} \arrow[dl] \arrow[dr] & & \\
 & Y^{\{0,2\}} \arrow[dr] & & Y^{\{1,2\}} \arrow[dl] & \\
 & & Y^{\{0,1,2\}} & & \\
Y^{\{0\}} \arrow[uur] \arrow[rr] & & Y^{\{0,1\}} \arrow[u] & & Y^{\{1\}} \arrow[uul] \arrow[ll]
\end{tikzcd}$$

Equivalently, we can regard an object of $\mathbf{pP}^N(G)$ as an $(n+1)$-cubical diagram with a point as initial vertex. For $N = \langle 2 \rangle$ this yields the following picture:

$$\begin{tikzcd}
 & Y^{\{2\}} \arrow[rr] \arrow[dd] & & Y^{\{0,2\}} \arrow[dd] \\
* \arrow[ur] \arrow[rr] \arrow[dd] & & Y^{\{0\}} \arrow[ur] \arrow[dd] & \\
 & Y^{\{1,2\}} \arrow[rr] & & Y^{\{0,1,2\}} \\
Y^{\{1\}} \arrow[ur] \arrow[rr] & & Y^{\{0,1\}} \arrow[ur] &
\end{tikzcd}$$

Definition 3.3.2 (*Structure presheaves of projective space*). We define the *structure presheaf of projective N-space* $\mathcal{O} = \mathcal{O}_{\mathbf{P}^N}$ to be the functor $A \mapsto M_N^A$ (for $A \neq \emptyset$); structure maps are given by inclusions, and M_N^A acts on itself by right translation.

If $N = \langle 2 \rangle$, we have the following picture for $\mathcal{O}_{\mathbf{P}^2}$:

$$
\begin{tikzcd}
& (\mathbf{N}\times\mathbf{N}\times\mathbf{Z})^0_+ \arrow[rr] \arrow[dd] & & (\mathbf{Z}\times\mathbf{N}\times\mathbf{Z})^0_+ \arrow[dd] \\
* \arrow[ur] \arrow[rr] \arrow[dd] & & (\mathbf{Z}\times\mathbf{N}\times\mathbf{N})^0_+ \arrow[ur] \arrow[dd] & \\
& (\mathbf{N}\times\mathbf{Z}\times\mathbf{Z})^0_+ \arrow[rr] & & (\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z})^0_+ \\
(\mathbf{N}\times\mathbf{Z}\times\mathbf{N})^0_+ \arrow[ur] \arrow[rr] & & (\mathbf{Z}\times\mathbf{Z}\times\mathbf{N})^0_+ \arrow[ur] &
\end{tikzcd}
$$

(The lower index "+" means adding a disjoint basepoint, and the upper index 0 denotes the subset of tuples with sum 0. That is, $(\mathbf{Z}\times\mathbf{Z}\times\mathbf{N})^0_+ = M_2^{\{0,1\}}$, and similarly for the other spaces in the diagram.)

For $N = \langle 1 \rangle$, the structure presheaf looks like this (with the same conventions for notation as before):

$$(\mathbf{Z}\times\mathbf{N})^0_+ \to (\mathbf{Z}\times\mathbf{Z})^0_+ \leftarrow (\mathbf{N}\times\mathbf{Z})^0_+.$$

One could think of a presheaf as a (non-linear) "module" over the structure presheaf $\mathcal{O}_{\mathbf{P}^N}$.

Definition 3.3.3 (*Homotopy sheaves on projective space*)**.** We define the category $\mathbf{P}^N(G)$ of (*G-equivariant quasi-coherent*) *homotopy sheaves* on projective N-space to be the full subcategory of $\mathbf{pP}^N(G)$ consisting of those objects Y which satisfy the following (*homotopy*) *sheaf condition*: for every inclusion $\sigma : A \to B$ of non-empty subsets of N, there is an object $\bar{Y}^A \in \mathbf{C}(G \wedge M^A)$ (cf. Section 2.1) and a weak equivalence $\bar{r}_\sigma : \bar{Y}^A \to Y^A$ such that the map

$$\bar{Y}^A[t_B^{-1}] \to Y^B$$

adjoint to the composite map $\bar{Y}^A \xrightarrow{\bar{r}_\sigma} Y^A \xrightarrow{Y^\flat_\sigma} Y^B$ is a weak equivalence.

This is a homotopy invariant non-linear analogue of the algebraic geometers' quasi-coherent sheaves (compare to the last paragraph of Section 3.1). We will abbreviate "homotopy sheaf" to "sheaf" in the sequel.

Standard model category arguments show that in the definition of the sheaf condition above, we could have worked with some *fixed* cofibrant replacement, or equivalently, we could have asked for the condition to be satisfied for *all* cofibrant replacements instead of just one. In particular, we can choose $\bar{Y}^A = Y^A$ of Y^A is cofibrant:

Corollary 3.3.4. *Suppose $Y \in \mathbf{pP}^N(G)$ is locally cofibrant in the sense that the space Y^A is cofibrant in $(G \wedge M^A)$-kTop$_*$ for all non-empty $A \subseteq N$. Then Y is a sheaf if and* *only if for all inclusions $\sigma: A \to B$ of non-empty subsets of N, the map $Y^\sharp_\sigma: Y^A[t_B^{-1}] \to Y^B$ (adjoint to the structure map $Y^\flat_\sigma: Y^A \to Y^B$) is a weak equivalence.*

Corollary 3.3.5 (Homotopy invariance of the sheaf condition)**.** *Suppose Y and Z are objects of $\mathbf{pP}^N(G)$. Assume that there is a weak equivalence $f: Y \to Z$ (i.e., all components of f are weak equivalences). Then Y is a sheaf if and only if Z is a sheaf.*

Remark 3.3.6. The structure presheaf $\mathcal{O}_{\mathbf{P}^N}$ defined above is in fact a sheaf, called the *structure sheaf of projective N-space*. This follows from the canonical isomorphism $M^A[t_B^{-1}] = M^A \wedge_{M^A} M^B \cong M^B$ and Corollary 3.3.4. Note that in this case the $\sharp$-type structure maps are even isomorphisms, not just weak equivalences.

3.4. Model structures

Our next goal is to establish two model structures on $\mathbf{pP}^N(G)$ sharing the same weak equivalences, but having a different class of cofibrations. Thinking of $\mathbf{pP}^N(G)$ as a generalized diagram category, these model structures are generalizations of those introduced for $kTop_*^{\langle N\rangle}$ (preceding Corollary 2.2.3). The interplay of the different notions of cofibrations and fibrations will be an important feature for handling finiteness conditions in $\mathbf{P}^N(G)$. Moreover, model structures facilitate the construction of categories with cofibrations and weak equivalences in the sense of [10].

As before, assume that G is a topological monoid with zero, and let N denote a non-empty finite set with power set $\langle N\rangle$. The symbol $\langle N\rangle_0$ means the set of non-empty subsets of N.

Suppose Y is an object of $\mathbf{pP}^N(G)$ and $A \subseteq N$ is not empty. We define the *twisted latching space* of Y at A, denoted $L_A Y$, by

$$L_A Y := \operatorname*{colim}_{\langle A\rangle_0^1} R(Y)$$

(colimit in $(G \wedge M^A)$-$kTop_*$) where $\langle A\rangle_0^1$ is the subcategory of non-empty proper subsets of A, and $R(Y)$ is the diagram $B \mapsto Y^B[t_A^{-1}]$ (this is a diagram in $(G \wedge M^A)$-$kTop_*$). If $\sigma: B \to C$ is an inclusion of proper subsets of A, i.e., a morphism in $\langle A\rangle_0^1$, the structure map $R(Y)(\sigma): R(Y)^B \to R(Y)^C$ is given by the composite

$$R(Y)^B = Y^B[t_A^{-1}] = Y^B \wedge_{M^B} M^A \cong (Y^B \wedge_{M^B} M^C) \wedge_{M^C} M^A$$

$$\xrightarrow{Y^\sharp_\sigma \wedge \mathrm{id}} Y^C \wedge_{M^C} M^A = Y^C[t_A^{-1}] = R(Y)^C,$$

where $Y^\sharp_\sigma: Y^B[t_C^{-1}] \to Y^C$ is adjoint to the structure map $Y^\flat_\sigma: Y^B \to Y^C$. It can be shown that this construction yields a commutative diagram in $(G \wedge M^A)$-$kTop_*$.

The latching space $L_A Y$ has a canonical map in $(G \wedge M^A)$-$kTop_*$ to Y^A, induced by the structure maps $Y^\sharp_\tau: Y^B[t_A^{-1}] \to Y^A$. If $\tau: B \to A$ is an inclusion of a proper subset, write $F_\tau = \cdot \wedge_{M^B} M^A: (G \wedge M^B)$-$kTop_* \to (G \wedge M^A)$-$kTop_*$, and let U_τ denote its right adjoint (restriction of action). There is a $(G \wedge M^B)$-equivariant map $Y^B \to U_\tau(L_A Y)$ given by the composite $Y^B \to U_\tau \circ F_\tau(Y^B) \to U_\tau(L_A Y)$ (the first map is the unit of the adjunction of F_τ and U_τ, the second map exists since $F_\tau(Y^B) = Y^B[t_A^{-1}]$ appears in the diagram defining the latching space, hence maps to $L_A Y$).

Proposition 3.4.1 (The c-structure of $\mathbf{pP}^{\mathrm{N}}(G)$). *The category $\mathbf{pP}^N(G)$ has the structure of a model category where a map is a weak equivalence (resp. fibration) if each of its components is a weak equivalence (resp. fibration) in its respective category. Furthermore, the map $Y \to Z$ is a cofibration if and only if the induced maps $L_A Z \cup_{L_A Y} Y^A \to Z^A$ are cofibrations in $G \wedge M^A$-$kTop_*$ for all non-empty $A \subseteq N$.*

Proof. Consider $\langle N \rangle_0$ as a direct category with degree function $d(A) := \#A$. With the above definition of (twisted) latching spaces, the proof of [3, 5.25] carries over word for word. $\square$

Corollary 3.4.2. *All objects of $\mathbf{pP}^N(G)$ are fibrant with respect to the c-structure.*

Example 3.4.3 (*The c-structure of* $\mathbf{pP}^1(G)$). *Let* $f : Y \to Z$ denote a map in $\mathbf{pP}^1(G)$, i.e., we have a commutative diagram of the following kind:

$$\begin{array}{ccccc} Y^{\{0\}} & \longrightarrow & Y^{\{0,1\}} & \longleftarrow & Y^{\{1\}} \\ {\scriptstyle f^{\{0\}}}\big\downarrow & & {\scriptstyle f^{\{0,1\}}}\big\downarrow & & \big\downarrow{\scriptstyle f^{\{1\}}} \\ Z^{\{0\}} & \longrightarrow & Z^{\{0,1\}} & \longleftarrow & Z^{\{1\}} \end{array}$$

Then f is a weak equivalence if and only if its components f^A are weak equivalences in $(G \wedge M^A)$-$kTop_*$ for all non-empty $A \subseteq \langle 1 \rangle$. The map f is a fibration if and only if f^A is a fibration in $(G \wedge M^A)$-$kTop_*$ for all non-empty $A \subseteq \langle 1 \rangle$. Finally, f is a cofibration if and only if $f^{\{0\}}$ is a cofibration in $(G \wedge M^{\{0\}})$-$kTop_*$, $f^{\{1\}}$ is a cofibration in $(G \wedge M^{\{1\}})$-$kTop_*$, and the induced map

$$\begin{aligned} & Y^{\{0,1\}} \cup_{L_{\{0,1\}} Y} L_{\{0,1\}} Z \\ & = Y^{\{0,1\}} \cup_{Y^{\{0\}}[t_1^{-1}] \vee Y^{\{1\}}[t_0^{-1}]} (Z^{\{0\}}[t_1^{-1}] \vee Z^{\{1\}}[t_0^{-1}]) \to Z^{\{0,1\}} \end{aligned}$$

is a cofibration in $(G \wedge M^{\{0,1\}})$-$kTop_*$. In particular, a sheaf Y is cofibrant if and only if $Y^{\{0\}} \in \mathbf{C}(G \wedge M^{\{0\}})$, $Y^{\{1\}} \in \mathbf{C}(G \wedge M^{\{1\}})$, and the map

$$Y^{\{0\}}[t_1^{-1}] \vee Y^{\{1\}}[t_0^{-1}] \to Y^{\{0,1\}}$$

is a cofibration in $(G \wedge M^{\{0,1\}})$-$kTop_*$. This model structure is used implicitly for the category $\mathbb{P}_f(G)'$ in [4, proof of 3.3(1)].

By duality we obtain a second model structure:

Proposition 3.4.4 (The f-structure of $\mathbf{pP}^{\mathrm{N}}(G)$). *The category $\mathbf{pP}^N(G)$ has the structure of a model category where a map is a weak equivalence (resp. cofibration) if each of its components is a weak equivalence (resp. cofibration) in its respective category.*

Both model structures share the same weak equivalences, called *h-equivalences*, and hence have the same homotopy category $Ho\mathbf{pP}^N(G)$.

We will need yet another notion of cofibrations which belongs to one of the model structures of the category $G\text{-}kTop_*^{\langle N\rangle}$.

Definition 3.4.5. A map $f:Y\to Z$ in $\mathbf{pP}^N(G)$ is a *weak cofibration* if it is an f-cofibration when considered as a map in $G\text{-}kTop_*^{\langle N\rangle}$, i.e., if all its components are cofibrations in $G\text{-}kTop_*$.

Any c-cofibration is an f-cofibration, and any f-cofibration is a weak cofibration since forgetting the M^A-actions preserves cofibrations by Proposition 2.1.1(2).

Definition 3.4.6. The presheaf Y is called *strongly cofibrant* if it is cofibrant with respect to the c-structure. Explicitly, Y is strongly cofibrant if and only if the map $L_AY\to Y^A$ is a cofibration in $G\wedge M^A\text{-}kTop_*$ for all non-empty $A\subseteq N$. An object $Y\in\mathbf{pP}^N(G)$ is said to be *locally cofibrant* if it is cofibrant with respect to the f-structure. Explicitly, Y is locally cofibrant if and only if $Y^A\in\mathbf{C}(G\wedge M^A)$ for all $A\subseteq N$. Finally, Y is called *weakly cofibrant* if it is f-cofibrant as an object of $G\text{-}kTop_*^{\langle N\rangle}$, i.e., if all its components are cofibrant in $G\text{-}kTop_*$.

Any strongly cofibrant presheaf is locally cofibrant, and a locally cofibrant presheaf is weakly cofibrant.

Let $Y\in\mathbf{pP}^N(G)$ be a presheaf, and fix $j\in N$. We can consider Y as an N-cubical diagram in $G\text{-}kTop_*$; then its jth face is an $N\setminus\{j\}$-cubical diagram consisting of those components Y^B with $j\in B\subseteq N$. More generally, a subset $C\subseteq N$ determines an $N\setminus C$-cubical diagram formed by those components Y^B with $C\subseteq B$.

Definition 3.4.7. Let $Y\in\mathbf{pP}^N(G)$ and $C\subseteq A\subseteq N$ be given. The *restricted latching space* $L_A^{+C}Y$ is defined as

$$L_A^{+C}Y:=\operatorname*{colim}_{\langle A\rangle^1_{0,C}}R(Y)$$

(colimit in $(G\wedge M^A)\text{-}kTop_*$) where $\langle A\rangle^1_{0,C}$ is the subcategory of non-empty proper subsets of A containing C, and $R(Y)$ is defined as in the case of (unrestricted) latching spaces L_A, i.e., $R(Y)$ is the diagram $B\mapsto Y^B[t_A^{-1}]$ (this is a diagram in $(G\wedge M^A)\text{-}kTop_*$) with structure maps as defined earlier.

In effect, the space $L_A^{+C}Y$ is the latching space at $A\setminus C$ of the restricted $(N\setminus C)$-cubical (twisted) diagram determined by C (given by $B\mapsto Y^{B\cup C}$).

The following lemma is a "twisted" version of the familiar fact that if a cube is "cofibrant as a cube", the same is true for all its faces. If $\tau:B\to A$ is an inclusion of a proper subset, write $F_\tau=\cdot\wedge_{M^B}M^A:(G\wedge M^B)\text{-}kTop_*\to(G\wedge M^A)\text{-}kTop_*$, and let U_τ denote its right adjoint (restriction of action).

Lemma 3.4.8. *Let $Y\in\mathbf{pP}^N(G)$, $C\subseteq A\subseteq N$ and $j\in A\setminus C$ be given*:

(1) *The restricted latching space comes equipped with a map $L_A^{+C}Y\to Y^A$.*

(2) *For $D \subseteq C$, there is a canonical map $L_A^{+C}Y \to L_A^{+D}Y$, and the composite $L_A^{+C}Y \to L_A^{+D}Y \to Y^A$ is the map of* (1). *In particular, there is a canonical map $L_A^{+C}Y \to L_AY$, and the composite $L_A^{+C}Y \to L_AY \to Y^A$ is the map of* (1).

(3) *Let σ denote the inclusion $A \setminus \{j\} \to A$. The following square is a pushout*:

$$\begin{array}{ccc} F_\sigma(L^{+C}_{A\setminus\{j\}}Y) & \longrightarrow & F_\sigma(Y^{A\setminus\{j\}}) \\ \big\downarrow & & \big\downarrow \\ L_A^{+C\amalg\{j\}}Y & \longrightarrow & L_A^{+C}Y \end{array}$$

(4) *If Y is strongly cofibrant, the natural map $L_A^{+\{j\}}Y \to Y^A$ of* (1) *is a cofibration.*

Proof. (1) The objects occurring in the definition of the restricted latching space are of the form $Y^C[t_A^{-1}]$. The $\sharp$-type structure maps $Y^C[t_A^{-1}] \to Y^A$ are compatible with the structure maps of $R(Y)$. Hence we have an induced map $L_A^{+j}Y \to Y^A$.

(2) The diagram used for defining $L_A^{+C}Y$ includes into the diagram used for defining $L_A^{+D}Y$, and the map from the former to Y^A is the restriction of the map from the latter to Y^A. Moreover $L_A^{+\emptyset}Y = L_AY$. Hence (2) holds.

(3) Since F_σ is a left adjoint, it commutes with colimits. By explicitly spelling out the definitions, one realizes that the pushout in the above square and $L_A^{+C}Y$ really are colimits of the same diagram, hence are isomorphic.

(4) By (2), we have a factorization $L_A^{+j}Y \to L_AY \to Y^A$. The second map is a cofibration since Y is strongly cofibrant. Part (3) asserts that the first map is a cobase change of the image of $L_{A\setminus\{j\}}Z \to Z^{A\setminus\{j\}}$ under F_σ. But this is a cofibration since Y is strongly cofibrant and F_σ preserves cofibrations. □

3.5. *Restriction and extension by zero*

In algebraic geometry, application of “Proj” to the $n+1$ projections

$$R[X_0, X_1, \ldots, X_n] \to R[X_0, X_1, \ldots, X_n]/\langle X_j \rangle \cong R[Y_0, Y_1, \ldots, Y_{n-1}]$$

gives rise to $n+1$ different closed immersions of projective $(n-1)$-space into projective n-space. Restriction along these immersions induces $n+1$ different functors mapping quasi-coherent sheaves on $\mathbf{P}_R^n$ to quasi-coherent sheaves on $\mathbf{P}_R^{n-1}$.

An analogous construction can be made in the non-linear context. It is closely related to the process of twisting sheaves (Lemma 3.6.9).

Definition 3.5.1. Given $j \in N$, we define the jth restriction functor

$$\rho_j = \rho_j^N : \mathbf{pP}^N(G) \to \mathbf{pP}^{N\setminus\{j\}}(G)$$

by the equation $\rho_j(Y)^A := Y^A \wedge_{M_N^A} M^A_{N\setminus\{j\}}$ using the canonical map $M_N^A \to M^A_{N\setminus\{j\}}$ of Example 3.2.4 (whose effect is to get rid of the generator t_j by dividing it out).

Example 3.5.2. There is a natural isomorphism $\rho_j^N(\mathcal{O}_{\mathbf{P}^N}) \cong \mathcal{O}_{\mathbf{P}^{N\setminus\{j\}}}$ since

$$\rho_j^N(\mathcal{O}_{\mathbf{P}^N})^A = M_N^A \wedge_{M_N^A} M_{N\setminus\{j\}}^A \cong M_{N\setminus\{j\}}^A$$

for all non-empty $A \subseteq N \setminus \{j\}$.

If we represent an object $Y \in \mathbf{pP}^n(G)$ by a diagram having the shape of an n-simplex, the restricted sheaf $\rho_j(Y)$ is given by the jth $(n-1)$-dimensional face of the diagram after dividing out the action of t_j.

Lemma 3.5.3. *The functor ρ_j^N has a right adjoint $\zeta_j^N : \mathbf{pP}^{N\setminus\{j\}}(G) \to \mathbf{pP}^N(G)$ called "extension by zero". It is given by*

$$\zeta(Y)^A = \begin{cases} \bar{Y}^A & \text{if } j \notin A, \\ * & \text{otherwise,} \end{cases}$$

where $\bar{Y}^A$ is Y^A as a space with M_N^A acting via the canonical map $M_N^A \to M_{N\setminus\{j\}}^A$ (cf. Example 3.2.4), i.e., with t_j acting trivially.

Proof. This follows from Lemma 2.1.2(1) applied to the monoid homomorphisms $M_N^A \to M_{N\setminus\{j\}}^A$ for non-empty $A \subseteq N \setminus \{j\}$. □

For an object $Y \in \mathbf{P}^1$, the sheaf $\zeta_2(Y)$ is described by the following diagram (together with the convention that the additional "indeterminate" t_2 acts trivially on all spaces):

$$\begin{tikzcd}
 & & * \arrow[dl] \arrow[dr] & & \\
 & * \arrow[dr] & & * \arrow[dl] & \\
 & & * & & \\
Y^{\{0\}} \arrow[uur] \arrow[rr] & & Y^{\{0,1\}} \arrow[u] & & Y^{\{1\}} \arrow[uul] \arrow[ll]
\end{tikzcd}$$

Lemma 3.5.4. *The restriction functors ρ_j^N preserve all (acyclic) f-cofibrations and in addition weak equivalences of locally cofibrant objects. Moreover, they map locally cofibrant objects of $\mathbf{P}^N(G)$ to objects of $\mathbf{P}^{N\setminus\{j\}}(G)$.*

Proof. The first two assertions follow from Lemma 2.1.2(2) applied componentwise.

Now assume Y is a locally cofibrant sheaf. Let $\sigma : A \to B$ denote an inclusion of non-empty subsets of $N \setminus \{j\}$. Since $\rho_j(Y)$ is locally cofibrant by the above, Corollary 3.3.4 asserts that it suffices to show that the map

$$\rho_j^N(Y)_\sigma^\sharp : \rho_j^N(Y)^A[t_B^{-1}] = \rho_j^N(Y)^A \wedge_{M_{N\setminus\{j\}}^A} M_{N\setminus\{j\}}^B \to \rho_j^N(Y)^B$$

is a weak homotopy equivalence. Tracing the definitions shows that it is obtained from the structure map $Y^{\sharp}_{\sigma}: Y^A[t_B^{-1}] \to Y^B$ by applying $\cdot \wedge_{M^B_N} M^B_{N\setminus\{j\}}$. But this functor is known to preserve weak equivalences between cofibrant spaces by Lemma 2.1.2(2), and $Y^{\sharp}_{\sigma}$ is a weak equivalence since Y is a sheaf. □

3.6. Twists and canonical sheaves

In algebraic geometry, twisting is one of the basic operations for quasi-coherent sheaves on projective space. Recall that

$$\Gamma_*(\mathcal{O}_{\mathbf{P}^n_R}) := \bigoplus_0^\infty \Gamma(\mathcal{O}(j)) = R[X_0, \dots, X_n]$$

is a polynomial ring. Its generators, the indeterminates X_j, induce natural maps $\mathcal{F}(k) \xrightarrow{\cdot X_j} \mathcal{F}(k+1)$ given by "multiplication with X_j" (where $\mathcal{F}$ denotes a quasi-coherent sheaf on $\mathbf{P}^n_R$). The cokernel of this map is the extension by zero of the restriction of $\mathcal{F}$ to an embedded $\mathbf{P}^{n-1}_R$.

In this section, we show how to transfer these constructions to the non-linear setting. The relation between twisting and restriction of sheaves (Lemma 3.6.9) enables us to do induction on the dimension n (Lemma 4.4.4).

The definition of the twist functor as given in [4] is not symmetric. It turns out that for twists and related constructions, it is convenient to have both symmetric and asymmetric descriptions, the latter arising from a choice of a total order on the indexing set. (In this case, it is enough to restrict attention to the standard ordered sets $[n]$.)

Definition 3.6.1 (*Tensor product of sheaves*)**.** Suppose Y is an object of $\mathbf{pP}^N(G)$ and Z is an object of $\mathbf{pP}^N$. The (*non-linear*) *tensor product* $Y \wedge_{\mathcal{O}} Z$ of Y and Z is the presheaf given by $A \mapsto Y^A \wedge_{M^A} Z^A$ with M^A acting from the right on Z^A. (This definition makes sense because M^A is abelian, hence acts from the left and from the right on Z^A.) The tensor product is an object of $\mathbf{pP}^N(G)$ (where G acts on the components of Y).

Definition 3.6.2 (*Symmetric description of twisting*)**.** The jth Serre *twisting sheaf* $\mathcal{O}_{\mathbf{P}^N}(j) = \mathcal{O}(j) \in \mathbf{P}^N$ is given by

$$A \mapsto \tilde{M}^A_N(j) \quad (A \neq \emptyset)$$

with $\flat$-type structure maps given by the inclusions $\mathcal{O}(j)^A = \tilde{M}^A_N(j) \subseteq \tilde{M}^B_N(j) = \mathcal{O}(j)^B$ for non-empty subsets $A \subseteq B \subseteq N$ (for notation cf. Definition 3.2.1). For an object $Y \in \mathbf{pP}^N(G)$ and $j \in \mathbf{Z}$, the jth *twist functor* $\theta_j = \theta^N_j : \mathbf{pP}^N(G) \to \mathbf{pP}^N(G)$ is defined as $\theta_j(Y) := Y \wedge_{\mathcal{O}} \mathcal{O}(j)$.

Definition 3.6.3 (*Asymmetric description of twisting*; *Hüttemann et al.* [4, 5.4])**.** The jth *twist functor* $\bar{\theta}_j = \bar{\theta}^n_j : \mathbf{pP}^n(G) \to \mathbf{pP}^n(G)$ (for $j \in \mathbf{Z}$) assigns to an object $Y \in \mathbf{pP}^n(G)$ the object $Z := \bar{\theta}_j(Y)$ in the following way: for all $A \subseteq [n]$, we define $Z^A := Y^A$, and for non-empty subsets $A \subseteq B$ of $[n]$ we define the structure map $Z^A \to Z^B$ as the composite

$$Z^A = Y^A \to Y^B \xrightarrow{t_m^j t_k^{-j}} Y^B = Z^B, \tag{$*$}$$

where $m:=\max(A)$ and $k:=\max(B)$. A morphism $f: Y \to \bar{Y}$ in $\mathbf{pP}^n(G)$ (with components f^A) gives rise to a morphism of twisted objects with the unchanged components f^A.

Obviously $\bar{\theta}_0^n = \mathrm{id}$. Moreover, the twist functor $\bar{\theta}_j$ restricts to an endofunctor of $\mathbf{P}^n(G)$ since $t_m^j t_k^{-j}$ is an invertible map and hence a weak homotopy equivalence.

To make the asymmetric definition of twisting less obscure, we include the diagrams for the projective line ($n=1$) and the projective plane ($n=2$). An object Y of $\mathbf{pP}^1(G)$ is a diagram $Y^{\{0\}} \to Y^{\{01\}} \leftarrow Y^{\{1\}}$, the twisted object $\bar{\theta}_j^1(Y)$ can be pictured as $Y^{\{0\}} \xrightarrow{t_0^j t_1^{-j}} Y^{\{01\}} \leftarrow Y^{\{1\}}$. (Here we abbreviated the above composition ($*$) to a single map.) In the second case, $Y \in \mathbf{pP}^2(G)$, the twisted object is given by the following diagram:

$$\begin{tikzcd}
 & & Y^{\{2\}} \arrow[dl] \arrow[dr] & & \\
 & Y^{\{0,2\}} \arrow[dr] & & Y^{\{1,2\}} \arrow[dl] & \\
 & & Y^{\{0,1,2\}} & & \\
Y^{\{0\}} \arrow[uur, "t_0^j t_2^{-j}"] \arrow[rr, "t_0^j t_1^{-j}"] & & Y^{\{0,1\}} \arrow[u, "t_1^j t_2^{-j}"] & & Y^{\{1\}} \arrow[ll] \arrow[uul, "t_1^j t_2^{-j}"']
\end{tikzcd}$$

The asymmetric description depends on a choice of order on the indexing set. All choices yield isomorphic sheaves by the next lemma.

Lemma 3.6.4 (Properties of the twist functor)**.** (1) *We have* $\mathcal{O}(i) \wedge_{\mathcal{O}} \mathcal{O}(j) \cong \mathcal{O}(i+j)$ *and* $\mathcal{O} = \mathcal{O}(0)$.

(2) *There are natural isomorphisms* $\theta_{j+k} \cong \theta_j \circ \theta_k$ *and* $\theta_0 \cong \mathrm{id}$. *In particular*, θ_k *is an equivalence of categories. Similarly*, $\bar{\theta}_{j+k} \cong \bar{\theta}_j \circ \bar{\theta}_k$ *and* $\bar{\theta}_0 = \mathrm{id}$.

(3) *For all* $Y \in \mathbf{pP}^n(G)$, *the presheaves* $\theta_j^n(Y)$ *and* $\bar{\theta}_j^n(Y)$ *are naturally isomorphic.*

(4) *If* $Y \in \mathbf{P}^N(G)$, *then* $\theta_j(Y) \in \mathbf{P}^N(G)$.

In algebraic geometry, the sheaf $\mathcal{O}_{\mathbf{P}_R^n}(1)$ has $n+1$ canonical sections $X_0, \ldots, X_n$. Each of these determines a natural map $\mathscr{F}(k) \xrightarrow{\cdot X_i} \mathscr{F}(k+1)$ where $\mathscr{F}$ is a module on $\mathbf{P}_R^n$. We have an analogous set of maps in the non-linear context (cf. the definition in [4] preceding 6.11):

Definition 3.6.5. Let $i \in N$ be given. The natural transformation

$$\kappa_i = \kappa_i^N : \theta_k^N \to \theta_{k+1}^N$$

is given by morphisms $\theta_k(Y) \to \theta_{k+1}(Y)$ for each object $Y \in \mathbf{pP}^N(G)$ described by

$$\theta_k(Y)^A \xrightarrow{t_i} \theta_{k+1}(Y)^A$$

for $\emptyset \neq A \subseteq N$.

Definition 3.6.6. The natural transformation

$$\bar{\kappa}_i = \bar{\kappa}_i^n : \bar{\theta}_k^n \to \bar{\theta}_{k+1}^n$$

is given by morphisms $\bar{\theta}_k(Y) \to \bar{\theta}_{k+1}(Y)$ for each object $Y \in \mathbf{pP}^n(G)$ described by

$$\bar{\theta}_k(Y)^A = Y^A \xrightarrow{t_i t_m^{-1}} Y^A = \bar{\theta}_{k+1}(Y)^A$$

with $m{:=}\max(A)$ (where $\emptyset \neq A \subseteq [n]$).

The transformations κ_j and $\bar{\kappa}_j$ correspond under the isomorphism of Lemma 3.6.4(3).

Definition 3.6.7 (*Hüttemann et al.* [4, 5.5])**.** For $K \in G\text{-}kTop_*$ we define the *canonical sheaf associated to* K as the object $\psi_0(K) = \psi_0^N(K) \in \mathbf{P}^N(G)$ given by

$$K \wedge \mathcal{O}_{\mathbf{P}^N} : A \mapsto K \wedge M_N^A$$

($\emptyset \neq A \subseteq N$) with structure maps induced by inclusions of submonoids. The assignment $K \mapsto \psi_0(K)$ is functorial in K. For convenience, we introduce the *twisted canonical sheaf functor* $\psi_j = \psi_j^N{:=}\theta_j^N \circ \psi_0^N$.

As a first example, we note that $\psi_0^N(S^0) \cong \mathcal{O}_{\mathbf{P}^N}$, the structure sheaf of projective N-space, and $\psi_j^N(S^0) \cong \mathcal{O}(j)$.

Recall that $\langle N \rangle_0$ is the set of non-empty subsets of N. Let $G\text{-}kTop_*^{\langle N \rangle_0}$ denote the functor category $Func(\langle N \rangle_0,\ G\text{-}kTop_*)$.

Lemma 3.6.8. *Let V denote the forgetful functor $\mathbf{pP}^N(G) \to G\text{-}kTop_*^{\langle N \rangle_0}$. The functor $\psi_{-k} : G\text{-}kTop_* \to \mathbf{pP}^N(G)$ is left adjoint to the functor* $\lim_{\leftarrow} \circ V \circ \theta_k$.

Proof. For $k = 0$ this can be deduced from adjointness of inverse limit and constant diagram functor. Since θ_k is an equivalence of categories with inverse given by θ_{-k} (Lemma 3.6.4(2)) the general case follows. □

The functor $\lim_{\leftarrow} \circ V$ (this is the case $k = 0$ from the lemma) is the literal translation of the algebraic geometers' global sections functor.

The following lemma (which establishes the connection between twisting and restriction) is one of the key ingredients for the splitting theorem. Recall that the natural maps $\kappa_j : \theta_k(Y) \to \theta_{k+1}(Y)$ are given by "multiplication with the indeterminate t_j"

(Definition 3.6.5), where Y is an arbitrary (non-linear) sheaf and θ denotes the twisting functor (Definition 3.6.2).

Lemma 3.6.9. *Suppose $Y \in \mathbf{pP}^N(G)$ is a locally cofibrant presheaf, and let $j \in N$ and $k \in \mathbf{Z}$ be given. Then $\kappa_j : \theta_k(Y) \to \theta_{k+1}(Y)$ is a weak cofibration, its (strict) cofibre is isomorphic to $\zeta_j \circ \rho_j \circ \theta_{k+1}(Y)$, and the projection $\theta_{k+1}(Y) \to \zeta_j \circ \rho_j \circ \theta_{k+1}(Y)$ is isomorphic to the $\theta_{k+1}(Y)$-component of the unit of the adjunction of ζ_j and ρ_j.*

Proof. We begin by showing that κ_j is a weak cofibration. By definition, we have to prove that for all non-empty $A \subseteq N$ its A-component

$$\kappa_j^A = \mathrm{id} \wedge (t_j) : Y^A \wedge_{M^A} \tilde{M}^A(k) \to Y^A \wedge_{M^A} \tilde{M}^A(k+1)$$

is a cofibration in $G\text{-}kTop_*$. Choose an element $e \in A$. We can factor the map

$$\tilde{M}^A(k) \xrightarrow{t_j} \tilde{M}^A(k+1)$$

in the following way:

$$\tilde{M}^A(k) \xrightarrow{t_e^{-k}} M^A \xrightarrow{t_j t_e^{-1}} M^A \xrightarrow{t_e^{k+1}} \tilde{M}^A(k+1).$$

The first and third map are isomorphisms of discrete M^A-spaces, the map in the middle is injective. Application of the functor $Y^A \wedge_{M^A} \cdot$ yields a factorization of κ_j^A as a composite of an isomorphism, a cofibration in $G\text{-}kTop_*$ (use Lemma 2.1.3) and an isomorphism again. Hence κ_j is a weak cofibration.

Since θ_k is an equivalence of categories (Lemma 3.6.4(2)) it commutes with restriction, extension by zero and taking cofibres. Thus it suffices to prove the remaining claims for $k = -1$ only.

Fix a non-empty subset $A \subseteq N$, and consider the A-component

$$\kappa_j^A : Y^A \wedge_{M_N^A} \tilde{M}_N^A(-1) \to Y^A \wedge_{M_N^A} M_N^A \cong Y^A$$

of κ_j. We claim that its cofibre is isomorphic to $\zeta_j \circ \rho_j(Y)^A = Y^A \wedge_{M_N^A} M_{N\setminus\{j\}}^A$. Now the functor $Y^A \wedge_{M_N^A} \cdot : M_N^A\text{-}kTop_* \to M_N^A\text{-}kTop_*$ commutes with taking cofibres since colimits commute among themselves. Thus it suffices to show that the cofibre of the map $\tilde{M}_N^A(-1) \xrightarrow{t_j} M_N^A$ is isomorphic to $M_{N\setminus\{j\}}^A$. But this is clear since its image is precisely the ideal $I_{j,N}^A$, and we know that $M_N^A / I_{j,N}^A \cong M_{N\setminus\{j\}}^A$ by Example 3.2.4. □

3.7. Global sections

We have seen above (Lemma 3.6.8) that there is a non-linear analogue of global sections of sheaves. However, it turns out that this functor is not suitable for K-theory calculations. Hence we introduce a functor Γ which (philosophically speaking) captures global sections and higher sheaf cohomology at the same time.

Important for the sequel is the fact that we can compute global sections of twisted canonical sheaves (Corollaries 3.7.4 and 3.7.5). On the level of K-theory spaces the canonical sheaf functor ψ_0 (Definition 3.6.7) provides a section of Γ.

Definition 3.7.1 (*Global sections of sheaves*; *Hüttemann et al.* [4, 5.1])**.** For any object $Y \in \mathbf{pP}^N(G)$ we define the global sections of Y, denoted

$$\Gamma(Y) = \Gamma^N(Y) \in G\text{-}kTop_*$$

as the iterated homotopy cofibre of Y in the sense of Definition 2.2.1 where Y is considered as a functor $\langle N \rangle \to G\text{-}kTop_*$ (with $\flat$-type structure maps).

For an object $Y \in \mathbf{pP}^1(G)$, the space $\Gamma(Y)$ is given by the mapping cone of the map $Y^{\{0\}} \vee Y^{\{1\}} \to Y^{\{0,1\}}$. An explicit model (as given in [4]) for this is

$$\Gamma(Y) = CY^{\{0\}} \cup_{Y^{\{0\}}} Y^{\{0,1\}} \cup_{Y^{\{1\}}} CY^{\{1\}},$$

where $CK = K \wedge I/(K \times \{0\})$ denotes the (reduced) cone on the pointed space K.

In what follows, we develop the elementary properties of the global sections functor. As a beginning, we note that suspension commutes with global sections, i.e.,

$$\Gamma(\Sigma Y) = \Gamma(S^1 \wedge Y) \cong S^1 \wedge \Gamma(Y) = \Sigma\Gamma(Y)$$

for all $Y \in \mathbf{P}^N(G)$ (this is a special case of Remark 2.2.2(2)).

Next, we consider global sections of a sheaf Y and its extension $\zeta_j(Y)$ (Lemma 3.5.3). In algebraic geometry, extension by zero does not change the cohomology groups of a sheaf. The analogous statement in the present context says that Y and $\zeta_j(Y)$ have *stably* the same global sections:

Lemma 3.7.2. *For an object $Y \in \mathbf{pP}^{N\setminus\{j\}}(G)$, the spaces $\Sigma\Gamma^{N\setminus\{j\}}(Y)$ and $\Gamma^N \circ \zeta_j(Y)$ are naturally isomorphic.*

Proof. By definition $\zeta_j(Y)^A = *$ if $j \in A$. Computing the homotopy cofibre of $\zeta_j(Y)$ in j-direction (cf. Remark 2.2.2(4)) results in an $N \setminus \{j\}$-cube Z. Its A-component is the mapping cone of $Y^A \to *$. But this is the suspension of Y^A. Hence Z is isomorphic to $\Sigma(Y)$, and since the global sections functor commutes with suspension, we infer that $\Gamma \circ \zeta_j(Y) \cong \Gamma(Z) \cong \Gamma(\Sigma Y) \cong \Sigma\Gamma(Y)$. □

Now, we want to compute global sections of twisted canonical sheaves. For $j \in \mathbf{Z}$ define an N-cube of spaces

$$\mathscr{W}_N(j) : \langle N \rangle \to kTop_*, \quad A \mapsto \tilde{M}_N^A(j),$$

where $\tilde{M}_N^A(j)$ has the discrete topology (cf. Definition 3.2.1). Structure maps are given by inclusions. As before, N denotes a non-empty finite set with $n+1$ elements.

Lemma 3.7.3. *If $j > -n-1$, the cube $\mathscr{W}_N(j)$ has contractible iterated homotopy cofibre, i.e., $\Gamma(\mathscr{W}_N(j)) \simeq *$.*

Proof. It suffices to consider ordered indexing sets $N=[n]$. For $k=-1,0,\dots,n$ define an $(n-k)$-cube

$$\begin{aligned} \mathscr{V}_k(j)&:\langle[n-k-1]\rangle\to kTop_*,\\ A&\mapsto\{(a_0,a_1,\dots,a_n)\in\tilde{M}_n^{A\coprod\{n-k,\dots,n\}}(j)\,|\,\forall\ell\geqslant n-k\colon a_\ell<0\}\end{aligned}$$

with structure maps given by inclusions. Note that $\mathscr{V}_{-1}(j)=\mathscr{W}_N(j)$, and $\mathscr{V}_n(j)$ is a single space consisting of the basepoint only (if $(a_0,a_1,\dots,a_n)$ is a non-basepoint in $\mathscr{V}_n(j)$, we have $j=\sum a_i\leqslant-(n+1)$ which is impossible by assumption on j). Now $\mathscr{V}_k(j)$ is weakly equivalent to the pointwise homotopy cofibre of the cube $\mathscr{V}_{k-1}(j)$ in $(n-k)$-direction since all spaces are discrete, all maps are injective, and $\mathscr{V}_k(j)$ is the strict cofibre of $\mathscr{V}_{k-1}(j)$ in $(n-k)$-direction. Hence $\Gamma(\mathscr{W}_N(j))\simeq\mathscr{V}_n(j)=*$. $\square$

Define, for $j\geqslant0$, the number $h(N,j){:=}\#\tilde{M}_N^{\emptyset}(j)-1$. Since $\tilde{M}_N^{\emptyset}(0)=M_N^{\emptyset}=S^0$, we have $h(N,0)=1$.

Corollary 3.7.4. (Global sections of canonical sheaves; Hüttemann et al. [4, 5.6]). *Suppose G is a topological monoid with zero which is cofibrant as an object of $kTop_*$. For $K\in\mathbf{C}(G)$ and $j\geqslant0$, there is a natural chain of weak equivalences connecting $\Sigma\circ\Gamma\circ\psi_j(K)$ and $\bigvee_{h(N,j)}\Sigma^{n+1}K$. In particular, we have a natural weak equivalence $\Sigma\circ\Gamma\circ\psi_0(K)\simeq\Sigma^{n+1}K$.*

Proof. It suffices to give a proof for $N=[n]$. First, note that we have $\Gamma(\psi_j(K))\cong\Gamma(\psi_j(K\wedge S^0))\cong K\wedge\Gamma(\psi_j(S^0))$. Since G is cofibrant in $k\,Top_*$ so is K (Proposition 2.1.1(3)), hence the functor $K\wedge\cdot$ is homotopy invariant. Consequently, it suffices to prove the claim for the special case $K=S^0$ (twisted structure sheaves).

Recall the definition of $\mathscr{W}_N(j)$ from Lemma 3.7.3. There is an obvious map of cubes $\psi_j(S^0)\to\mathscr{W}_N(j)$ (given by the identity map for $A\neq\emptyset$). For $N=[1]$, we have the following picture:

$$\begin{tikzcd}
 & (\mathbf{N}\times\mathbf{N})_+^j \arrow[rr] \arrow[dd] & & (\mathbf{Z}\times\mathbf{N})_+^j \arrow[dd] \\
* \arrow[ur] \arrow[rr, crossing over] \arrow[dd] & & (\mathbf{Z}\times\mathbf{N})_+^j \arrow[ur] & \\
 & (\mathbf{N}\times\mathbf{Z})_+^j \arrow[rr] & & (\mathbf{Z}\times\mathbf{Z})_+^j \\
(\mathbf{N}\times\mathbf{Z})_+^j \arrow[ur] \arrow[rr] & & (\mathbf{Z}\times\mathbf{Z})_+^j \arrow[ur] \arrow[from=uu, crossing over] &
\end{tikzcd}$$

(The lower index "+" means adding a disjoint basepoint, the upper index j denotes the subset of tuples with sum j.) The front face of this cube is $\mathcal{O}_{\mathbf{P}^1}(j)=\psi_j(S^0)$, the back face is $\mathscr{W}_{[1]}(j)$.

In the general case, note that all the components of the map $\psi_j(S^0) \to \mathscr{W}_N(j)$ are injective, and all components of the cubes are discrete spaces. Hence, the pointwise homotopy cofibre is weakly equivalent to the pointwise (strict) cofibre. Computation of the pointwise cofibre yields an $(n+1)$-cube $\mathscr{D}$ with $\mathscr{D}^\emptyset = \mathscr{W}_N^\emptyset(j) = \tilde{M}_n^\emptyset(j)$ (since $\psi_j(S^0)^\emptyset = *$), and $\mathscr{D}^A = *$ for $A \neq \emptyset$ (since $\psi_j(S^0)^A = \mathscr{W}_N^A(j)$ in this case). The discrete space $\mathscr{D}^\emptyset$ has $h(N,j)$ non-basepoint elements (by definition of that number), hence can be written as an $h(N,j)$-fold one-point union of zero spheres. Application of Γ to the cofibration sequence $\psi_j(S^0) \to \mathscr{W}_N(j) \to \mathscr{D}$ yields a sequence of maps

$$\Gamma(\psi_j(S^0)) \to \Gamma(\mathscr{W}_N(j)) \to \Gamma(\mathscr{D}) \cong \Sigma^{n+1}\mathscr{D}^\emptyset \cong \bigvee_{h(n,j)} \Sigma^{n+1} S^0. \qquad (*)$$

Since Γ commutes with pushouts, the last space of (*) is the cofibre of the map on the left. But this map is a cofibration by Corollary 2.2.3(1) since $\psi_j(S^0) \to \mathscr{W}_N(j)$ is an f-cofibration in $k\,Top_*^{\langle N \rangle}$. Hence its cofibre is weakly equivalent to its homotopy cofibre. By Lemma 3.7.3, the space in the middle is contractible, hence the homotopy cofibre is weakly equivalent to the suspension of $\Gamma(\psi_j(S^0))$ which finishes the proof. □

Now we treat negative twists. Define, for $j < 0$, the number

$$a(N,j) := \# \left\{ (a_i)_{i \in N} \in \mathbf{Z}^N \;\middle|\; \sum_{i \in N} a_i = j \text{ and } \forall i\colon a_i < 0 \right\}.$$

We have

$$a(N,j) = 0 \quad \text{for } -n-1 < j < 0,$$

$$a(N,-n-1) = 1,$$

which can be seen in the following way: suppose $(a_i)_{i \in N}$ is an element of the set occurring in the definition of $a(N,j)$. Then $j = \sum a_i \leqslant -(n+1)$. Hence such an element does not exist if $-n-1 < j$, and there is exactly one such element if $j = -n-1$. Using the notation of Lemma 3.7.3, we have $a(N,j) = \#\mathscr{V}_n(j) - 1$, the -1 being due to the basepoint.

Corollary 3.7.5 (Global sections of canonical sheaves; Hüttemann et al. [4, 5.6]). *Suppose G is a topological monoid with zero which is cofibrant as an object of $kTop_*$. For $K \in \mathbf{C}(G)$ and $j < 0$, there is a natural chain of weak equivalences connecting $\Gamma \circ \psi_j(K)$ and $\bigvee_{a(N,j)} K$. In particular, we have a natural weak equivalence $\Gamma \circ \psi_{-n-1}(K) \simeq K$, and for $-n-1 < j < 0$, the space $\Gamma \circ \psi_j(K)$ is contractible.*

Proof. As is the proof of Corollary 3.7.4 it suffices to give a proof for $N = [n]$ and $K = S^0$.

Recall the definition of $\mathscr{W}_N(j)$ from Lemma 3.7.3. For $j < 0$, comparing the definitions shows $\mathscr{O}_{\mathbf{P}^N}(j) = \mathscr{W}_N(j)$. Using the notation from the proof of Lemma 3.7.3,

$$\Gamma(\mathscr{O}_{\mathbf{P}^N}(j)) = \Gamma(\mathscr{W}_N(j)) \simeq \mathscr{V}_n(j).$$

As we have seen, this space is contractible for $j > -n-1$. In general, this space has $a(N,j)$ non-basepoints (this is just the definition of $a(N,j)$), hence can be written as an $a(N,j)$-fold one-point union of copies of S^0. If in particular $j = -n-1$, we have $\Gamma(\mathcal{O}_{\mathbf{P}^N}(-n-1)) \simeq S^0$. $\square$

3.8. Spread sheaves

Definition 3.8.1. Given $S \subseteq \mathbf{Z}$, a map f in $\mathbf{pP}^N(G)$ is called an *h_S-equivalence* if for all $s \in S$ the map $\Gamma \circ \theta_s(f)$ is a weak equivalence of spaces. A map of cubes $g \in G\text{-}kTop_*^{\langle N\rangle}$ is called an $h_{\{0\}}$-equivalence if $\Gamma(g)$ is a weak equivalence of G-spaces. (Recall that Γ is defined (Definition 2.2.1) as the iterated homotopy cofibre of cubical diagrams of pointed topological spaces.)

The purpose of this section is to establish a chain of $h_{\{0\}}$-equivalences in $\mathbf{P}^N(G)$ connecting $\Sigma^2 \circ \psi_0 \circ \Gamma(Y)$ and $\Sigma^{n+2}(Y)$.

Definition 3.8.2. Suppose Y is an object of $\mathbf{pP}^N(G)$, and C is a (possibly empty) subset of N. We define the *C-spreading of Y*, denoted $\mathrm{spr}^C(Y)$, to be the presheaf

$$A \mapsto \mathrm{spr}^C(Y)^A := Y^{C \cup A}$$

($\emptyset \neq A \subseteq N$) with ($\flat$-type) structure maps induced by those of Y. Here $Y^{C\cup A}$ is considered as a $G \wedge M^A$-equivariant space. Define $\widetilde{\mathrm{spr}}^C(Y) := \mathrm{spr}^C(Y)$ if $C \neq \emptyset$ and $\widetilde{\mathrm{spr}}^{\emptyset}(Y) := *$.

For $C \subseteq D \subseteq N$ there is a map of presheaves $\mathrm{spr}^C(Y) \to \mathrm{spr}^D(Y)$ with components induced by the structure maps of Y. The assignment $C \mapsto \mathrm{spr}^C(Y)$ for fixed Y is itself functorial, hence:

Lemma 3.8.3. *The cubical diagram $A \mapsto \Gamma \circ \mathrm{spr}^A(Y)$ in $G\text{-}kTop_*^{\langle N\rangle}$ is commutative.*

Example 3.8.4. We draw the diagram of Lemma 3.8.3 for the case $N = \langle 1\rangle$:

$$\begin{array}{ccc}
\Gamma\left(\begin{array}{ccc} * & \longrightarrow & Y^{\{0\}} \\ \downarrow & & \downarrow \\ Y^{\{1\}} & \longrightarrow & Y^{\{0,1\}} \end{array}\right) & \longrightarrow & \Gamma\left(\begin{array}{ccc} * & \longrightarrow & Y^{\{0\}} \\ \downarrow & & \downarrow \\ Y^{\{0,1\}} & \longrightarrow & Y^{\{0,1\}} \end{array}\right) \\
\downarrow & & \downarrow \\
\Gamma\left(\begin{array}{ccc} * & \longrightarrow & Y^{\{0,1\}} \\ \downarrow & & \downarrow \\ Y^{\{1\}} & \longrightarrow & Y^{\{0,1\}} \end{array}\right) & \longrightarrow & \Gamma\left(\begin{array}{ccc} * & \longrightarrow & Y^{\{0,1\}} \\ \downarrow & & \downarrow \\ Y^{\{0,1\}} & \longrightarrow & Y^{\{0,1\}} \end{array}\right)
\end{array}$$

As usual, the 0-direction is left to right, i.e., the small square in the upper right corner represents $\mathrm{spr}^{\{0\}}(Y)$, the small squares in the bottom row represent $\mathrm{spr}^{\{1\}}(Y)$ and $\mathrm{spr}^{\{0,1\}}(Y)$, respectively.

As each component of $\mathrm{spr}^C(Y)$ has an M^C-action, the space $\Gamma \circ \mathrm{spr}^C(Y)$ is an object of $G \wedge M^C$-$kTop_*$. Because of this and the previous lemma, it is possible to define a new presheaf $\sigma^N(Y) = \sigma(Y)$ by

$$A \mapsto \sigma^N(Y)^A := \Sigma^2 \circ \Gamma \circ \mathrm{spr}^A(Y) \quad (A \neq \emptyset).$$

In this way, we obtain a functor $\sigma^N = \sigma : \mathbf{pP}^N(G) \to \mathbf{pP}^N(G)$. (For $N = \langle 1 \rangle$ this is the double suspension of Σ' as defined in [4]. For a picture, replace the left upper square in the above example by a point, and suspend twice.)

Lemma 3.8.5. (1) *If Y is a weakly cofibrant object* (*Definition* 3.4.6) *of* $\mathbf{pP}^N(G)$, *there is a natural chain of h-equivalences connecting $\sigma(Y)$ and $\Sigma^{n+2}Y$. In particular, if Y is a weakly cofibrant sheaf, then $\sigma(Y) \in \mathbf{P}^N(G)$.*

(2) *The functor σ^N commutes with pushouts. It preserves h-equivalences and weak cofibrations* (*Definition* 3.4.5) *between weakly cofibrant objects.*

Proof. For a space $K \in kTop_*$, let $\mathrm{con}(K)$ denote the N-cube with a point as initial vertex and K everywhere else; structure maps are identity maps (away from the initial vertex). For $Y \in \mathbf{pP}^N(G)$ define a presheaf εY by $(\varepsilon Y)^C := \Gamma \circ \mathrm{con}(Y^C)$. It is h-equivalent to the n-fold suspension of Y (compute global sections in any direction and observe that the resulting cube has Y^C as initial vertex, and all other vertices are contractible). Thus it is sufficient to show that $\Sigma^2 \circ \varepsilon$ is h-equivalent to σ.

For $C \subseteq N$ and $Y \in \mathbf{P}^N(G)$ there is a map of cubes $f^C : \mathrm{con}(Y^C) \to \mathrm{spr}^C(Y)$, natural in C and Y, induced by the structure maps of Y.

In the case $N = \langle 1 \rangle$, this map has the following pictorial representation:

$$\begin{tikzcd} * \arrow[r] \arrow[d] & Y^C \arrow[d] \arrow[r, "f^C"] & * \arrow[r] \arrow[d] & Y^{\{0\} \cup C} \arrow[d] \\ Y^C \arrow[r] & Y^C & Y^{\{1\} \cup C} \arrow[r] & Y^{\{0,1\} \cup C} \end{tikzcd}$$

The square on the left depicts $\mathrm{con}(Y^C)$, the square on the right represents $\mathrm{spr}^C(Y)$.

Back to the general case, the maps f^C induce upon application of $\Sigma^2 \circ \Gamma$ a natural transformation $g : \Sigma^2 \circ \varepsilon \to \sigma$. We want to prove that g is an h-equivalence of functors, i.e., that all components of g are weak equivalences. Fix a non-empty subset $C \subseteq N$. By definition, the map g^C is given by $\Sigma^2 \circ \Gamma(f^C)$. Thus, it is enough to show that $\Gamma(f^C)$ has contractible mapping cone for all $C \neq \emptyset$. But Γ commutes with taking homotopy cofibres. Hence, it suffices to compute the pointwise mapping cone of f^C and show that the resulting cube Z has contractible iterated homotopy cofibre.

So define $Z := \mathrm{hocofibre}(f^C)$. Choose $i \in C$. By definition of the C-spreading we have $Z^\emptyset = *$ and $Z^A = \mathrm{hocofibre}(Y^C \to Y^{C \cup A})$ for $A \neq \emptyset$. This implies $Z^A \simeq Z^{A \amalg \{i\}}$ for all $A \subseteq N$ (even $Z^A = Z^{A \amalg \{i\}}$ if A is not the empty set). Hence, computing $\Gamma(Z)$ in i-direction yields a cube with contractible vertices. This proves that $\sigma(Y)$ and $\Sigma^{n+2}(Y)$ are weakly equivalent (with respect to h-equivalences).

Since the sheaf condition is homotopy invariant (Corollary 3.3.5), and since the suspension of a weakly cofibrant sheaf is a sheaf, we infer that σ maps weakly cofibrant sheaves to sheaves. This completes the proof of (1).

For (2) note that the functors spr^C commute with pushouts (since colimits are calculated pointwise). Since Γ is also compatible with pushouts (Corollary 2.2.3(3)), so is σ.

Let $f: Y \to Z$ denote a weak cofibration between weakly cofibrant objects. This means by definition that all components of f are cofibrations in $G\text{-}kTop_*$. Hence the same is true for $\mathrm{spr}^C(f)$, i.e., $\mathrm{spr}^C(f)$ is an f-cofibration in $G\text{-}kTop_*^{\langle N\rangle}$ between f-cofibrant objects of $G\text{-}kTop_*^{\langle N\rangle}$. Since Γ preserves these cofibrations by Corollary 2.2.3(1), we know that $\Gamma \circ \mathrm{spr}^C(f)$ is a cofibration in $G\text{-}kTop_*$, hence (by suspending twice) so is the C-component of $\sigma(f)$. This means by definition that $\sigma(f)$ is a weak cofibration.

Finally, σ preserves weak equivalences between weakly cofibrant objects since Σ^{n+2} does and both functors are connected by a chain of h-equivalences as shown in (1). □

We proceed with a technical lemma. Let $Y \in \mathbf{P}^N(G)$, and suppose B is a non-empty subset of N. Fix $i \in B$. There is a natural map

$$\gamma : \mathrm{spr}^B(Y) \to \delta_{\{i\}}(Y^B)$$

in $G\text{-}kTop_*^{\langle N\rangle}$ given by the identity id_{Y^B} on $\{i\}$-components (the functor $\delta_{\{i\}}$ has been defined in Definition 2.2.1). For $N = \langle 1\rangle$ and $i = 1 \in B$ this map has the following representation:

$$\begin{array}{ccc} * & \longrightarrow & Y^{\{0\}\cup B} \\ \downarrow & & \downarrow \\ Y^{\{1\}\cup B} & \longrightarrow & Y^{\{0,1\}\cup B} \end{array} \xrightarrow{\gamma} \begin{array}{ccc} * & \longrightarrow & * \\ \downarrow & & \downarrow \\ Y^{B} & \longrightarrow & * \end{array}$$

Lemma 3.8.6. *The map γ is an $h_{\{0\}}$-equivalence* (*Definition* 3.8.1).

Proof. Consider the cube Z which is the same as $\mathrm{spr}^B(Y)$ with $\{i\}$-component replaced by a single point, and let $o: Z \to \mathrm{spr}^B(Y)$ denote the inclusion map; it is the identity on all components except the $\{i\}$-component.

$$\begin{array}{ccc} * & \longrightarrow & Y^{\{0\}\cup B} \\ \downarrow & & \downarrow \\ * & \longrightarrow & Y^{\{0,1\}\cup B} \end{array} \xrightarrow{o} \begin{array}{ccc} * & \longrightarrow & Y^{\{0\}\cup B} \\ \downarrow & & \downarrow \\ Y^{\{1\}\cup B} & \longrightarrow & Y^{\{0,1\}\cup B} \end{array}$$

There is an obvious map from the pointwise homotopy cofibre of o to $\delta_{\{i\}}(Y^B)$ (given by the identity on $\{i\}$-components), and this map is an h-equivalence since all components of the homotopy cofibre, except the $\{i\}$-component, are contractible. Moreover, the composite $\mathrm{spr}^B(Y) \to \mathrm{hocofibre}\,(o) \to \delta_{\{i\}}(Y^B)$ is the map γ.

On application of Γ, we obtain the following diagram:

$$\begin{tikzcd}
\Gamma(Z) \arrow[r, "\Gamma(o)"] & \Gamma(\mathrm{spr}^B(Y)) \arrow[r, "p"] \arrow[d, equal] & \Gamma(\mathrm{hocofibre}(o)) \arrow[r, "\sim"] & \Gamma(\delta_{\{i\}}(Y^B)) \arrow[d, equal] \\
& \Gamma(\mathrm{spr}^B(Y)) \arrow[rr, "\Gamma(\gamma)"] & & \Gamma(\delta_{\{i\}}(Y^B))
\end{tikzcd}$$

Hence to show that $\Gamma(\gamma)$ is a weak equivalence, it suffices to prove that the map p is a weak equivalence. But the target of p is the homotopy cofibre of $\Gamma(o)$, hence it suffices to show $\Gamma(Z) \simeq *$. But this is clear since computing the homotopy cofibre of Z in i-direction results in a cube with contractible components. $\square$

Let $\Gamma \circ \widetilde{\mathrm{spr}}(Y)$ denote the presheaf $C \mapsto \Gamma \circ \widetilde{\mathrm{spr}}^C(Y)$, and recall the forgetful functor $V : \mathbf{pP}^N(G) \to G\text{-}kTop_*^{\langle N \rangle_0}$. Lemma 3.8.3 asserts that there is a canonical map $\Gamma(Y) \to \lim_{\leftarrow} \circ V(\Gamma \circ \widetilde{\mathrm{spr}}(Y))$. By passing to the adjoint map (Lemma 3.6.8) we obtain a natural map $\tau : \psi_0 \circ \Gamma(Y) \to \Gamma \circ \widetilde{\mathrm{spr}}(Y)$ and, by suspending twice, a natural transformation $\Sigma^2 \circ \psi_0 \circ \Gamma \to \sigma$.

We give a graphical representation of τ in the case $N = \langle 1 \rangle$:

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \Gamma(Y) \wedge M^{\{0\}} \arrow[d] & & * \arrow[r] \arrow[d] & \Gamma(\mathrm{spr}^{\{0\}}(Y)) \arrow[d] \\
\Gamma(Y) \wedge M^{\{1\}} \arrow[r] & \Gamma(Y) \wedge M^{\{0,1\}} \arrow[r, phantom, "\xrightarrow{\tau}"] & & \Gamma(\mathrm{spr}^{\{1\}}(Y)) \arrow[r] & \Gamma(\mathrm{spr}^{\{0,1\}}(Y))
\end{tikzcd}$$

The left square is the picture of $\psi_0 \circ \Gamma(Y)$, the right square symbolizes $\Gamma \circ \widetilde{\mathrm{spr}}(Y)$.

Lemma 3.8.7. *Suppose G is cofibrant as an object of $kTop_*$. Then the natural map $\Sigma^2 \circ \psi_0 \circ \Gamma(Y) \to \sigma(Y)$ constructed above is an $h_{\{0\}}$-equivalence for weakly cofibrant objects $Y \in \mathbf{P}^N(G)$.*

As an immediate consequence of this lemma and Lemma 3.8.5(1), we have:

Corollary 3.8.8. *Suppose G is cofibrant as an object of $kTop_*$. Then the two functors $\Sigma^2 \circ \psi_0 \circ \Gamma$ and Σ^{n+2}, restricted to weakly cofibrant objects, are connected by a chain of $h_{\{0\}}$-equivalences.*

Proof of Lemma 3.8.7. There is a canonical map $Y \wedge M^A \to \widetilde{\mathrm{spr}}^A(Y)$; its B-component is given by the composite of the structure maps $Y^B \to Y^{B \cup A}$ and the M^A-action on $Y^{B \cup A}$. The A-component of τ is the image of this map $Y \wedge M^A \to \widetilde{\mathrm{spr}}^A(Y)$ under Γ since $\psi_0 \circ \Gamma(Y)^A = \Gamma(Y) \wedge M^A \cong \Gamma(Y \wedge M^A)$. Let us agree to use the letter B as indexing set for the presheaves Y and $\widetilde{\mathrm{spr}}^A(Y)$, and we will write Γ_b to indicate that Γ belongs to this indexing set B. Similarly, let A denote the indexing set for the presheaves $\psi_0 \circ \Gamma_b(Y)$ and $\Gamma_b \circ \widetilde{\mathrm{spr}}(Y)$ with corresponding global sections functor Γ_a. Then the lemma asserts that

$$\Gamma_a \circ \Sigma^2(\tau) : \Gamma_a \circ \Sigma^2 \circ \psi_0 \circ \Gamma_b(Y) \to \Gamma_a \circ \sigma^N = \Gamma_a \circ \Sigma^2 \circ \Gamma_b \circ \widetilde{\mathrm{spr}}(Y)$$

is a weak homotopy equivalence.

Using the identification of τ^A from the beginning of the proof, we can combine both directions (A-direction and B-direction) into a "hypercube": the lemma is equivalent to the claim that the map of $N \amalg N$-cubes

$$\mathcal{O}_{\mathbf{P}^N} \tilde{\wedge} Y \to \widetilde{\mathrm{spr}}(Y) \tag{$*$}$$

becomes a weak equivalence after application of $\Sigma^2 \circ \Gamma$, where the source is the "external tensor product" of $\mathcal{O}_{\mathbf{P}^N}$ and Y, i.e., the $N \amalg N$-cube $A \amalg B \mapsto \mathcal{O}^A_{\mathbf{P}^N} \wedge Y^B$ (this is a diagram in G-$kTop_*$ because of the G-action of Y^B), and the target is the functor

$$\widetilde{\mathrm{spr}}(Y) : N \amalg N \to G\text{-}kTop_*, \quad A \amalg B(\widetilde{\mathrm{spr}}^A(Y))^B.$$

For $N = \langle 1 \rangle$, the source of the map $(*)$ is a four-dimensional cube. It has the following graphical representation:

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & * \arrow[d] \arrow[rr] & & * \arrow[r] \arrow[d] & M^{\{0\}} \wedge Y^{\{0\}} \arrow[d] \\
* \arrow[r] & * \arrow[d] & & M^{\{0\}} \wedge Y^{\{1\}} \arrow[r] & M^{\{0\}} \wedge Y^{\{0,1\}} \arrow[d] \\
* \arrow[r] \arrow[d] & M^{\{1\}} \wedge Y^{\{0\}} \arrow[d] \arrow[rr] & & * \arrow[r] \arrow[d] & M^{\{0,1\}} \wedge Y^{\{0\}} \arrow[d] \\
M^{\{1\}} \wedge Y^{\{1\}} \arrow[r] & M^{\{1\}} \wedge Y^{\{0,1\}} & & M^{\{0,1\}} \wedge Y^{\{1\}} \arrow[r] & M^{\{0,1\}} \wedge Y^{\{0,1\}}
\end{tikzcd}$$

Here the "B-direction" is encoded into the small squares, the A-direction is the large square. Similarly, the target of $(*)$ has the following picture:

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & * \arrow[d] \arrow[rr] & & * \arrow[r] \arrow[d] & Y^{\{0\} \cup \{0\}} \arrow[d] \\
* \arrow[r] & * \arrow[d] & & Y^{\{1\} \cup \{0\}} \arrow[r] & Y^{\{0,1\} \cup \{0\}} \arrow[d] \\
* \arrow[r] \arrow[d] & Y^{\{0\} \cup \{1\}} \arrow[d] \arrow[rr] & & * \arrow[r] \arrow[d] & Y^{\{0\} \cup \{0,1\}} \arrow[d] \\
Y^{\{1\} \cup \{1\}} \arrow[r] & Y^{\{0,1\} \cup \{1\}} & & Y^{\{1\} \cup \{0,1\}} \arrow[r] & Y^{\{0,1\} \cup \{0,1\}}
\end{tikzcd}$$

In informal language, the result of computing Γ "in B-direction" (i.e., evaluating Γ_b) is the map τ^A. But computing Γ_b first and then applying Γ_a is the same (up to natural isomorphism) as applying Γ to the "hypercube", which in turn is the same as first computing Γ_a, then applying Γ_b. Symbolically, this change of preference is depicted as follows for the source of $(*)$:

$$\begin{tikzcd}
* \arrow[r] \arrow[dd] & * \arrow[dd] & & & * \arrow[r] \arrow[dd] & M^{\{0\}} \wedge Y^{\{0\}} \arrow[dd] \\
& & {} \arrow[r] & {} & & \\
* \arrow[r] & * & & & M^{\{1\}} \wedge Y^{\{0\}} \arrow[r] & M^{\{0,1\}} \wedge Y^{\{0\}} \\
{} \arrow[d] & & & & {} \arrow[d] & \\
{} & & & & {} & \\
* \arrow[r] \arrow[dd] & M^{\{0\}} \wedge Y^{\{1\}} \arrow[dd] & & & * \arrow[r] \arrow[dd] & M^{\{0\}} \wedge Y^{\{0,1\}} \arrow[dd] \\
& & {} \arrow[r] & {} & & \\
M^{\{1\}} \wedge Y^{\{1\}} \arrow[r] & M^{\{0,1\}} \wedge Y^{\{1\}} & & & M^{\{1\}} \wedge Y^{\{0,1\}} \arrow[r] & M^{\{0,1\}} \wedge Y^{\{0,1\}}
\end{tikzcd}$$

This is really the same cubical diagram as before. The right upper small square is $\mathcal{O}_{\mathbf{P}^1} \wedge Y^{\{0\}}$, and similarly for the other small squares. There is a similar picture for the target of $(*)$:

$$\begin{tikzcd}
* \arrow[r] \arrow[dd] & * \arrow[dd] & & & * \arrow[r] \arrow[dd] & Y^{\{0\}\cup\{0\}} \arrow[dd] \\
& & {} \arrow[r] & {} & & \\
* \arrow[r] & * & & & Y^{\{0\}\cup\{1\}} \arrow[r] & Y^{\{0\}\cup\{0,1\}} \\
{} \arrow[d] & & & & {} \arrow[d] & \\
{} & & & & {} & \\
* \arrow[r] \arrow[dd] & Y^{\{1\}\cup\{0\}} \arrow[dd] & & & * \arrow[r] \arrow[dd] & Y^{\{0,1\}\cup\{0\}} \arrow[dd] \\
& & {} \arrow[r] & {} & & \\
Y^{\{1\}\cup\{1\}} \arrow[r] & Y^{\{1\}\cup\{0,1\}} & & & Y^{\{0,1\}\cup\{1\}} \arrow[r] & Y^{\{0,1\}\cup\{0,1\}}
\end{tikzcd}$$

Computing "in A-direction", i.e., evaluating Γ_a, results in a map of N-cubes. Its B-component is given by

$$\beta : \Gamma_a(\mathcal{O}_{\mathbf{P}^N} \wedge Y^B) \to \Gamma_a(\widetilde{\mathrm{spr}}(Y)^B).$$

We want to show that this map is a weak homotopy equivalence after two suspensions (for all non-empty $B \subseteq N$). Then application of $\Sigma^2 \circ \Gamma_b$ gives a weak homotopy equivalence (Corollary 2.2.3(2)), hence the double suspension of the map $(*)$ is an $h_{\{0\}}$-equivalence as claimed.

Now $\widetilde{\mathrm{spr}}(Y)^B$ is the same as $\mathrm{spr}^B(Y)$ which can be seen by tracing the definitions. The source of β is isomorphic to $\Gamma(\mathcal{O}_{\mathbf{P}^N}) \wedge Y^B$. Choose some $i \in B$. The map γ from Lemma 3.8.6 is an $h_{\{0\}}$-equivalence. Since weak homotopy equivalences satisfy the saturation axiom, it suffices to show that $\Gamma(\gamma) \circ \beta$ is a weak homotopy equivalence.

Define the map $\alpha : \mathcal{O}_{\mathbf{P}^N} \to \delta_{\{i\}}(S^0)$ by mapping all non-basepoints of $M^{\{i\}}$ into the non-basepoint of S^0. Picture for $N = \langle 1 \rangle$ and $i = 1$:

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & M^{\{0\}} \arrow[d] & {} \arrow[r, "\alpha"] & {} & * \arrow[r] \arrow[d] & * \arrow[d] \\
M^{\{1\}} \arrow[r] & M^{\{0,1\}} & & & S^0 \arrow[r] & *
\end{tikzcd}$$

Tracing the definitions shows that $\Gamma(\alpha \wedge \mathrm{id}_{Y^B}) \cong \Gamma(\gamma) \circ \beta$. Since Y^B is cofibrant as a G-space and hence as an object of $kTop_*$ (Proposition 2.1.1(3)), and since $\Gamma(\alpha \wedge \mathrm{id}) \cong \Gamma(\alpha) \wedge \mathrm{id}$, it suffices to show that $\Sigma^2(\alpha)$ is an $h_{\{0\}}$-equivalence.

Recall the cube $\mathscr{W} := \mathscr{W}_N(0)$ from Lemma 3.7.3. There is a map

$$\omega : \mathscr{W} \to \delta_\emptyset(S^0) \vee \delta_{\{i\}}(S^0)$$

defined similar to α and fitting into the following commutative diagram:

$$\begin{tikzcd}
\mathscr{O}_{\mathbf{P}^N} \arrow[r] \arrow[d, "\alpha"'] & \mathscr{W} \arrow[r] \arrow[d, "\omega"'] & \delta_\emptyset(S^0) \arrow[d, "\mathrm{id}"] \\
\delta_{\{i\}}(S^0) \arrow[r] & \delta_\emptyset(S^0) \vee \delta_{\{i\}}(S^0) \arrow[r] & \delta_\emptyset(S^0)
\end{tikzcd}$$

Note that the two horizontal maps on the left are f-cofibrations of objects in $kTop_*^{\langle N \rangle}$, and the two cubes on the right are the cofibres of these (strict cofibres). Hence by Corollary 2.2.3(1) application of $\Sigma^2 \circ \Gamma$ yields a map of two cofibre sequences:

$$\begin{tikzcd}
\Sigma^2 \circ \Gamma(\mathscr{O}_{\mathbf{P}^N}) \arrow[r] \arrow[d, "\Sigma^2 \circ \Gamma(\alpha)"'] & \Sigma^2 \circ \Gamma(\mathscr{W}) \arrow[r] \arrow[d, "\Sigma^2 \circ \Gamma(\omega)"'] & \Sigma^2 \circ \Gamma(\delta_\emptyset(S^0)) \arrow[d, "\mathrm{id}"] \\
\Sigma^2 \circ \Gamma(\delta_{\{i\}}(S^0)) \arrow[r] & \Sigma^2 \circ \Gamma(\delta_\emptyset(S^0) \vee \delta_{\{i\}}(S^0)) \arrow[r] & \Sigma^2 \circ \Gamma(\delta_\emptyset(S^0))
\end{tikzcd}$$

The two spaces in the middle are contractible (by Lemma 3.7.3 for the upper space, by computing Γ in i-direction for the lower space). Consequently, the vertical map in the middle induces an isomorphisms on homology groups. By the five lemma (applied to the long exact sequence of homology groups) this is true also for $\Sigma^2 \circ \Gamma(\alpha)$ which proves that $\Sigma^2(\alpha)$ is an $h_{\{0\}}$-equivalence. □

4. Algebraic *K*-theory

4.1. Finiteness in projective space

We define different finiteness conditions and show that they are well behaved in the sense that weak equivalences and formation of pushouts preserve finiteness. The main tool is the existence of the two model structures on $\mathbf{pP}^N(G)$ (cf. Section 3.4).

For the rest of the paper, we assume that N is a non-empty finite set with $n+1$ elements, and that G is a topological monoid with zero which is cofibrant as an object of $kTop_*$.

Definition 4.1.1. An object Y of $\mathbf{P}^N(G)$ is called *locally finite* if all its components Y^A are finite (and hence cofibrant) in their respective categories, i.e., if $Y^A \in \mathbf{C}_f(G \wedge M^A)$ for all non-empty $A \subseteq N$. The full subcategory of these objects is denoted by $\mathbf{P}^N_{\mathrm{f}}(G)$. An object $Y \in \mathbf{P}^N(G)$ is called *homotopy finite* if all its components are homotopy finite in their respective categories. We say that Y is *finitely dominated* if Y is a retract of a homotopy finite sheaf. Finally, we call Y *stably finitely dominated* if $\Sigma^k Y$ is finitely dominated for some $k \geqslant 0$. The full subcategories of locally cofibrant objects which are homotopy finite, finitely dominated and stably finitely dominated will, respectively, be denoted by $\mathbf{P}^N_{\mathrm{hf}}(G)$, $\mathbf{P}^N_{\mathrm{fd}}(G)$ and $\mathbf{P}^N_{\mathrm{sfd}}(G)$.

We will also need the full subcategory $\check{\mathbf{P}}^N_{\mathrm{f}}(G)$ of objects which are locally finite and strongly cofibrant. Finally, let $\bar{\mathbf{P}}^N_{\mathrm{sfd}}(G)$ denote the full subcategory of stably finitely dominated, weakly cofibrant objects.

We have defined a sequence

$$\check{\mathbf{P}}^N_{\mathrm{f}}(G) \subseteq \mathbf{P}^N_{\mathrm{f}}(G) \subseteq \mathbf{P}^N_{\mathrm{hf}}(G) \subseteq \mathbf{P}^N_{\mathrm{fd}}(G) \subseteq \mathbf{P}^N_{\mathrm{sfd}}(G) \subseteq \bar{\mathbf{P}}^N_{\mathrm{sfd}}(G)$$

of full subcategories of $\mathbf{pP}^N(G)$. It is the purpose of this paper to define and study the algebraic K-theory of $\mathbf{P}^N_{\mathrm{sfd}}(G)$. We will indicate how it compares to alternative definitions of the K-theory of projective space.

Lemma 4.1.2. *All the following conditions are equivalent for a locally cofibrant object $Y \in \mathbf{P}^N(G)$:*
(1) *For all $i \in N$, the component $Y^{\{i\}}$ is an object of $\mathbf{C}_{\mathrm{hf}}(G \wedge M^{\{i\}})$.*
(2) *There is a locally finite, strongly cofibrant object $Z \in \mathbf{P}^N(G)$ and a weak equivalence $Z \to Y$.*
(3) *There is a locally finite $Z \in \mathbf{P}^N(G)$ and a chain of weak equivalences in $\mathbf{pP}^N(G)$ connecting Y and Z.*
(4) *Y is homotopy finite.*

Proof. Condition (1) is a special case of (4). Conversely, assume (1) holds. Fix a non-empty subset $A \subseteq N$, and choose $i \in A$. By hypothesis $Y^{\{i\}} \in \mathbf{C}_{\mathrm{hf}}(G \wedge M^{\{i\}})$. Since Y is a sheaf, we know that Y^A and $Y^{\{i\}}[t_A^{-1}]$ are weakly equivalent (Corollary 3.3.4). But the latter space is homotopy finite since inverting the action of t_A preserves homotopy finiteness by Lemma 2.1.2(3). This holds for all A, hence Y is homotopy finite.

Condition (3) is a special case of (2). Moreover, (3) implies (4) since an object of $\mathbf{C}(G \wedge M^A)$ is homotopy finite if it is connected through a chain of weak equivalences in $(G \wedge M^A)$-$kTop_*$ to a homotopy finite object.

It remains to prove (4) $\Rightarrow$ (2). So assume (4) holds. By the Whitehead theorem [2, 4.24], there are spaces $\bar{Z}^A \in \mathbf{C}_{\mathrm{f}}(G \wedge M^A)$ and mutually inverse (M^A-equivariant) homotopy equivalences (in the strong sense) $\alpha^A : Y^A \to \bar{Z}^A$ and $\bar{\beta}^A : \bar{Z}^A \to Y^A$. Define $Z^{\{j\}} := \bar{Z}^{\{j\}}$ and $\beta^{\{j\}} := \bar{\beta}^{\{j\}}$ for all $j \in N$.

Let $A \subseteq N$, and suppose that by induction Z^B and β^B have already been constructed for all proper subsets $B \subset A$. Define a map $L_A Z \to \bar{Z}^A$ as the composite

$$L_A Z \longrightarrow L_A Y \longrightarrow Y^A \xrightarrow{\alpha^A} \bar{Z}^A,$$

where L_A denotes the Ath latching space functor (Section 3.4—this makes sense because the definition of $L_A Z$ involves only those spaces Z^B for proper non-empty subsets $B \subset A$). By choice of α^A and $\bar{\beta}^A$, the two maps $L_A Z \to L_A Y \to Y^A$ and $L_A Z \to \bar{Z}^A \xrightarrow{\bar{\beta}^A} Y^A$ are homotopic. Let Z^A denote the mapping cylinder of $L_A Z \to \bar{Z}$. Then a choice of a homotopy determines a weak equivalence $\beta^A : Z^A \to Y^A$, homotopic to $\bar{\beta}^A$, making the following diagram commute:

$$\begin{array}{ccc} L_A Z & \longrightarrow & Z^A \\ \downarrow & & \downarrow \beta^A \\ L_A Y & \longrightarrow & Y^A \end{array} \qquad (*)$$

We claim that $L_A Z$ (and consequently Z^A) is a finite cofibrant $(G \wedge M^A)$-space. Choose $j \in A$. By Lemma 3.4.8(3), we have a pushout square

$$\begin{array}{ccc} L_{A\setminus\{j\}} Z[t_j^{-1}] & \longrightarrow & Z^{A\setminus\{j\}}[t_j^{-1}] \\ \downarrow & & \downarrow \\ L_A^{+\{j\}} Z & \longrightarrow & L_A Z \end{array}$$

where the upper horizontal map is, by induction, a cofibration between finite spaces. Hence, it suffices to show that the restricted latching space $L_A^{+\{j\}} Z$ is a finite cofibrant $(G \wedge M^A)$-equivariant space. This can be shown by induction (note that the restricted latching space is the latching space of the jth face of Z, i.e., of a lower-dimensional cube).

The spaces Z^A assemble to a presheaf Z; $\flat$-type structure maps are the composites $Z^A \xrightarrow{s_0} Z^A[t_B^{-1}] \to L_B Z \to Z^B$ for the inclusion $A \subset B$, where s_0 denotes the unit of the adjunction of $\cdot \wedge_{M^A} M^B$ and the restriction functor from $(G \wedge M^B)$-$kTop_*$ to $(G \wedge M^A)$-$kTop_*$ (Lemma 2.1.2(1)). Since Z maps to Y via the weak equivalence β (with components β^A constructed above) it is a sheaf (Corollary 3.3.5). It is locally finite since its components Z^A are mapping cylinders of maps between finite spaces. Finally, Z is strongly cofibrant since the maps $L_A Z \to Z^A$ are inclusions into a mapping cylinder by construction of Z^A, hence are cofibrations. □

Lemma 4.1.3. *Suppose $\bar{Y}$ is a finitely dominated sheaf, and suppose that there is a strongly cofibrant sheaf Y and a weak equivalence $\pi_Y : Y \to \bar{Y}$. Then there exists a strongly cofibrant, homotopy finite sheaf Q such that Y is a retract of Q.*

Proof. By hypothesis, $\bar{Y}$ is a retract of a homotopy finite object $\bar{Q}$. Call retraction and section $\bar{r}$ and $\bar{s}$, respectively. Choose a strongly cofibrant $\tilde{Q}$ and a weak equivalence $\pi_Q : \tilde{Q} \to \bar{Q}$. The object $\tilde{Q}$ is homotopy finite since it maps to $\bar{Q}$ via a weak equivalence.

By 5.8 of [2] we can form $\pi_Y^{-1} \circ \bar{r} \circ \pi_Q$ in $Ho\mathbf{pP}^N(G)$, and since Y is c-fibrant we can represent this composite by a map $\tilde{r} : \tilde{Q} \to Y$ in $\mathbf{pP}^N(G)$ [2, 5.11]. Similarly, we can represent $\pi_Q^{-1} \circ \bar{s} \circ \pi_Y$ by a morphism $\tilde{s} : Y \to \tilde{Q}$. Since $\bar{r} \circ \bar{s} = \mathrm{id}_{\bar{Y}}$, we know that $\tilde{r} \circ \tilde{s} = \mathrm{id}_Y$ in $Ho\mathbf{pP}^N(G)$, i.e., the maps $\tilde{r} \circ \tilde{s}$ and id_Y are homotopic. Let Q denote the mapping cylinder of $\tilde{s}$. A choice of homotopy yields a map $Q \to Y$ with section given by the inclusion $Y \to Q$. Moreover, Q is homotopy finite since $\tilde{Q}$ is. □

Lemma 4.1.4. *The classes of homotopy finite, finitely dominated and weakly cofibrant stably finitely dominated objects of $\mathbf{P}^N(G)$ are closed under weak equivalences.*

Proof. For homotopy finite objects this holds by definition of homotopy finiteness. Suppose Z is connected by a chain of weak equivalences to the finitely dominated sheaf $\bar{Y}$. Choose a strongly cofibrant sheaf Y and a weak equivalence $\pi_Y : Y \to \bar{Y}$. According to Lemma 4.1.3 there exists a strongly cofibrant, homotopy finite sheaf Q such that Y is a retract of Q. Call section and retraction s and r, respectively.

The objects Z and Y are connected by a chain of weak equivalences. Since Z is strongly cofibrant and Y is fibrant with respect to the c-structure, this chain induces a weak equivalence $\mu : Y \to Z$ in $\mathbf{pP}^N(G)$.

Factor $s : Y \to Q$ as a c-cofibration $Y \to P$ followed by a weak equivalence $P \to Q$, and observe that P is homotopy finite since Q is. There is a map

$$\begin{tikzcd}
Z \arrow[d] & & & Y \arrow[lll, "\sim", "\mu"'] \arrow[dll] \arrow[dl, "s"] \arrow[dd, "\mathrm{id}_Y"] \\
Z \cup_Y P \arrow[d, dashed] & P \arrow[l, "\sim"'] \arrow[r, "\sim"'] & Q \arrow[dr, "r"] & \\
Z & & & Y \arrow[lll, "\sim", "\mu"']
\end{tikzcd}$$

$Z \cup_Y P \to Z$ induced by $\mathrm{id} : Z \to Z$ and the composite $P \to Q \xrightarrow{r} Y \xrightarrow{\mu} Z$; this map has a section given by the canonical map $Z \to Z \cup_Y P$. Hence Z is a retract of $Z \cup_Y P$. But by the gluing lemma (which is valid for cofibrant objects) the map $P \to Z \cup_Y P$ is a weak equivalence whence the pushout is homotopy finite. This shows that Z is finitely dominated as claimed.

Now assume Z is weakly cofibrant and weakly equivalent to a weakly cofibrant, stably finitely dominated object Y through a chain of weak equivalences. Choose a c-cofibrant object $\bar{Y}$ together with a weak equivalence $\bar{Y} \xrightarrow{\sim} Y$. Since $\bar{Y}$ is weakly equivalent to Z, we find a weak equivalence $\bar{Y} \to Z$. We have constructed a chain of weak equivalences $Y \leftarrow \bar{Y} \to Z$. All three spaces are weakly cofibrant, hence $\Sigma^k Y \leftarrow \Sigma^k \bar{Y} \to \Sigma^k Z$ is a chain of weak equivalences. But if k is large, the object on the left is finitely dominated by hypothesis, and the previous case asserts that $\Sigma^k(Z)$ is finitely dominated. □

Lemma 4.1.5. (1) *Let ? denote any of the subscripts* f, hf, fd *or* sfd. *Suppose* $Z \xleftarrow{\alpha} Y \xrightarrow{\gamma} P$ *is a diagram in* $\mathbf{P}^N_?(G)$, *and the map* $Y \to P$ *is an f-cofibration* (*Proposition* 3.4.4). *Then the pushout exists in* $\mathbf{P}^N_?(G)$, *and the map* $Z \to Z \cup_Y P$ *is an f-cofibration.*

(2) *If* $Z \leftarrow Y \to P$ *is a diagram in* $\bar{\mathbf{P}}^N_{\mathrm{sfd}}(G)$ *and the map* $Y \to P$ *is a weak cofibration* (*Definition* 3.4.5), *then the pushout exists in* $\bar{\mathbf{P}}^N_{\mathrm{sfd}}(G)$, *and the map* $Z \to Z \cup_Y P$ *is a weak cofibration.*

Proof. We prove (1) only, the other assertion being similar.

In all cases we can form the pushout in the ambient category $\mathbf{pP}^N(G)$. Since the class of cofibrations in $(G \wedge M^A)$-$kTop_*$ is closed under cobase changes, we know that the map $Z \to Z \cup_Y P$ is an f-cofibration; in particular, the pushout is locally cofibrant.

We have to show that the result satisfies the sheaf condition. By Corollary 3.3.4 it suffices to show that for all inclusions $\sigma : A \to B$ of non-empty subsets of N, the map

$$(Z \cup_Y P)^\sharp_\sigma : (Z \cup_Y P)^A[t_B^{-1}] \to (Z \cup_Y P)^B \qquad (*)$$

is a weak equivalence. But since pushouts are calculated pointwise in $\mathbf{pP}^N(G)$ and $\cdot[t_B^{-1}]$ commutes with pushouts, it is isomorphic to the map induced on pushouts of top and bottom row

$$\begin{tikzcd} Z^A[t_B^{-1}] \arrow[d, "Z^\sharp_\sigma"'] & Y^A[t_B^{-1}] \arrow[l] \arrow[r] \arrow[d, "Y^\sharp_\sigma"'] & P^A[t_B^{-1}] \arrow[d, "P^\sharp_\sigma"] \\ Z^B & Y^B \arrow[l] \arrow[r] & P^B \end{tikzcd}$$

where the two horizontal maps on the right are cofibrations since $\cdot[t_A^{-1}]$ preserves cofibrations by Lemma 2.1.2(2). All vertical maps are weak equivalences since Z, Y and P are sheaves. Hence the gluing lemma (in $(G \wedge M^B)$-$kTop_*$) asserts that $(*)$ is indeed a weak equivalence.

Standard model category arguments similar to those exhibited in the previous lemmas show that $Z \cup_Y P$ satisfies the correct finiteness condition. We omit the details. $\square$

4.2. Finiteness of global sections

Definition 4.2.1. For $j \in \mathbf{Z}$ and $k \in \mathbf{N}$, a *j-twisted k-cell* is the sheaf $\psi_j(\Delta^k_+ \wedge G)$; its boundary is given by $\psi_j(\partial\Delta^k_+ \wedge G)$ with the obvious inclusion into the cell. A sheaf $Y \in \mathbf{P}^N(G)$ is *globally finite* if it can be obtained from the zero sheaf by attaching finitely many twisted cells (not necessarily in order of increasing dimension).

Remark 4.2.2. Suppose $Y \in \mathbf{P}^N(G)$ is globally finite, and let $\sigma : A \to B$ denote an inclusion of non-empty subsets of N.
(1) The sheaf Y is locally finite and locally cofibrant.
(2) The structure maps $Y^\flat_\sigma : Y^A \to Y^B$ are injective.
(3) The structure maps $Y^\sharp_\sigma : Y^A[t_B^{-1}] \to Y^B$ are isomorphisms.
(4) If Y is globally finite, so are $\theta_k(Y)$ and $\bar{\theta}_k(Y)$.

The following lemma is implicit in [4, 5.2].

Lemma 4.2.3. *If $Y \in \mathbf{P}^N(G)$ is a globally finite sheaf, the space $\Gamma(Y)$ is stably finitely dominated as a G-space.*

Proof. For $Y = *$ we have nothing to prove. Now assume Y is obtained from Z by attaching a cell, i.e., $Y = Z \cup_{\psi_j(\partial\Delta^k_+ \wedge G)} \psi_j(\Delta^k_+ \wedge G)$ for some $j \in \mathbf{Z}$ and $k \geqslant 0$. Assume by induction that $\Gamma(Z)$ is a stably finitely dominated space. Since Γ commutes with pushouts (Corollary 2.2.3(3)) we know that the canonical map

$$\Gamma(Z) \cup_{\Gamma(\psi_j(\partial\Delta^k_+ \wedge G))} \Gamma(\psi_j(\Delta^k_+ \wedge G)) \to \Gamma(Y)$$

is an isomorphism. All spaces on the left are known to be stably finitely dominated (use the induction hypothesis for $\Gamma(Z)$, 3.7.4 and 3.7.5 for the other two spaces), and the map $\Gamma(\psi_j(\partial\Delta^k_+ \wedge G)) \to \Gamma(\psi_j(\Delta^k_+ \wedge G))$ is a cofibration since the functor $\cdot \wedge G : kTop_* \to G\text{-}kTop_*$ maps cofibrations in $kTop_*$ to cofibrations in $G\text{-}kTop_*$ (apply Lemma 2.1.2(2)), ψ_j maps cofibrations to f-cofibrations (apply Lemma 2.1.2(2) componentwise), and Γ preserves cofibrations by Corollary 2.2.3(1). Hence, the pushout is stably finitely dominated. □

Lemma 4.2.4 (Extending coherent sheaves to $\mathbf{P}^N$; Hüttemann et al. [4, 3.4]). *Given a finite space $T \in \mathbf{C}(G \wedge M^{\{n\}})$, there exists a globally finite sheaf $Y \in \mathbf{P}^n(G)$ with $Y^{\{n\}} = T$.*

Proof. We proceed by induction on the number of cells of T. For $T = *$ choose $Y = *$. Now assume $T = S \cup_{\partial C} C$ where $C = \Delta^\ell_+ \wedge G \wedge M^{\{n\}}$ is an equivariant cell. By induction we have a globally finite sheaf $Z \in \mathbf{P}^N$ with $Z^{\{n\}} = S$.

The boundary of C has a compact subspace $\partial C_0 := \partial\Delta^\ell_+ \wedge \{0,1\} \wedge \{0\}_+$. Let $\alpha^{\{n\}} : \partial C_0 \to Z^{\{n\}}$ denote the restriction of the attaching map. Fix $i \in [n] \setminus \{n\}$. By Remark 4.2.2(3) the maps $Z^{\{i\}}[t_n^{-1}] \to Z^{\{i,n\}}$ are isomorphisms. Identify the target with the colimit of the (second) telescope construction (Lemma 3.2.6(2)) for inverting the action of t_n on $Z^{\{i\}}$. Since all maps in the telescope are cofibrations in $kTop_*$ by Lemma 2.1.3 and Proposition 2.1.1(3), the composite

$$\partial C_0 \xrightarrow{\alpha^{\{n\}}} Z^{\{n\}} \longrightarrow Z^{\{i,n\}}$$

factors as $\partial C_0 \xrightarrow{\alpha^{\{i\}}} Z^{\{i\}} \xrightarrow{t_i^{k_i} t_n^{-k_i}} Z^{\{i,n\}}$ for some $k_i \geqslant 0$ (smallness of compact spaces; this is similar to Lemma 3.2.7). Since all k_i can be enlarged, we may assume that all k_i have the same value $k \geqslant 0$.

We claim that for all $A \subseteq [n]$ and $i,j \in A$ the diagram

$$\begin{tikzcd} \partial C_0 \arrow[r, "\alpha^{\{i\}}"] \arrow[d, "\alpha^{\{j\}}"'] & Z^{\{i\}} \arrow[d, "t_i^k t_a^{-k}"] \\ Z^{\{j\}} \arrow[r, "t_j^k t_a^{-k}"'] & Z^A \end{tikzcd} \qquad (*)$$

commutes where $a := \max(A)$; lower and right maps are structure maps in $\bar{\theta}_k(Z)$ (cf. Definition 3.6.3). To see this it is sufficient to show that both maps from upper left to lower right are coequalized by the map $Z^A \xrightarrow{t_a^k t_n^{-k}} Z^{[n]}$ since the latter is injective by Remark 4.2.2(2) and (4). So consider the following diagram:

$$\begin{tikzcd} \partial C_0 \arrow[r, "\alpha^{\{i\}}"] \arrow[d, "\alpha^{\{n\}}"'] & Z^{\{i\}} \arrow[r, "t_i^k t_a^{-k}"] \arrow[d, "t_i^k t_n^{-k}"'] & Z^A \arrow[d, "t_a^k t_n^{-k}"] \\ Z^{\{n\}} \arrow[r] & Z^{\{i,n\}} \arrow[r] & Z^{[n]} \end{tikzcd}$$

The left square commutes by construction of $\alpha^{\{i\}}$, the square on the right commutes since it consists of structure maps of $\bar{\theta}_k(Z)$. The composition of the lower maps is the structure map $Z^{\{n\}} \to Z^{[n]}$. Thus the composite $\partial C_0 \to Z^{[n]}$ from upper left to lower right is independent of i which proves commutativity of $(*)$.

Define $\alpha^A : \partial C_0 \to Z^A$ as the unique map from upper left to lower right in diagram $(*)$. By virtually the same argument as before (map everything into $Z^{[n]}$) we can show that the α^A are compatible with the structure maps of $\bar{\theta}_k(Z)$, i.e., they assemble to a map $\partial C_0 \to \lim_{\leftarrow} V(\bar{\theta}_k(Z))$, where V denotes the forgetful functor $\mathbf{pP}^N(G) \to G\text{-}kTop_*^{\langle N \rangle_0}$. Hence we obtain, by passing to the adjoint map, a morphism $f : \psi_{-k}(\partial C_0) \to Z$ (note that Lemma 3.6.8 applies since θ_k and $\bar{\theta}_k$ are isomorphic functors, cf. Lemma 3.6.4(3)), and by construction $f^{\{n\}}$ coincides with the given attaching map in T. Define Y by attaching an ℓ-cell with twist $-k$ to Z. Then Y is globally finite. On $\{n\}$-components, attaching the twisted cell amounts to attaching a free $(G \wedge M^{\{n\}})$-eqivariant cell to $Z^{\{n\}} = S$ along the given attaching map, hence $Y^{\{n\}} = T$. $\square$

Lemma 4.2.5 (Extending morphisms of sheaves; Hüttemann et al. [4, 3.5]). *Let Y and Z be objects of $\mathbf{P}^n(G)$. Suppose Z is globally finite, and suppose Y is locally finite and strongly cofibrant. Let $g : Y^{\{n\}} \to Z^{\{n\}}$ be a given $(G \wedge M^{\{n\}})$-equivariant map. Then there exists an integer $k \geqslant 0$ and a morphism $f : Y \to \bar{\theta}_k(Z)$ with $f^{\{n\}} = g$.*

Proof. The proof consists of three steps: first we construct maps $f^A : Y^A \to Z^A$ for $A \subseteq [n]$ with $n \in A$ which are compatible with the structure maps of Y and Z. (In other words: if Y and Z are considered as cubical diagrams, we construct a natural transformation from the n-face of Y to the n-face of Z.) The second step is to extend this partial map to a map of $(n+1)$-cubes (we have to twist Z to do this). In the last step, we check that the components f^A constructed before fit together.

Step 1: We proceed by induction on $s := \#A$. Start by defining $f^{\{n\}} := g$. If $A = \{j, n\}$, we claim that the map $Y^{\{n\}}[t_j^{-1}] \to Y^{\{j,n\}}$ is an acyclic cofibration. Indeed, by Lemma 3.4.8(4) we know that the map $L^{+\{n\}}_{\{j,n\}} Y \to Y^{\{j,n\}}$ is a cofibration. But $L^{+\{n\}}_{\{j,n\}} Y = Y^{\{n\}}[t_j^{-1}]$ by definition of restricted latching spaces, and the map is a $\sharp$-type structure map of Y. Since Y is a sheaf, it is a weak equivalence by Corollary 3.3.4. Hence, we can choose a (dotted) lift $f^{\{j,n\}}$ in the following diagram:

$$\begin{tikzcd}
Y^{\{n\}}[t_j^{-1}] \arrow[r, "f^{\{n\}}[t_j^{-1}]"] \arrow[d, "Y^{\sharp}_{\sigma}"'] & Z^{\{n\}}[t_j^{-1}] \arrow[r, "Z^{\sharp}_{\sigma}"] & Z^{\{j,n\}} \arrow[d] \\
Y^{\{j,n\}} \arrow[urr, dotted, "f^{\{j,n\}}"'] \arrow[rr] & & *
\end{tikzcd}$$

(Here σ denotes the inclusion $\{n\} \to \{j, n\}$.) This lift is compatible with the structure maps of Y and Z: by passing to adjoint maps, we obtain from the above diagram the following commutative square:

$$\begin{tikzcd}
Y^{\{n\}} \arrow[r, "f^{\{n\}}"] \arrow[d, "Y^{\flat}_{\sigma}"'] & Z^{\{n\}} \arrow[d, "Z^{\flat}_{\sigma}"] \\
Y^{\{j,n\}} \arrow[r, "f^{\{j,n\}}"] & Z^{\{j,n\}}
\end{tikzcd}$$

Now assume $s = \#A \geqslant 3$. We give a formulation of the induction hypothesis:

$(\mathrm{i})_s$ We have compatible maps f^B for all proper subsets B of A containing n, i.e., if σ denotes an inclusion $B \to C$ of proper subsets of A with $n \in B$, the following diagram commutes:

$$\begin{tikzcd} Y^B \arrow[r, "f^B"] \arrow[d, "Y^\flat_\sigma"'] & Z^B \arrow[d, "Z^\flat_\sigma"] \\ Y^C \arrow[r, "f^C"'] & Z^C \end{tikzcd}$$

$(\mathrm{ii})_s$ If B is a proper subset of C with $n \in B$, and C is a proper subset of A with $\#C = \#A - 1 = s - 1$, the canonical map $L_C^{+B} Y \to Y^C$ is an acyclic cofibration, and the source is cofibrant as an object of $(G \wedge M^C)$-$kTop_*$.

To start the induction, we still have to check condition $(\mathrm{ii})_3$, i.e., the case $\#A = 3$. But then necessarily $B = \{n\}$ and $C = \{j, n\}$ for some j, and we have checked above that the map $Y^{\{n\}}[t_j^{-1}] \to Y^{\{j,n\}}$ is an acyclic cofibration. Moreover, the source of this map is cofibrant in $(G \wedge M^{\{j,n\}})$-$kTop_*$ since $Y^{\{n\}}$ is cofibrant in $(G \wedge M^{\{n\}})$-$kTop_*$ (apply Lemma 2.1.2(2)).

Assume now that conditions $(\mathrm{i})_s$ and $(\mathrm{ii})_s$ are satisfied. Consider the following diagram (the restricted latching space functor $L_A^{+\{n\}}$ has been defined in Definition 3.4.7):

$$\begin{tikzcd} L_A^{+\{n\}} Y \arrow[r] \arrow[d] & L_A^{+\{n\}} Z \arrow[r] & Z^A \arrow[d] \\ Y^A \arrow[rr] \arrow[urr, dotted, "f^A"] & & * \end{tikzcd} \qquad (**)$$

The left arrow in the first row is induced by the maps f^B of $(\mathrm{i})_s$. The second map in the first row and the left vertical map are the canonical maps of Lemma 3.4.8(4). In particular, the left vertical map is a cofibration. To construct the dotted lift, it thus suffices to show that the left vertical map is a weak equivalence.

Choose a filtration of A

$$\{n\} = C_1 \subset C_2 \subset \cdots \subset C_{s-1} \subset C_s = A$$

with $\#C_i = i$. By iterated application of Lemma 3.4.8(2) we see that the map $L_A^{+C_{s-1}} Y \to Y^A$ factors as

$$L_A^{+C_{s-1}} Y \to L_A^{+C_{s-2}} Y \to \cdots \to L_A^{+C_1} Y \to Y^A. \qquad (*)$$

By Lemma 3.4.8(4) the last map is a cofibration. All the other maps in this factorization are acyclic cofibrations: fix i with $1 < i \leqslant s - 1$, and write $C_i = C_{i-1} \coprod \{j\}$. By induction hypothesis $(\mathrm{ii})_s$, the map $L_{A\backslash\{j\}}^{+C_{i-1}} Y \to Y^{A\backslash\{j\}}$ is an acyclic cofibration, hence so is $L_{A\backslash\{j\}}^{+C_{i-1}} Y[t_j^{-1}] \to Y^{A\backslash\{j\}}[t_j^{-1}]$ (Lemma 2.1.2(2) applies since the source is cofibrant by induction hypothesis). By Lemma 3.4.8(3) we have a pushout square

$$\begin{tikzcd} L^{+C_{i-1}}_{A\setminus\{j\}} Y[t_j^{-1}] \arrow[r] \arrow[d] & Y^{A\setminus\{j\}}[t_j^{-1}] \arrow[d] \\ L^{+C_i}_A Y \arrow[r] & L^{+C_{i-1}}_A Y \end{tikzcd}$$

which shows that the lower map (appearing also in $(*)$) is an acyclic cofibration.

Write $A = C_{s-1} \coprod \{j\}$. Then the leftmost space in $(*)$ is $L_A^{+C_{s-1}} Y = Y^{A\setminus\{j\}}[t_j^{-1}]$ by definition of restricted latching spaces. Since $Y^{A\setminus\{j\}}$ is cofibrant, this proves that all spaces appearing in $(*)$ are cofibrant. Moreover, the composite map $(*)$ is a $\sharp$-type structure map of Y, hence a weak equivalence. This shows that the rightmost map of $(*)$ is a weak equivalence.

The above arguments apply for all A with s elements, and for all filtrations. This proves $(\mathrm{ii})_{s+1}$.

We also have shown that the left vertical map in $(**)$ is an acyclic cofibration, hence the lift f^A exists. This map is compatible with the f^B constructed earlier since the structure maps of Y and Z are encoded in the restricted latching spaces. Hence $(\mathrm{i})_{s+1}$ holds.

Step 2: We commence now with the construction of maps f^A for $A \subseteq [n-1]$. Given such a set A, consider the composite map $Y^A \longrightarrow Y^{A\coprod\{n\}} \xrightarrow{f^{A\coprod\{n\}}} Z^{A\coprod\{n\}}$ where the second map is as constructed in step 1, and the first map is a $\flat$-type structure map of Y. Since Z is globally finite, we have $Z^{A\coprod\{n\}} \cong Z^A[t_n^{-1}]$, and finiteness of Y^A guarantees that we can factor this map as $Y^A \xrightarrow{f^A} Z^A \xrightarrow{t_m^{k_A}\cdot t_n^{-k_A}} Z^{A\coprod\{n\}}$ with $m = \max(A)$ and some $k_A \geqslant 0$ (apply Lemma 3.2.8). Since all k_A may be enlarged independently, we may assume that all k_A, for the various $A \subseteq [n-1]$, have the same value k.

Step 3: We claim that all the different maps f^A constructed in steps 1 and 2 assemble to a map of diagrams $Y \to \bar{\theta}_k(Z)$. So assume $B = A \coprod \{j\} \subseteq [n]$. We have to show that the square

$$\begin{tikzcd} Y^A \arrow[r, "f^A"] \arrow[d] & Z^A \arrow[d, "t_m^k t_\ell^{-k}"] \\ Y^B \arrow[r, "f^B"'] & Z^B \end{tikzcd} \qquad (*)$$

commutes (where $m = \max(A)$, $\ell = \max(B)$, and the right vertical arrow is the structure map of $\bar{\theta}_k(Z)$).

(a) If $n \in A$ this follows from the construction of the f^A in step 1 of this proof since $m = \ell$ in this case, and the right vertical map in $(*)$ coincides with the ($\flat$-type) structure map in Z.

(b) If $j = n$ (and, consequently, $\ell = n$) this follows from the construction of f^A in step 2.

(c) Now assume $n \notin B$, and consider the following cube:

$$\begin{tikzcd}
& Y^{A\amalg\{n\}} \arrow[rr, "f^{A\amalg\{n\}}"] \arrow[dd] & & Z^{A\amalg\{n\}} \arrow[dd] \\
Y^{A} \arrow[ur] \arrow[rr, "f^{A}"] \arrow[dd] & & Z^{A} \arrow[ur, "t_m^k t_n^{-k}"] \arrow[dd, "t_m^k t_\ell^{-k}"] & \\
& Y^{B\amalg\{n\}} \arrow[rr, "f^{B\amalg\{n\}}"] & & Z^{B\amalg\{n\}} \\
Y^{B} \arrow[ur] \arrow[rr, "f^{B}"] & & Z^{B} \arrow[ur, "t_\ell^k t_n^{-k}"'] &
\end{tikzcd}$$

The front face is diagram $(*)$. The back face of this cube is commutative by (a), for upper and lower face commutativity follows from (b). The left and right face commute by definition of the structure maps of Y and $\bar{\theta}_k(Z)$, respectively. This shows that the two composite maps from initial to terminal vertex, given by $Y^A \xrightarrow{f^A} Z^A \xrightarrow{t_m^k t_\ell^{-k}} Z^B \xrightarrow{t_\ell^k t_n^{-k}} Z^{B\amalg\{n\}}$ and $Y^A \to Y^B \xrightarrow{f^B} Z^B \xrightarrow{t_\ell^k t_n^{-k}} Z^{B\amalg\{n\}}$, are equal. But the last map in these compositions is injective since $\bar{\theta}_k(Z)$ is globally finite (Remark 4.2.2(2) and (4)). Hence the front face commutes as well. □

Proposition 4.2.6 (Finiteness of global sections; Hüttemann et al. [4, 5.2]). *The global sections functor Γ maps objects of $\bar{\mathbf{P}}^N_{\mathrm{sfd}}(G)$ to stably finitely dominated G-spaces.*

The proof will show that even for a locally finite sheaf Y, the space $\Gamma(Y)$ is, in general, only stably finitely dominated, not finitely dominated.

Proof of Proposition 4.2.6. Let P denote a weakly cofibrant object of $\mathbf{P}^N_{\mathrm{sfd}}(G)$. We can find a strongly cofibrant sheaf Z and a weak equivalence $Z \to P$. Since Z is stably finitely dominated (Lemma 4.1.4), we know that there exists $k \geqslant 0$ such that $\Sigma^k(Z)$ is a retract of a strongly cofibrant, homotopy finite sheaf Q (Lemma 4.1.3). By Lemma 4.1.2(2) we find a strongly cofibrant, locally finite sheaf Y and a weak equivalence $Y \to Q$. Now Γ preserves weak equivalences between weakly cofibrant objects (by Corollary 2.2.3(2)) and retractions (by functoriality), and commutes with suspensions. Hence it suffices to show that a strongly cofibrant, locally finite object Y is mapped to a stably finitely dominated space.

Assume $N = [n]$. We proceed by induction on the dimension; for $n = 0$, nothing has to be shown (since $\bar{\mathbf{P}}^0_{\mathrm{sfd}}(G) = \mathbf{C}_{\mathrm{sfd}}(G)$ and $\Gamma = \mathrm{id}$).

Now assume the statement is true for $n-1$, and let $Y \in \check{\mathbf{P}}^N_{\mathrm{f}}(G)$ be given. Choose a globally finite object Z with $Z^{\{n\}} = Y^{\{n\}}$ (Lemma 4.2.4). By Lemma 4.2.5, we can extend the identity map of $Y^{\{n\}}$ to a morphism $f : Y \to \bar{\theta}_k(Z)$ for some $k \geqslant 0$ such that $f^{\{n\}} = \mathrm{id}_{Y^{\{n\}}}$. For $A \subseteq [n]$ with $n \in A$, let $\sigma : \{n\} \to A$ denote the inclusion. Consider the following diagram:

$$\begin{tikzcd}
Y^{\{n\}}[t_A^{-1}] \arrow[r, "\mathrm{id}"] \arrow[d, "Y_\sigma^\sharp"'] & \bar{\theta}_k(Z)^{\{n\}}[t_A^{-1}] = Y^{\{n\}}[t_A^{-1}] \arrow[d, "\bar{\theta}(Z)_\sigma^\sharp"] \\
Y^A \arrow[r, "f^A"'] & \bar{\theta}_k(Z)^A
\end{tikzcd}$$

The vertical maps are the $\sharp$-type structure maps of Y and $\bar{\theta}_k(Z)$, respectively. They are weak equivalences since Y and $\bar{\theta}_k(Z)$ are locally cofibrant sheaves (Corollary 3.3.4). Hence the components f^A with $n \in A$ are weak equivalences.

Let C_f denote the (pointwise) mapping cone of f (i.e., the space C_f^A is given by the homotopy cofibre of the map f^A). Since Γ commutes with pushouts and preserves cofibrations, we have a cofibration sequence $\Gamma(Y) \xrightarrow{\Gamma(f)} \Gamma(\bar{\theta}_k(Z)) \to \Gamma(C_f)$. By Lemma 4.2.3 the space $\Gamma(\bar{\theta}_k(Z))$ is stably finitely dominated. Hence it suffices to show that the space on the right is stably finitely dominated. (By using a Puppe sequence type argument, one can show that if two out of three spaces in a cofibration sequence are stably finitely dominated, so is the third. We omit the details.)

Define $\bar{Y} \in \mathbf{P}^{n-1}(G)$ as the object $A \mapsto C_f^A$ where the spaces C_f^A are equipped with the restricted action (restriction along the inclusions $M_{n-1}^A \subset M_n^A$). (If we consider C_f as a diagram having the shape of an n-simplex, the presheaf $\bar{Y}$ is just the nth face of C_f.) There is an obvious map $o: C_f \to \zeta_n(\bar{Y})$ of diagrams in G-$kTop_*$ (not of sheaves) given by the identity and constant maps to the basepoint, respectively.

For $n = 2$, we have the following picture:

$$\begin{tikzcd}[column sep=tiny]
 & & C_f^{\{2\}} \arrow[dl] \arrow[dr] & & \\
 & C_f^{\{0,2\}} \arrow[dr] & & C_f^{\{1,2\}} \arrow[dl] & \\
 & & C_f^{\{0,1,2\}} & & \\
C_f^{\{0\}} \arrow[uur] \arrow[rr] & & C_f^{\{0,1\}} \arrow[u] & & C_f^{\{1\}} \arrow[ll] \arrow[uul]
\end{tikzcd}
\xrightarrow{\quad o \quad}
\begin{tikzcd}[column sep=tiny]
 & & * \arrow[dl] \arrow[dr] & & \\
 & * \arrow[dr] & & * \arrow[dl] & \\
 & & * & & \\
C_f^{\{0\}} \arrow[uur] \arrow[rr] & & C_f^{\{0,1\}} \arrow[u] & & C_f^{\{1\}} \arrow[ll] \arrow[uul]
\end{tikzcd}$$

This map is a weak equivalence in G-$kTop_*^{\langle N \rangle}$ by construction of f: it is given by identity maps (on A-components with $n \notin A$), or maps between contractible spaces (since f^A is a weak equivalence if $n \in A$ by the above). Moreover, $\bar{Y}$ is a sheaf since C_f is.

Caution: o is not a map in $\mathbf{pP}^N(G)$ since it fails to be equivariant with respect to the M_n^A-actions. For example, in the picture we have t_2 acting trivially on the spaces on the lower face in the target (by definition of extension by zero ζ_n), but acting non-trivially in the source; maps are identities on underlying spaces.

Recall that $\Gamma \circ \zeta_n(\bar{Y}) \cong \Sigma \circ \Gamma(\bar{Y})$ by Lemma 3.7.2. Hence the weak equivalence o shows that $\Gamma(C_f)$ is stably finitely dominated if $\bar{Y}$ is.

By its construction as the pointwise homotopy cofibre of a map between finite $(G \wedge M_n^A)$-spaces, we know that $\bar{Y}^A = (C_f)^A$ is a finite $(G \wedge M_n^A)$-space. Moreover, C_f is a sheaf. Hence if σ denotes the inclusion $A \to A \coprod \{n\}$, we have a weak equivalence

$$\bar{Y}^A[t_n^{-1}] = (C_f)^A[t_n^{-1}] \xrightarrow{(C_f)_\sigma^\sharp} (C_f)^{A \coprod \{n\}} \simeq *.$$

In this situation, we can apply Lemma 3.2.9 to conclude that for each $A \subseteq [n-1]$, the space $\Sigma\bar{Y}^A \vee \bar{Y}^A$ is homotopy finite as a $(G \wedge M_{n-1}^A)$-space, i.e., $\Sigma\bar{Y} \vee \bar{Y}$ is a homotopy finite object of $\mathbf{P}^{n-1}(G)$. This implies that the sheaf $\bar{Y} \in \mathbf{P}^{n-1}(G)$ (which is a retract of $\Sigma\bar{Y} \vee \bar{Y}$) is finitely dominated. In addition, $\bar{Y}$ is certainly weakly cofibrant since C_f is locally cofibrant as an object of $\mathbf{P}^n(G)$ (apply Proposition 2.1.1(2) to the components of $\bar{Y}$). Hence the induction hypothesis applies, asserting that $\Gamma(\bar{Y})$ is stably finitely dominated as claimed. □

4.3. K-theory structures

Lemma 4.3.1. *Let ? denote one of the subscripts* f, hf, fd *or* sfd.

(1) *The f-cofibrations* (Proposition 3.4.4) *and h-equivalences make* $\mathbf{P}_?^N(G)$ *into a category with cofibrations and weak equivalences satisfying the saturation axiom. The pointwise mapping cylinder construction equips* $\mathbf{P}_?^N(G)$ *with a cylinder functor satisfying the cylinder axiom.*

(2) *The c-cofibrations* (Proposition 3.4.1) *and h-equivalences make* $\check{\mathbf{P}}_\mathrm{f}^N(G)$ *into a category with cofibrations and weak equivalences satisfying the saturation axiom. The pointwise mapping cylinder construction equips* $\check{\mathbf{P}}_\mathrm{f}^N(G)$ *with a cylinder functor satisfying the cylinder axiom.*

Proof. (1) Axioms (Cof 1), (Cof 2) and (Weq 1) hold by definition, and (Cof 3) has been checked in Lemma 4.1.5(1). Axiom (Weq 2), the gluing lemma, follows from the fact that in any model category the gluing lemma is valid for cofibrant objects. The saturation axiom holds since $\mathbf{P}_?^N(G)$ is a full subcategory of a model category.

Concerning the cylinder functor the only non-trivial thing to verify is that the mapping cylinder construction respects finiteness. For $? = \mathrm{f}$, this follows from the fact that it preserves finiteness on each component. In all other cases, the mapping cylinder is weakly equivalent to the target of the map, hence is an object of $\mathbf{P}_?^N(G)$ by Lemma 4.1.4.

(2) This is similar to (1). □

Proposition 4.3.2 (Models for the K-theory of projective space; Hüttemann et al. [4, 3.3]).

(1) *The inclusion* $\check{\mathbf{P}}_\mathrm{f}^N(G) \to \mathbf{P}_\mathrm{f}^N(G)$ *induces an equivalence on* $\mathscr{S}_\bullet$*-constructions.*
(2) *The inclusion* $\mathbf{P}_\mathrm{f}^N(G) \to \mathbf{P}_\mathrm{hf}^N(G)$ *induces an equivalence on* $\mathscr{S}_\bullet$*-constructions.*
(3) *The inclusion* $\mathbf{P}_\mathrm{hf}^N(G) \to \mathbf{P}_\mathrm{fd}^N(G)$ *induces an isomorphism on all K-groups except possibly on* K_0.
(4) *The inclusion* $\mathbf{P}_\mathrm{fd}^N(G) \to \mathbf{P}_\mathrm{sfd}^N(G)$ *induces an equivalence on* $\mathscr{S}_\bullet$*-constructions.*

Proof. To prove (1) and (2) it suffices to check that the inclusions of $\check{\mathbf{P}}^N_{\mathrm{f}}(G)$ into $\mathbf{P}^N_{\mathrm{f}}(G)$ and $\mathbf{P}^N_{\mathrm{hf}}(G)$ have the approximation property [10, 1.6]. By definition both inclusions detect weak equivalences, i.e., satisfy (App 1).

Let ? denote the subscript f or hf, and let $f: Y \to Z$ be a map in $\mathbf{P}^N_?(G)$ where Y is an object of $\in \check{\mathbf{P}}^N_{\mathrm{f}}(G)$. By Lemma 4.1.2 there is an object $P \in \check{\mathbf{P}}^N_{\mathrm{f}}(G)$ and a weak equivalence $g: P \xrightarrow{\sim} Z$. We can form $g^{-1} \circ f$ in the homotopy category $Ho\mathbf{P}^N(G)$, and by Dwyer and Spalinski [2, Proposition 5.11] we find a map $h: Y \to P$ in $\mathbf{P}^N(G)$ representing it. Since $g \circ h$ and f have the same image in $Ho\mathbf{P}^N(G)$, we infer that $g \circ h$ and f are homotopic. A choice of a homotopy yields a factorization $Y \to Z_h \to Z$ of f where Z_h is the mapping cylinder of h. By construction Z_h is locally finite and strongly cofibrant and maps to Z via the weak equivalence $Z_h \xrightarrow{\sim} P \xrightarrow{\sim} Z$. This proves (App 2).

For (3), we follow the argument given in [4, 3.3(2)]: let Y be an object of $\mathbf{P}^N_{\mathrm{fd}}(G)$. Then Y is homotopy finite if and only if for all $i \in N$, the component $Y^{\{i\}}$ is a homotopy finite $(G \wedge M^{\{i\}})$-space (Lemma 4.1.2). We have maps of abelian groups

$$\pi_1|h\mathscr{S}_\bullet\mathbf{C}_{\mathrm{hf}}(G \wedge M^{\{i\}})| \to \pi_1|h\mathscr{S}_\bullet\mathbf{C}_{\mathrm{fd}}(G \wedge M^{\{i\}})|$$

with cokernel H_i. The space $Y^{\{i\}}$ gives rise to an element of $\pi_1|h\mathscr{S}_\bullet\mathbf{C}_{\mathrm{fd}}(G \wedge M^{\{i\}})|$ by the remarks in [10, p. 329], and $Y^{\{i\}}$ is homotopy finite if and only if the image of this element in H_i is zero. Thus the sheaf Y determines an element of $\prod_{i \in N} H_i$ which is zero if and only if Y is homotopy finite.

By the cofinality theorem of [9, 1.10.1], we obtain a fibration sequence

$$\Omega|h\mathscr{S}_\bullet\mathbf{P}^N_{\mathrm{hf}}(G)| \to \Omega|h\mathscr{S}_\bullet\mathbf{P}^N_{\mathrm{fd}}(G)| \to \prod_{i \in N} H_i,$$

where the abelian group on the right has the discrete topology. This proves (3).

For (4), let $\mathscr{C}_k$ denote the full subcategory of $\mathbf{P}^N_{\mathrm{sfd}}(G)$ of objects whose kth suspension is finitely dominated. Then $\mathbf{P}^N_{\mathrm{sfd}}(G) = \bigcup_0^\infty \mathscr{C}_k$ and $\mathscr{C}_0 = \mathbf{P}^N_{\mathrm{fd}}(G)$. We have inclusion maps $\mathscr{C}_k \to \mathscr{C}_{k+1}$ and suspensions $\mathscr{C}_{k+1} \xrightarrow{\Sigma} \mathscr{C}_k$. Since suspension (considered as an endofunctor) induces a self-equivalence of $\mathscr{S}_\bullet$-constructions by Waldhausen [10, 1.6.2], we know that the $\mathscr{S}_\bullet$-constructions of $\bigcup_0^\infty \mathscr{C}_k$ and $\mathscr{C}_0$ are homotopy equivalent. □

We have already introduced the notion of h_S-equivalences (Definition 3.8.1). The interplay with h-equivalences allows us to use the fibration theorem of [10]. Fix sets $T \subseteq S \subseteq \mathbf{Z}$.

Definition 4.3.3. We define $\mathbf{P}^{N,T}_{\mathrm{sfd}}(G)$ to be the full subcategory of $\mathbf{P}^N_{\mathrm{sfd}}(G)$ consisting of the objects Y such that $\Gamma \circ \theta_j(Y) \simeq *$ for all $j \in T$. Similarly, we define the category $\bar{\mathbf{P}}^{N,T}_{\mathrm{sfd}}(G)$ as the full subcategory of $\bar{\mathbf{P}}^N_{\mathrm{sfd}}(G)$ whose objects satisfy $\Gamma \circ \theta_j(Y) \simeq *$ for all $j \in T$.

Lemma 4.3.4. (1) *The axioms for a category with cofibrations and weak equivalences hold for* $\mathbf{P}^{N,T}_{\mathrm{sfd}}(G)$ *with respect to f-cofibrations and* h_S*-equivalences. The saturation* *axiom is satisfied. The mapping cylinder construction provides a cylinder functor, and the cylinder axiom holds.*

(2) *The axioms for a category with cofibrations and weak equivalences hold for* $\mathbf{P}^{N,T}_{\mathrm{sfd}}(G)$ *with respect to f-cofibrations and h-equivalences. The saturation axiom is satisfied. The mapping cylinder construction provides a cylinder functor, and the cylinder axiom holds.*

(3)*The axioms for a category with cofibrations and weak equivalences hold for* $\bar{\mathbf{P}}^{N,T}_{\mathrm{sfd}}(G)$ *with respect to weak cofibrations and* h_S*-equivalences. The saturation axiom is satisfied. The mapping cylinder construction provides a cylinder functor, and the cylinder axiom holds.*

Proof. We prove (1) only, the other cases being similar. Axioms (Cof 1), (Cof 2) and (Weq 1) hold by definition. The saturation axiom follows from the model category axioms. To prove (Cof 3), suppose we have a diagram $Z \leftarrow Y \rightarrow P$ with the right map an f-cofibration. Since $\mathbf{P}^{N}_{\mathrm{sfd}}(G)$ satisfies axiom (Cof 3) by Lemma 4.3.1(1), the pushout $Z \cup_Y P$ exists in $\mathbf{P}^{N}_{\mathrm{sfd}}(G)$, and the map $Z \rightarrow Z \cup_Y P$ is an f-cofibration. We are left to check that the pushout is an object of $\mathbf{P}^{N,T}_{\mathrm{sfd}}(G)$, i.e., that for all $t \in T$, the space $\Gamma \circ \theta_t(Z \cup_Y P)$ is weakly contractible. But Γ and θ_k commute with pushouts (the former by Corollary 2.2.3(3), the latter since it is an equivalence of categories by Lemma 3.6.4(2)). Moreover, every f-cofibration is a weak cofibration, i.e., a pointwise cofibration in $G\text{-}kTop_*^{\langle N\rangle}$ (cf. remark following Definition 3.4.5), and Γ maps weak cofibrations to cofibrations (Corollary 2.2.3(1)). Hence the right horizontal maps in the diagram

$$\begin{tikzcd}
\Gamma\circ\theta_t(Z) \arrow[d, "\sim"'] & \Gamma\circ\theta_t(Y) \arrow[l] \arrow[r] \arrow[d, "\sim"] & \Gamma\circ\theta_t(P) \arrow[d, "\sim"] \\
* & * \arrow[l] \arrow[r] & *
\end{tikzcd}$$

are cofibrations of G-spaces, and all three vertical arrows are weak equivalences since Z, Y and P are objects of $\mathbf{P}^{N,T}_{\mathrm{sfd}}(G)$. The gluing lemma in $G\text{-}kTop_*$ then asserts

$$\Gamma\circ\theta_t(Z\cup_Y P) \cong \Gamma\circ\theta_t(Z)\cup_{\Gamma\circ\theta_t(Y)}\Gamma\circ\theta_t(P) \simeq *.$$

The gluing lemma (axiom (Weq 2)) is proved with a similar argument.

The mapping cylinder construction is a cylinder functor by Lemma 4.3.1(1). Since all h-equivalences between weakly cofibrant objects are in particular h_S-equivalences, the projection from the cylinder is an h_S-equivalence. Hence the cylinder axiom holds. □

Recall the notion of an exact functor [10, 1.2]. Since we deal with several notions of weak equivalences, we say a functor is h-exact (or exact with respect to h-equivalences) if, in particular, it preserves h-equivalences. Similarly, we have h_S-exact functors. Again, if we have a weak equivalence of exact functors, and we want to specify the notion of weak equivalences, we speak of an h-equivalence of functors, or a weak equivalence with respect to h_S-equivalences.

Proposition 4.3.5. *The h_S-exact inclusion $\mathbf{P}^{N,T}_{\mathrm{sfd}}(G) \to \bar{\mathbf{P}}^{N,T}_{\mathrm{sfd}}(G)$ induces an equivalence on $\mathscr{S}_\bullet$-constructions with respect to h_S-equivalences.*

Proof. Exactness of the inclusion functor is obvious. We check that it has the approximation property. Axiom (App 1) holds by definition of weak equivalences (h_S-equivalences in both categories).

Given a map $f: Y \to Z$ with $Y \in \mathbf{P}^{N,T}_{\mathrm{sfd}}(G)$ and $Z \in \bar{\mathbf{P}}^{N,T}_{\mathrm{sfd}}(G)$, we can find a factorization $f = g \circ h$ with $h: Y \to P$ an f-cofibration (making P a locally cofibrant object) and an acyclic f-fibration $g: P \to Z$. From Lemma 4.1.4 we infer that P is stably finitely dominated, and since Γ preserves weak equivalences of weakly cofibrant objects (Corollary 2.2.3(2)) we conclude that $P \in \bar{\mathbf{P}}^{N,T}_{\mathrm{sfd}}(G)$. Application of the approximation theorem finishes the proof. □

Lemma 4.3.6. *Let* ? *denote any of the subscripts* f, hf, fd *or* sfd.

(1) *The functors Σ, $\bar{\theta}_k$ and θ_k (Section* 3.6*) are exact endofunctors of $\mathbf{P}^N_?(G)$.*

(2) *The canonical sheaf functors ψ_k (Section* 3.6*) are exact functors $\mathbf{C}_?(G) \to \mathbf{P}^N_?(G)$.*

(3) *The functor Γ: $\mathbf{P}^N_?(G) \to \mathbf{C}_{\mathrm{sfd}}(G)$ (Section* 3.7*) is exact.*

(4) *The functor $\sigma: \mathbf{P}^{N,T}_{\mathrm{sfd}}(G) \to \bar{\mathbf{P}}^{N,T}_{\mathrm{sfd}}(G)$ (Section* 3.8*) is h_S-exact (where $T \subseteq S \subseteq \mathbf{Z}$).*

Proof. For the proof we have to gather material from earlier lemmas.

(1) This follows from the fact that cofibrations and weak equivalences are defined pointwise. For the case of twisting functors use Lemma 3.6.4.

(2) The A-component of $\psi_j(K)$ is given by $K \wedge M^A$. Hence ψ_j is exact (apply Lemma 2.1.2(2) to the morphism $G \to G \wedge M^A$). Combining Lemma 2.1.2(3) with Lemma 4.1.2 it is easy to show that ψ_j preserves all finiteness notions.

(3) The functor Γ maps stably finitely dominated, locally cofibrant sheaves to stably finitely dominated G-spaces by Proposition 4.2.6. The other conditions have been checked in 2.2.3.

(4) This follows immediately from Lemma 3.8.5. □

Lemma 4.3.7 (Shifting lemma). *Suppose F, $G: \mathbf{P}^{N,T}_{\mathrm{sfd}}(G) \to \bar{\mathbf{P}}^{N,T}_{\mathrm{sfd}}(G)$ are h_S-exact functors, and $\tau: F \to G$ is a weak equivalence of functors (with respect to h_S-equivalences). Then the induced natural transformation*

$$\theta_{-k}(\tau_{\theta_k}): \theta_{-k} \circ F \circ \theta_k \to \theta_{-k} \circ G \circ \theta_k$$

is a weak equivalence of h_{k+S}-exact functors where $k+S$ denotes the set $\{k+s | s \in S\}$.

4.4. The splitting theorem

We now head for the main theorem of the paper. For $n = 1$ the splitting theorem has been proved in Section 6 of [4] (with a slightly different organization of the material). This section uses virtually everything of the preceding material.

Definition 4.4.1. Let G denote a topological monoid with zero. The algebraic K-theory of G-equivariant non-linear projective N-space is the algebraic K-theory in the sense of [10, 1.3] of the category $\mathbf{P}^N_{\mathrm{sfd}}(G)$ with respect to f-cofibrations and h-equivalences, i.e., the space $\Omega|h\mathscr{S}_\bullet\mathbf{P}^N_{\mathrm{sfd}}(G)|$.

The above definition displays, to the author's taste, the most natural choice of cofibrations. The finiteness condition, however, is forced upon us if we want to use the functor Γ since it maps locally finite sheaves to stably finitely dominated spaces.

From now on we will restrict ourselves to the stably finitely dominated case. We will only consider f-cofibrations (Proposition 3.4.4) unless otherwise stated. Let G denote a topological monoid with zero which is cofibrant as an object of $kTop_*$, and let N denote a non-empty finite set with $n+1$ elements.

Lemma 4.4.2. *For each number $k\in[n]$, the functors*

$$\psi_{-k}:\mathbf{C}_{\mathrm{sfd}}(G)\to\mathbf{P}^{N,[k-1]}_{\mathrm{sfd}}(G)\quad\textit{and}\quad\Gamma\circ\theta_k:\mathbf{P}^{N,[k-1]}_{\mathrm{sfd}}(G)\to\mathbf{C}_{\mathrm{sfd}}(G)$$

induce equivalences on $\mathscr{S}_\bullet$-constructions with respect to $h_{[k]}$-equivalences in $\mathbf{P}^{N,[k-1]}_{\mathrm{sfd}}(G)$. Explicitly, the map

$$|h\mathscr{S}_\bullet\mathbf{C}_{\mathrm{sfd}}(G)|\to|h_{[k]}\mathscr{S}_\bullet\mathbf{P}^{N,[k-1]}_{\mathrm{sfd}}(G)|$$

induced by ψ_{-k} is a homotopy equivalence, and similarly for $\Gamma\circ\theta_k$.

Proof. By Corollary 3.7.5, the spaces $\Gamma\circ\theta_j\circ\psi_{-k}(K)=\Gamma\circ\psi_{j-k}(K)$ are contractible for all $K\in G\text{-}kTop_*$ and $0\leqslant j<k$. Hence $\psi_{-k}:\mathbf{C}_{\mathrm{sfd}}(G)\to\mathbf{P}^N_{\mathrm{sfd}}(G)$ factors through $\mathbf{P}^{N,[k-1]}_{\mathrm{sfd}}(G)$, so ψ_{-k} induces a well-defined map in K-theory. Since Σ induces a self-equivalence [10, 1.6.2], it is sufficient to show that the map $\Sigma^2\circ\psi_{-k}$ induces an equivalence on $\mathscr{S}_\bullet$-constructions. Now $(\Gamma\circ\theta_k)\circ(\Sigma^2\circ\psi_{-k})\cong\Sigma^2\circ\Gamma\circ\psi_0\simeq\Sigma^{n+2}$ by Corollary 3.7.4, and the suspension functor induces an isomorphism on homotopy groups. Thus $\Sigma^2\circ\psi_{-k}$ induces an injection on homotopy groups, and $\Gamma\circ\theta_k$ induces a surjection.

It remains to prove that the functors $\Sigma^2\circ\psi_{-k}\circ\Gamma\circ\theta_k$ and Σ^{n+2} induce homotopic self-maps on the $\mathscr{S}_\bullet$-construction. By Proposition 4.3.5 it is sufficient to prove that they are weakly equivalent as exact functors $\mathbf{P}^{N,[k-1]}_{\mathrm{sfd}}(G)\to\bar{\mathbf{P}}^{N,[k-1]}_{\mathrm{sfd}}(G)$ with respect to $h_{[k]}$-equivalences (cofibrations are f-cofibrations in the source and weak cofibrations in the target). But by Corollary 3.8.8 and Lemma 4.3.6(4), the functors $\Sigma^2\circ\psi_0\circ\Gamma$ and Σ^{n+2} are connected by a chain of $h_{\{0\}}$-equivalences of functors. Hence

$$\begin{aligned}\Sigma^2\circ\psi_{-k}\circ\Gamma\circ\theta_k&\cong\theta_{-k}\circ\Sigma^2\circ\psi_0\circ\Gamma\circ\theta_k\\&\simeq\theta_{-k}\circ\Sigma^{n+2}\circ\theta_k\quad\text{(by Corollary 3.8.8)}\\&\cong\Sigma^{n+2}\end{aligned}$$

with respect to $h_{\{k\}}$-equivalences by the shifting lemma (Lemma 4.3.7). We claim that this is an $h_{[k]}$-equivalence of functors, i.e., for all $j\in[k]$, the components of the natural transformations involved in the chain are mapped to weak equivalences upon application of $\Gamma \circ \theta_j$. For $j = k$ this is what we have just checked. For $j < k$, observe that source and target of the components of the natural transformations are objects of $\bar{\mathbf{P}}^{N,[k-1]}_{\mathrm{sfd}}(G)$, hence are mapped to contractible spaces, and any map between contractible spaces is a weak homotopy equivalence. □

Lemma 4.4.3. *For $k \in [n]$, the map*

$$\iota \vee \psi_{-k} : \mathbf{P}^{N,[k]}_{\mathrm{sfd}}(G) \times \mathbf{C}_{\mathrm{sfd}}(G) \to \mathbf{P}^{N,[k-1]}_{\mathrm{sfd}}(G), \quad (Y,K) \mapsto Y \vee \psi_{-k}(K)$$

(*where $\iota : \mathbf{P}^{N,[k]}_{\mathrm{sfd}}(G) \to \mathbf{P}^{N,[k-1]}_{\mathrm{sfd}}(G)$ symbolizes the inclusion functor and $\vee$ denotes the coproduct in $\mathbf{P}^{N,[k-1]}_{\mathrm{sfd}}(G)$*) *induces an equivalence on $\mathscr{S}_\bullet$-constructions with respect to h-equivalences.*

Proof. For $n = 0$, i.e., if N is a set with one element, we know $k = 0$, $\mathbf{P}^{N,[-1]}_{\mathrm{sfd}}(G) = \mathbf{C}_{\mathrm{sfd}}(G)$, $\Gamma = \mathrm{id}$, $\psi_{-k} \cong \mathrm{id}$, and $\mathbf{P}^{N,[0]}_{\mathrm{sfd}}(G)$ is the subcategory of $\mathbf{C}_{\mathrm{sfd}}(G)$ of contractible spaces. Hence the assertion is true.

So assume N has at least two elements ($n \geqslant 1$), and let $\Sigma^2\mathbf{P}^{N,[k]}_{\mathrm{sfd}}(G)$ denote the full subcategory of $\mathbf{P}^{N,[k]}_{\mathrm{sfd}}(G)$ consisting of objects with simply connected components. This category inherits the structure of a category with cofibrations and weak equivalences. Similarly we can define all the other categories present in the following square:

$$\begin{array}{ccc}
h\mathscr{S}_\bullet\Sigma^2\mathbf{P}^{N,[k]}_{\mathrm{sfd}}(G) & \longrightarrow & h_{[k]}\mathscr{S}_\bullet\Sigma^2\mathbf{P}^{N,[k]}_{\mathrm{sfd}}(G) \\
\big\downarrow & & \big\downarrow \\
h\mathscr{S}_\bullet\Sigma^2\mathbf{P}^{N,[k-1]}_{\mathrm{sfd}}(G) & \longrightarrow & h_{[k]}\mathscr{S}_\bullet\Sigma^2\mathbf{P}^{N,[k-1]}_{\mathrm{sfd}}(G)
\end{array}$$

The $h_{[k]}$-equivalences satisfy the extension axiom in these modified categories. Hence we can apply the fibration theorem [10, 1.6.4] to conclude that the above square is homotopy cartesian with contractible upper right corner. This square has a canonical map into the square

$$\begin{array}{ccc}
h\mathscr{S}_\bullet\mathbf{P}^{N,[k]}_{\mathrm{sfd}}(G) & \longrightarrow & h_{[k]}\mathscr{S}_\bullet\mathbf{P}^{N,[k]}_{\mathrm{sfd}}(G) \\
\big\downarrow & & \big\downarrow \\
h\mathscr{S}_\bullet\mathbf{P}^{N,[k-1]}_{\mathrm{sfd}}(G) & \longrightarrow & h_{[k]}\mathscr{S}_\bullet\mathbf{P}^{N,[k-1]}_{\mathrm{sfd}}(G)
\end{array}$$

induced by inclusion of categories. There is a map going backwards induced by double suspension. But Σ^2, considered as an endofunctor, induces a self-equivalence on $\mathscr{S}_\bullet$-constructions [10, 1.6.2]. Hence the maps of squares are weak equivalences. This implies that the second square is homotopy cartesian with contractible upper right corner.

By the previous lemma, $\Sigma^n \circ \Gamma \circ \theta_k$ induces a homotopy equivalence from the lower right space to $h\mathscr{S}_\bullet\mathbf{C}_{\mathrm{sfd}}(G)$. Hence we obtain the following homotopy cartesian square:

$$\begin{array}{ccc}
h\mathscr{S}_\bullet\mathbf{P}^{N,[k]}_{\mathrm{sfd}}(G) & \longrightarrow & h_{[k]}\mathscr{S}_\bullet\mathbf{P}^{N,[k]}_{\mathrm{sfd}}(G) \simeq * \\
\big\downarrow & & \big\downarrow {\scriptstyle \Sigma^n\circ\Gamma\circ\theta_k} \\
h\mathscr{S}_\bullet\mathbf{P}^{N,[k-1]}_{\mathrm{sfd}}(G) & \xrightarrow[\Sigma^n\circ\Gamma\circ\theta_k]{} & h\mathscr{S}_\bullet\mathbf{C}_{\mathrm{sfd}}(G)
\end{array} \qquad (*)$$

As in the proof of the previous lemma, we see that the canonical sheaf functor ψ_{-k} induces a map $\mathbf{C}_{\mathrm{sfd}}(G) \to \mathbf{P}^{N,[k-1]}_{\mathrm{sfd}}(G)$. But there is a chain of weak equivalences of functors

$$\begin{aligned}
\Sigma^n \circ \Gamma \circ \theta_k \circ \psi_{-k} &\cong \Sigma \circ \Gamma \circ \theta_k \circ \psi_{-k} \circ \Sigma^{n-1} \\
&\cong \Sigma \circ \Gamma \circ \psi_0 \circ \Sigma^{n-1} \\
&\simeq \Sigma^{n+1} \circ \Sigma^{n-1} \quad \text{(by Corollary 3.7.4)} \\
&\cong \Sigma^{2n},
\end{aligned}$$

and Σ^{2n} induces, on $\mathscr{S}_\bullet$-construction, a map which is homotopic to the identity. This shows that ψ_{-k} induces a section-up-to-homotopy of the lower horizontal map.

We claim that the composite map

$$\begin{aligned}
|h\mathscr{S}_\bullet \mathbf{P}^{N,[k]}_{\mathrm{sfd}}(G)| \times |h\mathscr{S}_\bullet \mathbf{C}_{\mathrm{sfd}}(G)| &\xrightarrow{\imath \times \psi_{-k}} |h\mathscr{S}_\bullet \mathbf{P}^{N,[k-1]}_{\mathrm{sfd}}(G)| \times |h\mathscr{S}_\bullet \mathbf{P}^{N,[k-1]}_{\mathrm{sfd}}(G)| \\
&\xrightarrow{\vee} |h\mathscr{S}_\bullet \mathbf{P}^{N,[k-1]}_{\mathrm{sfd}}(G)|,
\end{aligned}$$

where $\imath$ is induced by the forgetful functor and $\vee$ denotes one-point union (the coproduct inducing the H-space structure) is a weak homotopy equivalence. Since source and target are connected, it suffices to check that the induced map $(\imath \vee \psi_{-k})_*$ on jth homotopy groups is an isomorphism for $j \geqslant 1$.

To see this, recall the homotopy cartesian square $(*)$ above. Since its upper right corner is contractible, we obtain a long exact sequence of homotopy groups (which are abelian because the $\mathscr{S}_\bullet$-construction has an abelian H-group structure induced by the coproduct)

$$\begin{aligned}
\cdots \longrightarrow \pi_j |h\mathscr{S}_\bullet \mathbf{P}^{N,[k]}_{\mathrm{sfd}}(G)| &\xrightarrow{\imath_*} \pi_j |h\mathscr{S}_\bullet \mathbf{P}^{N,[k-1]}_{\mathrm{sfd}}(G)| \xrightarrow{(\Sigma^n \circ \Gamma \circ \theta_k)_*} \pi_j |h\mathscr{S}_\bullet \mathbf{C}_{\mathrm{sfd}}(G)| \\
&\longrightarrow \pi_{j-1} |h\mathscr{S}_\bullet \mathbf{P}^{N,[k]}_{\mathrm{sfd}}(G)| \xrightarrow{\imath_*} \cdots
\end{aligned}$$

with maps induced by $\imath$ and $\Sigma^n \circ \Gamma \circ \theta_k$ as indicated. But we have shown above that the latter map have a section-up-to-homotopy, hence the long exact sequence gives rise (for $j \geqslant 1$) to short exact sequences

$$0 \longrightarrow \pi_j |h\mathscr{S}_\bullet \mathbf{P}^{N,[k]}_{\mathrm{sfd}}(G)| \xrightarrow{\imath_*} \pi_j |h\mathscr{S}_\bullet \mathbf{P}^{N,[k-1]}_{\mathrm{sfd}}(G)| \xrightarrow{(\Sigma^n \circ \Gamma \circ \theta_k)_*} \pi_j |h\mathscr{S}_\bullet \mathbf{C}_{\mathrm{sfd}}(G)| \longrightarrow 0,$$

which are split by the map induced from ψ_{-k}, i.e., for all $j \geqslant 1$ we have an isomorphism of abelian groups

$$\imath_* + (\psi_{-k})_* : \pi_j |h\mathscr{S}_\bullet \mathbf{P}^{N,[k]}_{\mathrm{sfd}}(G)| \oplus \pi_j |h\mathscr{S}_\bullet \mathbf{C}_{\mathrm{sfd}}(G)| \xrightarrow{\cong} \pi_j |h\mathscr{S}_\bullet \mathbf{P}^{N,[k-1]}_{\mathrm{sfd}}(G)|,$$

where "+" means the sum of group elements.

But the target is a homotopy group of an H-space. Hence the group structure is induced from the H-space structure [7, I.6, Corollary 10], i.e., from the coproduct in $\mathbf{P}^{N,[k-1]}_{\mathrm{sfd}}(G)$. Consequently, the isomorphism $\imath_* + (\psi_{-k})_*$ is the same map as $(\imath \vee \psi_{-k})_*$. □

Lemma 4.4.4. *Suppose $S \subseteq \mathbf{Z}$ contains $n+1$ successive integers. Then $|h\mathcal{S}_\bullet \mathbf{P}^{N,S}_{\mathrm{sfd}}(G)|$ is contractible. In particular, the map $|h\mathcal{S}_\bullet \mathbf{P}^{N}_{\mathrm{sfd}}(G)| \to |h_S\mathcal{S}_\bullet \mathbf{P}^{N}_{\mathrm{sfd}}(G)|$ is a homotopy equivalence.*

Proof. The second claim follows from the first by the fibration theorem [10, 1.6.4].

For $k \geqslant 0$, let $\mathscr{C}^k$ denote the full subcategory of $\mathbf{P}^{N,S}_{\mathrm{sfd}}(G)$ generated by the objects Y with $\Sigma^k(Y) \simeq *$. These categories inherit the structure of categories with cofibrations (f-cofibrations) and weak equivalences (h-equivalences).

Now $|h\mathcal{S}_\bullet \mathscr{C}^0| \simeq *$ since the map $Y \to *$ is a weak equivalence for all objects $Y \in \mathscr{C}^0$. Furthermore, suspension defines a map $\mathscr{C}^{k+1} \to \mathscr{C}^k$ which shows that the inclusion $\mathscr{C}^k \subseteq \mathscr{C}^{k+1}$ induces a homotopy equivalence on $\mathcal{S}_\bullet$-constructions. Hence the category $\mathscr{C}^\infty := \bigcup_0^\infty \mathscr{C}^k$ has contractible $\mathcal{S}_\bullet$-construction. We claim that $\mathbf{P}^{N,S}_{\mathrm{sfd}}(G) = \mathscr{C}^\infty$. By definition of $\mathscr{C}^k$, we have $\mathscr{C}^\infty \subseteq \mathbf{P}^{N,S}_{\mathrm{sfd}}(G)$. For the reverse inclusion, it suffices to prove the following assertion: if Y is an object of $\mathbf{P}^{N,S}_{\mathrm{sfd}}(G)$, some finite suspension of Y is acyclic (i.e., the map $\Sigma^\ell(Y) \to *$ is an h-equivalence for $\ell \geqslant 0$ large).

It is enough to consider ordered indexing sets $N = [n]$. We proceed by induction on the dimension n. The statement is clear for $n = 0$ (in which case $\mathbf{P}^{0}_{\mathrm{sfd}}(G) = \mathbf{C}_{\mathrm{sfd}}(G)$ and $\Gamma = \mathrm{id}$). So assume we have shown the result for $n-1$. Let $Y \in \mathbf{P}^{n,S}_{\mathrm{sfd}}$. Fix $j \in [n]$, and choose $k \in S$ with $k+1 \in S$. The functors θ_k and $\bar{\theta}_k$ are naturally isomorphic by Lemma 3.6.4(3), and the natural transformations κ_j and $\bar{\kappa}_j$ (defined in Definitions 3.6.5 and 3.6.6, respectively) correspond under this isomorphism. Hence Lemma 3.6.9 applies, asserting the existence of a cofibration sequence

$$\bar{\theta}_k(\Sigma^2(Y)) \xrightarrow{\bar{\kappa}_j} \bar{\theta}_{k+1}(\Sigma^2(Y)) \longrightarrow \zeta_j \circ \rho_j \circ \bar{\theta}_{k+1}(\Sigma^2(Y)),$$

which induces, by application of Γ, a cofibration sequence of spaces

$$\begin{aligned}\Gamma \circ \bar{\theta}_k(\Sigma^2(Y)) \to \Gamma \circ \bar{\theta}_{k+1}(\Sigma^2(Y)) \to \Gamma(\zeta_j \circ \rho_j \circ \bar{\theta}_{k+1}(\Sigma^2(Y)))\\ \cong \Sigma \circ \Gamma(\bar{\theta}_{k+1} \circ \rho_j(\Sigma^2(Y)))\end{aligned}$$

(we have used Lemma 3.7.2 and the fact that restriction commutes with twisting for the last isomorphism). Since suspension commutes with restriction and global sections (Remark 2.2.2(2)) all spaces in this sequence are simply connected. Furthermore, by hypothesis on Y the first two spaces are contractible, hence so is the third, which implies (by varying k) that the global sections of n successive twists of $\rho_j(\Sigma^2(Y))$ are contractible. In other words, the induction hypothesis applies to the object $\rho_j(\Sigma^2(Y))$. Thus we find that some finite suspension of $\rho_j(\Sigma^2(Y))$ is acyclic. Since twisting, suspension, and restriction commute, this means that $\rho_j \circ \bar{\theta}_{k+1}(\Sigma^\ell(Y)) \simeq *$ for some $\ell \geqslant 2$.

By Lemma 3.6.9, there is a cofibration sequence

$$\bar{\theta}_k(\Sigma^\ell(Y)) \xrightarrow{\bar{\kappa}_j} \bar{\theta}_{k+1}(\Sigma^\ell(Y)) \longrightarrow \zeta_j \circ \rho_j \circ \bar{\theta}_{k+1}(\Sigma^\ell(Y))$$

whose last term is acyclic by what we have just shown. From the long exact sequence in homology, applied componentwise, we infer that all components of $\bar{\kappa}_j$ are weak equivalences.

Now suppose we have a non-empty set $A \subseteq [n]$, and let $m := \max(A)$. The A-component of $\bar{\kappa}_j$ is given by $\Sigma^\ell(Y^A) \xrightarrow{t_j t_m^{-1}} \Sigma^\ell(Y^A)$ which is a weak homotopy equivalence by what we have shown above. This implies that all maps in the telescope construction, Lemma 3.2.6(2), for $\Sigma^\ell(Y^A)[t_j^{-1}]$ are weak homotopy equivalences. Consequently, the $(G \wedge M^A)$-equivariant inclusion of $\Sigma^\ell(Y^A)$ into $\Sigma^\ell(Y^A)[t_j^{-1}]$ (the unit of the adjunction described in Lemma 2.1.2(1), denoted s_0, is a weak homotopy equivalence.

Let σ denote the inclusion $A \to A \cup \{j\}$. The $\flat$-type structure map $\bar{\theta}_k(\Sigma^\ell(Y))^\flat_\sigma$ is given by the composite

$$\Sigma^\ell(Y^A) \xrightarrow{s_0} \Sigma^\ell(Y^A)[t_j^{-1}] \xrightarrow{\bar{\theta}_k(\Sigma^\ell(Y))^\sharp_\sigma} \Sigma^\ell(Y)^{A \cup \{j\}}.$$

The second map is a weak equivalence since $\bar{\theta}_k(\Sigma^\ell(Y))$ is a locally cofibrant sheaf (Corollary 3.3.4), and s_0 is a weak equivalence by the above argument. Hence $\bar{\theta}_k(\Sigma^\ell(Y))^\flat_\sigma$ is a weak homotopy equivalence.

This argument applies for all $j \in [n]$. Thus all $\flat$-type structure maps in $\bar{\theta}_k(\Sigma^\ell(Y))$ (away from the initial vertex) are weak homotopy equivalences. By Lemma 2.2.4, we conclude that $\bar{\theta}_k(\Sigma^\ell(Y))$ has contractible components since $\Gamma \circ \bar{\theta}_k(\Sigma^\ell(Y)) \simeq *$. □

Theorem 4.4.5. *Suppose N is a non-empty finite set with $n+1$ elements, and G is a topological monoid with zero which is cofibrant as an object of $kTop_*$. The map*

$$\bigvee_{k=0}^{n} \psi_{-k} : \underbrace{\mathbf{C}_{\mathrm{sfd}}(G) \times \cdots \times \mathbf{C}_{\mathrm{sfd}}(G)}_{n+1 \text{ factors}} \to \mathbf{P}^N_{\mathrm{sfd}}(G)$$

induces an equivalence on $\mathscr{S}_\bullet$-constructions with respect to h-equivalences and hence a weak homotopy equivalence

$$\underbrace{A^{\mathrm{sfd}}(*, G) \times \cdots \times A^{\mathrm{sfd}}(*, G)}_{n+1 \text{ factors}} \to \Omega|h\mathscr{S}_\bullet \mathbf{P}^N_{\mathrm{sfd}}(G)|,$$

where A^{sfd} denotes the version of Waldhausen's algebraic K-theory of spaces functor using stably finitely dominated spaces. In other words: the algebraic K-theory of G-equivariant non-linear projective N-space decomposes naturally as a product of $n+1$ copies of $A^{\mathrm{sfd}}(*, G)$.

Proof. The map can be written as the composite

$$\begin{aligned}
\mathbf{C}_{\mathrm{sfd}}(G)^{n+1} &\xrightarrow{* \times \mathrm{id}^{n+1}} \mathbf{P}^{N,[n]}_{\mathrm{sfd}}(G) \times \mathbf{C}_{\mathrm{sfd}}(G)^{n+1} \\
&\xrightarrow{(\iota \vee \psi_{-n}) \times \mathrm{id}^n} \mathbf{P}^{N,[n-1]}_{\mathrm{sfd}}(G) \times \mathbf{C}_{\mathrm{sfd}}(G)^{n} \\
&\xrightarrow{(\iota \vee \psi_{-n+1}) \times \mathrm{id}^{n-1}} \cdots \\
&\xrightarrow{(\iota \vee \psi_{-1}) \times \mathrm{id}} \mathbf{P}^{N,[0]}_{\mathrm{sfd}}(G) \times \mathbf{C}_{\mathrm{sfd}}(G) \\
&\xrightarrow{\iota \vee \psi_0} \mathbf{P}^{N}_{\mathrm{sfd}}(G),
\end{aligned}$$

which induces an equivalence on $\mathscr{S}_\bullet$-constructions by Lemmas 4.4.4 and 4.4.3. □

Since θ_{k_0} is an equivalence of categories, we see that for any $k_0 \in \mathbf{Z}$ the map

$$\bigvee_{k=0}^{n} \psi_{k_0-k} : \underbrace{\mathbf{C}_{\mathrm{sfd}}(G) \times \cdots \times \mathbf{C}_{\mathrm{sfd}}(G)}_{n+1 \text{ factors}} \to \mathbf{P}^N_{\mathrm{sfd}}(G)$$

induces an equivalence on $\mathscr{S}_\bullet$-constructions with respect to h-equivalences.

Corollary 4.4.6. *Let $k_0 \in \mathbf{Z}$ be any integer. The map*

$$(\Gamma \circ \theta_{-k_0}, \Gamma \circ \theta_{-k_0+1}, \ldots, \Gamma \circ \theta_{-k_0+n}) : \mathbf{P}^N_{\mathrm{sfd}}(G) \to \mathbf{C}_{\mathrm{sfd}}(G)^{n+1}$$

induces an equivalence on $\mathscr{S}_\bullet$-constructions with respect to h-equivalences.

Proof. The composite map $f := \Sigma \circ (\Gamma \circ \theta_{-k_0}, \Gamma \circ \theta_{-k_0+1}, \ldots, \Gamma \circ \theta_{-k_0+n}) \circ \bigvee_{k=0}^{n} \psi_{k_0-k}$ is given, up to homotopy, by the matrix

$$\begin{pmatrix} \Sigma^{n+1} & 0 & 0 & 0 & \ldots & 0 \\ ? & \Sigma^{n+1} & 0 & 0 & \ldots & 0 \\ ? & ? & \Sigma^{n+1} & 0 & \ldots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ ? & \ldots & & \ldots & ? & \Sigma^{n+1} \end{pmatrix},$$

where ? means “don’t care”, and 0 denotes null-homotopic maps (we have used Corollary 3.7.4 to identify the terms on the diagonal, and Corollary 3.7.5 for the zero entries). More precisely, this means that the map induced on homotopy groups is described by the above matrix. But since Σ induces a homotopy equivalence on $\mathscr{S}_\bullet$-constructions, the matrix is invertible (on the level of homotopy groups). Hence the map f induces a weak homotopy equivalence, hence so does the map of the corollary. $\square$

Acknowledgements

The author has to thank M. Bökstedt, J. Klein, S. Schwede and F. Waldhausen for many discussions. The model structures introduced in Section 3.4 result from joint work with O. Röndigs. The referee’s comments lead to a significantly improved exposition of the paper. All diagrams in the original manuscript are made with Paul Taylor’s diagrams macro package of TeX [8].

References

[1] A.K. Bousfield, E.M. Friedlander, Homotopy theory of Γ-spaces, spectra, and bisimplicial sets, in: M.G. Barratt, M.E. Mahowald (Eds.), Geometric Applications of Homotopy Theory II, Lecture Notes in Mathematics, Vol. 658, Springer, Berlin, 1978, pp. 80–130.

[2] W. Dwyer, J. Spalinski, Homotopy theory and model categories, I.M. James (Ed.), Handbook of Algebraic Topology, Elsevier, Amsterdam, 1995.

[3] M. Hovey, Model Categories, Mathematical Surveys and Monographs, Vol. 63, American Mathematical Society, Providence, RI, 1999.
[4] T. Hüttemann, J. Klein, W. Vogell, F. Waldhausen, B. Williams, The "Fundamental Theorem" for the algebraic K-theory of spaces: I, J. Pure Appl. Algebra. 160 (2001) 21–52.
[5] D.G. Quillen, Homotopical Algebra, Lecture Notes in Mathematics, Vol. 43, Springer, Berlin, 1967.
[6] D.G. Quillen, Higher algebraic K-theory: I, in: H. Bass (Ed.), Algebraic K-theory I: Higher K-theories, Lecture Notes in Mathematics, Vol. 341, Springer, Berlin, 1973, pp. 85–147.
[7] E.H. Spanier, Algebraic Topology, Springer, Berlin, 1966.
[8] P. Taylor, Commutative diagrams in TEX, available via anonymous FTP from theory.doc.ic.ac.uk as /`tex`/`contrib`/`Taylor`/`tex`/`diagrams.tex`.
[9] T. Thomason, T. Trobaugh, Higher algebraic K-theory of schemes and derived categories, in: P. Cartier (Ed.), The Grothendieck Festschrift III, Birkhäuser, Basel, 1990.
[10] F. Waldhausen, Algebraic K-theory of spaces, in: A. Ranicki (Ed.), Algebraic and Geometric Topology, Lecture Notes in Mathematics, Vol. 1126, Springer, Berlin, 1985, pp. 318–419.