Understanding the basics of measurements in Quantum Computation

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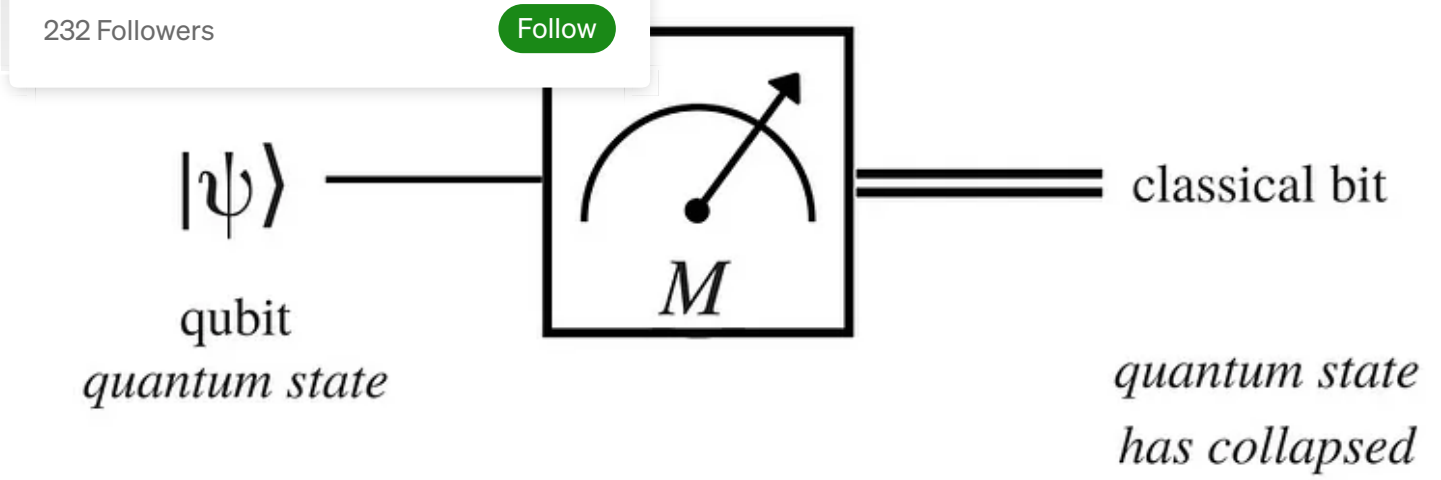


Measuring a Quantum bit

If you have had a deeper look into the theory of Quantum Computation,

chances are that you might have come across this term called *measurement.* On a top level, *measurement* is essentially what the figure above depicts: *some* operation on a qubit (some sort of superposition state of basis vectors|0> and |1>) to get a classical bit (the process of which is completely random).

Before you begin. If you are facing difficulty in understanding terms like qubit, superposition, basis vectors, I'll suggest reading this article as an introduction, the bra and ket notation, and this linear algebra review (and do **not** leave out Eigenvectors and Eigenspaces, orthogonality). Also, consult other sources about (complex) vector spaces, inner products, linear combinations, and related concepts.

Fine then, pre-requisites done. Let's get to the point!

Why this article?

When I set out to learn about quantum measurements, things were not very smooth. The math wasn't that tough, but the basic intuition of *why doing the things the way they are done* was missing, at least in my source. When I finally fit the different pieces together, I thought it would be helpful to carve out a different approach to understanding quantum measurements (a kind of top-bottom approach — *and that means reserving the math to the last while understanding the broader implications first*) and first understanding **why** do we need a distinction between the two kinds of measurements I shall discuss here- *general* and *projection* and *POVM*.

Wait! Didn't I say *two kinds* of measurements and named three. Well, don't worry. Interpret my statements as a sort of *quantum statements (*a word I

coined just now)with a certain degree of uncertainty attached to them. You'll ask questions and understand better that way (and that is specifically true for the entire article).

I'll begin with the pre-requisite postulates of quantum mechanics:

Postulate 1

In simple terms, if you have an **isolated** quantum system, there is a complex vector space with an inner product defined attached to that system, called the *state* space.

Intuitively, such a space is described to combine Euclidean (or mainly 2-dimensional spaces) calculus and notions of linear algebra to multi-dimensional spaces. This allows several operations like measurement of length and angles.

Read more about Hilbert spaces here

Postulate 2

Given a **closed** quantum system, the evolution of the system is described with a *unitary transformation.*

$$|\psi'\rangle = U|\psi\rangle \ .$$

Unitary transformation of a quantum state

In the above equation, U is a unitary matrix (or a matrix whose product with its adjoint yields the identity matrix). More generally, for continuous

time, this unitary transformation can be written as:

$$i\hbar\frac{d|\psi\rangle}{dt} = H|\psi\rangle.$$

You just got introduced to the Schrodinger's equation!

This is also a unitary transformation, though it requires a bit of understanding. Here, *H* is called the *Hamiltonian* operator, which is a Hermitian operator. Simply put, the adjoint of the Hermitian is the operator itself, i.e. *adj(H) = H.* Solving the above differential equation with respect to time, the following equation comes out:

$$|\psi(t_2)\rangle = \exp\left[\frac{-iH(t_2 - t_1)}{\hbar}\right]|\psi(t_1)\rangle$$

The term inside the *exp* part requires more explanation. Here, H is a Hermitian operator. Multiplication of a Hermitian operator by a scalar (which is essentially the ratio of difference of time and Dirac-h) preserves the Hermitian property. Without going into deeper math, anything of the form *exp(-i.K)* where K is a Hermitian operator yields a Unitary operator.

Back to measurements

I'll refrain from straightaway launching the formal discussion on measurements here. I would rather discuss a bigger picture and hope it will make the distinction between *general* and *projective* measurements much clearer.

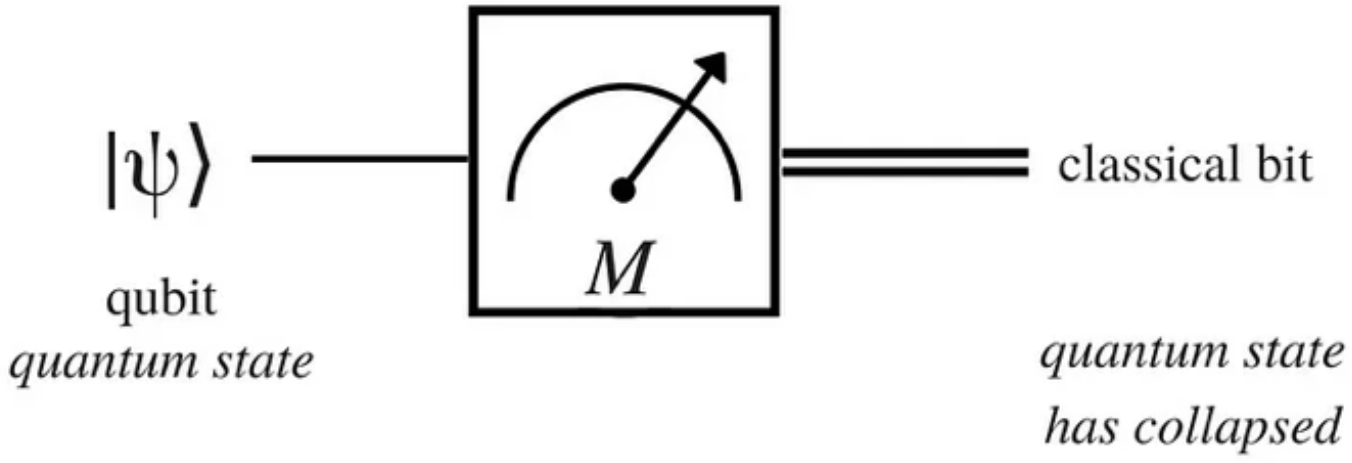
One important thing to realise so far:

> The postulates are applicable on closed, isolated systems.

And simply put, this is hardly a case ever in the real world. This implies your formal ideas about unitary evolution (postulate 2) isn't very correct. Your Hamiltonian doesn't give you a clean *unitary operator* that can perfectly describe the mechanics of the system under observation. Two very simple examples to illustrate this point:

1. Measurements destroy the quantum state in most cases.
2. Energy enters and leaves the system.

If you have some formal background with QM, you might know that 1 is essentially true. A system remains in its superposed state unless it is measured. *Example?* Consider the top image on this post.



We have a qubit (a superposed quantum state) formed by some *linear combination* of |0> and |1>. After measurement, it becomes a classical bit (0 or 1).

Now is the time to introduce a bit about *general, POVM* and *projective* measurements. Consider these systems of measurements as kind of black-boxes for now, so we might look at the bigger picture without being bogged down by the details.

Long story short, systems that are **closed** and are described by unitary time evolution by a Hamiltonian can be measured by *projective* measurements. Very clearly, systems are **not closed** in reality and hence are immeasurable using *projective* measurements. To measure such systems, we got two choices:

1. consider the **smaller, open** system being observed as a subpart of a **larger system** that is **closed.** In this case, the evolution of the **larger system** can be described by unitary evolution, and we can use the

projective measurements to measure things out. This would require adding **complete details** about the complete larger system (let's call it *environment*) to the Hamiltonian of the **smaller, open** system being observed. There's an inherent flaw here though. *Do you know **everything** about the environment in order to make it a completely closed system?* This is quite difficult to determine. This problem leads us to the second option.

2. let the system be **open** and develop some other technique of measurement from the known *projective measurements.*

It turns out that the POVM (*Positive Operator-Valued Measure*) is a *restriction* on the *projective measurements,* such that it encompasses everything except the *environment.* In short, if you take a *POVM,* you do not need to care about the environment anymore. Or in other words,

> We can get projective measurements from POVMs if we factor in the environment to make it a unitary time evolution, i.e. projections = POVMs + environment

Isn't this exactly what we were looking for? A way to measure the system without having to care about the environment. And *POVM* is the answer. There is one other subtle difference- though *POVMs* and *general measurements* look the same mathematically (as a matter of fact, *POVMs* are obtained by substituting a variable into the *general measurement equation*), there is an important difference between the two. However, in

order to show this, I need to introduce mathematical equations about these measurements.

General measurements

In Quantum measurement scenario, a measurement operator is essentially a matrix (rather a carefully chosen matrix) that mathematically manipulates the initial state of the system.

$$p(m) = \langle\psi| M_m^\dagger M_m |\psi\rangle$$

Probability of the measurement to be m

The above equation gives the probability of the measurement to output value *m*. If you are familiar with the bra and the ket notation, the leftmost symbol denotes the **transposed, complex conjugated** row vector of the original system state (original system state is the rightmost column vector). In the middle is the adjoint of the operator M on the left multiplied with the original operator M.

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi| M_m^\dagger M_m |\psi\rangle}}$$

Post-measurement state

This equation is simply the application of the operator to the current state, divided by the probability of the state occurring. **Do understand this important equation, as this will prove to be the essential difference between the *general* and the *POVM measurements*.**

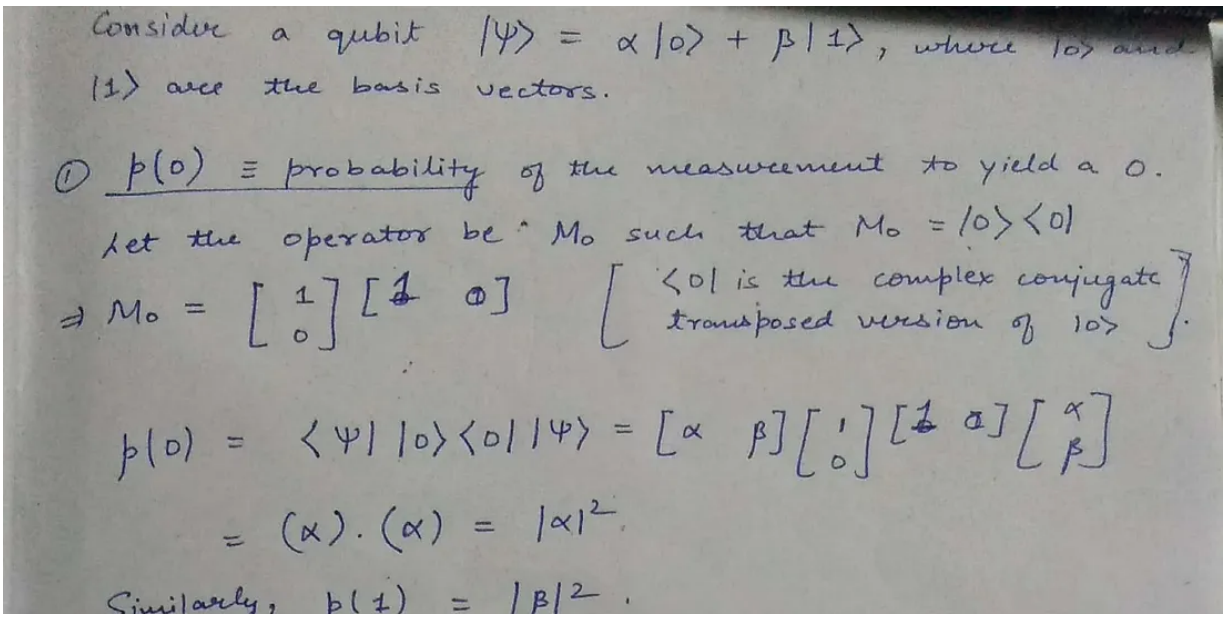
A couple of other equations depicting the nature of the operators we choose (basically summation over **all** possible outputs, if we take the product of adjoint and the original matrix, we end up with the identity matrix). The second equation is simply the basic postulate of probability, *summation over all probabilities is essentially one.*

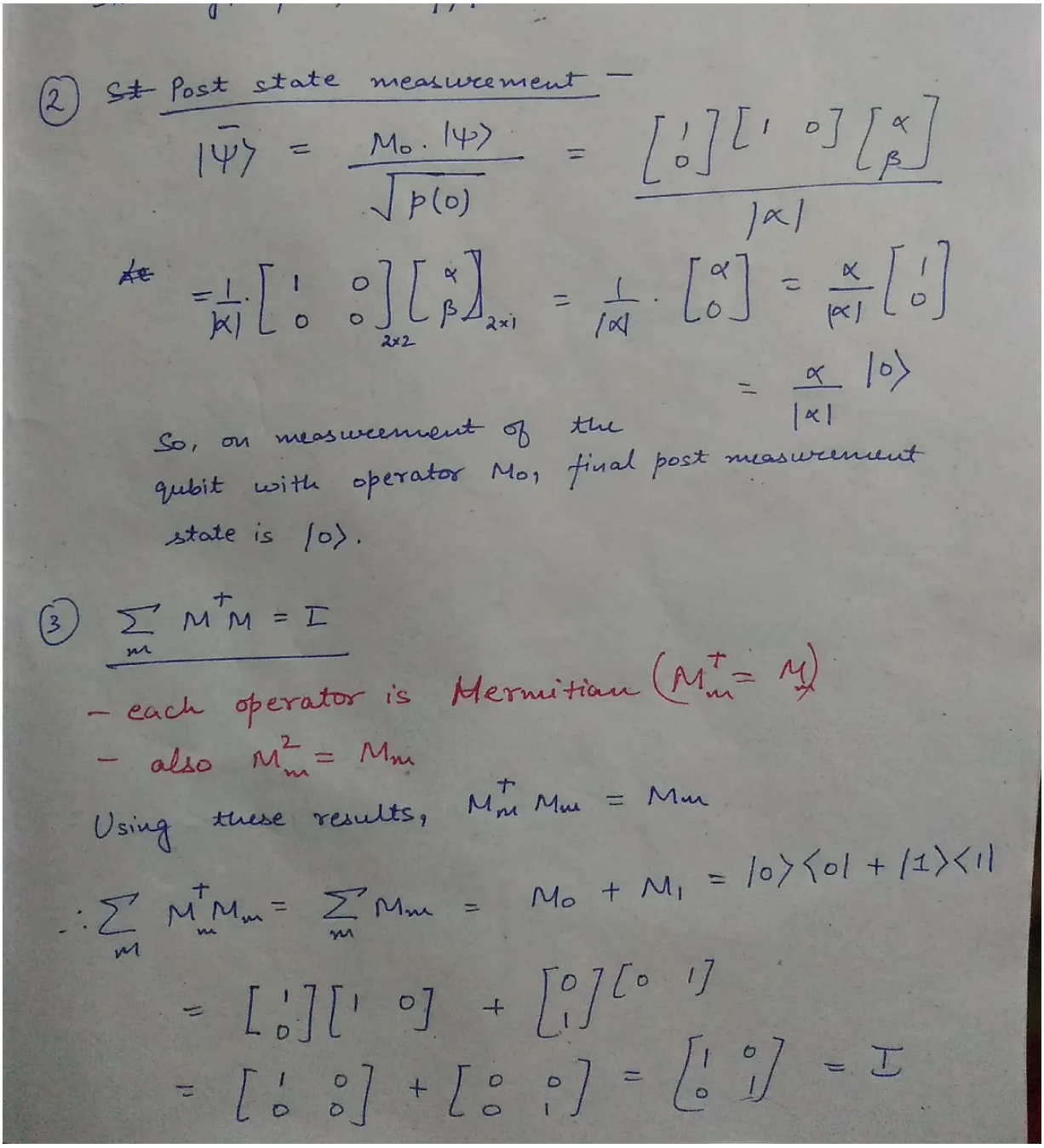
$$\sum_m M_m^\dagger M_m = I$$

The 'completeness equation'

$$1 = \sum_m p(m) = \sum_m \langle\psi| M_m^\dagger M_m |\psi\rangle$$

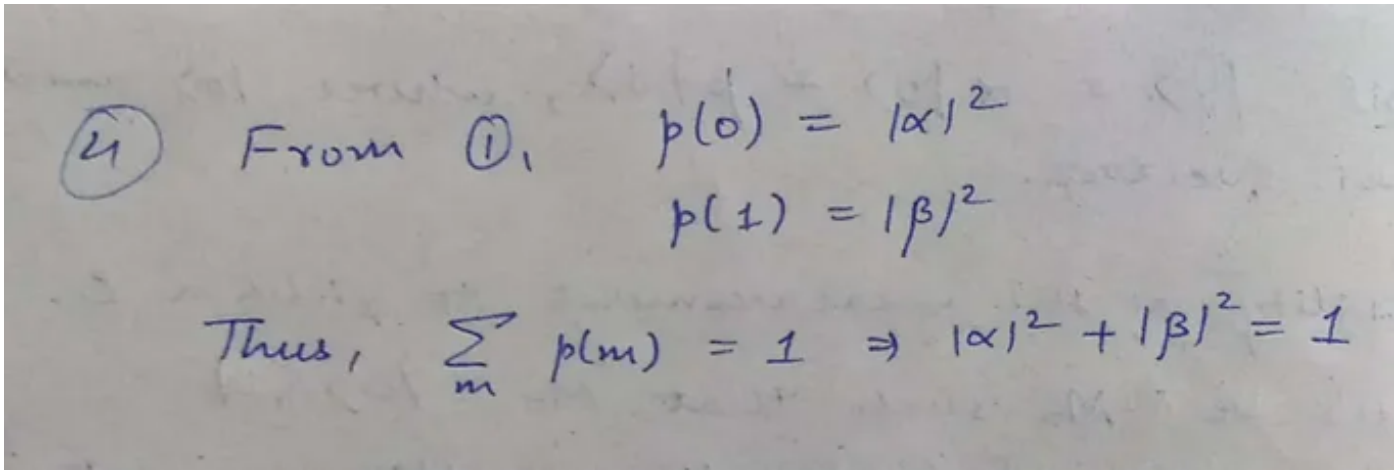
I'll take a break and show an example for these equations. It's a bit difficult to better represent equations here, thus I'll just write it on paper and post a screen here. I have shown the requisite calculations of the first three equations on a qubit.





To note are those two statements in red that simply calculations.

1. Each operator is *Hermitian*
2. The square of the operator is the operator itself.



An important result!!!

Projective measurements

Remember these are measurements for **closed systems undergoing unitary evolution,** and thus exploit one of the most basic intuitions- *eigenvalues and eigenvectors.*

Simply put, eigenvectors serve to *break* the operation of the operator into several *independent* vector directions. Hence, the action of an operator can now be observed independently and individually in different directions. Moreover, the action of the operator in these directions simply involves *stretching, squeezing, flipping (* or mathematically *scalar multiplication)* of the eigenvector.

It might take a while to let that sink in, but this is a very powerful property that can be exploited to the fullest.

$$M = \sum_m m P_m$$

The above equation is the **spectral decomposition** of the observable M. It is simply decomposing M into several *projections* into the *eigenspace* of M, multiplied by the eigenvalues *m.* Don't worry, I'll show how to do that.

Rest all equations from *general measurements* get fairly simplified:

$$p(m) = \langle \psi | P_m | \psi \rangle$$

Probability of the measurement yielding output as the eigenvalue m

$$\frac{P_m|\psi\rangle}{\sqrt{p(m)}}$$

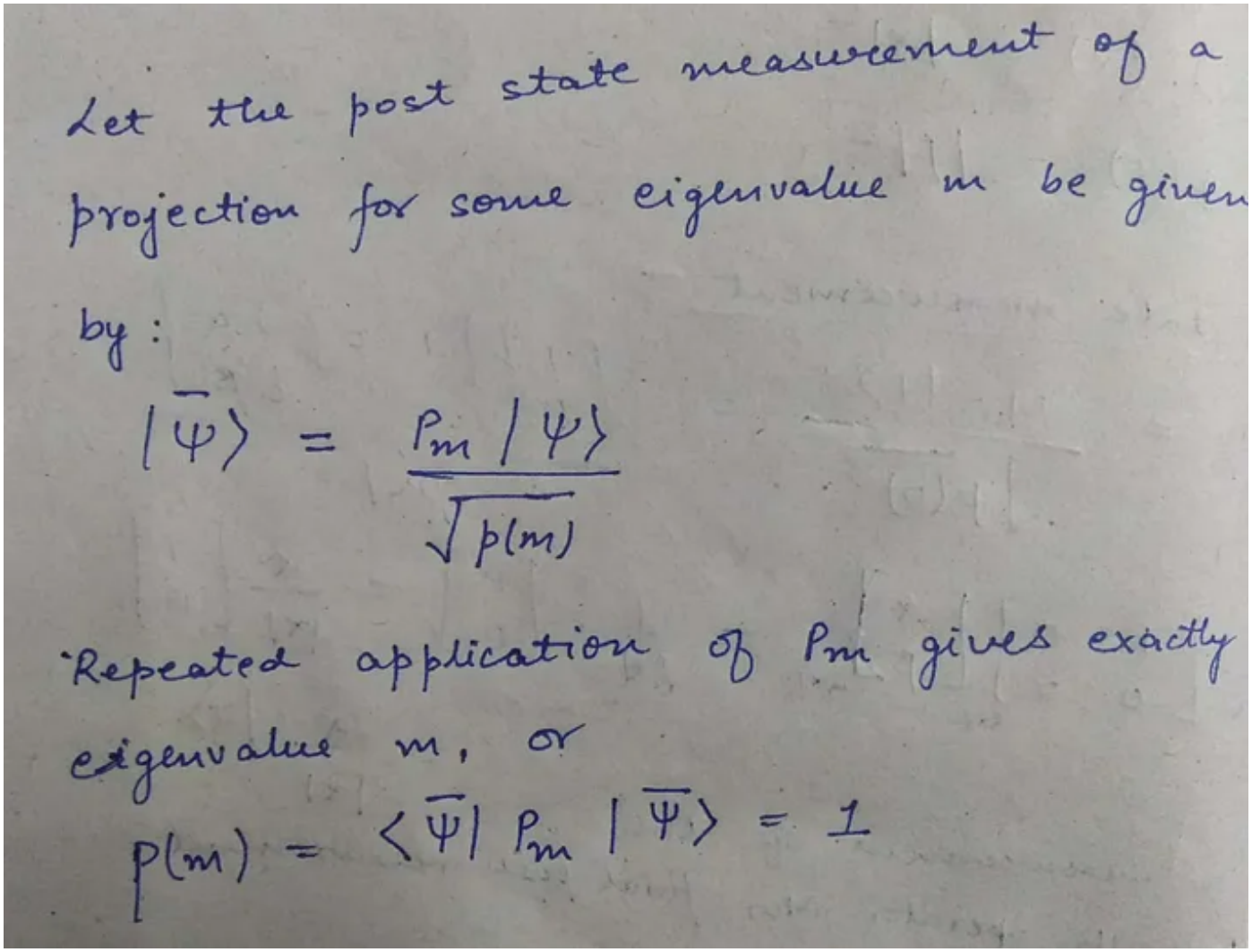
The post-measurement state calculation equation

Any operator that satisfies the requirements of *general measurements,* if follows one additional restriction, becomes a *projective measurement operator.*

$$M_m M_{m'} = \delta_{m,m'} M_m$$

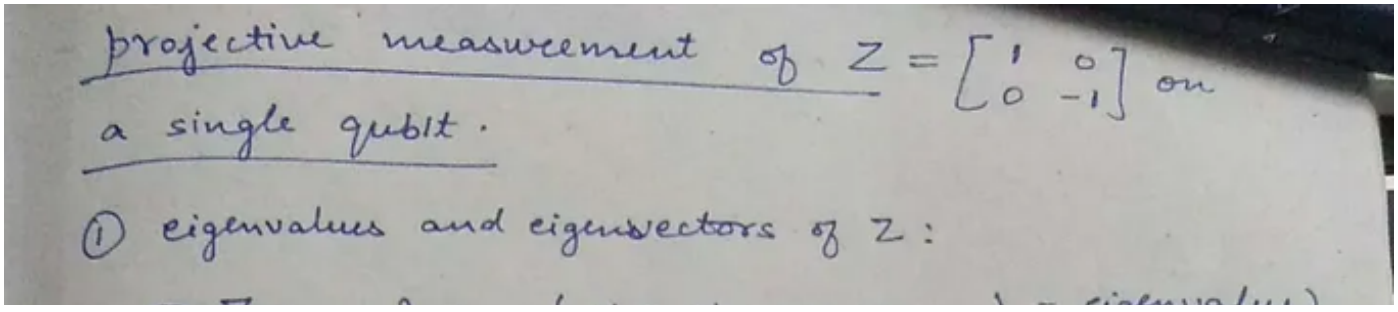
Additional restriction for being projective!

It is not difficult to connect the dots and remark here that this *additional restriction* is satisfied in only **closed systems,** or more specifically, systems in which measurements are **non-demolishing measurements,** i.e. they do not collapse the quantum state upon measuring. Why so? Let me give the mathematical solution:



It can be seen that repeated application of the *projection* doesn't alter the output. The probability of getting the same measurement again and again is 1. P*rojections* are, therefore, not what we can use in the real world.

Now let me demonstrate all mathematical aspects of a *projective* calculation in a single example:



(a) $Zv = \lambda v$ ($v \equiv$ eigenvector, $\lambda =$ eigenvalue)

(b) $\det(Z - \lambda I) = 0$ (I = identity (2×2) matrix)

$$\det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & -(1-\lambda) \end{bmatrix} = 0\right) \Rightarrow (1-\lambda)^2 = 0$$

$$\Rightarrow \lambda = \pm 1$$

Substituting in (a):

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 1\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \Rightarrow \begin{matrix} \beta = 0 \\ \alpha = 1 \end{matrix} \quad \text{or } |0\rangle$$

Similarly for $\lambda = -1 \Rightarrow \alpha = 0, \beta = 1$ or $|1\rangle$

Now, to form projections from eigenvector

$P_{+1} = |0\rangle\langle 0|$; $P_{-1} = |1\rangle\langle 1|$

Restricting the following projections such that —

~~(1) $+I(|0\rangle\langle 0|) + (-1)|1\rangle\langle 1| = I$ (completeness~~

completeness never involves m.

(1) $|0\rangle\langle 0| + |1\rangle\langle 1| = I$ (completeness)

(2) $M_m . M_{m'} = \delta_{m,m'} M_m$ (M_m are orthogonal projectors such that $M_m = \sum_m m . P_m$)

It can be verified both these results do hold.

Now, measurement of Z may result in +1 or −1 (eigenvalues). So probability:

$$p(+1) = \langle\psi| P_{+1} |\psi\rangle = [\alpha\ \beta]\begin{bmatrix} 1 \\ 0 \end{bmatrix}[0\ \ 1]\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$= (\alpha).(\beta) = \alpha\beta$$

Similarly $p(-1)$ can be calculated.

Note: The bottommost calculation wrongly computes using [0 1] instead of the previously assumed [1 0]. Thanks to Brian Droncheff for pointing this out in the comments.

The above example shows how much simplified the application of an operator Z became once it was decomposed into a set of vectors. This also demonstrates how eigenvectors can be analysed and treated independent of each other, making things much simpler.

POVM measurements

Mathematically, POVMs are formed from the given *general measurement probability distribution:*

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle.$$

Probability distribution of a general measurement

by simply doing the following substitution:

$$E_m \equiv M_m^\dagger M_m.$$

to get the following equations:

$$p(m) = \langle\psi|E_m|\psi\rangle$$

Probability distribution

$$\sum_m E_m = I$$

Completeness relation

And we are done? Well. No.

It might seem easy at first, but there is something hidden here. Fortunately, you already know that relation I asked you to remember for this particular section on distinction between *general measurements* and *POVMs.*

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$$

The post measurement state

Now if you start from knowledge of the original operator M, you are good to go. But what if you start from the initial knowledge about E? In that case, it is quite impossible to decompose E into another matrix M and it's adjoint. We thus lose M, and thus the ability to measure the post-measurement state. POVMs are amazing in cases where we just don't need to know anything about the post-measurement states. Also, it is for this reason for most standard texts prefer to develop POVMs from *general* measurement relations only.

Another dot to connect is the idea of *Projection = POVM + environment,* or that the POVMs are restrictions on the projections that do not take the environment into care. For this, have a look at the following equation:

$$E_m \equiv P_m^\dagger P_m = P_m.$$

POVMs = projections. The extreme right P(m) comes from the middle expression utilising the fact that the operator P is Hermitian and square of the operator is the operator itself (statements in red ink in my

handwritten example above in general measurements)

if and only if:

$$P_m P_{m'} = \delta_{mm'} P_m$$

The additional restriction for projections

This means that the talks about *POVM + environment = Projections* simply imply making the operators conform to the additional restriction described above. Additionally, this also implies that this same equation causes the *projections* to depend on the unitary time-evolution and isolation of the system, or in other words, give to the system sufficient responsiveness to its environment.

Conclusion

I hope this article helped you grab the basic intuitions behind these measurements and how they relate to each other in the bigger picture. Knowing these basics are keys to understand Postulate 3 of Quantum Mechanics.

Have a good day!

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