FINITE GEOMETRY
Summer Project Report

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Chapter 1

Finite Geometry

1.1 Introduction

The word "geometry," of Greek origin, means earth or land measure. As a branch of mathematics, geometry's standard definition concerns obtaining insights into shapes and the nature of space. However, the shape of geometry as a mathematical subject was dramatically set by Euclid's Elements which develops geometry deductively. Geometry now has changed significantly from what it was put forward by Euclid.

As the name suggests, finite geometry is the study of geometries in which every entity of the geometry is finite in number. (Entities like points, lines, planes...etc). It is a branch of combinatorics

According to Joe Malkevitch [5] following is the reason why it makes sense to think of finite geometries:

"A model of a line segment, such as a stick of thin spaghetti, can be subdivided into smaller and smaller pieces, suggesting that there are infinitely many points that make up the line segment or spaghetti stick. However, in the view of contemporary physics, the spaghetti is made up of atoms and the subatomic particles that make up these atoms, so one is led to the idea that there are only a finite number, albeit very large, of atoms in the universe. Would it make sense to investigate a geometry where the axioms or rules talked about the existence of only a finite number of points and lines? Would it make sense to talk about finite geometries?"

In finite geometry we ignore the ideas like continuity, distance, betweeness etc and only concentrate about incidence i.e. which point is on which line or

which line is on which plane etc.
Incidence System:
An incidence system is a triple (X, B, I) where X is a set, B is a set and $I \subseteq X \times B$. If $x \in X, b \in B$ and $(x, b) \in I$, then we write it as xIb and say " x is incident with b". If the set X has elements called points and set B has elements called blocks or lines then the structure is called a point-line incidence structure
Points and lines are undefined concepts or primitive notions in a point-line incidence structure.

1.1.1 Incidence system identified as a bipartite graph

:
The points of X and the lines of B can be considered as the vertices of a bipartite graph with the partition $X \uplus B$. There exists an edge between $x \in X$ and $b \in B$ iff xIb This graph is called the **levi graph** of the incidence structure.

1.1.2 Examples of incidence structures

:
-Any graph (whether it is simple or not) can be thought of as an incidence structure whose point set is the vertices of the graph and the edges joining vertices can be thought of as the set containing blocks. If the graph is simple, then distinct edges have distinct set of points incident on them. The converse is not true i.e. every incidence structure is not a graph as more than two points can be incident on a block. Infact every incidence structure can be viewed as a **hypergraph** which is the generalization of a graph in which an edge can join any number of vertices.

Definition 1. Two lines "l" and "m" of the incidence system are called **parallel** either if $l = m$ or there is no point of incidence system that is incident on both l and m.

Proposition 1. Parallelism is an equivalence relation on the set of all lines.

Proof. It is easy to see that the parallelism is a reflexive and symmetric relation. We have to prove transitivity. Let l_1, l_2, l_3 are three distinct lines of the incidence system such that $l_1 \parallel l_2$ and $l_2 \parallel l_3$. On the contrary if we assume that l_1 and l_3 are not parallel to each other, then there is a point P on both the lines. This means that there are two lines parallel to l_2 and passing through P which contradicts the parallel postulate. Hence $l_1 \parallel l_3$. □

1.2 The Real and general Euclidean Plane

:
Real euclidean plane is an incidence system with point set $\mathbb{R}^{\nvDash}$, set of blocks as all lines of the real euclidean plane and the incidence relation as “ ϵ'' (belongs to).
Some (NOT ALL) basic axioms followed by real euclidean plane are:

Collinearity : There are at least 3 non-collinear points.

Linearity Axiom : Any two distinct points are together incident with a unique line.

Parallel postulate : There are many ways to write the postulate. One of them which is very famous is the Playfair's axiom which is the modern version of parallel postulate given by Euclid. Given a line “l” and a point “P” not on the line, then there is a unique line passing through P and parallel to “l”.

Definition 2. A General Eucledian plane is an incidence system satisfying the above three axioms.
Example: The real euclidean plane whose point set is $\mathbb{R}^{\nvDash}$ and lines are subsets of $\mathbb{R}^{\nvDash}$ of the from $\{(x,y)|y = mx + c\}$ or $\{(x,y)|x = c\}$ where m and c belong to $\mathbb{R}$.

Theorem 1.2.1. The real euclidean plane constructed as mentioned above follows the three axioms of the general euclidean plane.

Proof. Since the point set is $\mathbb{R}^2$, we can find three points which are non-collinear like $(1,1),(1,2),(0,0)$. Hence the first axiom is followed. Given two points (x_1,y_1) and (x_2,y_2) , let us assume that the line $y = mx + c$ passes through the two points. Then $y_1 = mx_1 + c$ and $y_2 = mx_2 + c$. We are able to solve the two equations because $\mathbb{R}$ is a field and hence has -1 which can be multiplied to the second equation and then both equations can be added. We get $y_1 - y_2 = mx_1 - mx_2 = m(x_1 - x_2$. We wouldn't have been able to subtract the two equations if $\mathbb{R}$ was not a field. We have also used the distributive property of $\mathbb{R}$ as a field. From above equation we get that $m = \frac{y_1 - y_2}{x_1 - x_2}$. Hence when $x_1 \neq x_2$ then m is uniquely determined. From the value of m, we can find the value of c as $\frac{x_1 y_2 - x_2 y_1}{x_1 - x_2}$. When $x_1 = x_2$, it can be proved that $x = x_1$ is the only line through both points. (when both (x_1, y_1) and (x_1, y_2) lie on same line $y = mx + c$ then we get $y_1 = y_2$ which contradicts that the two points are different.). Hence the second axiom is also satisfied. When there are two lines $y = m_1 x + c_1$ and $y = m_2 x + c_2$. Then due to existence of $-m_1$ and m_2 , we can multiply second equation with $-m_1$, first with m_2 and then add both equations. Adding both equations gives us $m_2 y - m_1 y = c_1 - c_2$. By distributive law $y = \frac{c_1 - c_2}{m_1 - m_2}$. From value of y, the value of x can be uniquely determined as $x = \frac{y - c_1}{m_1}$. Hence lines intersect when $m_1 \neq m_2$. So parallel lines have $m_1 = m_2$. Given a line l, it is easy to check that a line parallel to l and passing through a particular point is uniquely determined. Hence the third axiom is also followed. □

Since we have only considered the properties of $\mathbb{R}$ as a field in the above proof, we can replace R by any field F to get a general euclidean plane.

Corollary 1.2.1.1. Take any field $\mathbb{F}$. The point set is F^2 and lines are the subsets of $\mathbb{F}^{\not\vDash}$ of the form $\{(x, y) | y = mx + c\}$ or $\{(x, y) | x = c\}$ where m and c belong to $\mathbb{F}$. This incidence system is a general euclidean plane called the euclidean plane over $\mathbb{F}$ ($EG(2, \mathbb{F})$)

Proof. F is a field and hence it has the points 0 and 1. Hence the non-collinear points are $(0, 0), (0, 1), (1, 1)$. The remaining proof is similar to above proof. □

Theorem 1.2.2. In any affine plane, any two lines l and l' have same number of points in them. In other words there exists a bijection between the points on l and points on l'.

Proof. l and l' are two distinct lines given to us. Hence there exists a point P present on l but not on l' and a point P' present on l' and absent on l. Join the two points to get the line PP'. For any line m let $P(m)$ denote the points on line m. Define the function $f : P(l) \to P(l')$ such that $R \mapsto R'$. For any point R on line l, there exists a unique line parallel to PP' and passing through R. By transitivity of parallelism, RR' has to intersect the line l' at some point, say R'. If R_1' and R_2' are images of points R_1 and R_2. If $R_1 \equiv R_2$

then since the line parallel to PP' and passing through R_1 (or) R_2 is unique and intersects l at a unique point, we can say that $R_1 \equiv R_2$. Hence the map is one - one. Again the map is onto because, for every point R' on the line l' there is a unique line parallel to PP' and passing through R' to intersect l in a unique point R. Hence the map is bijective. □

Theorem 1.2.3. Let (X, B, I) be any finite general euclidean plane. Then there exists a number $n \geqq 2$ such that the following hold :

Total number of points is n^2.

Total number of lines is $n^2 + n$.

Each line is incident with n points.

Each point is incident on $n + 1$ lines.

There are $n + 1$ parallel classes.

Each parallel class consists of n lines.

Proof. We saw in previous theorem that every line of the affine plane has "same number" of points. Let this number be n.
Claim: Each parallel class has n lines in it.
Let l be any line of the euclidean plane which is formed by joining any two of the three non-collinear points the affine plane has. Hence there exists at least one line m which is not parallel to l. Both l and m have n points on their line. Let the points on m be $\{M_1, M_2, ...M_n\}$. So l intersects the line m in one of the n points on m. Let l_i be the line passing through M_i and parallel to l. Then it is clear that l is one of $\{l_1, l_2, ..., l_n\}$. Without loss of generality, assume that $l = l_1$.Let l' be a line parallel to l. By the transitivity of the parallel lines, we can say that $l' \nparallel m$. Hence l' is parallel to l and intersects m in one of the n points on m. This means that l' is one of the n line in the set $\{l_1, l_2, ..., l_n\}$. So For the line l, $\{l_1, l_2, ..., l_n\}$ are the only lines parallel to l. Hence they form an parallel class and each such class has n lines in it.
Claim : If $P(l_i)$ denote the points on l_i, then $X = P(l_1) \dot{\cup} P(l_2) \dot{\cup} .... \dot{\cup} P(l_n)$
If P is any point of the affine plane then there is only one unique line which passes through P and is parralel to a line$l$. Hence P belongs to exactly one line of the parallel class of l. Hence the points on the lines partition the points of incidence system.
Claim : Total number of points is n^2.
Total number of points in X = (# lines in a parallel class)×(# points in each class) = $n \times n = n^2$.

Claim : Every point is incident on $n+1$ lines
Let P be a point of X. Then the number of points in X other than P are n^2-1 in number. Each line through P has $n-1$ points other than P itself. Every point $Q \neq P$ has a unique line passing through both P and Q. Hence all points are counted when we count the number of points on lines passing through a particular point. If r is the #lines through a given point, then $r\times$ (# points on each line except P) $= n^2-1$. So $r(n-1) = n^2-1$. Hence $r = n+1$
Claim : Total number of lines is $n(n+1)$ and number of parallel classes is $n+1$
(#points on a line) $\times$ (#lines) = (#points) $\times$ (#lines through each point). Hence $nx = n^2(n+1)$ where x is the total number of lines. Hence total number of lines is $n(n+1)$. So total number of parallel classes is $\frac{n(n+1)}{n} = n+1$.

□

Remark 1. In case of a euclidean plane over a finite field $\mathbb{F}_q$ $(EG(2,\mathbb{F}))$, the order of the euclidean plane is q.

Open Problem: Is it true that the order of any finite euclidean plane must be a prime power?

1.3 Isomorphism

Definition 3. Let (X,B,I) and (X',B',I') be two incidence structures.If there are two bijections $f: X \to X'$ and $g: B \to B'$ such that $\forall x \in X$ and $\forall b \in B$, $xIB \iff f(x)I'g(B)$. Then (f,g) is called an **isomorphism** from (X,B,I) to (X',B',I').

Definition 4. Let (X,B,I) be any incidence system. Define the function $sh : B \longrightarrow \mathcal{P}(\mathcal{X})$ (where $\mathcal{P}(\mathcal{X})$ is the power set of X) by $sh(b) = \{x \in X | xIb\}$. $sh(b)$ is called the **shadow of b**. The incidence structure (X,B,I) is called **simple** if the shadow function is one-one (or when distinct blocks have distinct images).

Remark 2. Let (X,B,I) be a simple incidence system. Let $B' \coloneqq \{sh(b) | b \in B\}$, Then the incidence structures (X,B,I) and $(X, sh(B), \in)$ are isomorphic.

Proof. Define the function $f : B \longrightarrow sh(B)$ such that $b \longmapsto sh(b)$. Since the shadow function is one-one, even the function f is one-one. By definition of

$sh(B)$, we can say that f is onto. Let xIb for some $b \in B$, then by definition of shadow function $x \in sh(b)$. Hence (f, g) is an isomorphism.
This means that we can identify the blocks with the points on them when the incidence structure is simple. □

Remark 3. Every Euclidean plane is simple.

Proof. Let b_1, b_2 be two distinct blocks of an incidence system. Assume that $sh(b_1) = sh(b_2) \implies \{x|xIb_1\} = \{x|xIb_2\}$. This means that every point of both the sets is incident on both b_1 and b_2. This can't happen if the sets b_1 and b_2 have more than or equal to 2 elements.(Since it violates the fact that there is a unique block between any two points.) Hence b_1 and b_2 are singleton sets.
Claim : Two blocks of the incidence system cannot have the same point incident on both of them when both blocks are singleton shadow sets.
Let b_1 and b_2 be two singleton blocks with same point x on both of them. Let x_1 be a point different from x.Then there is a line l through x_1 which is parallel to b_1. But this means that for the point x and line l, both b_1 and b_2 are parallel to l and pass through x. This contradicts the parallel postulate. Hence b_1 and b_2 can't be singleton sets.
Hence our assumption that b_1 and b_2 are distinct blocks is wrong. Hence $b_1 = b_2$. Hence the euclidean plane is simple.

□

Thus in studying Euclidean planes, we may assume without loss of generality that lines are point sets and incidence is $\in$

Proposition 2. A nullset or a singleton set cannot be a block of any affine plane.

Proof. **Claim**: Each line is incident with at least one point.
Conversely if it doesn't happen, then the empty line is parallel to all other lines. This means that all lines are mutually parallel. If there are more than or equal to 2 lines (other than the empty line) in the plane, then the linearity axiom is violated as there won't be any lines between the points in different classes. If there is only one line other than the empty line then collinearity axiom is violated. Hence an empty a null set can't be a block.
Claim There can't be a block which is a singleton set.
Conversely if it happens that there is a line l with only the point P_1 incident on it. Then by axiom(0), there exist points P_2 and P_3 such that all three

points are non-collinear. By axiom (1), there exists lines P_1P_2, P_2P_3, P_3P_1. By axiom (2), there is a line l_2 passing through P_2 and parallel to P_1P_3. l_1 is a singleton set and hence does not intersect l_2. This means that for the line l_2, there are two lines P_1P_3 and l_1 parallel to l_2 and passing through P_1. This contradicts axiom(2).

$\square$

1.3.1 Some Notations

Let π be a Euclidean plane, l_1, l_2 are lines, x, y are points of π. Then:

- $l_1 \parallel l_2$: l_1 is parallel to l_2.
- $l_1 \nparallel l_2$: l_1 is not parallel to l_2.
- $x \vee y$: Unique line incident with both x and y.
- $l_1 \wedge l_2$: Unique point incident with both l_1 and l_2.

1.4 Translation Planes

Alternate way to describe $EG(2, \mathbb{F})$:
Take a 2-dimensional vector space V over $\mathbb{F}$. Then the point set is the vectors in V and lines are the cosets of the 1-dimensional subspaces of V. This is true because the one dimensional subspaces of v are the lines $y = mx$ $(m \in \mathbb{F})$ or $x = 0$ through the origin. The cosets of this one dimensional subspaces will be the translates of the lines through the origin. Hence cosets are the lines of the form $y = mx + c$ or $x = c$.

1.4.1 Generalizing construction of Euclidean planes

Definition 5. Take a field $\mathbb{F}$ and let $m \geqq 1$. Let V be a $2m$-dimensional vector space over $\mathbb{F}$. Let Σ be a set of m- dimensional subspaces of V such that $\{M \backslash \{0\} | M \in \Sigma\}$ ia a partition of $V \backslash \{0\}$. Such a collection Σ is called a **spread in** V. The members of Σ are called the **components of the spread**.

Remark 4. Let V_i and V_j be two components of spread such that $i \neq j$. Then $dim(V_i \bigoplus V_j) = dim(V_i) + dim(V_j) = 2m$. Hence $V_i \bigoplus V_j = V$ for all components V_i and V_j.

Example: Let V be a 2- dimensional vector space over $\mathbb{F}_{\nVdash}$. Let the basis of the vector space be $B_v = \{v_1, v_2\}$. Then $V = \{av_1 + bv_2 | a, b \in \mathbb{F}_{\nVdash}\}$. Then

$|V| = 9$ and $V = \{0, v_1, v_2, 2v_1, 2v_2, v_1 + v_2, 2v_1 + v_2, v_1 + 2v_2\}_1 + 2v_2$, 2. Let Δ be the set of all one dimensional subspaces of V.
$\Delta = \{\langle v_1\rangle, \langle v_2\rangle, \langle v_1 + v_2\rangle, \langle 2v_1\rangle, \langle 2v_2\rangle, \langle 2v_1 + v_2\rangle, \langle v_1 + 2v_2\rangle\}$.
Also $sp(v_1 + v_2) = sp(2v_1 + 2v_2)$,$sp(v_1) = sp(2v_1)$, $sp(v_2) = sp(2v_2)$,$sp(2v_1 + v_2) = sp(v_1 + 2v_2)$. We can check that $V\backslash\{0\} = sp(v_1) \dot\cup sp(v_2) \dot\cup sp(v_1 + v_2) \dot\cup sp(v_1 + 2v_2) = V\backslash\{0\}$.
Hence $\Sigma = \{sp(v_1), sp(v_2), sp(v_1 + v_2), sp(2v_1 + v_2)\}$

Theorem 1.4.1. Given a spread Σ in V, consider the incidence structure $X(\Sigma)$ whose point set is the vectors in V and lines are the cosets of the subspaces in Σ. Then $X(\Sigma)$ is a euclidean plane. Euclidean planes thus obtained are called **translational planes**.

Proof. The lines of the incidence system are of the form $a + U$ where $a \in V$ and $U \in \Sigma$. Let x and y be different vectors of V. If $x \in a+U$ and $y \in a+U \implies x - y \in U$. Consider the coset $y + U$. Since $0, x - y$ is in U, we can say that the coset has y and x in it. But cosets of U partition the point set. Hence there is a unique coset containing the points x and y. So axiom(1) is followed.
The dimension of V is at-least $2m$. Hence there exist at least 2 elements say x , y in V such that $x, y \neq 0$ and $x \neq y$.
Case 1 : x and y belong to different spread members. Then $x, y, 0$ are three non-collinear points in the plane.
case 2 : Let x and y belong to the same spread member U. Then if U is the only spread member it means that $U = V$ and $dim(U) = dim(V) = 2m$ which contradicts that U is m-dimensional. Hence there are at-least two spread members. Call them U and U'. If $x \in U'$ such that $x \neq 0$, then x, y, z are three non-collinear points of the incidence system. Hence axiom(0) is satisfied.
Consider the line $a+U$ and a vector p not in it. Since the cosets of U partition the set of points, we can say that there exists an $a' \in V$ such that $p \in a' + U$, Since cosets are either disjoint or equal, we can say that $a' + U$ is the unique coset containing

□

Order of $X(\Sigma)$ and counting number of spread members

: In the above construction, assume that the vector space V is over the fild $\mathbb{F}_{\shortparallel}$. Then order of the euclidean plane is the number of points in each block (or) number of vectors in each coset. If $a + U$ is a coset such that $a \in V$ and $U \in \Sigma$, then $|a + U| = |U|$. Since U is an m- dimensional vector space, the

number of elements in U is q^m. Hence every coset or every line has q^m points on it. Hence the order of the euclidean plane is q^m
The number of non-zero vectors in V is $q^{2m}-1$. Each spread member has q^m-1 vectors in it. Since the spread members partition the non-zero elements of V, the number of spread members is $\frac{q^{2m}-1}{q^m-1} = q^m+1$.

- When $m = 1$, the euclidean plane obtained is $EG(2,\mathbb{F})$
- Even if $m \geqq 2$, the euclidean plane obtained as mentioned above may still be isomorphic to $EG(2,\mathbb{K})$ for some field $\mathbb{K}$.

Open Problem

: For any prime p, upto isomorphism $EG(2,\mathbb{F}_{\shortmid})$ is the unique euclidean plane of order p.

1.5 Uniqueness of euclidean planes of order $2,3,4$.

Proposition 3. Euclidean plane of order two is unique upto isomorphism.

Proof. $EG(2,\mathbb{F}_{\not\simeq})$ has 4 points, 6 lines , each line has two points , each point is on three lines and there are 3 parallel classes with 2 lines in each class.A block with two points is the same as a 2- subset of the point set. Total number of lines are $2(2+1)$ which is equal to number of 2- subsets of point set. Hence blocks are all possible 2- element subsets. Hence the euclidean plane is unique upto isomorphism.
summary of above proof: $(2+1)2 = \binom{2^2}{2}$ □

Theorem 1.5.1. Upto isomorphism, there is a unique eucliden plane of order 3.

Proof. Any euclidean plane of order 3 has 9 points, 12 lines, each point is in 4 lines. Each line has 3 points and there are 4 parallel classes each having 3 lines. Let the 4 parallel classes be $\{P,Q,R,S\}$. Pick any two parallel classes, say R and S. Let $R = \{r_1,r_2,r_3\}$ and $S = \{s_1,s_2,s_3\}$. r_i and s_j are intersecting lines for all $i,j = \{1,2,3\}$. These intersections give 9 points $r_1 \wedge s_j$ identified by (i,j). Hence all the points have been identified and $12-6 = 6$ lines are left to be identified. Each line has 3 points on it and any two points will uniquely

determine a line. The lines left are the transversals (set of positions meeting each row and each column in one position). The total number of transversals are $3 \times 2 \times 1 = 6$. There are 3 choices to select the first point (say x) of a transversal. Choosing a point right above x will give a vertical line which is already identified. Hence there are only two options for the second point of transversal. By axiom (1) , the third point is uniquly determined. The total number of transversals are $3 \times 2 \times 1 = 6$. This is the number of lines required. Hence all transversals are the remaining 6 lines. So the structure is identified upto isomorphism.
summary of above proof: $3(3+1) = (2 \times 3) + 3!$ □

Definition 6. A subset X of S_n is **sharply 2- transitive** if given any $x \neq x'$ and $y \neq y'$ in $\{1, 2, ..., n\}$ there is a unique $\pi \in X$ such that $\pi(x) = y$ and $\pi(x') = y'$.

Lemma 1.5.2. Any sharply 2- transitive susbset of S_n has size $n(n-1)$.

Proof. Let X be a sharply 2-transitive subset of S_n. Pick y, y' such that $y \neq y'$. This can be done in $n(n-1)$ ways. For each pair of (y, y') , there are $n(n-1)$ choices for the pair x, x'. Hence there are a total of $n(n-1) \times n(n-1)$ values of the pair $((x, y), (x', y'))$ each of which corresponds to a permutation on X. Given a permutation π on X , there are $n(n-1)$ pairs (x, x') such that $x \neq x'$. Since π is a one-one function, $\pi(x) \neq \pi(x')$ for each possible pair. Hence for each π there are $n(n-1)$ possible choices which correspond to the same π. So number of distinct permutations os $\frac{n^2(n-1)^2}{n(n-1)} = n(n-1)$. Hence $|X| = n(n-1)$.

□

Definition 7. A $2-(v, k, \lambda)$ design is an incidence system with v points, k points per block , λ blocks per pair of distinct points.

Theorem 1.5.3. A $2-(n^2, n, 1)$ design is an affine plane and conversely an affine plane is a $2-(n^2, n, 1)$ design. (Will be proved later)

Definition 8. The graph of a permutation $\sigma \in S_n$ is the set $\{(i, \sigma(i)) | i \in X\}$.

Lemma 1.5.4. Let $A \subset S_n$ be a sharply 2-transitive subset of S_n , then the incidence system with point set $X \times X$ and lines are the "rows" , "columns" and graphs of all permutations in A. In other words the lines are $\{(i, e) | i \in X\}$, $\{(e, j) | j \in X\}$ and $\{(i, \sigma(i)) | i \in X\}$ where e varies over X and σ varies over A. The incidence system obtained is a an affine plane of order n.

Proof. It is clear that there are n^2 points. Each "row" ,"column" and graph of a permutation in A have n points on them. Hence each line has n points on them. Now it is left to show that any two distinct points lie in exactly 1 block. Let $(x, y), (x', y')$ be two distinct points of the incidence system. If $x = x'$ then the unique line through the points is the line $\{(x, j)|j \in X\}$. Similarly the unique line through the two points when $y = y'$ is $\{(i, y)|i \in X\}$. (by definition of sharply 2-transitive subsets, we can say that there are no lines which are graphs of A in above two cases. If $x \neq x'$ and $y \neq y'$, then by definition of 2-transitive subset, there exists a unique $\sigma \in A$ such that $\sigma(x) = y$ and $\sigma(x) = y$. Hence both points lie on the graph of $\sigma \in A$ □

Lemma 1.5.5. If $A \subseteq S_n$ is sharply 2-transitive and σ,τ are in S_n, then $B = \sigma A \tau$ is again sharply 2- transitive. In this case we say that A **and** B **are isomorphic sharply** 2**-transitive subsets**.

Proof. Let $x \neq x'$ and $y \neq y'$. $\tau and \sigma^{-1}$ belong to S_n.Since A is a sharply 2-transitive set, there exists a unique permutation $\pi \in A$ which maps $\tau(x)$ to $\sigma^{-1}(y)$ and $\tau(x')$ to $\sigma^{-1}(y')$. Consider $\sigma\pi\tau \in X$, $\sigma\pi\tau(x) = y$ and $\sigma\pi\tau(x') = y'$. Hence there is a unique permutation which takes x to x' and y to y'. Hence B is also a sharply 2-transitive set. □

Lemma 1.5.6. Isomorphic sharply 2-transitive subsets of S_n correspond to isomorphic affine planes of order n.

Proof. Let A and B be isomorphic 2-transitive subsets such that $B = \sigma A \tau$. We know that they both have the same point set. The "horizontal" lines and "vertical" lines are also the same. We have to show that the set of graphs of A is the same as the set of graphs of set B. Consider the line $\{(i, \theta(i))|\theta \in A, i \in X\}$. This line can be identified by any two points lying on it. Consider the points $P \equiv (i_1, \theta(i_1))$ and $Q \equiv (i_2, \theta(i_2))$. ($i_1 \neq i_2$) . Since B is a sharply 2- transitive set, there exists $\sigma\pi\tau \in B$ which takes i_1 to $\theta(i_1)$ and i_2 to $\theta(i_2)$ .Hence the lines $\{(i, \theta(i))|\theta \in A, i \in X\}$ and $\{(i, \sigma\pi\tau(i))|\sigma\tau\pi \in B, i \in X\}$ are the same. Hence the set of all graphs of both sets is also the same. Hence the structures are isomorphic affine planes of order n. □

Definition 9. A sharply 2-transitive subset A is called standard if the identity permutation $I \in A$.

Remark 5. Every 2-sharply transitive set is isomorphic to a standard one.

Theorem 1.5.7. Affine plane of order 4 is unique upto isomorphism.

Proof. Without loss of generality, assume A to be the standard 2-transitive subset of of S_4. Since $I \in A$, there can't be any other permutation (in A) which fixes more than one element of $\{1,2,3,4\}$. Hence all $\pi \in A\backslash\{0\}$ fix at most one element of $\{1,2,3,4\}$. Let $X_4 = \{\pi \in A|\pi(4) = 4\} \subseteq A$. Let f be the function $f : X_4 \longrightarrow \{1,2,3\}$ such that $\pi \longmapsto \pi(1)$.
Claim:If $\pi_1(1) = \pi_2(1)$ then $\pi_1 = \pi_2$.
Since 4 is already fixed, $f(\pi_1) = f(\pi_2) = \pi(1) \neq 1$. Conversely, let us assume that $\pi_1 \neq \pi_2$. This means that $\pi_1(2) \neq \pi_2(2)$ or $\pi_1(3) \neq \pi_2(3)$. Without loss of generality let $\pi_1(2) \neq \pi_2(2)$. Let the map π_1 be $1 \mapsto a, 2 \mapsto c_1$,$3 \mapsto d_1, 4 \mapsto 4$. Let the map π_2 be $1 \mapsto a, 2 \mapsto c_2,$,$3 \mapsto d_2, 4 \mapsto 4.a = 2$ or 3. Let $a = 2$, then $c_1 = 3$ and $d_1 = 1$. Since we assumed that $\pi_1(2) \neq \pi_2(2)$ i.e. $c_1 \neq c_2$. Hence $c_2 = 1$ and $d_2 = 3$. This way π_2 fixes 2 elements which contradicts that it belongs to A. Similarly when $a = 3$, we can arrive at a contradiction. Hence $\pi_1 = \pi_2$.
Clearly the map f is onto. Hence the map is a bijection. So $|X_4| = 3$. Hence $|X_4\backslash\{I\}| = 2$. Similarly $|X_3| = |X_2| = |X_1| = 3$ and $|X_3\backslash\{I\}| = |X_2\backslash\{I\}| = |X_1\backslash\{I\}| = 2$. So total number of permutations which fix exactly one element are $2 + 2 + 2 + 2 = 8$. Hence there are 8 elements of A which fix exactly one element. But $A \subseteq S_n$ and S_n has 8 3-cycles.Hence all 3 - cycles are contained in A. We know from earlier lemma that $|A| = n(n-1) = 4 \times 3 = 12$. We have identified eight 3-cycles and one identity map. Remaining elements are $12 - 8 - 1 = 3$. Hence 3 elements of A do not fix any element. These elements are either the 4-cycles or the product of disjoint cycles. But no 4 cycle of the form $(t_1, t_2, t_3, t_4), t_i \in \{1,2,3,4\}$ can be contained in A since A already contains (t_1, t_2, t_3). So we are left with product of disjoint cycles. There are only 3 permutations which are product of disjoint cycles in S_n. They are $(1,2)(3,4)$, $(1,3)(2,4)$ and $(1,4)(2,3)$.Hence all these three elements are in A.
All elements identified as being in the set are elements of set $A_4 \subseteq S_4$.
Claim : A_4 and $S_4\backslash A_4$ are the only 2-transitive subsets of S_4.
If $B = \sigma A\tau$ such that σ and τ are in A_4. Then $B = A_4$ since A_4 is a group under composition. If σ and τ are such that $\sigma^{-1} = \tau$, then since A_4 contains I, B also will be a 2-transitive set containing 1.Let both σ, τ be from $S_4 - A_4$ such that they are not inverse to each other. Then, there is no identity in the two transitive subset. Hence all 2-cycles must be present in B. Since 2 cycles are present there should be no 3 cycles or product of disjoint cycles. Hence this set is $S_4\backslash A_4$.
These two are the only two transitive subsets of S_4 and they are isomor-

phic. Hence their corresponding affine planes are unique. Hence there exists a unique affine plane of order 4.

□

1.6 Full Automorpism group of $EG(2,\mathbb{F})$

Let $(P,\mathcal{B},I)$ and $(P',\mathcal{B}',I')$ be two isomorphic incidence systems. Hence there exists the isomorphism (f,g) such that $f : P \longrightarrow P'$ and $g : \mathcal{B} \longrightarrow \mathcal{B}'$ are bijections which preserve incidence i.e. $\forall x \in P, \forall B \in \mathcal{B}, xIB \Longleftrightarrow f(x)I'g(B)$. In other words (f,g) are bijections such that for $B \in \mathcal{B}$, $g(B)$ is a block in $\mathcal{B}'$ such that the shadow of $g(B)$ is the image of the shadow of B under f. If both systems are simple (so we can identify blocks with their shadows), then $\forall B \in \mathcal{B}$, $g(B) = \{f(x)|x \in B\}$

Let (f,g) is an isomorphism. the function f is given. Let g map $B \in \mathcal{B}$ to $g(B)$. Then $g(B)$ can be identified with its shadow since both structures are simple. $sh(g(B)) = \{y \in \mathcal{B}'|yI'g(B)\} = \{f(x) \in \mathcal{B}'|f(x)I'g(B)\} = \{f(x)|xIB\} \equiv g(B)$. Hence $g(B)$ is uniquely determined with a given f.

So when both systems are simple , g is uniquely determined by f , so we can forget about g and identify an isomorphism to be a bijection $f : P \longrightarrow P'$such that

- Image under f of any block is again a block.
- Every block of the second system is the image of a block under f.

Definition 10. An **automorphism** of an incidence system χ is an isomorphism from χ to χ.

Clearly , the set of all automorphisms of χ is a group under composition i.e. $((f_1,g_1),(f_2,g_2)) \coloneqq (f_1 \circ f_2, g_1 \circ g_2)$. The set of all automorphisms of an incidence system χ is given by $Aut(\chi)$.

Remark 6. When χ is simple, $Aut(\chi)$ may be viewed as the subgroup of $Sym(P)$. This is because the system is simple and hence an isomorphism can be uniquely determined by f which is a permutation of P. It is easy to prove that all such automorphisms form a group under composition.

Theorem 1.6.1. Automorphism group of $AG(2,\mathbb{F}_2)$ is S_4.

Proof. $AG(2, \mathbb{F}_2)$ has 4 points. Identify them with the points $X = \{1, 2, 3, 4\}$. Let f be any permutation on X. Define the function g from set of blocks to blocks as $\{i, j\} \longmapsto \{f(i), f(j)\}$.The above map is both one one and onto since f is bijective. Hence every permutation on X gives a bijection between blocks. If $x \in b$ then by definition of $g(b)$, $f(x) \in g(b)$. ($x \in X$ and b is a block .) Hence (f, g) is an automorphism and the automorphism group is isomorphic to S_4.
From above description, we can say that lines are all 2 subsets of the point set. □

Definition 11. An action of a group G on a set X, with $|X| \geqq 2$ is said to be **doubly transitive group action** if for $x, x', y, y' \in X$, given any two pairs, (x, x') and (y, y') with $x \neq x'$ and $y \neq y'$, there exists a $g \in G$ such that $y = g.x$ and $y' = g.x'$.

Lemma 1.6.2. Let $\pi = EG(2, 3)$. Then $Aut(\pi)$ acts doubly transitively on the set of 4 parallel classes.

Proof. Let let the parallel classes be $X = \{P_1, P_2, P_3, P_4\}$. Let $\sigma \in Aut(\pi)$, then σ is an isomorphism from π to π which respects parallelism. Hence $\sigma(P_i) = P_j$ where $i, j = \{1, 2, 3, 4\}$. Define $\pi_\sigma : X \longrightarrow X$ such that $P_i \longmapsto \pi_\sigma(P_i) = \sigma.P_i = \sigma(P_i) = \{\sigma(p) | p \in P_i\}$. Easy to verify that $\pi_I(P_i) = I(P_i) = P_i$ and $\pi_{\sigma_1} \circ \pi_{\sigma_1}(P_i) = \sigma_1(\sigma_2(P_i)) = \sigma_1 \circ \sigma_2(P_i) = \pi_{\sigma_1 \circ \sigma_2}(P_i)$. Hence G acts on X.
Claim: Action of $EG(2, \mathbb{F})$ on X is doubly transitive.
In the uniqueness proof of $EG(2, \mathbb{F})$, we saw that once two equivalence classes are chosen, other lines are automatically determined. If (P_x, P'_x) and (P_y, P'_y) are given such that $P_x \neq P'_x$ and $P_y \neq P'_y$, we should show that there exists a $\sigma \in Aut(\pi)$ such that $\sigma.(P_y, P'_y) = (P_x, P'_x)$ that is $P_x = \sigma(P_y)$ and $P'_x = \sigma(P'_y)$. Let $P_x = \{p_{x1}, p_{x2}, p_{x3}\}$ and $P_y = \{p_{y1}, p_{y2}, p_{y3}\}$. Also $P'_x = \{p_{x'1}, p_{x'2}, p_{x'3}\}$ and $P'_y = \{p_{y'1}, p_{y'2}, p_{y'3}\}$. Consider the map $p_{xi} \wedge p_{x'i} \longmapsto p_{yi} \wedge p_{y'i}$. This will be a bijection between the point set.The $EG(2, 3)$ constructed using any two parallel classes is unique upto isomorphism. Hence the action is doubly transitive. □

Remark 7. $Aut(\pi)$ acts doubly transitively on the set of 4 parallel classes is the same as saying $Aut(\pi)$ acts transitively on the pairs of elements of set of parallel classes.

So $Aut(\pi)$ acts transitively on the set of all ordered pairs with distinct parallel classes as co-ordinates. There are $4 \times 3 = 12$ pairs of ordered parallel classes with co-ordinates which are distinct parallel classes. Let (P_x, P_y) be such a pair. Then $stab_{(P_x,P_y)} = \{\sigma \in Aut(\pi) | \sigma(P_x) = P_x, \sigma(P_y) = P_y\}$

Claim: The index of $stab_{(P_x,P_y)}$ in $Aut(\pi)$ is 12. In other words, the number of cosets of $stab_{P_x,P_y}$ in $Aut(\pi)$ is 12. By orbit stabilizer lemma,we can say thatwhat we want to prove is that the number of orbits of (P_x, P_y) is 12.

$orb_{P_x,P_y} = \{\sigma.(P_x, P_y) | \sigma \in Aut(\pi)\}$
By doubly transitive action of $Aut(\pi)$ on X , we can say that the number of elements in the above orbit is 12. Hence index of the stabilizer in $Aut(\pi)$ is 12.

Lemma 1.6.3. Consider the ordered pair of parallel classes (R, C). The stabilizer of (R, C) is isomorphic to $S_3 \times S_3$.

Proof. this is the set of all permutations which take parallel class R to R and also S to S. We saw earlier that two sets of parallel classes uniquely determine an $EG(2, \mathbb{F_{\not{k}}})$. Hence permuting R and S individually gives an automorphism which satisfies required condition. Hence it is isomorphic to $S_3 \times S_3$. □

Lemma 1.6.4. $Stab_{(R,C)}$ is a subgroup of index 12 in $Aut(\pi)$.

Proof. $Aut(\pi)$ acts on 12 ordered pairs of parallel classes. Stabilizer of any such pair is of index 12 and the stabilizer is isomorphic to $S_3 \times S_3$.
Hence $[Aut(\pi) : S_3 \times S_3] = 12$. So $\frac{|Aut(\pi)|}{|S_3 \times S_3|} = 12$ (# cosets). So $|Aut(\pi)| = 12 \times (3! \times 3!)$ □

Lemma 1.6.5. If χ_1 and χ_2 are isomorphic incidence systems, then $Aut(\chi_1)$ and $Aut(\chi_2)$ are also isomorphic.

Proof. Let $P(\chi_i)$ = points on χ_i and $B(\chi_i)$ = blocks od the incidence system χ_i. If χ_1 and χ_2 are two isomorphic incidence systems then there exist bijections $f : P(\chi_1) \longrightarrow P(\chi_2)$ and $g : B(\chi_1) \longrightarrow B(\chi_2)$. together, (f, g) is a bijection from χ_1 to χ_2.
Let $(\theta_1, \theta_2) \in Aut(\chi_1)$ then $\theta_1 : P(\chi_1) \longrightarrow P(\chi_2)$ and $\theta_2 : B(\chi_1) \longrightarrow B(\chi_2)$.Let ϕ be a function from $Aut(\chi_1)$ to $Aut(\chi_2)$ such that
$(\theta_1, \theta_2) \longmapsto (f, g)^{-1} \circ (\theta_1, \theta_2) \circ (f, g)$ and
$\phi((\theta_1, \theta_2) \circ (\theta_1', \theta_2')) = \phi(\theta_1 \circ \theta_1', \theta_2 \circ \theta_2') = (f, g)^{-1} \circ (\theta_1 \circ \theta_1', \theta_2 \circ \theta_2') \circ (f, g)$
$= (f, g)^{-1} \circ ((\theta_1, \theta_1') \circ (\theta_2, \theta_2')) \circ (f, g)$

$= (f,g)^{-1} \circ ((\theta_1, \theta_1') \circ (f,g) \circ (f,g)^{-1} \circ (\theta_2, \theta_2')) \circ (f,g)$
$= \phi(\theta_1, \theta_2) \circ \phi(\theta_1', \theta_2')$
Hence ϕ is a homomorphism and it is easy to see that the map $\psi : Aut(\chi_2) \longrightarrow Aut(\chi_1)$ such that $(\theta_1', theta_2') \longmapsto (f,g) \circ (\theta_1', \theta_2')(f,g)^{-1}$ is the inverse of ϕ. Hence ϕ is a bijective homomorphism. So $Aut(\chi_1) \cong Aut(\chi_2)$

□

Remark 8. Above lemma expalains why it makes sense to ask about **the** automorphism group of an incidence system.

1.6.1 Examples of automorphisms of $EG(2, \mathbb{F})$

1. The group of all translations by vector in $V = \mathbb{F}^2$ i.e. the group $(A, \circ) = \{f_b : V \longrightarrow V | f(v) = v + b, b \in V\}$. This group is isomorphic to $(V, +)$ under the map $w \longmapsto f_w$. Every translation defines an automorphism of $EG(2, \mathbb{F})$.
2. The group of all invertible linear transformations from V to V occurs as a subgroup of $EG(2, \mathbb{F})$.
3. The group $Aut(\mathbb{F})$ of all field automorphisms of $\mathbb{F}$ occurs as a subgroup.

1.6.2 Semi-direct product

Semi direct product is the generalization of the idea of direct product. Semi-direct product is of two types -inner and outer semi-direct product. As in case of direct product, both semi direct products are equivalent.

Definition 12. Let H and K be disjoint subgroups of G such that $H \trianglelefteq G$ such that $\frac{G}{H} \cong K$. Then we say that G is a **split extension of** H **by** K. It is written as $G : H : K$.

Definition 13. Given a group G, $H \leqslant G$, $N \trianglelefteq G$. We say that G is the (inner) semi-direct product of N and H (or) G splits over N by H if any one of the following statements hold :

- $G = NH$ such that $N \cap H = \{1\}$
- $\forall g \in G$, $\exists$ unique $n \in N$ and $h \in H$ such that $g = nh$
- $\forall g \in G$, $\exists$ unique $n \in N$ and $h \in H$ such that $g = hn$
- there exists a homomorphism from G to H whose kernel is N.

It is written as $G = N \rtimes H$

Definition 14. Given any two groups N, H and a group homomorphism $\phi : H \longrightarrow Aut(N)$ then we can construct a new group called (outer) semi-direct product of N and H with respect to ϕ defined as follows :

- underlying set is the cartesian product $N \times H$.
- the new group, i.e. the outer product $N \rtimes_\phi H$ has the product of elements as $(n_1, h_1)\dot{}(n_2, h_2) = (n_1\phi_{h_1}(n_2), h_1h_2)$ for all $n_1, n_2 \in N$ and $h_1, h_2 \in H$

It is easy to verify that $N \rtimes_\phi H$ is a group under above multiplication rule.

Note

1. In case of the outer product, it is easy to see that (e_N, e_H) is the identity element. The inverse of the element $(n, h) \in N \rtimes_\phi H$ is $(\phi_{h^{-1}}(n^{-1}), h^{-1})$.
2. The subgroup $\{(n, e_H)|n \in N\} \cong N$. Similarly the subgroup $\{(e_N, h)|h \in H\} \cong H$.
3. The group $N \rtimes_\phi H$ can be viewed as the inner semi direct product of $\{(n, e_H)|n \in N\} \cong N$ and $\{(e_N, h)|h \in H\} \cong H$.
4. Conversely, if G is a group , $N \trianglelefteq G$ and $H \leqslant G$ such that every element of g may be written uniquely in the form $g = nh$ where $n \in N$ and $h \in H$. ϕ is the map which takes $h \in H$ to $\phi_h \in Aut(N)$ where $\phi_h : N \longmapsto N$ takes $x \in N$ to hxh^{-1}. Under above conditions, $G \cong N \rtimes_\phi H$ under the map $\psi : G \longrightarrow N \rtimes_\phi H$ such that $g \longmapsto (n, h)$ where n, h are such that $g = nh$.
5. If we take the homomorphism ϕ as the trivial homomorphism i.e. it takes every element of H to the identity automorphism of $Aut(N)$, then semi-direct product becomes the direct product.
   Hence direct product is a special case of the semi-direct product.

Group generated by $(V, +)$ and $GL(2, \mathbb{F})$

We know that $(V, +) = \{f_b : V \longrightarrow V|v \longmapsto v + b, b \in V\}$ and $GL(2, \mathbb{F}) = \{g_A : V \longrightarrow V|det(A) \neq 0\}$ It is easy to verify that the elements of the group formed by $(V, +)$ and $GL(2, \mathbb{F})$ is the group $H = \{h_{A,b} : V \longrightarrow V, v \longmapsto vA + b \mid det(A) \neq 0, b \in V\}$. Also it is easy to check that the inverse of $h_{A,b} = f_b \circ g_A$ is $h^{-1}_{A,b} = g_{A^{-1}} \circ f_{-b}$. Using above points we can verify that $(V, +)$ is the normal subgroup of H. Also no translation is a linear transformation. Hence $(V,+)$ and $GL(2,\mathbb{F})$ are disjoint.

Summary

- $(V,+) \leqslant H$
- $GL(2,\mathbb{F}) \trianglelefteq H$
- $(V,+) \cap GL(2,\mathbb{F}) = \Phi$
- $H = (V,+).(GL(2,\mathbb{F}))$

Hence H is the semi- direct product of $(V,+)$ and $GL(2,\mathbb{F})$
$K = AGL(2,\mathbb{F}) = sp((V,+), GL(2,\mathbb{F}))$

Consider the group generated by K and $Aut(\mathbb{F})$. Let the automorphism subgroup of $EG(2,\mathbb{F})$ which is isomorphic to $Aut(\mathbb{F})$ be the set $\{\theta_f : \mathbb{F}^2 \longrightarrow \mathbb{F}^2 \mid f \in Aut(\mathbb{F})\}$
If f_b is an element of $(V,+)$ and g_A is an invertible linear transformation, then the elements of $AGL(2,\mathbb{F})$ are of the forms $h_{b,A} = f_b \circ g_A$ and $h_{A,b} = g_A \circ f_b$. Without loss of generality , we can assume all elements to be of the form $h_{b,A}$. Then the subgroup generated by K and $Aut(\mathbb{F})$ has elements of the type $h_{b,A} \circ \theta_f$ such that $v \longmapsto f(v)A + b$, $\theta_f \circ h_{b,A}$ such that $v \longmapsto f(vA) + f(b)$. . We can also show that the elements of $sp(K, Aut(\mathbb{F})$ are of the forms γ and γ' such that the maps are $v \longmapsto f(vA) + c$ and $v \longmapsto f(v)A + c$ respectively.($A \in GL(2,\mathbb{F}), c \in V and f \in Aut(\mathbb{F})$.Easy to prove that $K \trianglelefteq sp(K, Aut(\mathbb{F}))$

The set $\{\theta_f : \mathbb{F}^2 \longrightarrow \mathbb{F}^2 \mid f \in Aut(\mathbb{F})\}$ has elements θ_f which take v to $f(v) = vA + b$.
$\theta_f(v_1 + v_2) = f(v_1 + v_2) \neq f(v_1) + f(v_2)$. So θ_f can't be a homomorphism, hence it is not an automorphism.

1.7 Bruck Nets

Definition 15. Incidence system is called a **partial linear space** if

- any line is incident with at least two points
- any two distinct points are together incident with **at most** one line.

Definition 16. Incidence system is called a **linear space** if

- any line is incident with at least two points

- any two distinct points are together incident with **exactly** one line.

The definition of parallel is same as earlier. Lines are parallel either when they are equal or when no point is incident on both the lines. Note that every linear space is also a partial linear space but the converse is not true, i.e. every partial linear space is not a linear space.

Definition 17. Bruck Net is a partial linear space which follows playfair's axiom.

So "parallel" is an equivalence relation on the set of lines. Consider the equivalence class $[l]$. For a point A not on l, there is a unique line l_A which is parallel to l and passing through A. Hence every point is on a unique line of the equivalence class. Hence the set of points on the lines partition the point set of the bruck net.

Definition 18. Degree of a bruck net is the number of parallel classes (r). It will be called a bruck net of degree "r''.Also every point is lying on exactly one line from each parallel class.

Remark 9. A bruck net of degree 1 is just an arbitrary partition.

Bruck net of degree 2

Bruck net of degree 2 has two parallel classes. Let the parallel classes be $[p]$ and $[q]$. Let $[p] = \{p_1, p_2, ...p_m\}$ for some $m \in \mathbb{N}$ and let $[q] = \{q_1, q_2, ...q_n\}$ for some $n \in \mathbb{N}$. Also 2 lines are incident with each point.
Claim: Number of lines in the parallel class $[p]$ is n and similarly number of lines in $[q]$ is m
Consider any line $p_i \in [p]$. Then any line which can intersect p_i are lines of the class $[q]$. Each line q_i has to intersect p_i in a unique line. Hence the number of points on p_i is n. Similarly the number of points on q_i is m. Summarizing what we showed as a proposition , we get:

Proposition 4. If χ is a finite bruck net of degree two, then there exists two numbers $m \geqq 2$, $n \geqq 2$ such that one parallel class has m lines of size n each and other parallel class consists of n lines of size m each.

Corollary 1.7.0.1. Let χ be a finite bruck net of degree $r \geqq 3$. Then there is a number $n \geqq 2$ such that each line has size n, each parallel class has n lines and total number of points is n^2.

Proof. Let the number of parallel classes be r. Let the parallel classes be $[P_1], [P_2], ...[P_r]$. Each point P on $p_i \in [P_i]$ should be incident on r lines. Exactly one line from each equivalence class passes through P (because two lines from the same equivalence class cannot intersect at P and since points on lines of equivalence class partition the point set, we can say that every point is on exactly one line of each parallel class). Since $r \geqq 3$, there exist at least three parallel classes - $[l], [k], [o]$. Let $|[l]| = m$ and each line of $[l]$ has n points on them. Every line k' of $[k]$ is intersected at m different points on k' by the m lines of $[l]$. Since every point of the line k' in $[k]$ should have a unique line through parallel class $[l]$, we can say that number of points on k' is m and number of lines is n
So $|[l]| = m, |[k]| = n, \#(pointsonl' \in [l]) = n$ and $\#(pointsonk' \in [k]) = m$
Now consider the parallel classes $[k]$. $k' \in [k]$ has m points on it and the class has n lines in it. Using the argument we used above we can say that the $|[o]| = m$ and number of points on $o' \in [o]$ is n.
Same argument for parallel classes $[l]$ and $[o]$ gives that $|[o]| = n$ and number of points on o' is m.
From above two lines , it can be concluded that $m = n$.
Hence the number of points on each line = number of lines in a parallel class = n. This means that the total number of points is $n \times n = n^2$.

□

Proposition 5. Given $m \geqq 2$, $n \geqq 2$. Show that upto isomorphism there is a unique bruck net of degree 2 with given values of m and n.

Proof. Given m and n. Consider the two parallel classes $[l]$ and $[m]$ where $[l] = \{l_1, l_2, ...l_m\}$ and $[k] = \{k_1, k_2, ...k_n\}$. From an argument given previously, we can say that each line of $[l]$ has n points on it and each line of $[k]$ has m points on it. l_i and k_j for $1 \leqq i \leqq m, 1 \leqq j \leqq n$. These two lines determine all mn points of the bruck net. Hence if we take two different points sets of two bruck nets χ_1 and χ_2, then the point set is $\{{}^1l_i \wedge^1 k_j | 1 \leqq i \leqq m, 1 \leqq j \leqq n\}$ and $\{{}^2l_i \wedge^2 k_j | 1 \leqq i \leqq m, 1 \leqq j \leqq n\}$. Consider the map ${}^1l_i \wedge^1 k_j \longmapsto^2 l_i \wedge^2 k_j$. It is clearly a bijection which takes blocks to blocks. Hence the incidence systems χ_1 and χ_2 are isomorphic. □

Proposition 6. χ is a finite bruck net of degree $r \geqq 3$ and order $n \geqq 2$. Then $r \leqq n + 1$.

Proof. Consider any point of χ. Call it x. There are $n^2 - 1$ points other than x. There are r lines through x.Consider the blocks through x without the point

x, then we get r disjoint sets/blocks each of which has $n-1$ points other than x. So the number of points covered by these r lines are $r(n-1)$(all lines may not be covered as a bruck net is a partial linear space.)Hence $r(n-1) \leqq n^2 - 1 \implies r \leqq n+1$ □

Remark 10. $r = n+1$: Then there are $n+1$ equivalence classes, each pointt is on $n+1$ lines , each line has n points on it and there are n^2 points in total. Hence the bruck net becomes an affine plane when $r = n+1$.It is easy to see that when equality holds, then bruck net is a linear space.

Corollary 1.7.0.2. Let π be an affine plane of order n, then for any $r \leqq n+1$, retaining only r classes of the $n+1$ classes, we get a bruck net of order n and degree r.

More generally

If χ is a bruck net of degree r and order n, $r' \leqq r$, then retaining retaining any r' of the parallel classes of χ, we get a bruck net $\mathcal{Y}$ of degree r' and order n.

What we have proved so far can be written as a theorem which is

Theorem 1.7.1. In a finite bruck net there exists $r \geqq 3$ and $n \geqq 2$ such that the following are true :

1. there are "r" lines incident with every point.
2. there are "r" parallel classes.
3. there are "n" points on every line.
4. there are "n" lines in each parallel class
5. there are "n^2" points in total
6. there are "rn" lines in total
7. two lines in distinct parallel classes intersect in a unique point.

Definition 19. For $n \geqq 2$, let $\mathcal{N}(n)$ denote the largest number r such that there is a bruck net of degree r and order n.
Thus $\mathcal{N}(n) \leqq n+1$

Remark 11. $\mathcal{N}(n) = n+1$ iff $\exists$ euclidean plane of order n. In particular $\mathcal{N}(n) = n+1$ for any prime power n.

Example

$n \geqq 2$. $(G,+)$ - any group of order n. (Group may or may not be abelian.)
Consider the incidence system $\chi = \chi(G)$ with points set $G \times G$ and lines $l_1 \equiv \{(x,c)|x \in G\}$, $l_2 \equiv \{(c,x)|x \in G\}$ and $l_3 \equiv \{(x,c-x)|x \in G\}$ where $c \in G$. Then $\chi(G)$ is a bruck net of order n and degree 3.

Proof. The point set is $G \times G$. Hence number of points is n^2. The lines l_1, l_2 clearly have n points each on the lines. Consider the line l_3 now. Let $(x_1, c - x_1) = (x_2, c - x_2)$. Then $x_1 = x_2$. Hence there are n distinct points on l_3 also . Hence each line has n points on them.
The equivalence class for the line l_1 is $[l_1] = \{\{(x,c)|x \in G\}, \{(x,c')|x \in G, c \neq c' \in G\}\}$.Now $[l_2] = \{\{(c,x)|x \in G\}, \{(c',x)|x \in G, c \neq c' \in G\}\}$ and $[l_3] = \{\{(x,c-x)|x \in G\}, \{(x,c'-x)|x \in G, c \neq c' \in G\}\}$. All possible lines have been covered. Hence the bruck net has three parallel classes. So the bruck net is of degree 3. □

Remark 12. Note that $\mathcal{N}(2) = 3$, $\mathcal{N}(3) = 4$, $\mathcal{N}(5) = 6$. (Because there exist affine planes of order 2,3,4,5).

Some Conjectures

We can say that $3 \leq \mathcal{N}(n) \leq n + 1 \forall n \geq 2$. Hence for $n = 6$, we can say that $3 \leq \mathcal{N}(6) \leq 7$.
Euler conjectured that $\mathcal{N}(6) = 3$. (He actually asked a question equivalent to this.) This conjecture was proved to be true in the beginning of 20th century. It proves that there does not exist an affine plane of order 6.
Euler conjectured that if $n \equiv 2 (\mod 4)$ then $\mathcal{N}(n) = 3$. This conjecture was proved wrong in 1960s by Bose , Shikande and Parker. They used proof by construction to show that $\mathcal{N}(n) \geq 4 \forall n \geq 7$.
Erdos proved that $\mathcal{N}(n) \longrightarrow 0$ as $n \longrightarrow \infty$
Open Question: Is it true for instance, that $\mathcal{N}(n) \geq n^{\frac{1}{2}}$ for all sufficiently large n.

1.8 Block Designs

Definition 20. A **2-design** with v, k, λ (in short a $2 - (v, k, \lambda)$ design) is an incidence system with v points , k points per block , λ blocks per pair of distinct points.

Examples

- Any affine plane of order n is a $2-(n^2, n, 1)$ design
- Any projective plane of order n is a $2-(n^2+n+1, n+1, 1)$ design.

Conversely, any $2-(n^2, n, 1)$ design is an affine plane of order n and any $2-(n^2+n+1, n+1, 1)$ design is a projective plane of order n. $(n \geqq 2)$

Theorem 1.8.1. Integrity condition / Divisibility condition
If X is a $2-(v, k, \lambda)$ design then there is a number r such that every point of X is incident with exactly r blocks. This number r is given by the formula : $r \times (k-1) = \lambda \times (v-1)$. So $(k-1) \mid \lambda(v-1)$.
Also the total number of blocks of $X(b)$ is given by the formula $bk = rv$. Thus $b = \frac{rv}{k} = \frac{\lambda v(v-1)}{k(k-1)}$. So $k(k-1) \mid \lambda v(v-1)$.
Where b, v, r, k, λ are the auxiliary parameters of 2- design.

Proof. Fix a point x. Let $r(x)$ be the number of blocks through x. We have to show that $r(x)$ is independent of the point x and $r(x) = \frac{\lambda(v-1)}{k-1}$
To show this, we count in two ways the ordered pair (y, B) wher $y \neq x$ is a point and B is a block incident with both x and y.
First count: We can choose y in $v-1$ ways. Then we have λ choices for B. So total number of points is $\lambda \times (v-1)$.
Second count: Choose B first, $r(x)$ choices for B and $k-1$ choices for y. So total points is $r(x)(k-1)$
Equating both answers , we get $r(x) = \frac{\lambda \times (v-1)}{k-1}$. So $r(x)$ is independent of x.
Similarly if we count the pairs (x, B) where x is a point, B is a block incident with x. Then we get $bk = rv$. □

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Example

Easy to verify that the following incidence system is a 2- design :
Take a set of v points(Point set). Take all subsets of size k as blocks.

Definition 21. A 2- design is called **trivial** if $v = k$ (or) $r = \lambda$. The 2- design is called **non - trivial** otherwise.
Note that for a non-trivial 2-design , $r \geq \lambda$. The number $n = r - \lambda$ is called the **order of the** 2**- design**.

Definition 22. Let X be a finite incidence system with v points and b blocks. Then the incidence matrix N of X is the $v \times b$ matrix with rows indexed by the points of X, such that for any point x and block B the (x, B)th entry of N is:
$N(x, B) = 1$ if xIB and $N(x, B) = 0$ otherwise.

Notations

$I_{v\times v}$ is the $v \times v$ identity matrix
$J_{m,n}$ is the $m \times n$ matrix all whose entries are 1.

Lemma 1.8.2. Let N be the incidence matrix of a 2- design. Then it is easy to verify that

1. $J_{v,b}N = kJ_{v,b}$
2. $NJ_{v,b} = rJ_{v,b}$
3. $NN^T = (r - \lambda)I_v + \lambda J_{v,v}$

Theorem 1.8.3. Fisher's inequality
The parameters of any non-trivial design satisfy $b \geqq v$ or equivalently $r \geqq k$. Also the following are equivalent for any non- trivial 2-design X:

1. $b = v$
2. $r = k$
3. any two distinct blocks are together incident with exactly λ blocks , and
4. the dual X^* of X is also a $2 - (v, k, \lambda)$ design.

Proof. Following is the prof by Bose.
Let N be the incidence matrix of the given non-trivial 2-design. Then $NN^T = (r - \lambda)I_v + \lambda J_{v,v}$. Since it is non-trivial, $r > \lambda$ and hence $(r - \lambda)I_v > 0$ and also $\lambda J_v \geq 0$. So $NNT > 0$.
In particular, NN^T is a $v \times v$ non-singular matrix. So rank NN^T = rank N $= v$. So $b \geq v$.
Now suppose that $b = v$. Hence $r = k$. Thus in this case, (1) and (2) hold. Need to show that (3) and (4) hold.
If (4) holds i.e. X^* is a $2 - (v^*, k^*, \lambda^*)$ design. Then $v^* = b = v$, $k^* = r = k$. Since $\lambda^*(v^* - 1) = r^*(k^* - 1)$. It follows that $\lambda^* = \lambda$. So (3) holds.
Thus enough to prove that (1) $\equiv$ (2) implies (4).

N is a $v \times v$ matrix, so rank $(N) = v \implies N$ is non-singular.
By the pervious lemma, we have $JN = NJ = kJ$, $NN^T = (k - \lambda)I + \lambda J$ (all are square matrices of order v.)
We get that $N^T = (k - \lambda)N^{-1} + \lambda N^{-1}J$. But $NJ = kJ \implies J = kN^{-1}J \implies N^{-1}J = \frac{1}{k}J$
So $N^T = (k - \lambda)N^{-1} + \frac{\lambda}{k}J$.
So $NN^T = (k - \lambda)I + \frac{\lambda}{k}JN = (k - \lambda)I + \frac{\lambda}{k}kJ$.
So $N^TN = (k - \lambda)I + \lambda J$. Also $N^TJ = (k - \lambda)N^{-1}J + \frac{\lambda}{k}J^2$.
i.e. $N^TJ = \frac{k-\lambda}{k} + \frac{\lambda v}{k}J = \frac{\lambda(v-1)+k}{k}J = \frac{k(k-1)+k}{k}J = kJ$.
So $N^TN = (k - \lambda)I_v + \lambda J_{v,v}$. So N^T is a zero-one matrix satisfying the conclusion of lemma.
So NT is the incidence matrix of a $2-(v, k, \lambda)$ design. But N^T is the incidence matrix of X^*. So X^* is a $2 - (v, k, \lambda)$ design. □

Definition 23. A non-trivial 2- design satisfying equality equality in Fisher is called a square 2- design (because the incidence matrix is a square matrix).
Note that non trivial $2 - (v, k, \lambda)$ design is a square 2 design iff parameters satisfy $k(k - 1) = \lambda(v - 1)$.
Thus projective planes are square 2-designs.

Definition 24. A 2- design is callled steiner 2- design if it has $\lambda = 1$
Example: Finite projective planes are square steiner 2- designs.

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