Abel Summation and Character mod q

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1 Partial summation formula

Partial summation technique also called as Abel Summation was first introduced by Niels Henrik Abel.

Theorem 1. *Let* $(a_n)_{n=1}^{\infty}$ *be a sequence of complex numbers and* $f(t)$ *a differentiable function for* $t \geq 0$*.Set* $A(x) = \Sigma a_n$ *for* n*.Then*

$$\Sigma(a_n \times f(n)) = A(x)f(x) - \int_1^x A(t)f^{'}(t)$$

Proof: We can write a_n as $a_n = A(n) - A(n-1)$ and let $k \leq x < k+1$ where k is a natural number.

$$\begin{gathered}
\Sigma_{n\leq x}(a_n f(n)) = \Sigma_{n\leq k}(a_n f(n)) \\
\Sigma_{n\leq k}(A(n) - A(n-1))f(n) = \Sigma_{n\leq k}(A(n)f(n)) - \Sigma_{n\leq k}(A(n-1)f(n)) \\
= \Sigma_{n\leq k}(A(n)f(n)) - \Sigma_{n\leq k}(A(n-1)f(n)) \\
= \Sigma_{n\leq k}(A(n)f(n)) - \Sigma_{n\leq k-1}(A(n)f(n+1)) \\
= A(k)f(k) + \Sigma_{n\leq k-1}A(n)(f(n) - f(n+1))
\end{gathered}$$

but $f(n) - f(n+1)$ can be written as $-\int_n^{n+1} f'(t)dt$
So, $\Sigma_{n\leq k-1}A(n)(f(n) - f(n+1)) = -\Sigma_{1\leq k-1}A(n)\int_n^{n+1} f'(t)dt$
But $A(t) = A(n)$ when $n \leq t \leq n+1$.Hence

$$\begin{gathered}
\Sigma_{1\leq k-1}A(n)\int_n^{n+1} f'(t)dt = \int_1^k A(t)f'(t)dt \\
\int_1^k A(t)f'(t)dt = \int_1^x A(t)f'(t)dt - \int_k^x A(t)f'(t)dt \int_k^x A(t)f'(t)dt = A(k)\int_k^x f'(t)dt \\
= A(k)f(x) - A(k)f(k) = A(x)f(x) - A(k)f(k)
\end{gathered}$$

Hence we get

$$\boxed{\Sigma(a_n \times f(n)) = A(x)f(x) - \int_1^x A(t)f^{'}(t)}$$

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1.1 Big O notation

Big O notation (with a capital letter O, not a zero) is also called Landau's symbol.It describes the asymptotic behaviour of functions. It tells how fast a function is growing or declining.O is used because rate of growth of function is also called its order.
If f(x) and g(x) are defined on some subset of real numbers we write f(x)=O(g(x)) if and only if $\exists N$ and c such that

$$|f(x)| \leq c|g(x)| \forall x > N$$

If "a" is some real number, then

$$f(x) = O(g(x)) for x \to a$$
if and only if $\exists d > 0$ and c such that
$$|f(x)| \leq c|g(x)| \forall x \text{ with } |x - a| < d$$

1.1.1 Examples on Abel Summation

Example 1 Show that $H_x = \Sigma_{n \leq x}(1/n) = \gamma + \log x + O(1/x)$
Proof: Here we take $a_n = 1$ and $f(x) = 1/x$ and $A(x) = \Sigma_{n \leq x} 1 = \lceil x \rceil$
We now use the partial summation formula that we derived in section 1

$\Sigma_{n \leq x}(1.(1/n)) = \lceil x \rceil.(1/x) - \int_1^x [t](-1/t^2)dt$
$\Sigma_{n \leq x}(1.\frac{1}{n}) = \frac{\lceil x \rceil}{x} + \int_1^x \frac{\lceil t \rceil}{t^2}dt$ and $\lceil x \rceil = x - \{x\}$
$\Sigma_{n \leq x}(1.\frac{1}{n}) = \frac{x - \{x\}}{x} + \int_1^x \frac{t - \{t\}}{t^2}dt$
$\Sigma_{n \leq x}(1.\frac{1}{n}) = 1 - \frac{\{x\}}{x} + \log x - \int_1^x \frac{\{t\}}{t^2}dt$
$\int_1^x \frac{\{t\}}{t^2}dt$ is convergent as $x \to \infty$
Thus we can write,$\int_1^x \frac{\{t\}}{t^2} = \int_1^\infty \frac{\{t\}}{t^2} - \int_x^\infty \frac{\{t\}}{t^2}$
for large x $\int_x^\infty \frac{\{t\}}{t^2} = O(\frac{1}{x})$ and $\int_1^\infty \frac{\{t\}}{t^2} = O(\frac{1}{x})$ is a constant(c) as the integral is convergent.
So, $H_x = 1 - \frac{\lceil x \rceil}{x} + \log x - (c - O(\frac{1}{x}))$
$H_x = (1 - c) - \frac{\lceil x \rceil}{x} + \log x + O(\frac{1}{x})$
but $\frac{\lceil x \rceil}{x} = O(\frac{1}{x})$ and take $1 - c = \gamma$
$H_x = \gamma + \log x + O(\frac{1}{x})$ where γ is the Euler's constant.

Example 2 Show that $\Sigma_{n \leq x} \log n = x \log x - x + O(\log x)$
Proof: Here we take a_n as 1 and $f(t) = \log t$. Applying Abel's lemma, we get
$\Sigma_{n \leq x} \log n = \lceil x \rceil \log x - \int_1^x \frac{\lceil t \rceil}{t}dt$
take $\lceil x \rceil = x + O(1)$
$\Sigma_{n \leq x} \log n = (x + O(1)) \log x - \int_1^x \frac{(t + O(1))dt}{t}$
$\Sigma_{n \leq x} \log n = x \log x + O(\log x) - (x - 1) + O(\log x)$
$\Sigma_{n \leq x} \log n = x \log x - x + O(\log x) + 1$
$+1$ can be ignored as $\log x > 1$
$\Sigma_{n \leq x} \log n = x \log x - x + O(\log x)$

Example 3 Show that $\Sigma n \leq x\frac{1}{n} = \log x + O(1)$. In fact show that $\lim_{x\to\infty}(\Sigma_{n\leq x}(\frac{1}{n}) - \log x)$
Proof: $\Sigma n \leq x\frac{1}{n} = \lceil x\rceil \frac{1}{x} - \int_1^x \lceil t\rceil(-\frac{1}{t^2})dt$
take $\lceil x\rceil = x + O(1)$
$\Sigma n \leq x\frac{1}{n} = \frac{x+O(1)}{x} + \int_1^x \frac{t+O(1)}{t^2}dt = 1 + O(\frac{1}{x}) + \log x + \int_1^x \frac{O(1)dt}{t^2}$
$\int_1^x \frac{O(1)dt}{t^2}$ is a convergent integral. It can be written as $O(1)$
So, $\Sigma n \leq x\frac{1}{n} = \log x + O(1)$
For the second part $\Sigma_{n\leq x}(\frac{1}{n} - \log x$ can be written as $\Sigma_{n\leq x}\frac{1}{n} - \int_1^x \frac{dt}{t}$
$\Sigma_{n\leq x}(\frac{1}{n} - \int_n^{n+1}\frac{dt}{t}) = \Sigma_{n\leq x}(\frac{1}{n} - (\log t)_n^{n+1}) = \Sigma_{n\leq x}(\frac{1}{n} - \log\frac{n+1}{n})$
We know that $\log(1+\frac{1}{n}) = \frac{1}{n} - \frac{1}{2}\frac{1}{n^2} + \frac{1}{3}\frac{1}{n^3} - ......$
Using the above expansion we get $\Sigma_{n\leq x}(\frac{1}{n} - \log\frac{n+1}{n}) = \Sigma_{n\leq x}(\frac{1}{2}\frac{1}{n^2} - \frac{1}{3}\frac{1}{n^3} + ...)$
The latter term has a limit as $n \to \infty$
Hence the given summation has a limit as $n \to \infty$

Example 4 Show that $\Sigma_{n\leq x}d(n) = x\log x + O(x)$. $d(x)$ here represents the number of divisors of $n \leq x$.
Proof: $d(n) = \Sigma_{d|n}1$ then $\Sigma_{n\leq x}d(n) = \Sigma_{d\leq x}(\Sigma_{n\leq x}1)$ when $d|n$
if$d|n$ then $n = dt$ for some t.Then $t \leq \frac{x}{d}$.Hence the double summation can be replaced by $\Sigma_{d\leq x}\lceil\frac{x}{d}\rceil = \Sigma_{d\leq x}(\frac{x}{d} + O(1))$ as $\lceil\frac{x}{d}\rceil = \frac{x}{d} + O(1)$
$\Sigma_{d\leq x}(\frac{x}{d} + O(1)) = x\Sigma_{d\leq x}\frac{1}{d} + \Sigma_{d\leq x}O(1)$
We have seen earlier that $\Sigma_{d\leq x}\frac{1}{d} = \log x + O(1)$
Using this we can write $\Sigma_{d\leq x}(\frac{x}{d} + O(1)) = x(\log x + O(1)) + xO(1)$
$\Sigma_{d\leq x}(\frac{x}{d} + O(1)) = x\log x + O(x) + O(x) = x\log x + O(x)$

Example 5 Suppose $A(x) = O(x^\delta)$. Show that for $s > \delta$, $\Sigma_{n=1}^{\infty}\frac{a_n}{n^s} = s\int_1^\infty \frac{A(t)dt}{t^{s+1}}$ .Hence the Dirichlet series converges for $s > \delta$
Proof:Apply Abel summation by taking $f(x) = x^{-s}$
Hence $\Sigma_{n\leq x}a_n f(n) = A(x)f(x) - \int_1^x A(x)f'(t)dt$
$\Sigma_{n\leq x}a_n f(n) = O(x^\delta)(x^{-s}) - \int_1^x O(x^\delta)(-s^{-(s+1)})tdt$
$\Sigma_{n\leq x}a_n f(n) = \frac{O(x^\delta)}{x^s} + s\int_1^x \frac{O(x^\delta)dt}{t^{s+1}}$
The former term in the above line tends to 0 as $x \to \infty$ only when $s > \delta$. The latter term has a bounded numerator and the denominator tends to infinity as $x \to \infty$
Hence when $x \to \infty$, $\Sigma_{n\leq x}a_n f(n) = s\int_1^x \frac{O(x^\delta)dt}{t^{s+1}}$
The right hand side is a convergent integral only when $s > \delta$

Example 6 Show that for $s > 1$; $\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}dx}{x^{s+1}}$
Show that $\lim_{s\to 1^+}(s-1)\zeta(s) = 1$
Proof:$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}dx}{x^{s+1}}$
$\lim_{s\to 1^+}(\frac{s}{s-1} - s\int_1^\infty \frac{\{x\}dx}{x^{s+1}})$
$\lim_{s\to 1^+}(s-1)(\frac{s}{s-1} - s\int_1^\infty \frac{\{x\}dx}{x^{s+1}})$
$\lim_{s\to 1^+}(s-1)\zeta(s) = s - s(s-1)\int_1^\infty(\frac{1}{x^s} - \frac{\lceil x\rceil}{x^{s+1}})dx$
The integral in above line is always convergent and the term $s-1$ tends to 0. Hence the limit value is 1 .The integral converges for $s > 0$

1.2 Bernoulli's numbers

These numbers were discovered by Jacob Bernoulli.These numbers hold a deep relation with Riemann zeta function(this function is related to prime numbers) If $S_n(n)$ refers to n_{th} powers of natural numbers then S_n is written as $\Sigma_{k\leq p}\frac{B_k}{k!}\frac{p!}{(p+1-k)!}n^{p+1-k}$ where each B_n is called a Bernoulli's number and they are independent of p.

Modern Definition: Series expansion of $\frac{z}{e^z-1} = \Sigma_{k=0}^{\infty}B_k\frac{z^k}{k!}$ has the cofficients of its terms as Bernoulli numbers.

Bernoulli Polynomials:Consider the sequence of polynomials $b_r(x)$ defined recursively as

$b_0(x) = 1$

$b'_r = rb_{r-1}(x)r \geq 1$

$\int_0^1 b_r(x)dx = 0$

For example B_3 is given by $x^3 - \frac{3x^2}{2} + \frac{1x}{2}$

We get the Bernoulli number when we substitute 0 in place of x

Question: Prove that $F(x,t) = \Sigma_{r=0}^{\infty}b_r(x)\frac{t^r}{r!} = \frac{te^{xt}}{e^t-1}$

$\frac{dF(x,t)}{dx} = \Sigma_{r=1}^{\infty}b'_r(x)\frac{t^r}{r!}$

$\Sigma_{r=1}^{\infty}b'_r(x)\frac{t^r}{r!} = \Sigma_{r=1}^{\infty}b_{r-1}(x)\frac{t^{r-1}}{(r-1)!t}$

Hence $\frac{dF(x,t)}{dx} = t \times F(x,t)$

So $t = \frac{F'(x,t)}{F(x,t)}$ which means $tdx = d(\log(F(x,t)))$

hence $F(x,t) = e^{tx+c(t)}$ Given that $\int_0^1 F(x,t)dx = 1$

so, $\int_0^1 e^{tx+c(t)}dt = e^{c(t)}\frac{(e^t-1)}{t} = 1$

so $e^{c(t)} = \frac{t}{e^t-1}$

Therefore $F(x,t) = e^{c(t)}.e^{tx} = \frac{e^{tx}.t}{(e^t-1)}$

Question: Show that $B_{2r+1} = 0$ for $r \geq 1$ Proof: $\frac{t}{2} + \Sigma_{r=0}^{\infty}b_r(0)\frac{t^r}{r!}$

$= \frac{t}{2} + \frac{t}{e^t-1} = \frac{t(e^t+1)}{2(e^t-1)}$

The right hand side is an even function.So $b_r(0) = 0$ when r is odd. So $B_r(0) = 0$ when $r = 1n + 1$ form or odd.

1.3 Characters mod q

$(\mathbb{Z}/q\mathbb{Z})$: This denotes the set containing all possible remainders when a number is divided by q.

$(\mathbb{Z}/q\mathbb{Z})^*$: This is a multiplicative group that is every element x in the set has an element y such that $xy \equiv 1 mod q$

So $xy - qt = 1$ for some t. This can have integer solutions for y and t only if $\gcd(x,q) = 1$ Hence this set contains all elements co-prime to q.Hence the order of this group(number of elements in group) is $\varphi(q)$ that is the Euler-phi function.

Characters: They are a homomorphism from a multiplicative group $(\mathbb{Z}/q\mathbb{Z})^*$ to the multiplicative group of complex numbers $\mathbb{C}^*$

We know that the Euler theorem states that $a^{\varphi(q)} \equiv 1 mod q$

Hence $\chi^{\varphi(q)(a)} = 1 \forall a in (\mathbb{Z}/q\mathbb{Z})^*$. Thus $\chi(a)$ is the $\varphi(a)^{th}$ root of 1.

We extend χ to all numbers by defining it as follows:

$\chi(n) = \chi(n mod q)$ if $\gcd(n,q) = 1$ otherwise it is given the value 0 which does not

belong to the set $\mathbb{C}^*$
From the above definition of χ and using the fact that it is a homomorphism we can say that it is a completely multiplicative function that is $f(a*b) = f(a)*f(b)$. The operation we consider here is composition of functions.
QuestionWe define L-series as L(s,χ) $= \Sigma_{n=1}^{\infty} \frac{\chi(n)}{n^s}$
Prove that for$Re(s) > 1$,$L(s,\chi) = \prod_p (1 - \frac{\chi(p)}{p^s})^{-1}$ where product is over primes p.
Proof: $L = \Sigma_{n=1}^{\infty} \frac{\chi(n)}{n^s}$
$L - L\chi(2)/2^s = \Sigma_{n=1}^{\infty} \frac{\chi(2n)}{(2n)^s}$ So all even number denominator terms are removed. This is true as χ is multiplicative.
Similarly subtracting $L\chi(3)/3^s$ will remove all the fractions whose denominator has multiples of 3.
$L \prod (1 - \frac{\chi(n)}{n^s}) = 1$
So $L = \prod (1 - \frac{\chi(n)}{n^s})^{-1}$

References

[1] Silverman, Joseph H. (2012). *A Friendly Introduction to Number Theory.* Pearson Inc.

[2] M Ram Murthy . *Problems in analytic number theory*

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