Lotka Volterra Dynamics

Padma Tanikella

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1 The Lotka-Volterra Model

The non-linear ODE model is defined as:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x - \beta xy \quad \text{(prey: growth minus predation)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \delta xy - \gamma y \quad \text{(predator: growth from eating minus death)}$$

Where: - x or $x(t)$ denotes prey population - y or $y(t)$ denotes predator population - α represents prey birth rate - β represents predation rate - δ represents predator birth rate (from consuming prey) - γ represents predator death rate

Equilibria: Find points which satisfy $\dot{x} = 0$ and $\dot{y} = 0$ simultaneously.

- From $\dot{x} = 0$: $x(\alpha - \beta y) = 0 \Rightarrow x = 0$ or $y = \alpha/\beta$.
- From $\dot{y} = 0$: $y(\delta x - \gamma) = 0 \Rightarrow y = 0$ or $x = \gamma/\delta$.

Combining these gives two equilibria:

- $(0, 0)$: extinction of predator and prey (saddle)
- $(\gamma/\delta, \alpha/\beta)$: coexistence of predator and prey (center)

The **nullclines** are are curves where one derivative equals zero:

- x-nullclines satisfy $\dot{x} = 0$ i.e. where prey do not change: $x = 0$ and $y = \alpha/\beta$
- y-nullclines satisfy $\dot{y} = 0$ where predators do not change i.e. : $y = 0$ and $x = \gamma/\delta$
- The nullclines divide the plane into four regions. The signs of $(\dot{x}, \dot{y})$ in each of the four regions are summerized in the table below.

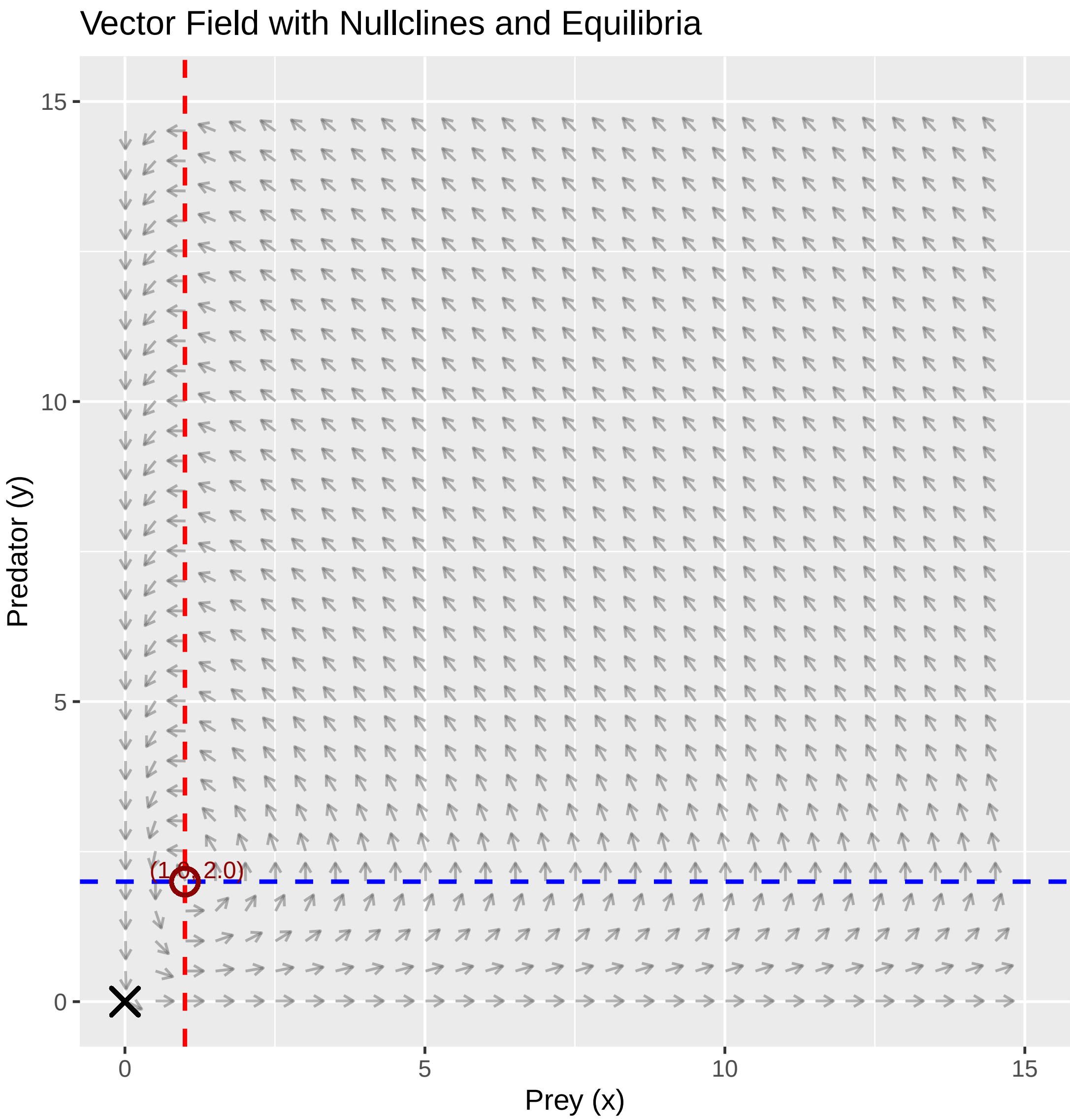
Region	Condition	$(\dot{x}, \dot{y})$	Arrow Direction
I	$x > \gamma/\delta$, $y > \alpha/\beta$	$(-, +)$	Up-left
II	$x < \gamma/\delta$, $y > \alpha/\beta$	$(-, -)$	Down-left
III	$x < \gamma/\delta$, $y < \alpha/\beta$	$(+, -)$	Down-right
IV	$x > \gamma/\delta$, $y < \alpha/\beta$	$(+, +)$	Up-right

We now define the Lotka–Volterra right-hand side for use with `deSolve::ode`.

Note that each point (x, y) on the phase plane represents a state of the system i.e. a specific combination of prey and predator populations at some moment in time.

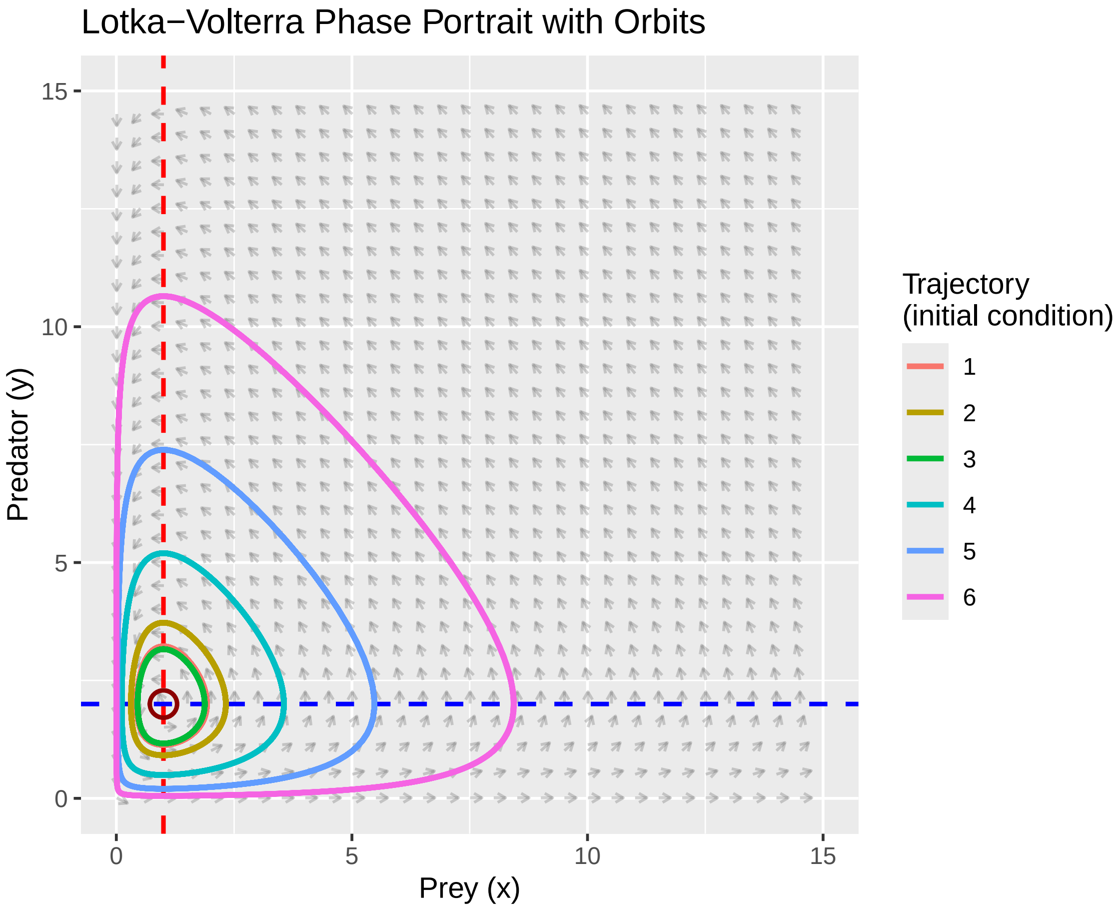
To construct the phase portrait, we first compute the vector field at each point (x, y), i.e. we compute $(\dot{x}, \dot{y})$, and draw a small arrow in that direction.

Overlay the analytically derived nullclines and the equilibrium points on top of the vector field.



A **trajectory** (orbit) is the path traced by $(x(t), y(t))$ for a given initial condition. To visualize these orbits, we numerically integrate the system from several starting points.

The closed orbits around the center represent **periodic predator–prey cycles**, with amplitudes depending on the initial condition.



Trajectory Type	Has Loop?	Behavior
Interior $(x > 0, y > 0)$	Yes	Closed periodic orbits around interior equilibrium
Positive x-axis $(y = 0, x > 0)$	No	Moves rightward to $+\infty$
Positive y-axis $(x = 0, y > 0)$	No	Moves downward to origin
Origin	No	Fixed point (equilibrium)