Quantitative Tauberian theorems

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EECS Theory Seminar

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These slides are available at
`https://t-powell.github.io/talks`

Structure of talk

This will be a talk of two halves:

1. Short introduction to applied proof theory.
2. Outline of new area of application (with some unpublished results).

Please feel free to interrupt and ask questions!

Applied Proof Theory: A 30 Minute Introduction

What is applied proof theory?

There is a famous quote due to G. Kreisel (*A Survey of Proof Theory II*):

> *"What more do we know when we know that a theorem can be proved by limited means than if we merely know that it is true?"*

In other words, the **proof** of a theorem gives us much more information than the mere **truth** of that theorem.

Applied proof theory is a branch of logic that uses proof theoretic techniques to exploit this phenomenon.

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Everyone does applied proof theory

PROBLEM. Give me an upper bound on the *n*th prime number p_n.

1. What is p_n? I know it exists because of Euclid…
2. Specifically, given $p_1, \ldots, p_{n-1}$, I know that $N := p_1 \cdot \ldots \cdot p_{n-1} + 1$ contains a *new* prime factor q, and so $p_n \leq q \leq N$.
3. In other words, the sequence $\{p_n\}$ satisfies

$$p_n \leq p_1 \cdot \ldots \cdot p_{n-1} + 1 \leq (p_{n-1})^{n-1}$$

4. By induction, it follows that e.g. $p_n < 2^{2^n}$.

This is a simple example of applied proof theory in action! From the **proof** that there are infinitely many primes, we have inferred a **bound** on the *n*th prime.

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… but it's not always that simple

Theorem (Littlewood 1914)

The functions of integers

(a) $\psi(x) - x$, *and*

(b) $\pi(x) - li(x)$

change signs infinitely often, where $\pi(x)$ *is the number of prime* $\leq x$, $\psi(x)$ *is the is logarithm of the l.c.m. of numbers* $\leq x$ *and* $li(x) = \int_0^x (1/\log(u))du$.

The original proof is utterly nonconstructive, using among other things a **case distinction on the Riemann hypothesis**. At the time, no numerical value of x for which $\pi(x) > li(x)$ was known.

In 1952, Kreisel analysed this proof and extracted recursive bounds for sign changes (*On the interpretation of non-finitist proofs, Part II*):

> *"Concerning the bound … note that it is to be expected from our principle, since if the conclusion … holds when the Riemann hypothesis is true, it should also hold when the Riemann hypothesis is nearly true: not all zeros need lie on* $\sigma = \frac{1}{2}$, *but only those whose imaginary part lies below a certain bound … and they need not lie on the line* $\sigma = \frac{1}{2}$, *but near it"*

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What applied proof theory looks like today

Theorem (Kirk and Sims, *Bulletin of the Polish Academy of Sciences* 1999)

Suppose that C is a closed subset of a uniformly convex Banach space and $T : C \to C$ is asymptotically nonexpansive with $\mathrm{int}(\mathrm{fix}(T)) \neq \emptyset$. Then for each $x \in C$ the sequence $\{T^n x\}$ converges to a fixed point of T.

Theorem (P., *Journal of Mathematical Analysis and Applications* 2019)

Let $T : C \to C$ be a nonexpansive mapping in L_p for $2 \leq p < \infty$, and suppose that $B_r[q] \subset \mathrm{fix}(T)$ for some $q \in L_p$ and $r > 0$. Suppose that $x \in C$ and $\|x - q\| < K$, and define $x_n := T^n x$. Then for any $\varepsilon > 0$ we have

$$\forall n \geq f(\varepsilon)(\|Tx_n - x_n\| \leq \varepsilon)$$

where

$$f(\varepsilon) := \left\lceil \frac{p \cdot 2^{3p+1} \cdot K^{p+2}}{\varepsilon^p \cdot r^2} \right\rceil$$

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How does proof theory come in to play?

We obtained a bound on the nth prime from Euclid's proof without any special techniques. However, serious applications usually involve some of the following, either implicitly or explicitly:

- proof interpretations, particularly Gödel's Dialectica,
- computability and complexity in higher types,
- logical relations (particularly *majorizability*),
- formal systems and type theory.

Typically, one also needs to do some serious *mathematics* as well!

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Soft analysis, hard analysis, and the finite convergence principle

23 May, 2007 in expository, math.CA, math.CO, math.LO, opinion | Tags: finite convergence principle, hard analysis, pigeonhole principle, proof theory, Ramsey theory, soft analysis

In the field of analysis, it is common to make a distinction between "hard", "quantitative", or "finitary" analysis on one hand, and "soft", "qualitative", or "infinitary" analysis on the other. "Hard analysis" is mostly concerned with finite quantities (e.g. the cardinality of finite sets, the measure of bounded sets, the value of convergent integrals, the norm of finite-dimensional vectors, etc.) and their *quantitative* properties (in particular, upper and lower bounds). "Soft analysis", on the other hand, tends to deal with more infinitary objects (e.g. sequences, measurable sets and functions, σ-algebras, Banach spaces, etc.) and their *qualitative* properties (convergence, boundedness, integrability, completeness, compactness, etc.). To put it more symbolically, hard analysis is the mathematics of ε, N, $O()$, and $\leq$[1]; soft analysis is the mathematics of 0, ∞, $\in$, and $\to$.

At first glance, the two types of analysis look very different; they deal with different types of objects, ask different types of questions, and seem to use different techniques in their proofs. They even use[2] different axioms of mathematics; the axiom of infinity, the axiom of choice, and the Dedekind completeness axiom for the real numbers are often invoked in soft analysis, but rarely in hard analysis. (As a consequence, there are occasionally some finitary results that can be proven easily by soft analysis but are in fact *impossible* to prove via hard analysis methods; the Paris-Harrington theorem gives a famous example.) Because of all these differences, it is common for analysts to specialise in only one of the two types of analysis. For instance, as a general rule (and with notable exceptions), discrete mathematicians, computer scientists, real-variable harmonic analysts, and analytic number theorists tend to rely on "hard analysis" tools, whereas ~~functional analysts,~~ operator algebraists, abstract harmonic analysts, and ergodic theorists tend to rely on "soft analysis" tools. (PDE is an interesting intermediate case in which *both* types of analysis are popular and useful, though many practitioners of PDE still prefer to primarily use just one of the two types. Another interesting transition occurs on the interface between point-set topology, which largely uses soft analysis, and metric geometry, which largely uses hard analysis. Also, the ineffective bounds which crop up from time to time in analytic number

The correspondence principle

(emphasis mine)

> *"It is fairly well known that the results obtained by hard and soft analysis respectively can be connected to each other by various "correspondence principles" or "compactness principles". It is however my belief that the relationship between the two types of analysis is in fact much closer than just this ... "*

> *"I wish to illustrate this point here using a simple but not terribly well known result, which I shall call the "finite convergence principle" ... It is the finitary analogue of an utterly trivial infinitary result – namely, that every bounded monotone sequence converges – but sometimes, a careful analysis of a trivial result can be surprisingly revealing, as I hope to demonstrate here."*

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An even more utterly trivial infinitary result: The drinkers paradox

In any pub there is someone such that if they are drinking, then everyone is drinking

ALTERNATIVELY:

$$\exists x \in P\ (D(x) \to \forall y \in P\ D(y))$$

Proof.

Either everyone is drinking, so we can pick $x := c$ to be some canonical drinker $c \in P$ OR there is at least someone $y \in D$ *not* drinking, in which case we pick $x := y$. $\square$

In a pub with infinitely many drinkers, this becomes computationally problematic… There is no effective way of finding x.

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Let's finitise it!

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\Leftrightarrow\ &\underline{\forall f : P \to P\ \exists x \in P\ (D(x) \to D(fx))} \quad (*)
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We can now solve x in f: Either fx is drinking, so we can set $x := c$, OR fx is not drinking, in which case set $x := fc$.

Original DP: *In any pub there is a person x such that if they are drinking, then everyone is drinking*

Finitary DP: *Given a pub and any function f, there is a person $x \in \{c, fc\}$ such that if they are drinking, then person fx is drinking*

The formula $(*)$ corresponds to the classical Dialectica interpretation of the original DP! The witnesses $\{c, fc\}$ give rise to the corresponding Herbrand disjunction.

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\Leftrightarrow\ & \exists x \in P\ \forall y \in P\ (D(x) \to D(y)) \\
\Leftrightarrow\ & \neg\neg\exists x \in P\ \forall y \in P\ (D(x) \to D(y)) \\
\Leftrightarrow\ & \neg\forall x \in P\ \exists y \in P\ \neg(D(x) \to D(y)) \\
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\Leftrightarrow\ & \underline{\forall f : P \to P\ \exists x \in P\ (D(x) \to D(fx))} \quad (*)
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We can now solve x in f: Either fx is drinking, so we can set $x := c$, OR fx is not drinking, in which case set $x := fc$.

Original DP: *In any pub there is a person x such that if they are drinking, then everyone is drinking*

Finitary DP: *Given a pub and any function f, there is a person $x \in \{c, fc\}$ such that if they are drinking, then person fx is drinking*

The formula $(*)$ corresponds to the classical Dialectica interpretation of the original DP! The witnesses $\{c, fc\}$ give rise to the corresponding Herbrand disjunction.

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Tao's example

Monotone convergence principle (MCP): Let $\{x_n\}$ be an increasing sequence in $[0,1]$. Then for any $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that $|x_m - x_n| \leq \varepsilon$ for all $m, n \geq N$.

Finite convergence principle (FCP): If $\varepsilon > 0$ and $f : \mathbb{N} \to \mathbb{N}$ and

$$0 \leq x_0 \leq \ldots \leq x_M \leq 1$$

is such that M is sufficiently large depending of ε and f, then there exists $0 \leq N \leq N + f(N) \leq M$ such that $|x_m - x_n| \leq \varepsilon$ for all $N \leq m, n \leq N + f(N)$.

Two interesting observations:

1. FCP $\approx$ classical Dialectica intepretation of MCP
2. By analysing the proof of MCP we can extract a bound on M, which is $\tilde{f}^{\lfloor 1/\varepsilon \rfloor}(0)$ for $\tilde{f}(x) := x + f(x)$.

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Why we are interested in finitary theorems I

> *"So, we've now extracted a quantitative finitary equivalent of the infinitary principle that every bounded monotone sequence converges. But can we actually use this finite convergence principle for some non-trivial finitary application? The answer is a definite yes: the finite convergence principle (implicitly) underlies the famous Szemerédi regularity lemma, which is a major tool in graph theory, and also underlies some rather less well known regularity lemmas, such as the arithmetic regularity lemma of Green. More generally, this principle seems to often arise in any finitary application in which tower-exponential bounds are inevitably involved."*

Quantitative, finitary versions of mathematical principles are of interest in their own right, and play a role in mathematics **entirely independently of proof theory**.

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Why we are interested in finitary theorems II

There are lots of purely existential theorems which use infinitary principles as lemmas i.e.

$$\text{infinitary principle} \Rightarrow \exists x\, A(x)$$

On the face of it, these proofs are *nonconstructive*, and we have no way of finding x.

But there is a formal way (Dialectica interpretation) to replace the infinitary principle with its finitary counterpart.

$$\text{finitary principle} \Rightarrow \exists x \leq t\, A(x)$$

Typically, we can then use a bound for the finitary principle to compute a bound on x.

Remember Kreisel:

> *"if the conclusion … holds when the Riemann hypothesis is true, it should also hold when the Riemann hypothesis is nearly true"*

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The following is lifted directly from Kohlenbach (*Some logical metatheorems with applications in functional analysis, Trans. Amer. Math. Soc 2005*):

Corollary

Let P (resp. K) be a $\mathcal{A}^\omega$ definable Polish space (resp. compact metric space) and $B_\forall$, $C_\exists$ be as before $\forall$- resp. $\exists$-formulas. If $\mathcal{A}^\omega[X, d, W]$ proves that

$$\forall x \in P \, \forall y \in K \, \forall z^X, f : X \to X \, (f \text{ n. e. } \wedge \mathrm{Fix}(f) \neq \emptyset \wedge \forall u \in \mathbb{N} \, B_\forall \Rightarrow \exists v \in \mathbb{N} \, C_\exists),$$

then we can extract from the proof a computable functional $\Phi : \mathbb{N}^\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ (on representatives $r_x : \mathbb{N} \to \mathbb{N}$ of elements $x \in P$) such that for all $r_x \in \mathbb{N}^\mathbb{N}$, $b \in \mathbb{N}$

$$\forall y \in K \, \forall z \in X, f : X \to X \, (f \text{ n. e. } \wedge \forall u \leq \Phi(r_x, b) \, B_\forall \Rightarrow \exists v \leq \Phi(r_x, b) \, C_\exists)$$

holds in any (nonempty) hyperbolic space (X, d, W) whose metric is bounded by $b \in \mathbb{N}$.

What can we achieve with applied proof theory?

1. Computational information from proofs (including those which are at first glance completely nonconstructive).

2. Finitary formulations of infinitary principles, complete with relevant numerical data.

3. Logical metatheorems and abstract variants of proofs in the literature, which explain and generalise mathematical phenomena.

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What makes an area of mathematics amenable to proof theoretic techniques?

1. Numerical information is relevant in that area.

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Commercial Break!

https://prooftheory.blog/

The Proof Theory Blog

$\vdash \exists x\,(D(x) \to \forall y D(y))$

HOME ABOUT CONTRIBUTE RESOURCES

What proof mining is about, Part I

Posted on June 6, 2020 by Andrei Sipoş

This is the first post in a series designed to introduce a working logician to the field of proof mining (for when you'll want to know more than this hopefully gentle introduction will provide, the standard reference is the monograph of Kohlenbach). It will mainly focus on its motivations and logico-mathematical content, with historical information...

Read more

https://prooftheory.blog/

Continuity in System T

Posted on May 15, 2020 by Thomas Powell

Continuity is one of the most important concepts in higher-order computability. Informally, a functional is continuous if we only require a finite part of its input to compute a finite part of its output. Though there are a number of ways of making this precise, at type level 2, that is, for functionals $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$, continuity has a simple syntactic characterisation. We say that F is continuous if it satisfies:

$$(\forall \alpha : \mathbb{N}^{\mathbb{N}})(\exists N : \mathbb{N})(\forall \beta : \mathbb{N}^{\mathbb{N}})((\forall i < N)(\alpha_i = \beta_i) \Rightarrow F\alpha = F\beta)$$

It is a well known fact that all type 2 functionals definable in Gödel's System T are continuous in the above sense. At first glance, the reasons for this seem clear: We know that such functionals are *computable*, and therefore only ever evaluate their input α on a finite number of arguments before terminating with an output. But despite this, the literature contains a multitude of genuinely distinct proofs of this apparently simple result, and moreover, brand new proofs continue to appear! This would suggest that continuity in System T is not quite as straightforward seems...

Strong normalization and continuous models

Continuity in System T was something I used to take for granted. After all, everybody knows that System T is strongly normalizing. Therefore if F is represented by the term $t : (0 \to 0) \to 0$, and we plug in some argument α, then $t\alpha$ reduces to some numerical $\underline{n}$ in a finite number of steps. In particular, α can be queried only a finite number of times, which is the same as saying that t, or equivalently the functional F it represents, is continuous.

> *Continuity of T-definable functionals is* ***obvious*** *because $t\alpha$ reduces to a numeral in a finite*

Invertibility for propositional logic

However, one can also sometimes replace the need for (2) above by globally modifying subproofs. One successful such method is known as invertibility. For classical propositional logic, this allows us to sidestep the contraction issue altogether, e.g. with the following reduction:

$$
\text{cut}\ \dfrac{\overset{P}{\Gamma \to \Delta, A \land B} \qquad \overset{Q}{\Gamma', A \land B \to \Delta'}}{\Gamma, \Gamma' \to \Delta, \Delta'}
\quad \mapsto \quad
\text{c}\ \dfrac{\text{cut}\ \dfrac{\overset{P_A^{-1}}{\Gamma \to \Delta, A} \qquad \text{cut}\ \dfrac{\overset{P_B^{-1}}{\Gamma \to \Delta, B} \qquad \overset{Q^{-1}}{\Gamma', A, B \to \Delta'}}{\Gamma, \Gamma', A \to \Delta, \Delta'}}{\Gamma, \Gamma, \Gamma' \to \Delta, \Delta, \Delta'}}{\Gamma, \Gamma' \to \Delta, \Delta'}
$$

New Applications in Tauberian Theory

Recall Abel's theorem

Let $\{a_n\}$ be a sequence of reals, and suppose that the power series

$$F(x) := \sum_{i=0}^{\infty} a_i x^i$$

converges on $|x| < 1$. Then whenever

$$\sum_{i=0}^{\infty} a_i = s$$

it follows that

$$F(x) \to s \quad \text{as} \quad x \nearrow 1.$$

This is a classical result in real analysis called **Abel's theorem** (N.b. it also holds in the complex setting). You can use it to prove that e.g.

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{i+1} = \ln(2).$$

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Does the converse of Abel's theorem hold?

NO.

For a counterexample, define $F : (-1, 1) \to \mathbb{R}$ by

$$F(x) = \frac{1}{1+x} = \sum_{i=0}^{\infty} (-1)^i x^i$$

Then

$$F(x) \to \frac{1}{2} \quad \text{as} \quad x \nearrow 1$$

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Tauber's theorem fixes this

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Tauberian theorems

The basic structure of Tauber's theorem is:

$$\text{Let} \quad F(x) = \sum_{i=0}^{\infty} a_i x^i$$

Then if we know

(A) Something about the behaviour of $F(x)$ as $x \nearrow 1$

(B) Something about the growth of $\{a_n\}$ as $n \to \infty$

Then we can conclude

(C) Something about the convergence of $\sum_{i=0}^{\infty} a_i$.

This basic idea has been **vastly generalised** e.g. for

$$F(s) := \int_1^{\infty} a(t) t^{-s} \, dt$$

and has grown into a general area of research known as **Tauberian Theory**.

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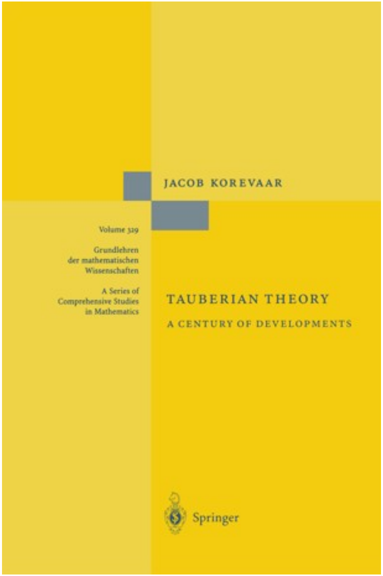
There is now a whole textbook (published 2004, 501 pages)



Tauberian theorems have an interesting structure

$$\boxed{\text{convergence}} + \boxed{\text{growth condition}} \Rightarrow \boxed{\text{convergence}}$$

Can we finitise these theorems in some way? e.g.

$$\boxed{\text{finitary convergence}} + \boxed{\text{approximate growth}} \Rightarrow \boxed{\text{finitary convergence}}$$

Can we formulate the latter in a quantitative way?

This would appear to be a **considerable challenge**, as the proofs of many Tauberian theorems are based on complicated analytic techniques.

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People have already studied quantitative aspects of Tauberian theorems

BEST L_1 APPROXIMATION AND THE REMAINDER IN LITTLEWOOD'S THEOREM [1])

BY

JACOB KOREVAAR

(Communicated by Prof. H. D. KLOOSTERMAN at the meeting of March 28, 1953)

1. *Introduction and results.* Let $f(x)$ be continuous on $a \leqslant x \leqslant b$ and satisfy a LIPSCHITZ condition of order 1:

(1. 1) $\quad |f(x_1) - f(x_2)| \leqslant A|x_1 - x_2|$ for all x_1, x_2 on $a \leqslant x \leqslant b$.

D. JACKSON [2] has shown that for such an $f(x)$ there are a constant D and a sequence of polynomials $p_n(x)$ of degree n, $n = 1, 2, \ldots$, such that

$$\max_{a \leqslant x \leqslant b} |f(x) - p_n(x)| < D/n.$$

In this paper we consider approximation to functions $f(x)$ which are continuous on $a \leqslant x \leqslant b$ except for a finite number of jump discontinuities, and which satisfy a LIPSCHITZ condition (1. 1) on each of the subintervals of $a \leqslant x \leqslant b$ on which they are continuous ("functions of class $J(a, b)$"). It follows from results by NIKOLSKY [7] that for any such function $f(x)$ there still are a constant D_1 and a sequence of polynomials $p_n(x)$ of degree n such that

$$(1.\ 2) \qquad \int_a^b |f(x) - p_n(x)|\, dx < D_1/n, \qquad (n = 1, 2, \ldots).$$

We shall prove that this sequence of polynomials $p_n(x) = \sum c_{nk} x^k$ can be chosen in such a way that moreover

$$(1.\ 3) \qquad |c_{nk}| < D_2^n \qquad (k = 0, 1, \ldots, n)$$

Cauchy variants of Abelian and Tauberian theorems

From now on, $\{a_n\}$ is a sequence of reals, $F(x) := \sum_{i=0}^{\infty} a_i x^i$ and $s_n := \sum_{i=0}^{n} a_i$.

The following are presented in P., *A note on the finitization of Abelian and Tauberian theorems*, but may well feature in some form elsewhere in the literature...

Theorem (Abel's theorem, Cauchy variant)

Suppose that

- $\{s_n\}$ *is Cauchy,*
- $\{x_m\} \in [0, 1)$ *satisfies* $\lim_{m\to\infty} x_m = 1$.

Then $\lim_{m,n\to\infty} |F(x_m) - s_n| = 0$.

Theorem (Tauber's theorem, Cauchy variant)

Suppose that

- $\{F(v_m)\}$ *is Cauchy, where* $v_m := 1 - \frac{1}{m}$,
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A quantitative analysis of Abel's theorem

Theorem (Finite Abel's theorem, P. 2020)

Let $\{a_n\}$ and $\{x_k\}$ be arbitrary sequences of reals, and $L \in \mathbb{N}$ a bound for $\{|s_n|\}$. Fix some $\varepsilon \in \mathbb{Q}_+$ and $g : \mathbb{N} \to \mathbb{N}$. Suppose that $N_1, N_2 \in \mathbb{N}$ and $p \geq 1$ are such that

$$|s_i - s_n| \leq \frac{\varepsilon}{4} \ \text{ and } \ \frac{1}{p} \leq 1 - x_m \leq \frac{\varepsilon}{8LN_1}$$

for all $i, n \in [N_1; \max\{N + g(N), l\}]$ and all $m \in [N_2; N + g(N)]$ where

$$N := \max\{N_1, N_2\} \ \text{ and } \ l := p \cdot \left\lceil \ln\left(\frac{8Lp}{\varepsilon}\right)\right\rceil$$

Then we have $|F(x_m) - s_n| \leq \varepsilon$ for all $m, n \in [N; N + g(N)]$.

Corollary

Let ϕ be a rate of Cauchy convergence for $\{s_n\}$ and ψ a rate of convergence for $x_m \nearrow 1$. Then

$$\Phi(\varepsilon) := \max\left\{\phi(\varepsilon/4), \psi\left(\frac{\varepsilon}{8L\phi(\varepsilon/4)}\right)\right\}$$

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Let $\{a_n\}$ be an arbitrary sequence of reals, and L a bound for $\{|a_n|\}$. Define $v_m := 1 - \frac{1}{m}$, and fix some $\varepsilon \in \mathbb{Q}_+$ and $g : \mathbb{N} \to \mathbb{N}$. Suppose that $N_1, N_2 \in \mathbb{N}$ are such that

$$i|a_i| \leq \frac{\varepsilon}{8} \quad \text{and} \quad |F(v_m) - F(v_n)| \leq \frac{\varepsilon}{4}$$

for all $i \in [N_1; l]$ and all $m, n \in [N_2; N + g(N)]$ where

$$N := \max\left\{\frac{2LN_1^2}{\varepsilon}, N_2\right\} \quad \text{and} \quad p \cdot \left\lceil \ln\left(\frac{4Lp}{\varepsilon}\right)\right\rceil \quad \text{for} \quad p := N + g(N)$$

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These results (and much more) appear in:

A note on the finitization of Abelian and Tauberian theorems

Thomas Powell

Abstract

We present finitary formulations of two well known results concerning infinite series, namely Abel's theorem, which establishes that if a series converges to some limit then its Abel sum converges to the same limit, and Tauber's theorem, which presents a simple condition under which the converse holds. Our approach is inspired by proof theory, and in particular Gödel's functional interpretation, which we use to establish quantitative version of both of these results.

1 Introduction

In an essay of 2007 [17] (later published as part of [18]) T. Tao discussed the so-called *correspondence principle* between 'soft' and 'hard' analysis, whereby many *infinitary* notions from analysis can be given an equivalent *finitary* formulation. An important instance of this phenomenon is provided by the simple concept of Cauchy convergence of a sequence $\{c_n\}$:

$$\forall \varepsilon > 0 \; \exists N \; \forall m, n \geq N \; (|c_m - c_n| \leq \varepsilon).$$

This corresponds to the finitary notion of $\{c_n\}$ being *metastable*, which is given by the following formula:

$$\forall \varepsilon > 0 \; \forall g : \mathbb{N} \to \mathbb{N} \; \exists N \; \forall m, n \in [N; N + g(N)] \; (|c_m - c_n| \leq \varepsilon), \tag{1}$$

where $[N; N+k] := \{N, N+1, \ldots, N+k-1, N+k\}$. Roughly speaking, a sequence $\{c_n\}$ is metastable if for any given error $\varepsilon > 0$ it contains a finite regions of stability of any 'size', where size is represented by the function $g : \mathbb{N} \to \mathbb{N}$.

The equivalence of Cauchy convergence and metastability is established via purely logical reasoning, and indeed, as was quickly observed, the correspondence principle as presented in [17] has deep connections with proof theory. More specifically, the finitary variant of an infinitary statement is typically closely related to its *classical Dialectica interpretation* [1], which provides a general method for obtaining quantitative versions of mathematical theorems.

Finitary formulations of infinitary properties play a central role in the *proof mining* program developed by U. Kohlenbach from the early 90s [7]. Here, it is often the case that a given mathematical theorem has, in general, no computable realizer (for Cauchy convergence this is demonstrated by the existence of so-called *Specker sequences* [16], which will be discussed further in Section 3). On the other hand, the corresponding finitary formulation can typically not only be realized, but a realizer can be directly extracted from a proof that the original property holds. The extraction of a computable bound $\Omega(\varepsilon, g)$ on N in (1) – a so-called *rate*

Tauber's theorem was first extended by Littlewood (1911)

THE CONVERSE OF ABEL'S THEOREM ON POWER SERIES

By J. E. LITTLEWOOD.

[Received September 28th, 1910.—Read November 10th, 1910.—
Revised December, 1910.*]

Introduction.

Abel's theorem states that if $\sum_0^\infty a_n$ is convergent, then $\lim \sum_0^\infty a_n x^n$ exists as $x \to 1$ by real values, and is equal to Σa_n. The converse theorem, however, that the existence of $\lim_{x\to 1} \Sigma a_n x^n$ implies the convergence of Σa_n, is very far from being true; for example, either the Cesàro or the Borel summability of Σa_n suffices for the existence of Abel's limit. It is known, however, that the existence of this limit, *combined with certain conditions satisfied by the a's*, does imply the convergence of Σa_n. Three such sets of conditions, for example, are:

(*a*)† the a's are all positive;

(*b*) the order of a_n has a certain upper limit;

(*c*)‡ the function $\Sigma a_n x^n$ is regular at the point $x = 1$ and $a_n \to 0$.

In the present paper we are concerned with the problems arising out of case (*b*), where the only additional restriction on the a's is an upper limit to the order of a_n. The theorem of this case is due to M. Tauber.§ The result is remarkable and apparently paradoxical in view of Abel's theorem, for it may be expressed roughly by saying that if Σa_n is not

Littlewood Tauberian theorem

Let $\{a_n\}$ be a sequence of reals, and suppose that the power series

$$F(x) := \sum_{i=0}^{\infty} a_i x^i$$

converges on $|x| < 1$. Then whenever

$$F(x) \to s \quad \text{as} \quad x \nearrow 1 \quad \text{AND} \quad |na_n| \leq C$$

for some constant C, it follows that

$$\sum_{i=0}^{\infty} a_i = s$$

One of Littlewood's first major results. In *A Mathematical Education* he writes (of this period)

> *"On looking back this time seems to me to mark my arrival at a reasonably assured judgement and taste, the end of my "education". I soon began my 35-year collaboration with Hardy."*

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One of first papers of this collaboration (1914):

TAUBERIAN THEOREMS CONCERNING POWER SERIES AND DIRICHLET'S SERIES WHOSE COEFFICIENTS ARE POSITIVE*

By G. H. HARDY *and* J. E. LITTLEWOOD.

[Received October 3rd, 1913.—Read November 13th, 1913.]

1. The general nature of the theorems contained in this paper resembles that of the "Tauberian" theorems which we have proved in a series of recent papers.† They have, however, a character of their own, in that they are concerned primarily with series of positive terms.

Let
$$f(x) = \Sigma a_n x^n$$
be a power series convergent for $|x| < 1$. We shall consider only positive values of x less than 1.

Let
$$s_n = a_0 + a_1 + \ldots + a_n,$$
$$L(u) = (\log u)^{a_1} (\log \log u)^{a_2} \ldots,$$
where the a's are real. Then it is known that, *if*
$$s_n \sim A n^a L(n),$$
where $A \neq 0$, *as* $n \to \infty$, *the indices* $a, a_1, a_2, \ldots$ *being such that* $n^a L(n)$ *tends to a positive limit or to infinity, then*

The Hardy-Littlewood Tauberian theorem

Let $\{a_n\}$ be a sequence of reals, and suppose that $\sum_{i=0}^{\infty} a_i x^i$ converges for $|x| < 1$. Then whenever

$$(1-x)\sum_{i=0}^{\infty} a_i x^i \to s \quad \text{as} \quad x \nearrow 1 \;\; \text{AND} \;\; a_n \geq -C$$

for some constant C, it follows that

$$\frac{1}{n}\sum_{i=0}^{n} a_i \to s \quad \text{as} \quad n \to \infty$$

They later used this result to give a new proof of the *prime number theorem*:

$$\pi(x) \sim \frac{x}{\ln(x)}$$

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A quantitative analysis of the Littlewood Tauberian theorem

Theorem (Finite Littlewood theorem, P. unpublished)

Suppose that $\{a_n\}$ *satisfies* $n|a_n| \leq C$ *and* L *is a bound for* $|F(x)|$ *on* $(0,1)$. *Then there are constants* K_1 *and* K_2 *such that whenever* N_1 *satisfies*

$$\left| \sum_{k=0}^{\infty} a_k (e^{-ik/m} - e^{-jk/n}) \right| \leq \frac{\varepsilon}{4K_2^{C/\varepsilon}}$$

for all $(i,m),(j,n) \in [1;d] \times [dN_1; dN + g(dN)]$, *where*

$$N := \left\lceil \frac{4LK_2^{C/\varepsilon}}{\varepsilon} \right\rceil \cdot N_1 \quad \text{and} \quad d := \frac{K_1 C}{\varepsilon}$$

then we have $\left| \sum_{k=0}^{\infty} a_k e^{-k/m} - \sum_{k=0}^{n-1} a_k \right| \leq \varepsilon$ *for all* $m,n \in [N; N+g(N)]$.

On rates of convergence

Corollary

Suppose that $\{a_n\}$ *satisfies* $n|a_n| \leq C$ *and* L *is a bound for* $|F(x)|$ *on* $(0,1)$. *Let* $\phi : (0, \varepsilon) \to \mathbb{N}$ *be a rate of Cauchy convergence for* $F(x)$ *in the sense that*

$$\forall \varepsilon > 0 \; \forall x \in [e^{-1/\phi(\varepsilon)}, 1) \; (|F(x) - F(y)| \leq \varepsilon).$$

Then a rate of Cauchy convergence $\psi : (0, \infty) \to \mathbb{N}$ *for* $\{s_n\}$ *i.e.*

$$\forall \varepsilon > 0 \; \forall n \geq \psi(\varepsilon) \; (|s_m - s_n| \leq \varepsilon)$$

is given by

$$\psi_{C,L}(\varepsilon) := Lu \cdot \phi\left(\frac{1}{u}\right) \text{ for } u := \left\lceil \frac{D^{C/\varepsilon}}{\varepsilon} \right\rceil$$

for a suitable constant D.

A special case

Suppose that

$$|F(x) - s| \leq K(1-x)$$

for some constant K.

Then the correponding rate of Cauchy convergence for $F(x) \to s$ is

$$\phi(\varepsilon) = \left\lceil \frac{2K}{\varepsilon} \right\rceil$$

Plugging this in to our corollary yields the following rate of Cauchy convergence for the partial sums $\{s_n\}$:

$$\psi(\varepsilon) := LD_1^{C/\varepsilon}$$

for suitable D_1.

Rearranging, we can show that

$$|s_N - s| \leq \frac{C_1}{\ln(N)}$$

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How does this compare to known remainder theorems?

The above results are used to obtain a best possible estimate of the remainder in LITTLEWOOD's TAUBERian theorem [5] for power series. Let

$$|na_n| < K_1, \qquad (n = 1, 2, \ldots), \tag{1.5}$$

and let $\sum a_n x^n \to s$ as $x \uparrow 1$. Then LITTLEWOOD's theorem asserts that $s_n = a_0 + a_1 + \ldots + a_n \to s$ as $n \to \infty$. What can we say about the order of $|s - s_n|$ if something is known about $|\sum a_n x^n - s|$ on $0 < x < 1$? To take a simple case, assume that

$$|\sum a_n x^n - s| < K_2(1 - x), \qquad (0 < x < 1). \tag{1.6}$$

Using the above approximation theory for the case $m = 0$ it is shown that (1. 5) and (1. 6) together imply that there is a constant C such that

$$|s - s_n| < C/\log(n + 2), \qquad (n = 0, 1, \ldots), \tag{1.7}$$

where C depends only on K_1 and K_2. From the theory for $m = 1$ it follows that

$$|s - (s_0 + s_1 + \ldots + s_n)/(n + 1)| < C_1/\{\log(n + 2)\}^2, \quad (n = 0, 1, \ldots), \tag{1.8}$$

$C_1 = C_1(K_1, K_2)$, etc. The estimates (1. 7) and (1. 8) etc. are best possible (see [4] and section 5). They improve earlier results by POSTNIKOV (who proved $|s - s_n| < C(\log n)^{-\frac{1}{4}}$ for $n > n_0$, see [8]) and the author [4]. Using the methods of the present paper it can be shown that for a fairly extensive class of functions $\omega(u)$ which $\downarrow 0$ as $u \downarrow 0$ the hypotheses (1. 5) and

$$|\sum a_n x^n - s| < \omega(1 - x) \qquad (0 < x < 1) \tag{1.9}$$

imply

$$|s - s_n| < C/|\log \omega(1/n)| \qquad (n > n_0). \tag{1.10}$$

Open questions

1. Can we finitize the Hardy-Littlewood theorem?
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3. Extracted numerical data matches well with known results. Can we produce new "remainder theorems" which don't have any precedent in the literature?
4. Are there abstract proof theoretic metatheorems which describe and generalise certain phenomena is Tauberian theory?

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