Some recent work in proof mining

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Minisymposium on Proof and Computation in Mathematics

DMV Annual Meeting (held online)
14 September 2020

These slides are available at
`https://t-powell.github.io/talks`

Disclaimer…

This is a presentation of work in progress.

Not all definitions, theorems etc. are in their final form.

Contractive mappings

In this talk, X is a Banach space.

Let $E \subseteq X$ and consider a mapping $T : E \to X$. Then T is contractive if $\forall x, y \in E$:

$$x \neq y \Rightarrow \|Tx - Ty\| < \|x - y\|$$

A *modulus* of contractivity for T (cf. [Kohlenbach and Oliva, 2003]) is a function $\tau : (0, \infty) \to (0, \infty)$ such that $\forall x, y \in E$:

$$\|x - y\| \geq \varepsilon \Rightarrow \|Tx - Ty\| + \tau(\varepsilon) \leq \|x - y\|$$

Contractive mappins with a modulus generalise certain notions from the literature e.g. *weakly contractive* mappings:

A mapping T is weakly contractive if there exists a continuous and strictly increasing function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ such that $\forall x, y \in E$:

$$\|Tx - Ty\| \leq \|x - y\| - \psi(\|x - y\|)$$

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Convergence theorems for weakly contractive mappings

The convergence properties of weakly contractive mappings are widely studied, and explicit rates of convergence are typically provided. For example

cf. Theorem 3.1 of [Alber and Guerre-Delabriere, 1997]

Let $T : E \to X$ be a weakly contractive mapping w.r.t ψ, with a fixed point $q \in E$. Let the sequence $\{x_n\}$ satisfy $x_{n+1} = Tx_n$. Then $\|x_n - q\| \to 0$, and moreover we have the estimate

$$\|x_n - q\| \leq \Psi^{-1}(\Psi(\|x_0 - q\|) - (n-1))$$

where Ψ denotes the function

$$\Psi(s) = \int^s \frac{dt}{\psi(t)}$$

In this project, we are interested in generalisations of weakly contractive mappings for which

- convergence to a fixpoint is harder to prove
- rates of convergence are not given/known.

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Example: Total asymptotically weakly contractive mappings

The following definition is due to [Alber et al., 2006].

A mapping $T : E \to X$ is *total asymptotically weakly contractive* if there exist nonnegative sequences of reals $\{k_n^{(1)}\}$ and $\{k_n^{(2)}\}$ with $k_n^{(1)}, k_n^{(2)} \to 0$ together with a pair of continuous and strictly increasing functions $\psi, \phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = \psi(0) = 0$ such that $\forall x, y \in E$:

$$\|T^n x - T^n y\| \leq \|x - y\| + k_n^{(1)} \phi(\|x - y\|) - \psi(\|x - y\|) + k_n^{(2)}$$

NOTE. In the case that $k_n^{(1)} = k_n^{(2)} = 0$ we reobtain the class of ordinary weakly contractive mappings.

Cf. Theorem 4.1 of [Alber et al., 2006]

Let $T : E \to X$ be a total asymptotically weakly contractive mapping with some fixed point $q \in E$. Let the sequence $\{x_n\}$ satisfy

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n$$

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An abstract formulation of asymptotic contractivity

We say that a sequence $\{A_n\}$ of mappings with $A_n : E \to X$ is asymptotically contractive with moduli $\tau : (0, \infty) \to (0, \infty)$ and $\sigma : (0, \infty) \to \mathbb{N}$ if for all $x, y \in E$ and any $\varepsilon, \delta, K > 0$ we have

$$\|x - y\| \in [\varepsilon, K] \wedge n \geq \sigma(\delta, K) \Rightarrow \|A_n x - A_n y\| + \tau(\varepsilon) \leq \|x - y\| + \delta$$

Example. If $T : E \to X$ is total asymptotically weakly contractive w.r.t. $\{k_n^{(1)}\}, \{k_n^{(2)}\}$, ϕ and ψ and f, g are rates of convergence for $k_n^{(1)} \to 0$, $k_n^{(2)} \to 0$ respectively, then $\{T^n\}$ is asymptotically contractive in the above sense with

- $\tau(\varepsilon) := \psi(\varepsilon)$
- $\sigma(\delta, K) := \max\left\{f\left(\frac{\delta}{2\phi(K)}\right), g\left(\frac{\delta}{2}\right)\right\}$

However, our abstract formulation covers a number of other notions in the literature.

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How to establish convergence of asymptotically contractive mappings

Take the following simple version of asymptotic contractivity:

$$\|T^n x - T^n y\| \leq \|x - y\| - \psi(\|x - y\|) + k_n$$

where ψ satisfies the usual properties and $k_n \to 0$. Suppose that $Tq = q$ and that

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for $\sum \alpha_n = \infty$. We now prove that $\|x_n - q\| \to 0$.

STEP 1. We observe that

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STEP 2. Prove more generally that whenever some sequence $\{\mu_n\}$ of nonnegative reals satisfies the recurrence

$$\mu_{n+1} \leq \mu_n - \alpha_n \psi(\mu_n) + \alpha_n k_n$$

where $k_n \to 0$, then also $\mu_n \to 0$.

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A key combinatorial lemma

The following is similar to Lemma 3.4 of [Kohlenbach and Powell, 2020], but there are nevertheless a number of important differences.

Theorem

Let $\{\mu_n\}$ be a sequence of nonnegative reals bounded above by L. Suppose that $\{\alpha_n\}$ is a sequence in $(0, \alpha]$ and $r : \mathbb{N} \times (0, \infty) \to \mathbb{N}$ a function satisfying

$$\sum_{n=k}^{r(k,x)} \alpha_n > x \quad \text{for all } k \in \mathbb{N}, \;\; x \in (0, \infty).$$

Finally, let $\psi : [0, \infty) \to [0, \infty)$ be a monotone function with $\psi(0) = 0$ and $\psi(t) > 0$ for $t > 0$ and suppose there exists some $N : (0, \infty) \to \mathbb{N}$ such that for any $\delta > 0$ we have

$$\mu_{n+1} \leq \mu_n - \alpha_n \psi(\mu_n) + \alpha_n \delta$$

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Then $\mu_n \to 0$ with rate of convergence

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A key combinatorial lemma

The following is similar to Lemma 3.4 of [Kohlenbach and Powell, 2020], but there are nevertheless a number of important differences.

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Let $\{\mu_n\}$ be a sequence of nonnegative reals bounded above by L. Suppose that $\{\alpha_n\}$ is a sequence in $(0, \alpha]$ and $r : \mathbb{N} \times (0, \infty) \to \mathbb{N}$ a function satisfying

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Consider the simple case where

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for all $n \in \mathbb{N}$, which is all that is usually required for ordinary weakly contractive mappings.
A rate of convergence in this case is just

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Let us now try to find some function $f : \mathbb{N} \to (0, \infty)$ such that $n \in \mathbb{N}$ we have

$$\Phi(f(n)) = n \quad \text{and thus} \quad \mu_n \leq f(n).$$

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Reformulating the rate of convergence

Assuming that $r(k,x)$ returns the least n with $\sum_{i=k}^{n} \alpha_i > x$, then $\Phi(f(n)) = n$ implies that

$$\sum_{i=0}^{n-1} \alpha_i \leq 2\int_{f(n)/2}^{\mu_0} \frac{dt}{\psi(t)} < \sum_{i=0}^{n} \alpha_i$$

and so in particular

$$\frac{1}{2}\sum_{i=0}^{n-1} \alpha_i \leq \Psi(\mu_0) - \Psi(f(n)/2)$$

for $\Psi(s) := \int^{s} \frac{dt}{\psi(t)}$.

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$$f(n) \leq 2\Psi^{-1}\left(\Psi(\mu_0) - \frac{1}{2}\sum_{i=0}^{n-1} \alpha_i\right)$$

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A direct comparison with known bounds

In other words, in the case where

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Compare with:

cf. Theorem 3.1 of [Alber and Guerre-Delabriere, 1997]

Let $T : E \to X$ be a weakly contractive mapping w.r.t ψ, with a fixed point $q \in E$. Let the sequence $\{x_n\}$ satisfy $x_{n+1} = Tx_n$. Then $\|x_n - q\| \to 0$, and moreover we have the estimate

$$\|x_n - q\| \leq \Psi^{-1}(\Psi(\|x_0 - q\|) - (n-1))$$

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A new quantitative result: Example 1 (apologies in advance...)

Theorem

Suppose that the sequence of mappings $\{A_n\} : E \to X$ is asymptotically contractive with moduli $\tau : (0, \infty) \to (0, \infty)$ and $\sigma : (0, \infty) \to \mathbb{N}$. Let $\{x_n\}$ be some sequence satisfying

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where $\{\alpha\}$ is a sequence of reals in $(0, \alpha]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ with rate of divergence r. Finally, suppose that $q \in E$ satisfies

$$||A_n q - q|| \to 0$$

with rate of convergence g, and that $\{||x_n - q||\}$ is bounded above by L.

Then $||x_n - q|| \to 0$ with rate of convergence

$$\Phi_{\tau,\sigma,\alpha,r,g,L}(\varepsilon) := r\left(\max\{\sigma(\delta, K), g(\delta)\}, \frac{2L}{\tau(\varepsilon/2)}\right)$$

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Further results

There are many related notions of asymptotic contractivity which can be approached in a similar way.

Let X be a uniformly smooth space, and $J : X \to X^*$ the single valued normalized duality mapping satisfying

$$\langle x, Jx\rangle = \|x\|^2 = \|Jx\|^2$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing operation.

We say that a sequence $\{A_n\}$ of mappings with $A_n : E \to X$ is asymptotically d-contractive with moduli $\tau : (0,\infty) \to (0,\infty)$ and $\sigma : (0,\infty) \to \mathbb{N}$ if for all $x, y \in E$ and any $\varepsilon, \delta, K > 0$ we have

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This generalises the notion of a d-weakly contractive mapping, studied in e.g. [Chidume et al., 2002].

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Let X be uniformly smooth, and suppose that the sequence of mappings $\{A_n\} : E \to X$ is asymptotically d-contractive with moduli $\tau : (0,\infty) \to (0,\infty)$ and $\sigma : (0,\infty) \to \mathbb{N}$. Let $\{x_n\}$ be some sequence satisfying

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n$$

where $\{\alpha\}$ is a sequence of reals in $(0,\alpha]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ with rate of divergence r and $\alpha_n \to 0$ with rate of convergence f. Finally, suppose that $q \in E$ satisfies

$$\|A_n q - q\| \to 0$$

with rate of convergence g, and that $\{\|x_n - q\|\}$, $\|q\|$ and $\{\|x_n - A_n x_n\|\}$ are all bounded above by L.
Then $\|x_n - q\| \to 0$ with rate of convergence

$$\Phi(\varepsilon) := r\left(\max\left\{\sigma\left(\frac{\delta}{6}, K\right), f\left(\frac{\delta}{6}\right), g\left(\frac{\omega_X((2+\alpha)L, \delta/6L)}{L}\right)\right\}, \frac{2L}{\tau(\varepsilon/\sqrt{2})}\right)$$

where δ is given by

$$\delta := \frac{1}{2}\min\left\{\tau\left(\frac{\varepsilon}{\sqrt{2}}\right), \frac{\varepsilon^2}{\alpha}\right\}$$

and ω_X is a modulus of uniform continuity on bounded subsets for J.

Conclusion

- There are many interesting classes of weakly contractive mappings, with corresponding convergence results.
- In simple cases, explicit rates of convergence are given.
- We consider abstract formulations of mappings of weakly contractive type, and provide general rates of convergence (which in simple cases match up with those in the literature).
- This is work in progress - by the end we will hopefully have several metatheorems which in each case generalise and unify a class of existing convergence theorems.

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